Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Lower Bound of Concurrence and Distillation for Arbitrary Dimensional Bipartite Quantum States

by

Ming-Jing Zhao, Xue-Na Zhu, Shao-Ming Fei, and Xianqing Li-Jost

Preprint no.: 59 2011

Lower Bound of Concurrence and Distillation for Arbitrary Dimensional Bipartite Quantum States

Ming-Jing Zhao[1], Xue-Na Zhu[2], Shao-Ming Fei[1,3], and Xianqing Li-Jost[1]
[1]*Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*
[2]*Department of Mathematics, School of Science, South China University of Technology, Guangzhou 510640, China*
[3]*School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*

We present an analytical lower bound of concurrence for arbitrary dimensional bipartite quantum states. This lower bound may be used to improve all the known lower bounds of concurrence. Moreover, the lower bound gives rise to an operational sufficient criterion of distillability of quantum entanglement. The significance of our result is illustrated by quantitative evaluation of entanglement for entangled states that fail to be identified by the usual concurrence estimation method, and by showing the distillability of mixed states that can not be recognized by other distillability criteria.

PACS numbers: 03.67.Mn, 03.65.Ud

Introduction. — Quantum entanglement is a striking feature of quantum systems and plays essential roles in some physical processes such as quantum phase transitions in various interacting quantum many-body systems. Quantum entangled states are the key physical resources in many quantum information processing. An important issue in the theory of quantum entanglement is to recognize and quantify the entanglement for a given quantum state. Concurrence is one of the most important measures of quantum entanglement [1–6]. For mixed two-qubit states, an analytical formula of concurrence has been derived [1]. For general high dimensional case, due to the extremizations involved in the computation, only a few analytic formulas of concurrence have been found for some special symmetric states [7].

To estimate the concurrence for general mixed states, efforts have been made toward the analytical lower bounds of concurrence. In Ref. [8] a lower bound of concurrence that can be tightened by numerical optimization over some parameters has been derived. In Ref. [9] analytic lower bounds of concurrence for any dimensional mixed bipartite quantum states have been presented by using the positive partial transposition (PPT) and realignment separability criteria. These bounds are exact for some special classes of states and can be used to detect many bound entangled states. In Ref. [10] another lower bound of concurrence for even dimensional bipartite states has been presented from a new separability criterion Ref. [11]. A lower bound of concurrence based on local uncertainty relations criterion is derived in Ref. [12]. This bound is further optimized in Ref. [13]. In Refs. [14, 15] the authors presented lower bounds of concurrence for bipartite systems in terms of a different approach, which has a close relationship with the distillability of bipartite quantum states. In Ref. [16] an explicit analytical lower bound of concurrence is obtained by using positive maps, which is better than the ones in Refs. [9, 10] in detecting some quantum entanglement. These bounds give rise to a good quantitative estimation of concurrence. They are also supplementary in detecting quantum entanglement. In this letter we present a tight lower bound of concurrence which may be used to improve all these existing lower bounds of concurrence. Detailed examples are given to show the improvement of the lower bounds of concurrence in Refs. [9, 15].

Due to the influence of the environment, generally maximally entangled pure states could evolve into mixed ones. To get the ideal resource for quantum information processing, a possible approach is distillation [17, 18]. Nevertheless operational necessary and sufficient criterion of distillability has not been found yet. The reduction criterion [19] and the lower bound in Ref. [15] provide sufficient conditions for distillability. It turns out that our lower bound of concurrence also gives rise to a sufficient condition of distillability that may improve the existing distillability criteria.

Lower bound of concurrence. — Let H_A and H_B be m and n dimensional vector spaces respectively. A pure bipartite state $|\psi\rangle \in H_A \otimes H_B$ has the form,

$$|\psi\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}|ij\rangle, \tag{1}$$

where $a_{ij} \in \mathbb{C}$, $\sum_{ij}|a_{ij}|^2 = 1$, $\{|ij\rangle\}$ is the basis of $H_A \otimes H_B$. The concurrence of $|\psi\rangle$ is given by

$$C(|\psi\rangle) = \sqrt{2(1 - tr\rho_A^2)}, \tag{2}$$

where $\rho_A = tr_B|\psi\rangle\langle\psi|$ is the reduced density matrix, Tr_B stands for the partial trace over the space H_B [1–6]. It can be further written as [4],

$$C(|\psi\rangle) = 2\sqrt{\sum_{i<k}^{m}\sum_{j<l}^{n}|a_{ij}a_{kl} - a_{il}a_{kj}|^2}. \tag{3}$$

The concurrence is extended to mixed state ρ by the convex roof $C(\rho) = \min\sum_i p_i C(|\psi_i\rangle)$ for all possible ensemble realizations $\rho = \sum p_i|\psi_i\rangle\langle\psi_i|$.

To evaluate $C(\rho)$, we project high dimensional states to "lower dimensional" ones. For a given bipartite

$m\otimes n$ pure state (1), we define "$s\otimes t$" (unnormalized) pure state $|\psi\rangle_{s\otimes t}=\sum_{i_1}^{i_s}\sum_{j=j_1}^{j_t}a_{ij}|ij\rangle$, $1<s<m$, $1<t<n$. We denote the concurrence of $|\psi\rangle_{s\otimes t}$ by $C(|\psi\rangle_{s\otimes t})=2\sqrt{\sum_{i_p<i_k}^{s}\sum_{j_q<j_l}^{t}|a_{i_pj_q}a_{i_kj_l}-a_{i_pj_l}a_{i_kj_q}|^2}$. There are $\binom{m-2}{s-2}\times\binom{n-2}{t-2}$ different $s\otimes t$ states $|\psi\rangle_{s\otimes t}$ for a given $|\psi\rangle$, where $\binom{m-2}{s-2}$ and $\binom{n-2}{t-2}$ are the binomial coefficients. For a mixed state ρ, we define its "$s\otimes t$" mixed states $\rho_{s\otimes t}=A\otimes B\rho A^\dagger\otimes B^\dagger$, where $A=\sum_{i_p=1}^{s}|i_p\rangle\langle i_p|$ and $B=\sum_{j_q=1}^{t}|j_q\rangle\langle j_q|$, $1<s<m$, $1<t<n$. $\rho_{s\otimes t}$ has the following form,

$$\rho_{s\otimes t}=\begin{pmatrix}
\rho_{i_1j_1,i_1j_1} & \cdots & \rho_{i_1j_1,i_1j_t} & \rho_{i_1j_1,i_2j_1} & \cdots & \rho_{i_1j_1,i_2j_t} & \cdots & \rho_{i_1j_1,i_sj_1} & \cdots & \rho_{i_1j_1,i_sj_t}\\
\cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots\\
\rho_{i_1j_t,i_1j_1} & \cdots & \rho_{i_1j_t,i_1j_t} & \rho_{i_1j_t,i_2j_1} & \cdots & \rho_{i_1j_t,i_2j_t} & \cdots & \rho_{i_1j_t,i_sj_1} & \cdots & \rho_{i_1j_t,i_sj_t}\\
\rho_{i_2j_1,i_1j_1} & \cdots & \rho_{i_2j_1,i_1j_t} & \rho_{i_2j_1,i_2j_1} & \cdots & \rho_{i_2j_1,i_2j_t} & \cdots & \rho_{i_2j_1,i_sj_1} & \cdots & \rho_{i_2j_1,i_sj_t}\\
\cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots\\
\rho_{i_2j_t,i_1j_1} & \cdots & \rho_{i_2j_t,i_1j_t} & \rho_{i_2j_t,i_2j_1} & \cdots & \rho_{i_2j_t,i_2j_t} & \cdots & \rho_{i_2j_t,i_sj_1} & \cdots & \rho_{i_2j_t,i_sj_t}\\
\cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots\\
\rho_{i_sj_1,i_1j_1} & \cdots & \rho_{i_sj_1,i_1j_t} & \rho_{i_sj_1,i_2j_1} & \cdots & \rho_{i_sj_1,i_2j_t} & \cdots & \rho_{i_sj_1,i_sj_1} & \cdots & \rho_{i_sj_1,i_sj_t}\\
\cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots\\
\rho_{i_sj_t,i_1j_1} & \cdots & \rho_{i_sj_t,i_1j_t} & \rho_{i_sj_t,i_2j_1} & \cdots & \rho_{i_sj_t,i_2j_t} & \cdots & \rho_{i_sj_t,i_sj_1} & \cdots & \rho_{i_sj_t,i_sj_t}
\end{pmatrix}, \tag{4}$$

which are unnormalized bipartite $s\otimes t$ mixed states. The concurrence of $\rho_{s\otimes t}$ is defined by $C(\rho_{s\otimes t})\equiv\min\sum_i p_iC(|\psi_i\rangle_{s\otimes t})$, minimized over all possible $s\otimes t$ pure state decompositions of $\rho_{s\otimes t}=\sum_i p_i|\psi_i\rangle_{s\otimes t}\langle\psi_i|$, with $\sum_i p_i=tr(\rho_{s\otimes t})$.

Theorem 1 *For bipartite mixed state $\rho\in H_A\otimes H_B$,*

$$C^2(\rho)\geq c_{st}\sum_{P_{st}}C^2(\rho_{s\otimes t})\equiv\tau_{s\otimes t}(\rho), \tag{5}$$

where $c_{st}=[\binom{m-2}{s-2}\times\binom{n-2}{t-2}]^{-1}$, $\sum_{P_{st}}$ stands for summing over all possible $s\otimes t$ mixed states, $\tau_{s\otimes t}(\rho)$ denotes the lower bound of $C(\rho)$ with respect to the $s\otimes t$ subspace.

Proof. It is straightforward to prove that the concurrence of a pure state $|\psi_i\rangle$ and the concurrence of $|\psi_i\rangle_{s\otimes t}$ have the following relation,

$$C^2(|\psi_i\rangle)=c_{st}\sum_{P_{st}}C^2(|\psi_i\rangle_{s\otimes t}). \tag{6}$$

Therefore for bipartite mixed state $\rho=\sum p_i|\psi_i\rangle\langle\psi_i|$, we have

$$\begin{aligned}
C(\rho)&=\min\textstyle\sum_i p_iC(|\psi_i\rangle)\\
&=\sqrt{c_{st}}\min\textstyle\sum_i p_i\left(\sum_{P_{st}}C^2(|\psi_i\rangle_{s\otimes t})\right)^{\frac{1}{2}}\\
&\geq\sqrt{c_{st}}\min\left[\textstyle\sum_{P_{st}}\left(\sum_i p_iC(|\psi_i\rangle_{s\otimes t})\right)^2\right]^{\frac{1}{2}}\\
&\geq\sqrt{c_{st}}\left[\textstyle\sum_{P_{st}}\left(\min\sum_i p_iC(|\psi_i\rangle_{s\otimes t})\right)^2\right]^{\frac{1}{2}}\\
&=\sqrt{c_{st}}\left[\textstyle\sum_{P_{st}}C^2(\rho_{s\otimes t})\right]^{\frac{1}{2}},
\end{aligned} \tag{7}$$

where relation $(\sum_j(\sum_i x_{ij})^2)^{\frac{1}{2}}\leq\sum_i(\sum_j x_{ij}^2)^{\frac{1}{2}}$ has been used in the first inequality, the first three minimizations run over all possible pure state decompositions of the mixed state ρ, while the last minimization runs over all $s\otimes t$ pure state decompositions of $\rho_{s\otimes t}=\sum_i p_i|\psi_i\rangle_{s\otimes t}\langle\psi_i|$. □

The lower bound of concurrence of ρ in Eq. (5) is given by the concurrence of sub-matrix $\rho_{s\otimes t}$. Choosing different s and t would result in different lower bounds. Generally we have

Corollary 1

$$C^2(\rho)\geq\sum_{s=2}^{m}\sum_{t=2}^{n}p_{st}\tau_{s\otimes t}(\rho), \tag{8}$$

where $0\leq p_{st}\leq 1$, $\sum_{s=2}^{m}\sum_{t=2}^{n}p_{st}=1$.

Theorem 1 not only provides a lower bound of concurrence, but also shows the hierarchy among all the concurrences of the lower dimensional bipartite mixed states $\rho_{s\otimes t}$. The concurrence of $m\otimes n$ mixed state ρ is bounded by all the concurrences of the $(m-1)\otimes(n-1)$ mixed states associated to ρ, while the concurrence of the $(m-1)\otimes(n-1)$ mixed state is bounded by that of all the $(m-2)\otimes(n-2)$ related mixed states, and so on.

Lower bound of concurrence from lower bounds. — Theorem 1 may be used to improve all the known lower bounds of concurrence. Assume $g(\rho)$ is any lower bound of concurrence, $C(\rho)\geq g(\rho)$. Then for a given mixed state ρ, the concurrence of the projected lower dimensional mixed state $\rho_{s\otimes t}$ satisfies $C(\rho_{s\otimes t})=tr(\rho_{s\otimes t})C((tr\rho_{s\otimes t})^{-1}\rho_{s\otimes t})\geq tr(\rho_{s\otimes t})g((tr\rho_{s\otimes t})^{-1}\rho_{s\otimes t})$. Subsequently $C^2(\rho)\geq c_{st}\sum_{P_{st}}C^2(\rho_{s\otimes t})\geq c_{st}\sum_{P_{st}}(tr\rho_{s\otimes t})^2g^2((tr\rho_{s\otimes t})^{-1}\rho_{s\otimes t})$. Here if one chooses $\rho_{s\otimes t}$ to be the given mixed state ρ itself, the inequality reduces to $C(\rho)\geq g(\rho)$ again. Generally the lower bound $g(\rho)$ may be improved if one takes into account all the lower dimensional mixed states $\rho_{s\otimes t}$. Let g_i, $i=1,...,k$, be a set of lower bounds of concurrence. From Theorem 1 and Corollary 1 we have

Theorem 2 *For bipartite mixed state* $\rho \in H_A \otimes H_B$,

$$C^2(\rho) \geq \sum_{s=2}^{m}\sum_{t=2}^{n}\sum_{i=1}^{k}\sum_{P_{st}} p_{st}c_{st}q_i(tr\rho_{s\otimes t})^2 g_i^2\left((tr\rho_{s\otimes t})^{-1}\rho_{s\otimes t}\right), \qquad (9)$$

with $0 \leq p_{st} \leq 1$, $\sum_{s=2}^{m}\sum_{t=2}^{n} p_{st} = 1$, $0 \leq q_i \leq 1$, $\sum_{i=1}^{k} q_i = 1$.

Therefore from any existing lower bounds of concurrence g_i, $i=1,...,k$, one can get a new lower bound (9), which satisfies $C(\rho) \geq \max(g_1(\rho),...,g_k(\rho))$.

Generally, let $|\psi\rangle = \sum_i \lambda_i|ii\rangle$ be an arbitrary pure state in Schmidt decomposition, and $f(\rho)$ a real-valued and convex function such that $f(|\psi\rangle\langle\psi|) \leq 2\sum_{i<j}\lambda_i\lambda_j$ for any pure state $|\psi\rangle$. Then f gives a lower bound of concurrence for $n\otimes n$ mixed states ρ, $C(\rho) \geq \sqrt{\frac{2}{n(n-1)}}f(\rho)$ [10]. In particular, concerning the partial transposition related lower bound, we obtain the following lower bound.

Corollary 2 *For any bipartite mixed state* $\rho \in H_A\otimes H_B$,

$$C^2(\rho) \geq \frac{2c_{tt}}{t(t-1)}\sum_{P_{tt}}(||\rho_{t\otimes t}^{T_A}|| - tr(\rho_{t\otimes t}))^2 \equiv \kappa_{t\otimes t}(\rho), \qquad (10)$$

where T_A *stands for partial transpose with respect to the Hilbert space* H_A.

Combining the result in Ref. [9], we can get another lower bound.

Corollary 3 *For any bipartite mixed state* $\rho \in H_A\otimes H_B$,

$$C^2(\rho) \geq \frac{2c_{st}}{t(t-1)}\sum_{P_{st}}(\max(||\rho_{s\otimes t}^{T_A}||,\ ||R(\rho_{s\otimes t})||) - tr(\rho_{s\otimes t}))^2 \equiv \zeta_{s\otimes t}(\rho), \qquad (11)$$

where $t < s$, $R(\rho)$ *stands for the realigned matrix of* ρ.

The lower bound $\tau_{s\otimes t}(\rho)$ is better than the lower bound in Ref. [9], since the latter is just a special case of the lower bound $\zeta_{t\otimes t}(\rho)$ with $s=m$, $t=n$, $n\leq m$ in the Corollary 3. As an example, we consider the mixed state $\varrho_0 = \frac{p}{3}(|00\rangle+|11\rangle+|22\rangle)(\langle 00|+\langle 11|+\langle 22|)+(1-p)|33\rangle\langle 33|$ with $0<p<1$. Its concurrence is $C(\varrho_0) = \frac{4}{3}p$. The lower bound in Ref. [9] gives $C(\varrho_0) \geq \frac{2}{3}p^2$. Our lower bound shows that $\tau_{3\otimes 3}(\varrho_0) = \frac{10}{9}p^2 > \frac{2}{3}p^2$.

In the following we highlight the advantages of the lower bounds $\tau_{s\otimes t}(\rho)$, $\kappa_{t\otimes t}(\rho)$ and $\zeta_{s\otimes t}(\rho)$ by detailed analysis. First, the lower bound $\tau_{s\otimes t}(\rho)$ is tight since the lower bound $\tau_{2\otimes 2}(\rho)$ provides exact value for some mixed states. Second, the lower bound $\tau_{s\otimes t}(\rho)$ is strictly stronger than the lower bound $\tau(\rho)$ in Ref. [15] because $\tau(\rho)$ is just a special case of our lower bound with $s=t=2$. Since in two-qubit case, positive partial transpose (PPT) implies separability, the lower bound $\tau_{2\otimes 2}$ of any PPT entangled state ρ is zero. Therefore in this case $\tau_{2\otimes 2}$ can not detect any PPT entanglement. However, the lower bound $\tau_{s\otimes t}$ and $\zeta_{s\otimes t}$ can detect some PPT entangled states for $t>2$.

Example 1. Let's consider the $4\otimes 4$ mixed state, $\varrho_1 = p\sigma_\alpha + (1-p)|33\rangle\langle 33|$ with $0<p<1$, $\sigma_\alpha = \frac{2}{7}|\psi^+\rangle\langle\psi^+| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-$, $\sigma_+ = \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|)$, $\sigma_- = \frac{1}{3}(|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|)$, $|\psi^+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$. The state σ_α is separable for $2\leq\alpha\leq 3$, bound (PPT) entangled for $3<\alpha\leq 4$, and free entangled for $4<\alpha\leq 5$ [20]. For PPT entangled case, $\tau_{2\otimes 2}(\varrho_1)=0$ and $C^2(\varrho_1)\geq 0$ from the lower bound in Ref. [15]. While our lower bound gives $C^2(\varrho_1) \geq \tau_{3\otimes 3}(\varrho_1) \geq \zeta_{3\otimes 3}(\varrho_1) \geq \frac{p^2}{5292}(2\sqrt{3\alpha^2-15\alpha+19}-2)^2$, which demonstrates the ability of our lower bounds in detecting PPT entanglement.

Example 2. For non-PPT (NPT) entangled state

$$\begin{aligned}\varrho_2 = {} & \tfrac{p}{6}(|00\rangle\langle 00| + |01\rangle\langle 01| + |02\rangle\langle 02| + |10\rangle\langle 10| \\ & + |11\rangle\langle 11| + |12\rangle\langle 12|) - \tfrac{p}{6}(|00\rangle\langle 12| + |01\rangle\langle 12| \\ & + |12\rangle\langle 00| + |12\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|) \\ & + \tfrac{1-p}{2}(|22\rangle\langle 22| + |33\rangle\langle 33|),\end{aligned} \qquad (12)$$

it can be verified that $\tau_{2\otimes 2}(\varrho_2) = 0$ and $\tau_{3\otimes 3}(\varrho_2) \geq \kappa_{3\otimes 3}(\varrho_2) = \frac{p^2}{54} > 0$. Hence $\tau_{3\otimes 3}$ provides a better lower bound of concurrence for the NPT entangled state ϱ_2.

We remark that the lower bound $\kappa_{2\otimes 2}(\rho)=0$ is equivalent to $\tau_{2\otimes 2}(\rho)=0$. That is, $C(\rho_{2\otimes 2})=0$ is equivalent to $||\rho_{2\otimes 2}^{T_A}|| - tr(\rho_{2\otimes 2}) = 0$ for every two-qubit state $\rho_{2\otimes 2}$ of ρ. This is because two-qubit state $\rho_{2\otimes 2}$ is PPT if and only if it is separable, i.e. zero concurrence. Therefore $\kappa_{2\otimes 2}$ and $\tau_{2\otimes 2}$ could detect entanglement equivalently, up to a normalization. They both present sufficient conditions for distillability of quantum entanglement. Generally $\kappa_{2\otimes 2}$ can be more easily computed than $\tau_{2\otimes 2}$.

Lower bound of concurrence and distillation. — Distillation is an important protocol to improve the quantum entanglement against the decoherence due to noisy channels in information processing. Theoretically an $m \otimes n$ mixed state ρ is distillable if and only if there are some projectors A and B that map the N-copy state $\rho^{\otimes N}$ to two-dimensional ones such that the state $A \otimes B \rho^{\otimes N} A \otimes B$ is entangled [18]. Practically it is quite difficult to judge whether an entangled mixed state is distillable in general. From our lower bound of concurrence, we can derive the following distillability criterion:

Theorem 3 *Any bipartite mixed state ρ is distillable if $\tau_{2\otimes 3}(\rho^{\otimes N}) > 0$ for some positive integer N.*

Proof. The conclusion in Ref. [18] is also true for projectors A and B that map the N-copy state $\rho^{\otimes N}$ to one two-dimensional space and one three-dimensional space respectively. An $m \otimes n$ mixed state ρ is distillable if $A \otimes B \rho^{\otimes N} A \otimes B$ is entangled. This is due to that a mixed $2 \otimes 3$ bipartite state ρ is entangled if and only if it is distillable. Therefore if $\tau_{2\otimes 3}(\rho^{\otimes N}) > 0$ for any positive integer N, there will exist at least one "$2 \otimes 3$ state" from $\rho^{\otimes N}$, which has nonzero concurrence. Hence ρ is distillable. □

The sufficient distillability condition in Ref. [15] says that if $\tau_{2\otimes 2}(\rho^{\otimes N}) > 0$ for a certain positive integer N, then ρ is distillable. Our sufficient condition for distillability in Theorem 3 is stronger than the one in Ref. [15]. The reason is that if $\tau_{2\otimes 2}(\rho) > 0$, then $\tau_{2\otimes 3}(\rho) > 0$. But the converse is not true, since the $2 \otimes 2$ submatrix mapped from the $2 \otimes 3$ state may be not included in the states in the bound $\tau_{2\otimes 2}$.

For example, let us consider the mixed state ϱ_2 in Eq. (12) in example 2. There we have showed that $\tau_{2\otimes 2}(\varrho_2) = 0$. We now show that $\tau_{2\otimes 2}(\varrho_2^{\otimes N}) = 0$ for arbitrary positive integer N. To do this we only need to prove that all the $2 \otimes 2$ quantum states from $\varrho_2^{\otimes N}$ are PPT for arbitrary positive integer N. From the relations $\varrho_2^{\otimes N} = \varrho_2^{\otimes N-1} \otimes \varrho_2$ and $(C \otimes D)^{T_A} = C^{T_A} \otimes D^{T_A}$ for operators C and D acting on $H_A \otimes H_B$, we have that all the principal minors of order 2 from ϱ_2 and $\varrho_2^{\otimes N-1}$ are PPT, therefore all the $2 \otimes 2$ states from $\varrho_2^{\otimes N}$ are PPT and separable for any positive integer N. This implies $\tau_{2\otimes 2}(\varrho_2^{\otimes N}) = 0$. Hence from the sufficient condition of distillability in Ref. [15], one does not know whether the state ϱ_2 is distillable. In fact, it is straightforward to verify that the state ϱ_2 does not violate the reduction criterion [19] either, so the reduction criterion can neither detect its distillability.

But from the fact that $\tau_{2\otimes 3}(\varrho_2) = \tau_{3\otimes 3}(\varrho_2) > 0$, we can assert that state ϱ_2 is distillable. Hence our sufficient condition of distillability is strictly stronger than that in Ref. [15].

Conclusions. — In summary, we have proposed a novel method to construct hierarchy lower bounds of concurrence for arbitrary dimensional bipartite mixed states, in terms of the concurrences of all the lower dimensional mixed states extracted from the given mixed states. This lower bound is no worse than any known lower bounds and may be used to improve all the existing lower bounds of concurrence. The lower bound is tight and has advantages in detecting PPT entanglement. Moreover, our lower bound also gives rise to a sufficient condition for distillability. This sufficient condition is complement to the reduction criterion and is strictly stronger than the one in Ref. [15]. Although our lower bound depends on the concurrence of the lower dimensional mixed states and no analytical formulas of concurrence for the lower dimensional mixed states are ready yet, our lower bound still exhibits its excellence in entanglement characterization and distillation.

Acknowledgments M.J. Zhao would like to thank Y.C. Ou and B.B. Hua for valuable discussions. This work is supported by the NSFC 10875081 and PHR201007107.

[1] W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[2] A. Uhlmann, Phys. Rev. A **62**, 032307 (2000).
[3] P. Rungta, V. Bužek, C.M. Caves, M. Hillery, and G.J. Milburn, Phys. Rev. A **64**, 042315 (2001).
[4] S. Albeverio and S.M. Fei, J. Opt. B: Quantum Semiclass. Opt. **3**, 223 (2001).
[5] D.A. Meyer and N.R. Wallach, J. Math. Phys. (N.Y.) **43**, 4273 (2002).
[6] A.R.R. Carvalho, F. Mintert, and A. Buchleitner, Phys. Rev. Lett. **93**, 230501 (2004).
[7] P. Rungta and C.M. Caves, Phys Rev A **67**, 012307 (2003).
K. Chen, S. Albeverio, and S.M. Fei, Rep. Math. Phys. **58**, 325 (2006);
S.M. Fei, Z.X. Wang, and H. Zhao, Phys. Lett. A **329**, 414 (2004);
[8] F. Mintert, M. Kuś, and A. Buchleitner, Phys. Rev. Lett. **92**, 167902 (2004).
[9] K. Chen, S. Albeverio, and S.M. Fei, Phys. Rev. Lett. **95**, 040504 (2005).
[10] H.P. Breuer, J. Phys. A: Math. Gen. **39**, 11847 (2006).
[11] H.P. Breuer, Phys. Rev. Lett. 97, 080501(2006).
[12] J.I. de Vicente, Phys. Rev. A **75**, 052320 (2007).
[13] C.J. Zhang, Y.S. Zhang, S. Zhang, and G.C. Guo. Phys. Rev. A **76**, 012334 (2007).
[14] E. Gerjuoy, Phys. Rev. A **67**, 052308 (2003).
[15] Y.C. Ou, H. Fan, and S.M. Fei, Phys. Rev. A **78**, 012311 (2008).
[16] X.S. Li, X.H. Gao, and S.M. Fei, Phys. Rev. A **83**, 034303 (2011).
[17] C.H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J.A. Smolin, and W.K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).
[18] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
[19] M. Horodecki and P. Horodecki, Phys. Rev. Lett. **59**, 4206 (1999).
[20] P. Horodecki, M. Horodecki, and R. Horodecki, Phys. Rev. Lett. **82**, 1056 (1999).