Stability of solutions to abstract evolution equations with delay

by

Alexander Ramm

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A.G. Ramm
Department of Mathematics
Kansas State University, Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract
An equation \( \dot{u} = A(t)u + B(t)F(t, u(t-\tau)) \), \( u(t) = v(t), -\tau \leq t \leq 0 \) is considered, \( A(t) \) and \( B(t) \) are linear operators in a Hilbert space \( H \), \( \dot{u} = \frac{du}{dt}, F : H \to H \) is a non-linear operator, \( \tau > 0 \) is a constant. Under some assumption on \( A(t), B(t) \) and \( F(t, u) \) sufficient conditions are given for the solution \( u(t) \) to exist globally, i.e, for all \( t \geq 0 \), to be globally bounded, and to tend to zero as \( t \to \infty \).

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1 Introduction
Consider an abstract evolution problem
\[
\dot{u} = A(t)u + B(t)F(t, u(t-\tau)), \quad u(t) = v(t), \quad -\tau \leq t \leq 0
\]
where \( u(t) \in H, H \) is a Hilbert space, \( A(t) \) and \( B(t) \) are linear operators in \( H, F(t, u) \) is a nonlinear operator in \( H, \tau > 0 \) is a constant.
Let us assume that \( A(t) \) is closed densely defined operator, \( D(A(t)) = D(A), D(A) \) is the domain of \( A(t) \), independent of \( t, \)
\[
\text{Re}(A(t)u, u) \leq -\gamma(t)(u, u), \quad ||B(t)|| \leq b(t), \quad ||F(t, u)|| \leq \alpha(t, g), \quad g := ||u(t)||.
\]
We assume that problem (1)-(2) has a unique local solution. Sufficient conditions for this one can find in the literature, see, e.g., [1].

We assume that the function \( \alpha(t, g) \geq 0 \) satisfies a local Lipschitz condition with respect to \( g \) and is continuous with respect to \( t \) on \([-\tau, \infty)\), functions \( b(t) \) and \( \gamma(t) \) are continuous on \([-\tau, \infty)\).

Our aim is to give sufficient conditions for global existence, global boundedness, and stability of the solution to problem (1)-(2).

There is a large literature on functional differential equations, see [1]-[4], and references therein. The method we propose is new. A version of this method was used in a study of the Dynamical Systems Method (DSM) for solving operator equations, see [5]-[7].

Our approach is as follows: multiply equation (1) by \( u(t) \) in \( H \) and take real part to get

\[
\text{Re}(\dot{u}, u) = \text{Re}(A(t)u(t), u(t)) + \text{Re}(B(t)F(t, u(t), u(t-\tau), u)).
\]

Let \( g(t) := ||u(t)|| \). Then equation (6) yields an inequality

\[
g\dot{g} \leq -\gamma(t)g^2 + b(t)\alpha(t, g(t-\tau))g.
\]

Since \( g(t) \geq 0 \), inequality (7) implies

\[
\dot{g}(t) \leq -\gamma(t)g(t) + b(t)\alpha(t, g(t-\tau)), \quad g(t) := ||v(t)||, \quad \tau \leq t \leq 0.
\]

Indeed, at the points at which \( g(t) > 0 \), inequality (7) is equivalent to (8) and \( \dot{g}(t) = \text{Re}(\dot{u}, \frac{u(t)}{||u(t)||}) \).

If \( g(t) = 0 \) on an open interval, \( t \in (a, b) \), then \( \dot{g}(t) = 0, t \in (a, b) \), and inequality (8) holds since \( b(t) \geq 0 \) and \( \alpha(t, g) \geq 0 \).

If \( g(s) = 0 \) but in any neighborhood \((s-\delta, s) \cup (s, s+\delta), g(t) \neq 0 \), provided that \( \delta > 0 \) is sufficiently small, then by \( \dot{g}(s) \) we understand derivative from the right:

\[
\dot{g}(s) = \lim_{h \to 0^+} g(s+h)h^{-1} = ||\dot{u}(s)||.
\]

Inequality (8) then follows from (7) by continuity as \( t \to s + 0 \).

The following lemma is a key to our results.

**Lemma 1.** If there exists a function \( \mu(t) > 0 \), defined for all \( t \geq 0 \), such that

\[
b(t)\alpha \left( t, \frac{1}{\mu(t-\tau)} \right) \mu(t) \leq \gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)},
\]

and

\[
\mu(0)g(0) \leq 1,
\]

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then any solution \( g(t) \geq 0 \) to inequality (8) satisfies the following inequality:

\[
0 \leq g(t) \leq \frac{1}{\mu(t)}, \quad \forall t \geq 0.
\] (12)

**Remark 1.** Since \( \mu(t) \) is defined on all of \( \mathbb{R}_+ = [0, \infty) \), inequality (12) implies that \( g(t) \geq 0 \) is defined on all \( \mathbb{R}_+ \). Moreover, if \( \lim_{t \to \infty} \mu(t) = +\infty \), then \( \lim_{t \to \infty} g(t) = 0 \). In section 2, we show how to choose \( \mu(t) \) and to use Lemma 1 in order to obtain estimates for the solution to problem (1)-(2).

**Proof of Lemma 1.** Let us write inequality (8) as

\[
\dot{g}(t) \leq l(g) := -\gamma(t)g(t) + b(t)\alpha(t, g(t - \tau)).
\] (13)

Then inequalities (10)-(11) can be written as

\[
l \left( \frac{1}{\mu(t)} \right) \leq \frac{d\mu^{-1}(t)}{dt}, \quad \mu^{-1}(0) \geq g(0).
\] (14)

Let \( w_n \) solve the problem

\[
\dot{w}_n = l(w_n) - \frac{1}{n}, \quad w_n(0) = g(0) = v(0), \quad w_n(t) = v(t), \quad -\tau \leq t \leq 0,
\] (15)

\( n = 1, 2, \ldots \)

Let us prove that

\[
w_n(t) \leq \mu^{-1}(t), \quad \forall t \geq 0.
\] (16)

Since \( \lim_{n \to \infty} w_n = w \), where

\[
\dot{w} = l(w), \quad w(t) = v(t), \quad -\tau \leq t \leq 0,
\] (17)

it follows from (16) and (17) that

\[
w(t) \leq \mu^{-1}(t), \quad \forall t \geq 0.
\] (18)

To prove (16), note that if \( w_n(0) < \mu^{-1}(0) \), then there exists an interval \((0, t_1), \ t_1 > 0\), such that \( w_n(t) < \mu^{-1}(t) \) when \( t \in [0, t_1) \). If \( w_n(0) = \mu^{-1}(0) \), then inequality (14) and equation (15) imply that

\[
w_n(0) = \mu^{-1}(0), \quad \dot{w}_n(0) < \left. \frac{d\mu^{-1}(t)}{dt} \right|_{t=0}.
\]

Therefore, in this case there exists a number \( t_1 > 0 \) such that on the interval \((0, t_1)\) one has

\[
w_n(t) < \mu^{-1}(t), \quad 0 < t < t_1.
\] (19)
Let us prove that \( t_1 = \infty \) in both cases. Assume the contrary. Then at some point \( s < \infty \), one has \( w_n(s) = \mu^{-1}(s) \) and
\[
\dot{w}_n(t) \leq \frac{d\mu^{-1}(t)}{dt} \bigg|_{t=s}.
\] (20)

At the point \( s \) the following inequalities hold:
\[
\dot{w}_n(s) = l(w_n(s)) - \frac{1}{n} < l(\mu^{-1}(s)) \leq \frac{d\mu^{-1}(t)}{dt} \bigg|_{t=s}.
\] (21)

By continuity, one has
\[
\dot{w}_n(t) \leq \frac{d\mu^{-1}(t)}{dt}, \quad s - \delta \leq t \leq s,
\] (22)
for a sufficiently small \( \delta \).

Integrate (22) on the interval \([s-\delta, s]\) and get
\[
w_n(s) - w_n(s-\delta) < \mu^{-1}(s) - \mu^{-1}(s-\delta).
\] (23)

Since \( w_n(s) = \mu^{-1}(s) \), inequality (23) implies
\[
\mu^{-1}(s - \delta) < w_n(s - \delta).
\] (24)

This is a contradiction, and it proves that \( t_1 = \infty \). Consequently,
\[
w_n(t) < \mu^{-1}(t), \quad \forall t > 0.
\] (25)

Passing to the limit \( n \rightarrow \infty \) in (25), one gets (18).

A similar argument proves that
\[
g(t) \leq w(t), \quad \forall t \geq 0.
\] (26)

Combining inequalities (18) and (26), one obtains (12).

Lemma 1 is proved.

\[\square\]

2 Estimates of solutions to evolution problem

Let us apply Lemma 1 to the solution of problem (1) - (2).

In order to choose \( \mu(t) \), let us assume that
\[
\gamma(t) = \gamma = \text{const} > 0, \quad b(t) \leq \frac{\gamma}{2}, \quad \alpha(t, g) \leq c_0 g^p,
\] (27)
where \( c_0 > 0 \) and \( p > 1 \) are constants, and \( b(t) \geq 0, \alpha(t, g) \geq 0 \).
Let us choose
\[ \mu(t) = \lambda e^{\nu t}, \]
where \( \lambda \) and \( \nu \) are positive constants.

Inequalities (10) and (11) hold if
\[ \frac{\gamma}{2} c_0 \lambda^{-(p-1)} e^{-\nu(t-\tau)+\nu t} \leq \gamma - \nu, \] (28)
and
\[ \lambda g(0) \leq 1. \] (29)

Choose
\[ \lambda = \frac{1}{g(0)}. \]
Then inequality (29) holds. Choose \( \nu = \frac{\gamma}{2} \). Then inequality (28) holds if
\[ c_0 g^{p-1}(0)e^{\nu\tau} \leq 1. \] (30)

Inequality (30) holds if \( c_0 \) is sufficiently small, or if \( g(0) \) is sufficiently small.

We have proved the following theorem.

**Theorem 1.** Assume that (3) holds with \( \gamma(t) = \gamma = \text{const} > 0 \), (4) holds with \( b(t) \leq \frac{\gamma}{2} \), (27) and (30) hold. Then the solution to problem (1)-(2) satisfies inequality
\[ ||u(t)|| \leq g^{p-1}(0)e^{-\gamma t/2}, \quad \forall t \geq 0. \] (31)

Estimate (31) of Theorem 1 implies exponential stability of the solution to problem (1)-(2).

Consider now the case when \( \gamma(t) \) tends to zero as \( t \to \infty \).

Assume that
\[ \gamma(t) = \frac{c_1}{(1+t)^{m_1}}, \quad b(t) \leq \frac{c_2}{(1+t)^{m_2}}, \quad \alpha(t, g) \leq \frac{c_3}{(1+t)^{m_3}} g^p, \] (32)
where \( c_j, m_j > 0, j = 1, 2, 3 \), and \( p > 1 \) are constants.

Choose \( \mu(t) \) of the form
\[ \mu(t) = \lambda (1 + t + \tau)^\nu, \quad \lambda, \nu > 0, \] (33)
where \( \lambda \) and \( \nu \) are positive constants.

Inequalities (10) and (11) hold if
\[ \frac{c_2}{(1+t)^{m_2}} \frac{c_3}{(1+t)^{m_3}} \lambda^{p-1} (1+t)^{(p-1)\nu} \leq \frac{c_1}{(1+t)^{m_1}} - \frac{\nu}{1+t}, \] (34)
\[ \lambda g(0) \leq 1. \] (35)
Inequality (35) holds if \( \lambda = \frac{1}{g(0)} \).

Assume that

\[
m_2 + m_3 + (p - 1)\nu \geq 1, \quad m_1 \leq 1.
\]

Then inequality (34) holds for all \( t \geq 0 \) provided that

\[
c_2c_3g^{p-1}(0) \leq c_1 - \nu.
\]

Inequality (37) holds if \( \nu < c_1 \) and \( c_2c_3 \) is sufficiently small. If these conditions are satisfied then, by Lemma 1, one gets

\[
\|u(t)\| \leq \frac{\|u(0)\|}{(1 + t + \tau)^\nu}, \quad \forall t \geq 0.
\]

We have proved the following theorem

**Theorem 2.** Assume that (32) and (36) hold, \( \lambda = \frac{1}{g(0)} \), \( \nu < c_1 \), and \( c_2c_3 \) is sufficiently small so that (37) holds. Then the solution to problem (1)-(2) exists for all \( t \geq 0 \), and estimate (38) holds.

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