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by

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Abstract

An equation $\dot{u} = A(t)u + B(t)F(t, u(t-\tau))$, $u(t) = v(t)$, $-\tau \le t \le 0$ is considered, $A(t)$ and $B(t)$ are linear operators in a Hilbert space H, $\dot{u} = \frac{du}{dt}$, $F : H \to H$ is a non-linear operator, $\tau > 0$ is a constant. Under some assumption on $A(t), B(t)$ and $F(t, u)$ sufficient condittions are given for the solution $u(t)$ to exist globally, i.e, for all $t \ge 0$, to be globally bounded, and to tend to zero as $t \to \infty$.

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1 Introduction

Consider an abstract evolution problem

$$\dot{u} = A(t)u + B(t)F(t, u(t-\tau)), \tag{1}$$
$$u(t) = v(t), \quad -\tau \le t \le 0 \tag{2}$$

where $u(t) \in H$, H is a Hilbert space, $A(t)$ and $B(t)$ are linear operators in H, $F(t, u)$ is a nonlinear operator in H, $\tau > 0$ is a constant.

Let us assume that $A(t)$ is closed densely defined operator, $D(A(t)) = D(A)$, $D(A)$ is the domain of $A(t)$, independent of t,

$$\text{Re}(A(t)u, u) \le -\gamma(t)(u, u), \tag{3}$$
$$||B(t)|| \le b(t), \tag{4}$$
$$||F(t, u)|| \le \alpha(t, g), \qquad g := ||u(t)||. \tag{5}$$

We assume that problem (1)-(2) has a unique local solution. Sufficient conditions for this one can find in the literature, see, e.g., [1].

We assume that the function $\alpha(t,g) \geq 0$ satisfies a local Lipschitz condition with respect to g and is continuous with respect to t on $[-\tau,\infty)$, functions $b(t)$ and $\gamma(t)$ are continuous on $[-\tau,\infty)$.

Our aim is to give sufficient conditions for global existence, global boundedness, and stability of the solution to problem (1)-(2).

There is a large literature on functional differential equations, see [1]-[4], and references therein. The method we propose is new. A version of this method was used in a study of the Dynamical Systems Method (DSM) for solving operator equations, see [5]-[7].

Our approach is as follows: multiply equation (1) by $u(t)$ in H and take real part to get

$$\mathrm{Re}(\dot{u},u) = \mathrm{Re}(A(t)u(t),u(t)) + \mathrm{Re}(B(t)F(t,u(t-\tau),u). \tag{6}$$

Let $g(t) := ||u(t)||$. Then equation (6) yields an inequality

$$g\dot{g} \leq -\gamma(t)g^2 + b(t)\alpha(t,g(t-\tau))g. \tag{7}$$

Since $g(t) \geq 0$, inequality (7) implies

$$\dot{g}(t) \leq -\gamma(t)g(t) + b(t)\alpha(t,g(t-\tau)), \qquad g(t) := ||v(t)||, \qquad \tau \leq t \leq 0. \tag{8}$$

Indeed, at the points at which $g(t) > 0$, inequality (7) is equivalent to (8) and $\dot{g}(t) = \mathrm{Re}(\dot{u}, \frac{u(t)}{||u(t)||})$.

If $g(t) = 0$ on an open interval, $t \in (a,b)$, then $\dot{g}(t) = 0$, $t \in (a,b)$, and inequality (8) holds since $b(t) \geq 0$ and $\alpha(t,g) \geq 0$.

If $g(s) = 0$ but in any neighborhood $(s-\delta,s)\cup(s,s+\delta)$, $g(t) \neq 0$, provided that $\delta > 0$ is sufficiently small, then by $\dot{g}(s)$ we understand derivative from the right:

$$\dot{g}(s) = \lim_{h\to+0} g(s+h)h^{-1} = ||\dot{u}(s)||. \tag{9}$$

Inequality (8) then follows from (7) by continuity as $t \to s+0$.

The following lemma is a key to our results.

Lemma 1. *If there exists a function $\mu(t) > 0$, defined for all $t \geq 0$, such that*

$$b(t)\alpha\left(t, \frac{1}{\mu(t-\tau)}\right)\mu(t) \leq \gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}, \tag{10}$$

and

$$\mu(0)g(0) \leq 1, \tag{11}$$

then any solution $g(t) \geq 0$ to inequality (8) *satisfies the following inequality:*

$$0 \leq g(t) \leq \frac{1}{\mu(t)}, \qquad \forall t \geq 0. \tag{12}$$

Remark 1. *Since $\mu(t)$ is defined on all of $\mathbb{R}_+ = [0, \infty)$, inequality* (12) *implies that $g(t) \geq 0$ is defined on all R_+. Moreover, if $\lim_{t\to\infty} \mu(t) = +\infty$, then $\lim_{t\to\infty} g(t) = 0$ In section 2, we show how to choose $\mu(t)$ and to use Lemma 1 in order to obtain estimates for the solution to problem* (1)-(2).

Proof of Lemma 1. Let us write inequality (8) as

$$\dot{g}(t) \leq l(g) := -\gamma(t)g(t) + b(t)\alpha(t, g(t-\tau)). \tag{13}$$

Then inequalities (10)-(11) can be written as

$$l\left(\frac{1}{\mu(t)}\right) \leq \frac{d\mu^{-1}(t)}{dt}, \qquad \mu^{-1}(0) \geq g(0). \tag{14}$$

Let w_n solve the problem

$$\dot{w}_n = l(w_n) - \frac{1}{n}, \quad w_n(0) = g(0) = v(0),\ w_n(t) = v(t), -\tau \leq t \leq 0, \tag{15}$$

$n = 1, 2, ...$

Let us prove that

$$w_n(t) \leq \mu^{-1}(t), \qquad \forall t \geq 0. \tag{16}$$

Since $\lim_{n\to\infty} w_n = w$, where

$$\dot{w} = l(w), \qquad w(t) = v(t), \qquad -\tau \leq t \leq 0, \tag{17}$$

it follows from (16) and (17) that

$$w(t) \leq \mu^{-1}(t), \qquad \forall t \geq 0. \tag{18}$$

To prove (16), note that if $w_n(0) < \mu^{-1}(0)$, then there exists an interval $(0, t_1)$, $t_1 > 0$, such that $w_n(t) < \mu^{-1}(t)$ when $t \in [0, t_1)$. If $w_n(0) = \mu^{-1}(0)$, then inequality (14) and equation (15) imply that

$$w_n(0) = \mu^{-1}(0), \qquad \dot{w}_n(0) < \left.\frac{d\mu^{-1}(t)}{dt}\right|_{t=0}.$$

Therefore, in this case there exists a number $t_1 > 0$ such that on the interval $(0, t_1)$ one has

$$w_n(t) < \mu^{-1}(t), \qquad 0 < t < t_1. \tag{19}$$

Let us prove that $t_1 = \infty$ in both cases. Assume the contrary. Then at some point $s < \infty$, one has $w_n(s) = \mu^{-1}(s)$ and

$$w_n(t) \leq \mu^{-1}(t), \qquad \text{for } t < s. \tag{20}$$

At the point s the following inequalities hold:

$$\dot{w}_n(s) = l(w_n(s)) - \frac{1}{n} < l(\mu^{-1}(s)) \leq \left. \frac{d\mu^{-1}(t)}{dt} \right|_{t=s}. \tag{21}$$

By continuity, one has

$$\dot{w}_n(t) \leq \frac{d\mu^{-1}(t)}{dt}, \qquad s - \delta \leq t \leq s, \tag{22}$$

for a sufficiently small δ.

Integrate (22) on the interval $[s-\delta, s]$ and get

$$w_n(s) - w_n(s-\delta) < \mu^{-1}(s) - \mu^{-1}(s-\delta). \tag{23}$$

Since $w_n(s) = \mu^{-1}(s)$, inequality (23) implies

$$\mu^{-1}(s-\delta) < w_n(s-\delta). \tag{24}$$

This is a contradiction, and it proves that $t_1 = \infty$. Consequently,

$$w_n(t) < \mu^{-1}(t), \qquad \forall t > 0. \tag{25}$$

Passing to the limit $n \to \infty$ in (25), one gets (18).

A similar argument proves that

$$g(t) \leq w(t), \qquad \forall t \geq 0. \tag{26}$$

Combining inequalities (18) and (26), one obtains (12).

Lemma 1 is proved. □

2 Estimates of solutions to evolution problem

Let us apply Lemma 1 to the solution of problem (1) - (2).

In order to choose $\mu(t)$, let us assume that

$$\gamma(t) = \gamma = const > 0, \qquad b(t) \leq \frac{\gamma}{2}, \qquad \alpha(t,g) \leq c_0 g^p, \tag{27}$$

where $c_0 > 0$ and $p > 1$ are constants, and $b(t) \geq 0$, $\alpha(t,g) \geq 0$.

Let us choose

$$\mu(t) = \lambda e^{\nu t},$$

where λ and ν are positive constants.

Inequalities (10) and (11) hold if

$$\frac{\gamma}{2} c_0 \lambda^{-(p-1)} e^{-p\nu(t-\tau)+\nu t} \le \gamma - \nu, \tag{28}$$

and

$$\lambda g(0) \le 1. \tag{29}$$

Choose

$$\lambda = \frac{1}{g(0)}.$$

Then inequality (29) holds. Choose $\nu = \frac{\gamma}{2}$. Then inequality (28) holds if

$$c_0 g^{p-1}(0) e^{p\nu\tau} \le 1. \tag{30}$$

Inequality (30) holds if c_0 is sufficiently small, or if $g(0)$ is sufficiently small.

We have proved the following theorem.

Theorem 1. *Assume that* (3) *holds with* $\gamma(t) = \gamma = const > 0$, (4) *holds with* $b(t) \le \frac{\gamma}{2}$, (27) *and* (30) *hold. Then the solution to problem* (1)-(2) *satisfies inequality*

$$||u(t)|| \le g^{p-1}(0) e^{-\gamma t/2}, \qquad \forall t \ge 0. \tag{31}$$

Estimate (31) of Theorem 1 implies *exponential stability* of the solution to problem (1)-(2).

Consider now the case when $\gamma(t)$ tends to zero as $t \to \infty$.

Assume that

$$\gamma(t) = \frac{c_1}{(1+t)^{m_1}}, \quad b(t) \le \frac{c_2}{(1+t)^{m_2}}, \quad \alpha(t,g) \le \frac{c_3}{(1+t)^{m_3}} g^p, \tag{32}$$

where $c_j, m_j > 0$, $j = 1,2,3$, and $p > 1$ are constants.

Choose $\mu(t)$ of the form

$$\mu(t) = \lambda(1+t+\tau)^{\nu}, \qquad \lambda, \nu > 0, \tag{33}$$

where λ and ν are positive constants.

Inequalities (10) and (11) hold if

$$\frac{c_2}{(1+t)^{m_2}} \frac{c_3}{(1+t)^{m_3}} \frac{1}{\lambda^{p-1}(1+t)^{(p-1)\nu}} \le \frac{c_1}{(1+t)^{m_1}} - \frac{\nu}{1+t}, \tag{34}$$

$$\lambda g(0) \le 1. \tag{35}$$

Inequality (35) holds if $\lambda = \frac{1}{g(0)}$.

Assume that

$$m_2 + m_3 + (p-1)\nu \geq 1, \qquad m_1 \leq 1. \tag{36}$$

Then inequality (34) holds for all $t \geq 0$ provided that

$$c_2 c_3 g^{p-1}(0) \leq c_1 - \nu. \tag{37}$$

Inequality (37) holds if $\nu < c_1$ and c_2c_3 is sufficiently small. If these conditions are satisfied then, by Lemma 1, one gets

$$||u(t)|| \leq \frac{||u(0)||}{(1+t+\tau)^\nu}, \qquad \forall t \geq 0. \tag{38}$$

We have proved the folowing theorem

Theorem 2. *Assume that* (32) *and* (36) *hold,* $\lambda = \frac{1}{g(0)}$, $\nu < c_1$, *and* c_2c_3 *is sufficiently small so that* (37) *holds. Then the solution to problem* (1)-(2) *exists for all* $t \geq 0$, *and estimate* (38) *holds.*

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