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Wave scattering by many small bodies

by

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Abstract

Wave scattering by many ($M = M(a)$) small bodies, at the boundary of which an interface boundary condition is imposed, is studied.

Smallness of the bodies means that $ka << 1$, where a is the characteristic dimension of the body and $k = \frac{2\pi}{\lambda}$ is the wave number in the medium in which small bodies are embedded.

Equation for the effective field is derived in the limit as $a \to 0$, $M(a) \to \infty$, at a suitable rate.

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1 Introduction

There is a large literature on "homogenization", which deals with the properties of the medium in which other materials is distributed. Quite often it is assumed that the medium is periodic, and homogenization is considered in the framework of G-convergence ([1],[2]). In most cases, one considers elliptic or parabolic problems with elliptic operators positive-difinite and having discrete spectrum.

The author has developed a theory of wave scattering by many small particles embedded in an inhomogeneous medium ([5]-[10]). One of the pratically important consequences of his theory was a derivation of the equation for the effective (self-consistent) field in the limiting medium, obtained in the limit $a \to 0$, $M = M(a) \to \infty$, where a is the characteristic size of a small particle, and $M(a)$ is the total number of the embedded particles.

The theory was developed for boundary conditions (bc) on the surfaces of small bodies, which include the Dirichlet bc, $u|_{S_m} = 0$, where S_m is the surface

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of the m-th particle D_m, the impedance bc, $\zeta_m u|_{S_m} = u_N|_{S_m}$, where N is the unit normal to S_m, pointing out of D_m, ζ_m is the boundary impedance, and the Neumann bc, $u_N|_{S_m} = 0$.

In this paper, we develop similar theory for the interface bc:

$$\rho_m u_N^+ = u_N^-, \quad u^+ = u^- \quad \text{on } S_m, 1 \le m \le M. \tag{1}$$

Here ρ_m is a constant, +(-) denotes the limit of $\frac{\partial u}{\partial N}$, from inside (outside) of D_m. Our approach is completely different from the approach developed in homogenization theory. Our results are of interest also in the case when the number of scatterers is not large, so the homogenization theory is not applicable.

Let us formulate the scattering problem we are treating.

$$\text{Let } \Omega := \bigcup_{m=1}^{M} D_m, \quad \Omega' = \mathbb{R}^3 \backslash \Omega,$$

$$(\nabla^2 + k^2)u = 0 \quad \text{in } \Omega', \tag{2}$$

$$(\nabla^2 + k_m^2)u = 0 \quad \text{in } D_m, \quad 1 \le m \le M, \tag{3}$$

$$u = u_0 + v, u_0 = e^{ik\alpha\cdot x}, \quad \alpha \in S^2, S^2 \text{ is a unit sphere in } \mathbb{R}^3, \tag{4}$$

$$r\left(\frac{\partial v}{\partial r} - ikv\right) = o(1), \quad r \to \infty. \tag{5}$$

We assume that ρ_m and k_m^2 are positive constants, and the surfaces S_m are smooth. A sufficient smoothness condition is $S_m \in C^{1,\mu}$, $\mu \in (0,1)$, where S_m in local coordinates is given by a continuously differentiable function whose first derivatives are Hölder-continuous with exponent μ.

We assume that $x_m \in D_m$ is a point inside D_m, $a = \frac{1}{2}\text{diam} D_m$, $d = O(a^{\frac{1}{3}})$ is the distance between the neighboring particles, $\mathcal{N}(\Delta) = \sum_{x_m \in \Delta} 1$, is the number of particles in an arbitrary open set Δ, the domains D_m are not intersecting, and

$$\mathcal{N}(\Delta) = \frac{1}{V}\int_{\Delta} N(x)dx[1 + o(1)], \quad a \to 0, \tag{6}$$

where $N(x) \ge 0$ is a function which is at our disposal, V is the volume of one small body, $V = O(a^3)$. If D_m are balls of radius a, then $V = \frac{4\pi a^3}{3}$.

It is proved in [3] that problem (1)-(5) has a unique solution.

We study wave scattering by a single small body in Section 2. In other words, we study in Section 2 problem (1)-(5) with $M = 1$. The basic results of this Section are formulated in Theorem 1.

In section 3 wave scattering by many small bodies is considered. The basic results of this Section are formulated in Theorem 2. We always assume that

$$ka << 1, \qquad d = O(a^{\frac{1}{3}}). \tag{7}$$

2 Wave scattering by one small body

Let us look for the solution to problem (1)-(5) with $M = 1$ of the form

$$u(x) = u_0(x) + \int_S g(x,t)\sigma(t)dt + \varkappa \int_D g(x,y)u(y)dy, \tag{8}$$

where $S = S_1$, $D = D_1$,

$$\varkappa := k_1^2 - k^2, \qquad g(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{9}$$

and $\sigma(t)$ is to be found so that conditions (1) are satisfied. For any $\sigma \in C^{0,\mu_1}$, $\mu_1 \in (0,1)$, the solution to (8) satisfies equations (2), (3) with $m = 1$, (4) and (5). This is easily checked by a direct calculation. The second condition (1) is also satisfied. To satisfy the first condition (1), with $\rho_1 = \rho$, one has to satisfy the following equation

$$(\rho - 1)u_{0_N} + \rho\frac{A\sigma + \sigma}{2} - \frac{A\sigma - \sigma}{2} + (\rho - 1)\frac{\partial}{\partial N_s}Bu = 0, \tag{10}$$

where

$$A\sigma = 2\int_S \frac{\partial g(s,t)}{\partial N_S}\sigma(t)dt, \quad Bu = \varkappa \int_D g(x,y)u(y)dy, \tag{11}$$

and we have used the well-known formulas for the limiting values of the normal derivatives of the single-layer potential $T\sigma := \int_S g(x,t)\sigma(t)dt$ on S from inside and outside D.

Let us rewrite (10) as

$$\sigma = \lambda A\sigma + 2\lambda B_1 u + 2\lambda u_{0_N}, \tag{12}$$

where

$$\lambda = \frac{1-\rho}{1+\rho}, \quad B_1 u = \varkappa \frac{\partial}{\partial N_s}\int_D g(x,y)u(y)dy \tag{13}$$

Let us now use the first assumption (7). One has:

$$g(s,t) = g_0(s,t)(1 + O(ka)), \quad a \to 0; \quad g_0(s,t) = \frac{1}{4\pi|s-t|}, \tag{14}$$

$$\frac{\partial}{\partial N_s}\frac{e^{ik|s-t|}}{4\pi|s-t|} = \frac{\partial g_0}{\partial N_s}(1 + O((ka)^2)), \quad a \to 0, \tag{15}$$

$$\text{so } A = A_0(1 + O((ka)^2)), \quad a \to 0; A_0 := A|_{k=0}, \tag{16}$$

$$B = B_0(1 + O(ka)), \quad B_0 u = \varkappa \int_D g_0(x,y)u(y)dy, \tag{17}$$

$$B_1 u = \varkappa \int_D \frac{\partial g_0(s,y)}{\partial N}u(y)dy(1 + O(k^2a^2)) := \varkappa B_{10}u(1 + O(k^2a^2)). \tag{18}$$

It follows from (8) that

$$u(x) = u_0(x) + \frac{e^{ik|x-x_1|}}{|x-x_1|}\left(\frac{1}{4\pi}\int_S e^{-ik\beta\cdot t}\sigma(t)dt + \frac{\varkappa}{4\pi}u_1V_1\right), \quad |x-x_1| >> a, \tag{19}$$

where V_1 is the volume of $D = D_1$, $V_1 = vol(D_1) := |D_1|$, $u_1 := u(x_1)$, $x_1 = 0$ is the origin, $\beta := \frac{x-x_1}{|x-x_1|}$.
We did not keep the factor $e^{-ik\beta\cdot x}$ in the integral over D because $e^{-ik\beta\cdot x} = 1 + O(ka)$, and

$$\int_D e^{-ik\beta\cdot y}u(y)dy = u_1V_1(1+O(ka)), \quad a \to 0. \tag{20}$$

However, it will be proved that this factor under the surface integral can not be dropped because

$$\int_S e^{-ik\beta\cdot t}\sigma(t)dt = \int_S \sigma(t)dt - ik\beta_p\int_S t_p\sigma(t)dt + O(a^4), \tag{21}$$

where over the repeated indices here and throughout this paper summation is understood, and the second integral in the right-hand side of (21) is $O(a^3)$, as $a \to 0$, i.e., it is of the same order of smallness as the the first integral $Q := \int_S \sigma(t)dt$. The last statement will be proved later.
With the notations

$$Q := \int_S \sigma(t)dt, \quad Q_1 := \int_S e^{-ik\beta\cdot t}\sigma(t)dt, \tag{22}$$

the expression

$$A(\beta,\alpha) := \frac{Q_1}{4\pi} + \frac{\varkappa}{4\pi}u_1V_1, \quad V_1 := V := |D|, \tag{23}$$

is the scattering amplitude, α is the unit vector in the direction of the incident wave $u_0 = e^{ik\alpha\cdot x}$, β is the unit vector in the direction of the scattered wave.
Let us prove that

$$-ik\beta_p\int_S t_p\sigma(t)dt = O(a^3) \tag{24}$$

and therefore, the second integral in the right-hand side of (21) cannot be dropped.
It follows from (8) that

$$u(x) \sim u_0(x) + g(x,x_1)Q_1 + \varkappa g(x,x_1)u(x_1)V_1, \quad |x-x_1| \geq d \gg a, \tag{25}$$

where $\sim$ means asymptotic equivalence as $a \to 0$.
Formula (25) can be used for calculating $u(x)$ if two quantities Q_1 and $u_1 :=$

$u(x_1)$ are found.
Let us derive asymptotic formulas for these quantities as $a \to 0$. Integrate equation (12) over S and get

$$Q = 2\lambda \int_S u_{0_N} ds + \lambda \int_S A\sigma dt + 2\lambda \int_S B_1 u ds, \tag{26}$$

Use formulas (14)-(18), the following formula (see [4], p.96):

$$\int_S A_0 \sigma ds = -\int_S \sigma ds, \tag{27}$$

and the Divergence theorem, to rewrite (26) as

$$Q = 2\lambda \int_D \nabla^2 u_0 dx - \lambda Q + 2\lambda\varkappa \int_D dx \nabla_x^2 \int_D g(x,y)u(y)dy. \tag{28}$$

Since

$$\nabla^2 u_0 = -k^2 u_0; \quad \nabla_x^2 g(x,y) = -k^2 g(x,y) - \delta(x-y), \tag{29}$$

equation (28) takes the form

$$(1+\lambda)Q = 2\lambda \nabla^2 u_0(x_1) V_1 - 2\lambda k^2 \varkappa \int_D dx \int_D g(x,y)u dy - 2\lambda\varkappa \int_D u(x)dx \tag{30}$$

Let us use the following estimates:

$$\int_D u(x)dx = u_1 V_1 (1+o(1)), \quad a \to 0; \quad u_1 := u(x_1), \tag{31}$$

$$\int_D dx \int_D g(x,y)u(y)dy = \int_D dy u(y) \int_D dx g(x,y) = O(a^5), \tag{32}$$

$$\int_D g(x,y)dx = O(a^2), \quad \forall y \in D. \tag{33}$$

From (30)-(33) it follows that

$$Q \sim \frac{2\lambda}{1+\lambda} V_1 \nabla^2 u_{01} - \frac{2\lambda\varkappa}{1+\lambda} V_1 u_1, \quad a \to 0, \tag{34}$$

where

$$\nabla^2 u_{01} = \nabla^2 u_0(x)|_{x=x_1}. \tag{35}$$

Let us now integrate equation (8) over D and use estimate (31) to obtain

$$u_1 V_1 = u_{01} V_1 + \int_S dt \sigma(t) \int_D g(x,t)dx + \varkappa \int_D dy u(y) \int_D g(x,y)dx. \tag{36}$$

If D is a ball of radius a, then one can easily check that

$$\int_D g(x,t)dx \sim \int_D g_0(x,t)dx = \frac{a^2}{3}, \quad |t| = a, \quad a \to 0. \tag{37}$$

In general,

$$\int_D g(x,y)dx = O(a^2), \quad y \in D, \quad a \to 0. \tag{38}$$

If D is a ball of radius a, then equations (36)-(38) imply

$$u_1 = u_{01} + Q\frac{a^2}{3\frac{4\pi a^3}{3}} + \varkappa u_1 O(a^2), \quad a \to 0. \tag{39}$$

Consequently,

$$u_1 \sim u_{01} + O(a^2), \quad a \to 0, \tag{40}$$

because $Q = O(a^3)$.

Indeed, from (34) and (40) one gets

$$Q \sim V_1(1-\rho)[\nabla^2 u_{01} - \varkappa u_{01}], \tag{41}$$

where we took into account that

$$\frac{2\lambda}{1+\lambda} = 1 - \rho, \tag{42}$$

the relation $u_1 \sim u_{01}$ as $a \to 0$, see (40), and neglected the terms of higher order of smallness. It follows from (41) that

$$Q = O(a^3). \tag{43}$$

From (40) and (41) one obtains

$$u_1 \sim u_{01}, \quad a \to 0. \tag{44}$$

Let us now estimate Q_1. One has

$$Q_1 = \int_S \sigma(t)dt - ik\beta_p \int_S t_p\sigma(t)dt, \tag{45}$$

up to terms of higher order of smallness as $a \to 0$, and summation is understood over the repeated indices. It turns out that the integral

$$I := \int_S t_p\sigma(t)dt \tag{46}$$

is of the same order, $O(a^3)$, as $Q = \int_S \sigma(t)dt$.
Let us check that the integral

$$J := \int_S dt t_p \frac{\partial}{\partial N} \int_D g(t,y)u(y)dy = O(a^4)$$

as $a \to 0$, and, therefore, can be neglected when one estimates I. Indeed, $u = O(1)$, $\int_D \frac{\partial}{\partial N} g(t,y)dy = O(a)$, and $\int_S t_p dt = O(a^3)$. Thus, $J = O(a^4)$.

Define the function σ_q, $q = 1,2,3$, as the unique solution to the equation

$$\sigma_q = \lambda A \sigma_q - 2\lambda N_q. \tag{47}$$

Since $\lambda = (1-\rho)/(1+\rho)$, and $\rho > 0$, one concludes that $\lambda \in (-1,1)$, and it is known (see, e.g., [4]) that the operator A is compact in $L^2(S)$ and does not have characteristic values in the interval $(-1,1)$. This and the Fredholm alternative imply that equation (47) has a solution and this solution is unique.

Note that $\int_S \sigma_q(t)dt = O(a^3)$. To prove this, integrate equation (47) over S, take into account formula (27), the relation $(A - A_0)\sigma_q = O(a^3)$, and obtain

$$(1+\lambda)\int_S \sigma_q(t)dt = -2\lambda \int_S N_q dt + O(a^3) = O(a^3),$$

because $\int_S N_q dt = 0$ by the Divergence theorem.

Define the matrix

$$\beta_{pq} := \beta_{pq}(\lambda) := V_1^{-1} \int_S t_p \sigma_q(t)dt, \qquad p,q = 1,2,3. \tag{48}$$

This matrix is similar to the matrix β_{pq} defined in [4], p. 62, by a similar formula with $\lambda = 1$. In this case β_{pq} is the magnetic polarizability tensor of a superconductor D placed in a homogeneous magnetic field directed along the unit Cartesian coordinate vector e_q (see [4], p. 62). In [4] analytic formulas are given for calculating β_{pq} with an arbitrary accuracy.

One may neglect the term $B_1 u$ in equation (12) because this term is $O(a^4)$, take into account definition (48), and get

$$\int_S t_p \sigma(t)dt = -\beta_{pq}\frac{\partial u_0}{\partial x_q} V, \tag{49}$$

where $V := V_1$, and summation is done over q. Consequently, one can rewrite (45) as

$$Q_1 = (1-\rho)V_1[\nabla^2(u_0(x_1) - \varkappa u_0(x_1)] + ik\beta_{pq}\frac{\partial u_0}{\partial x_q}\beta_p V_1, \quad \beta := \frac{x - x_1}{|x - x_1|}, \tag{50}$$

and $(x)_p := x \cdot e_p$ is the $p-$th Cartesian coordinate of vector x.

Formula (19) can be written as

$$u(x) = u_0(x)+g(x,x_1)\Big((1-\rho)[\nabla^2 u_0(x_1)-\varkappa u_0(x_1)]+ik\beta_{pq}\frac{\partial u_0(x_1)}{\partial x_q}\beta_p+\varkappa u_0(x_1)\Big)V_1, \tag{51}$$

where summation is understood over repeated indices, and $|x - x_1| >> a$.

Formulas (41),(43),(44) are valid for small D of arbitrary shape. Let us formulate the results of this Section in a theorem.

Theorem 1. *Assume that* $ka \ll 1$. *The scattering problem* (1)-(5) *has a unique solution. This solution has the form* (8) *and can be calculated by formula* (51) *in the region* $|x - x_1| >> a$ *up to the terms of order* $O(a^4)$ *as* $a \to 0$, *where* $a = 0.5 diam D$, $\varkappa = k_1^2 - k^2$, $V_1 = vol D$, $\beta = \frac{x-x_1}{|x-x_1|}$, *and* β_{pq} *is defined in* (48).

3 Wave scattering by many small bodies

Assume for simplicity that the distribution of small bodies is given by formula (6), and that there are $M = M(a)$ non-intersecting small bodies D_m of size a. For simplicity we assume that D_m is a ball of radius a, centered at x_m. There is an essential novel feature in the theory, compared with the problems investigated in [5],[6], [9], where the scattered field was much larger, as $a \to 0$. For example, for the impedance boundary condition, $u_N = \zeta u$ on S, the scattered field is $O(a^2)$, and for the Dirichlet boundary condition, $u = 0$ on S, the scattered field is $O(a)$.

For the Neumann boundary condition the scattered field is $O(a^3)$. We have the same order of smallness of the scattered field, $O(a^3)$, in the problem we study, because $V_1 = O(a^3)$. The basic role in this section is played by formula (51). We assume that the distance d between neighboring bodies (particles) is much larger than a, $d >> a$. This assumption effectively means that the function $N(x)$ in (6) has to be small, $N(x) << 1$. Indeed, if on a segment of unit length there are small particles placed at a distance d between neighboring particles, then there are $O(\frac{1}{d})$ particles on this unit segment, and $O(\frac{1}{d^3})$ in a unit cube C_1. Since $V = O(a^3)$, by formula (6) one gets

$$\frac{1}{O(a^3)}\int_{C_1} N(x)dx = O(\frac{1}{d^3}).$$

Therefore $d >> a$ can hold only if $(\int_{C_1} N(x)dx)^{\frac{1}{3}} << 1$.

Let us look for the (unique) solution to problem (1)-(5) with $1 \le m \le M = M(a)$ of the form

$$u(x) = u_0(x) + \sum_{m=1}^{M}\int_{S_m} g(x,t)\sigma_m(t)dt + \sum_{m=1}^{M}\varkappa_m\int_{D_m} g(x,y)u(y)dy. \tag{52}$$

Keeping the main terms in equation , as $a \to 0$, one gets (53) as

$$u(x) = u_0(x) + \sum_{m=1}^{M} g(x, x_m) \left(Q_m - ik \frac{(x - x_m)_p}{|x - x_m|} \int_{S_m} t_p \sigma_m(t) dt \right) +$$
$$+ \sum_{m=1}^{M} \varkappa_m g(x, x_m) u_e(x_m) V_m, \qquad Q_m := \int_{S_m} \sigma_m(t) dt, \quad a \to 0, \quad (53)$$

where we have used formula (51) for the scattered field by every small particle replacing u_0 by the effective field u_e, acting on every particle, and taking into account that $\beta := \beta_m := \frac{x - x_m}{|x - x_m|}$. By $(x - x_m)_p$ the p-th component of vector $(x - x_m)$ is denoted.

The effective (self-consisting) field u_e, acting on j-th particle, is defined as:

$$u_e(x) = u_0(x) + \sum_{m=1, m\neq j}^{M} g(x, x_m) \big((1 - \rho_m)[\nabla^2 u_e(x_m) - \varkappa_m u_e(x_m)] +$$
$$ik\beta_{pq}^{(m)} \frac{\partial u_e}{\partial x_q} \frac{(x - x_m)_p}{|x - x_m|}\Big) V_m + \sum_{m=1, m\neq j}^{M} \varkappa_m g(x, x_m) u_e(x_m) V_m, \quad |x - x_j| \sim a$$
$$(54)$$

Setting $x = x_j$ in (54) one gets a linear algebraic system for the unknowns $u_j := u_e(x_j), 1 \leq j \leq M$, and $\frac{\partial u_e(x_j)}{\partial x_p}$. Differentiating (54) with respect to $x_p, p = 1, 2, 3$, and then setting $x = x_j$, one obtains a complete set of linear algebraic systems for the $4M$ unknowns u_j and $\frac{\partial u_e(x_j)}{\partial x_p}$, $1 \leq j \leq M$, $1 \leq p \leq 3$. This linear algebraic system one gets if one solves by a collocation method the following integral equation

$$u(x) = u_0(x) + \int_D g(x, y) \Big[(1 - \rho)(\nabla^2 - K^2(y) + k^2) u(y) +$$
$$ik\beta_{pq}(y, \lambda) \frac{\partial u(y)}{\partial y_q} \frac{(x - y)_p}{|x - y|} + (K^2(y) - k^2) u(y)\Big] N(y) dy. \quad (55)$$

Equation (55) is a non-local integrodifferential equation for the limiting effective field in the medium in which many small bodies are embedded. In the derivation of this equation from equation (54) we assume that $\rho_m = \rho$ does not depend on m, took into account that $\varkappa_m^2$ becomes in the limit $K^2(y) - k^2$, and denoted by $K^2(y)$ a continuous function in D such that $K^2(x_m) = k_m^2$. As $a \to 0$ the function $K^2(y)$ is uniquely defined because the set $\{x_m\}_{m=1}^{M(a)}$ becomes dense in D as $a \to 0$. The function $\beta_{pq}(y, \lambda)$ is defined as

$$\beta_{pq}(y, \lambda) = \lim_{a \to 0} \frac{\sum_{x_m \in \Delta_p} \beta_{pq}^{(m)}}{\mathcal{N}(\Delta_p)},$$

where $y = y_p \in \Delta_p$.

To derive (55) from (54) we argue as follows. Consider a partition of D into a union centered at the points y_p of P non-intersecting cubes Δ_p, of size $b(a)$, $b(a) >> d$, so that each cube contains many small bodies, $lim_{a\to 0} b(a) = 0$. Write each sum in (54) as follows (we do it for the first sum, for example):

$$\sum_{m\neq j} g(x, x_m)(1-\rho_m)[\nabla^2 u_e(x_m) - \kappa_m u_e(x_m)]V_m$$
$$= \sum_{p=1}^{P} g(x, y_p)(1-\rho_p)[\nabla^2 u_e(y_p) - \kappa_p u_e(y_p)]V_m \sum_{x_m\in\Delta_p} 1$$
$$= \sum_{p=1}^{P} g(x, y_p)(1-\rho_p)[\nabla^2 u_e(y_p) - \kappa_p u_e(y_p)]N(y_p)|\Delta_p|(1+o(1)), \tag{56}$$

where we have used formula (6), took into account that diam $\Delta_p \to 0$ as $a \to 0$, wrote formula (6) as follows:

$$V \sum_{x_m\in\Delta_p} 1 = V\mathcal{N}(\Delta_p) = N(y_p)|\Delta_p|(1+o(1)), \quad a\to 0, \tag{57}$$

and used the Riemann integrability of the functions involved, which holds, for example, if these functions are continuous. By ρ_p we denote the value $\rho(y_p)$, where $\rho(y)$ is a continuous function.

The sum in (56) is the Riemann sum for the integral

$$\int_D g(x,y)(1-\rho(y))[\nabla^2 u(y) - K^2(y)u(y) + k^2 u(y)]N(y)dy. \tag{58}$$

Similarly one treats the other sums in (56).

Let us formulate the results of this Section as a theorem.

Theorem 2. *Assume that* (6) *and* (7) *hold. Then, as $a \to 0$, the effective field, defined by* (54)*, has a limit $u(x)$ which solves equation* (55).

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