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Abstract

Consider an abstract evolution problem in a Hilbert space H

$$\dot{u} = A(t)u + G(t,u) + f(t), \quad u(0) = u_0, \tag{1}$$

where $A(t)$ is a linear, closed, densely defined operator in H with domain independent of $t \geq 0$, $G(t,u)$ is a nonlinear operator such that $||G(t,u)|| \leq a(t)||u||^p$, $p = const > 1$, $||f(t)|| \leq b(t)$. We allow the spectrum of $A(t)$ to be in the right half-plane $\mathrm{Re}(\lambda) < \lambda_0(t)$, $\lambda_0(t) > 0$, but assume that $lim_{t\to\infty}\lambda_0(t) = 0$.
Under suitable assumption on $a(t)$ and $b(t)$ we prove boundedness of $||u(t)||$ as $t \to \infty$. If $f(t) = 0$, the Lyapunov stability of the zero solution to problem (1) with $u_0 = 0$ is established. For $f \neq 0$, sufficient conditions for Lyapunov stability are given. The novel point in the paper is the possibility for the linear operator $A(t)$ to have spectrum in the half-plane $\mathrm{Re}(\lambda) < \lambda_0(t)$ with $\lambda_0(t) > 0$ and $lim_{t\to\infty}\lambda_0(t) = 0$ at a suitable rate.

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1 Introduction and the results.

The main results of this paper are formulated in Lemma 1 and Theorem 2. Lemma 1 is proved in Section 2. In Section 3 an example of applications of our result is given.

There is a large literature [1, 2, 3] on the stability of the solutions to differential equation of the form

$$\dot{u} = A(t)u + G(t,u) + f(t), \quad u(0) = u_0; \quad \dot{u} = \frac{du}{dt}, \quad t \geq 0, \tag{1}$$

where $A(t)$ is a linear, closed, densely defined in a Hilbert space H operator, with domain $D(A(t))$ independent of t, $G(t,u)$ is a nonlinear operator,

$$||G(t,u)|| \leq a(t)||u||^p, \quad p > 1, \quad t \geq 0, \tag{2}$$

$f(t)$ is a function on $\mathbb{R}_+ = [0,\infty)$ with values in H,

$$||f(t)|| \leq b(t), \quad t \geq 0, \tag{3}$$

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We assume that $a(t)$ and $b(t)$ are non-negative continuous functions, that $G(t, u)$ is locally Lipschitz with respect to u in the ball $B(u_0, R) = \{u : ||u-u_0|| \leq R\}$ and is a continuous in the operator norm function of t. We assume that problem (1) has a unique local solution.

This assumption holds, for example, if $A(t)$ is a generator of C_0 semigroup. In this case, problem (1) is equivalent to

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)G(s,u(s))ds + \int_0^t U(t,s)f ds := T(u), \tag{4}$$

where

$$\frac{\partial U(t,s)}{\partial t} = A(t)U(t,s), \quad U(s,s) = I, \quad t \geq s, \tag{5}$$

I is the identity operator, and $U(t,s)$ is a bounded operator in H:

$$U(t,s) = I + \int_s^t A(\tau)U(\tau,s)d\tau, \quad t \geq s. \tag{6}$$

Indeed, a standard calculation shows that u, defined in (4), solved problem (1):

$$\begin{aligned} \frac{d}{dt}\int_0^t U(t,s)f(s)ds &= U(t,t)f + \int_0^t \frac{\partial U(t,s)}{\partial t} f(s)ds \\ &= f(t) + \int_0^t A(t)U(t,s)f ds \\ &= f(t) + A(t)\int_0^t U(t,s)f(s)ds, \end{aligned}$$

$$\begin{aligned} &\frac{d}{dt}U(t,0)u_0 = A(t)U(t,0)u_0, \quad U(0,0)u_0 = u_0, \\ &\frac{d}{dt}\int_0^t U(t,s)G(s,u(s))ds = G(t,u(t)) + A(t)\int_0^t U(t,s)G(s,u(s))ds. \end{aligned}$$

It follows that u, defined in (4), solves problem (1). If problem (1) is equivalent to problem (4) and $U(t,s)$ are bounded operators, then the operator $T(u)$ in (4) is locally Lipschitz since $G(t,u)$ has this property. Consequently, problem (4) (and therefore problem (1)) has a unique local solution. This solution is global if it satisfies a uniform with respect to t bound

$$\sup_t ||u(t)|| \leq c, \tag{7}$$

where the supremum is taken over all $t \in [0, T)$ for which $u(t)$ exists (see, e.g., [5]).

Let us derive bound (7). This derivation is based on an application of a new nonlinear differential inequality, stated in Lemma 1, below. Our basic result, Theorem 2, is also based on this inequality.

Take inner product of (1) with $u(t)$, then take real part of the resulting equation, and denote $||u(t)|| = g(t)$. The result is:

$$g\dot{g} \leq \mathrm{Re}(A(t)u,u) + |(G(t,u),u)| + b(t)g(t). \tag{8}$$

Assume that

$$\mathrm{Re}(A(t)u,u) \le \gamma(t)g^2, \quad ||G(t,u)|| \le \alpha(t,g), \tag{9}$$

where $\gamma(t) \ge 0$ and $\alpha(t,g) \ge 0$ are continuous functions of t, and

$$\alpha(t,g) \le a(t)g^p, \quad p > 1. \tag{10}$$

Then inequality (8) implies

$$\dot{g} \le \gamma(t)g + a(t)g^p + b(t), \quad g(0) = ||u_0||. \tag{11}$$

We assume that

$$a(t) > 0, \quad \dot{a}(t) < 0, \tag{12}$$

and $\gamma(t)$ and $b(t)$ tend to zero monotonically as $t \to \infty$.

We need the following lemma, which is proved in Section 2.

Lemma 1 *If there exists a function*

$$\mu(t) > 0, \quad \dot{\mu}(t) < 0, \quad \lim_{t\to\infty} \mu(t) = d > 0, \tag{13}$$

and

$$\gamma(t) + a(t)\mu^{-p+1}(t) + b(t)\mu(t) + \frac{\dot{\mu}}{\mu} \le 0, \quad \mu(0)g(0) \le 1, \tag{14}$$

then any non-negative solution g to (11) *satisfies inequality*

$$0 \le g(t) \le \mu^{-1}(t) \le \frac{1}{d}. \tag{15}$$

Applying Lemma 1 with $\mu(t) = d + q(t)$, $d = const > 0$, $q(t) > 0$, $\dot{q}(t) < 0$, one gets from (14) the following inequalities:

$$[d + q(t)]\gamma(t) + a(t)[d + q(t)]^{2-p} + b(t)[d + q(t)]^2 \le -\dot{q}(t) = |\dot{q}(t)|, \tag{16}$$

$$[d + q(0)]g(0) \le 1. \tag{17}$$

If $g(0) \ne 0$ and $[d + q(0)] \le g^{-1}(0)$, then (17) holds. If $g(0) = 0$, then (17) holds for any d and $q(0)$. Fix d and $q(0)$ such that (17) holds. Then (16) holds if $q(t)$ is such that

$$C[\gamma(t) + a(t) + b(t)] \le |\dot{q}(t)|. \tag{18}$$

Here $C = const > 0$,

$$C = \max\{d + q(0), [d + q(0)]^2, [d + q(0)]^{2-p}, d^{2-p}\}. \tag{19}$$

We have used monotone decay of $q(t)$ and the inequalities: $[d + q(t)]^{2-p} \le [d + q(0)]^{2-p}$ if $2 \ge p$, and $[d + q(t)]^{2-p} \le d^{2-p}$ if $2 \le p$.

To satisfy inequality (18), one may choose $q(t)$ using the relation

$$-q(t) + q(0) = C\int_0^t [\gamma(s) + a(s) + b(s)]ds := CQ(t), \tag{20}$$

or

$$q(t) = q(0) - CQ(t). \tag{21}$$

If the number

$$Q_\infty := \int_0^\infty [\gamma(t) + a(t) + b(t)]dt$$

is sufficiently small, then one can choose the constant $q(0)$, so that the function $q(t)$, defined in (21), is positive for all $t \geq 0$.

Applying inequality (15), we obtain the following theorem:

Theorem 2 *Assume that $u(t)$ solves problem* (1) *and inequalities* (2),(3),(9), (10), *hold. If the function $q(t)$, defined in* (21), *with C defined in* (19), *is positive for all $t \geq 0$, then $u(t)$ exists for all $t \geq 0$, and is globally bounded:*

$$||u(t)|| \leq \frac{1}{d + q(t)} \leq \frac{1}{d}, \qquad \forall t \geq 0. \tag{22}$$

Remark. Let $\epsilon > 0$ be a fixed small number. If $g(0) = ||u(0)|| \leq \delta$, where $\delta > 0$ is a sufficiently small number, then one can choose $d = \delta^{-1}$, define $\epsilon = \delta$, and obtain:

$$||u(t)|| \leq \epsilon, \qquad \forall t \geq 0. \tag{23}$$

The statement in the above Remark shows that the solution to evolution problem (1) is Lyapunov stable.

2 Proof of Lemma 1.

Consider the problem:

$$\dot{v}_n = \gamma(t)v_n + a(t)v_n^p + f(t) + \frac{1}{n}, \quad v_n(0) = g(0), \tag{24}$$

where $n > 0$ is an integer. This problem has a unique local solution. Clearly, $\dot{v}_n > \dot{g}(0)$. Therefore, there is an interval $(0, T)$, such that

$$v_n(t) > g(t), \quad 0 < t < T, \tag{25}$$

where $T > 0$ is some number. The interval $[0, T)$ is the maximal interval of the existence of the local solution to (24), and if

$$\sup_{t \geq 0} |v_n(t)| < \infty, \tag{26}$$

then $T = \infty$.

Indeed, if inequality (26) holds but $T < \infty$, then one can solve the problem

$$\dot{w} = \gamma(t)w + a(t)w^p + f(t) + \frac{1}{n}, \quad w(T - \frac{l}{2}) = v_n(T - \frac{l}{2}), \tag{27}$$

where l is the length of the interval of the local existence of the solution to problem (27). If (26) holds, then the length l does not depend on the choice of the Cauchy data. Thus,

w exists on the interval $(T-\frac{l}{2}, T+\frac{l}{2})$ and is equal to $v_n(t)$ on the interval $(T-\frac{l}{2}, T)$. Consequently, by the uniqueness of the solution to the Cauchy problem for equation (24), $v_n(t)$ is defined on the interval $[0, T+\frac{l}{2})$, so T is not the maximal interval of the existence of v_n. This contradiction shows that $T=\infty$, as claimed, and inequality (25) holds for any $t>0$.

Passing to the limit $n \to \infty$ in (25), one obtains

$$0 \le g(t) \le v(t), \qquad \forall t \ge 0, \tag{28}$$

where

$$\dot{v} = \gamma(t)v + a(t)v^p + f(t), \quad v(0) = g(0). \tag{29}$$

Inequality (14) can be written as

$$-\frac{\dot{\mu}}{\mu^2} = \left(\frac{1}{\mu}\right)^{\cdot} \ge \frac{\gamma(t)}{\mu(t)} + \frac{a(t)}{\mu^p(t)} + b(t), \qquad \frac{1}{\mu(0)} \ge v(0), \quad t \ge 0. \tag{30}$$

By the argument similar to the given above, one obtains

$$\frac{1}{\mu(t)} \ge v(t) \ge g(t) \ge 0, \quad \forall t \ge 0. \tag{31}$$

From (31) the conclusion (15) of Lemma 1 follows, because $\frac{1}{\mu(t)} \le \frac{1}{d}$.

Lemma 1 is proved. $\Box$

A lemma, similar to Lemma 1, but with $\gamma(t) < 0$, was proved in [4] by a different argument. In [2] and [5] one can find proofs of some comparison results for ordinary differential equations. The ideas of these proofs are close to the idea of our proof of Lemma 1.

The *principally novel* result in our paper is Theorem 2, because it gives sufficient conditions for Lyapunov stability of the solution to evolution problem (1) under the assumptions which allow the linear operator $A(t)$ to have spectrum in the half-plane $\operatorname{Re} z > 0$. Such a result is possible to obtain because this spectrum tends sufficiently fast to the imaginary axis as $t \to \infty$.

3 Example

Let us illustrate our result by a simple example. As H we take $\mathbb{R}^2$, as $A(t)$ we take $m(t)I$, where I is the unit matrix, $m(t) = c_1(1+t)^{-m_1}$, $G(t,u)$ is a quadratic nonlinearity, $\|G(t,u)\| \le n(t)g^2$, so $p=2$, $g^2 = x^2(t) + y^2(t)$, vector u has two components, $u(t) := \{x(t), y(t)\}$, $n(t) = c_2(1+t)^{-m_2}$, constants $c_j, m_j > 0$ will be chosen later. Inequality (11) with $b(t) = 0$ takes the form

$$\dot{g} \le c_1(1+t)^{-m_1} g + c_2(1+t)^{-m_2} g^2, \quad \forall t \ge 0. \tag{32}$$

Choose $\mu(t) = d + c_3(1+t)^{-m_3}$, where the constants $d, c_3, m_3 > 0$ will be fixed later. Then (14) (with $b(t)=0$) takes the form:

$$\frac{c_1}{(1+t)^{m_1}} + \frac{c_2}{(1+t)^{m_2}[d + c_3(1+t)^{-m_3}]} \le \frac{m_3 c_3}{(1+t)^{m_3+1}[d + c_3(1+t)^{-m_3}]}, \quad \forall t \ge 0, \tag{33}$$

and

$$(d + c_3)g(0) \le 1. \tag{34}$$

Let us check that inequalities (33) and (34) can be satisfied if the parameters d, c_j, m_j are properly chosen. Assume that

$$m_3 + 1 \le \min\{m_1, m_2\}. \tag{35}$$

Then (33) holds if

$$c_1 + c_2 d^{-1} \le c_3 m_3 (d + c_3)^{-1}. \tag{36}$$

Inequality (34) holds if

$$d + c_3 = \frac{1}{g(0)}. \tag{37}$$

Choose $c_3 = d$. Then

$$d = \frac{1}{2g(0)}, \qquad g(0) = \|u(0)\|. \tag{38}$$

Let $g(0) \le \delta$, where $\delta > 0$ is a small number. Then $d \ge (2\delta)^{-1}$, and inequality (36) holds if

$$c_1 + 2\delta c_2 \le c_3 m_3 (2d)^{-1} \le c_3 m_3 \delta. \tag{39}$$

This inequality holds if c_1, c_2 are sufficiently small. For example, let

$$c_1 \le 0.5 c_3 m_3 \delta, \qquad c_2 \le 0.25 c_3 m_3. \tag{40}$$

Then inequalities (33) and (34) hold, and Lemma 1 yields:

$$0 \le \|u(t)\| \le [d + c_3(1+t)^{-m_3}]^{-1} \le d^{-1}, \qquad \forall t \ge 0. \tag{41}$$

This estimate obviously yields global boundedness of $\|u(t)\|$, and also Lyapunov stability. Indeed, if $\|u(0)\| \le \delta$, and $\delta > 0$ is sufficiently small, then $d \ge (2\delta)^{-1}$ can be chosen sufficntly large, and consequently, inequlity (41) yields the estimate $\|u(t)\| \le \epsilon$, $\forall t \ge 0$, where $\epsilon = 2\delta$ is arbitrary small if $\delta > 0$ is sufficiently small.

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