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THE BOUNDARY VALUE PROBLEM FOR DIRAC-HARMONIC MAPS

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Abstract. Dirac-harmonic maps are a mathematical version (with commuting variables only) of the solutions of the field equations of the non-linear supersymmetric sigma model of quantum field theory. We explain this structure, including the appropriate boundary conditions, in a geometric framework. The main results of our paper are concerned with the analytic regularity theory of such Dirac-harmonic maps. We study Dirac-harmonic maps from a Riemannian surface to an arbitrary compact Riemannian manifold. We show that a weakly Dirac-harmonic map is smooth in the interior of the domain. We also prove regularity results for Dirac-harmonic maps at the boundary when they solve an appropriate boundary value problem which is the mathematical interpretation of the D-branes of superstring theory.

1. Introduction

In [6], a variational problem has been introduced that is an analogue with ordinary, that is, commuting fields of the non-linear supersymmetric sigma model of quantum field theory. Of course, this model is no longer supersymmetric, but it does share the other symmetries of the sigma model, in particular conformal invariance. Also, this model has a surprisingly subtle geometric and analytic structure. In the present paper, we explore some further geometric and analytic aspects. In particular, we look at boundary conditions that are of the type of the D-branes of superstring theory and involve the chirality operator of a spin structure. After a careful geometric derivation of these boundary conditions, we shall provide the analytic regularity theory for solutions of the field equations at such a boundary.

Let us now describe the model in some more detail. For the non-linear supersymmetric sigma model of quantum field theory (see e.g. [8] or [21] for mathematical background), one considers a map

$$Y : M^s \to N \tag{1.1}$$

from a (2|2)-dimensional supermanifold M^s to some Riemannian manifold N. With local even coordinates x^1, x^2 and odd (i.e., anticommuting) coordinates θ^1, θ^2, the action is

$$S = \int \frac{1}{4}\epsilon^{\alpha\beta}\langle D_\alpha Y, D_\beta Y\rangle d^2x d\theta^2 d\theta^1 \tag{1.2}$$

where ϵ is the usual antisymmetric tensor, the brackets $\langle ., .\rangle$ denote the Riemannian metric on N (by conformal invariance, we may assume that the domain metric is flat), and $d\theta$ indicates that a Berezin integral has to be taken.

Y has the following expansion

$$Y = \phi(x) + \psi_\alpha(x)\theta^\alpha + F(x)\theta^1\theta^2. \tag{1.3}$$

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Here, ϕ is an ordinary map from the ordinary manifold M underlying the supermanifold M^s into N; in fact, M, since 2-dimensional, is considered as a Riemann surface. ψ is an anticommuting spinor with values in the pull-back tangent bundle $\phi^{-1}TN$. In fact, ψ is a real Euclidean Majorana spinor w.r.t. a real 2-dimensional Euclidean representation of the Clifford algebra $Cl(2,0)$. The field F is needed to close the supersymmetry algebra off-shell, but will not be of importance for our subsequent purposes.

Using this expansion and carrying out the θ-integral, the Lagrangian density in (1.2) becomes

$$\frac{1}{2}\|d\phi\|^2 + \frac{1}{2}\langle\psi, D\!\!\!/\psi\rangle - \frac{1}{12}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\langle\psi_\alpha, R(\psi_\beta,\psi_\gamma)\psi_\delta\rangle. \tag{1.4}$$

$D\!\!\!/$ is the Dirac operator along the map ϕ; it involves the ordinary Dirac operator $\partial\!\!\!/$ of M and the Levi-Civita connection of N (see e.g. [6, 21]). $\|.\|$ indicates again the metric of N, and R is its curvature. In fact, the curvature term arises from the Berezin integration of the F-term, and again, we shall not need it in the sequel.

The reason why the spinor field ψ is taken as odd is that for an even ψ, $\langle\psi, D\!\!\!/\psi\rangle$ would vanish upon integration by parts. This in turn results from the fact that we are working with a Clifford algebra ($Cl(2,0)$ in the present case) with a real representation. Were the representation imaginary, in contrast, the integral of $\langle\psi, D\!\!\!/\psi\rangle$ would vanish for an odd, but no longer for an even ψ. Of course, $Cl(2,0)$ does not have such a representation, but the Clifford algebra $Cl(0,2)$ does. This is the basis of the model of [6].

To be concrete, consider the representation of $Cl(0,2)$ with

$$e_1 \to \gamma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad e_2 \to \gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.5}$$

acting on spinors. For a spinor field $\omega : \mathbb{R}^2 \to \mathbb{C}^2$, we then have the Dirac operator

$$\partial\!\!\!/\omega = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial\omega_1}{\partial x_1} \\ \frac{\partial\omega_2}{\partial x_1} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial\omega_1}{\partial x_2} \\ \frac{\partial\omega_2}{\partial x_2} \end{pmatrix} = 2i\begin{pmatrix} \frac{\partial\omega_2}{\partial z} \\ \frac{\partial\omega_1}{\partial \bar{z}} \end{pmatrix}, \tag{1.6}$$

that is, the Cauchy-Riemann operator. Let ω and ψ be two spinor fields with compact support on $\mathbb{R}^2$, we then have the integration by parts formula

$$\int \langle\omega, \partial\!\!\!/\psi\rangle = \int \langle\partial\!\!\!/\omega, \psi\rangle, \tag{1.7}$$

that is, $\partial\!\!\!/$ is formally self-adjoint.

We can thus introduce the model of [6]. Let M be a Riemann spin surface, ΣM the spinor bundle over M, N a compact Riemannian manifold without boundary. Let ϕ be a map from M to N, ψ a section of the bundle $\Sigma M \otimes \phi^{-1}TN$. Let $\widetilde{\nabla}$ be the connection induced from those on ΣM and $\phi^{-1}TN$. The Dirac operator $D\!\!\!/$ along the map ϕ is defined by $D\!\!\!/\psi := \gamma_\alpha \cdot \widetilde{\nabla}_{\gamma_\alpha}\psi$, where γ_α is a local orthonormal frame on M. We consider the functional

$$L(\phi,\psi) := \int_M (\|d\phi\|^2 + \langle\psi, D\!\!\!/\psi\rangle). \tag{1.8}$$

Except for the curvature term (which we do not need as we are not concerned with supersymmetry), the Lagrangian density here is formally the same as in (1.4). However, in (1.8), all fields are commuting.

The critical points (ϕ,ψ) of (1.8) are called Dirac-harmonic maps from M to N. They constitute the object of our study in this paper.

The focus of our paper is on boundary conditions and boundary regularity for such Dirac-harmonic maps. The first issue is the identification of the correct boundary conditions. In a certain sense, we are translating the boundary conditions of the non-linear supersymmetric sigma model, see [1, 2], into a geometric framework. Our Riemannian geometric perspective will clarify some geometric aspects. Let M thus be a Riemann surface with boundary ∂M. This boundary should be mapped to a D-brane. Geometrically, this means that we have a submanifold $\mathcal{S}$ of N, and $\phi(\partial M)$ should be contained in $\mathcal{S}$ in such a way that it is critical for (1.8) w.r.t. to all such boundary values. This simply means that, in the absence of the field ψ, $\phi(\partial M)$ should meet $\mathcal{S}$ orthogonally. In the harmonic map literature, this is called a free boundary condition with support $\mathcal{S}$. In analytic terms, this is a combination of Dirichlet and Neumann boundary conditions. Analytically, this is usually treated by some reflection method, see e.g. [13, 20, 26]. That is, one doubles M to $\bar{M}$ by reflection across the boundary ∂M and extends ϕ to $\bar{M}$ by reflection across the submanifold $\mathcal{S}$. This clarifies the geometric meaning of the tensor R utilized in [1, 2], as we shall explain in more detail below. In any case, the reflection across $\mathcal{S}$ is particularly well controlled when $\mathcal{S}$ is a *totally geodesic* submanifold of N. This condition is also required (in different terminology) in [1, 2]. In fact, we shall not need this condition for the formulation of the boundary condition, nor for the proof of continuity of our solutions, but we shall need to require it in order to get higher regularity of solutions at the boundary.

As our model couples the harmonic map equation to a Dirac type equation, besides the regularity theory for harmonic maps, also the one for solutions of Dirac equations, in the interior and at the boundary, is relevant. Some pertinent references are [3, 4, 5, 9, 23]. In our setting, for the spinor ψ we shall need a chirality boundary condition (first introduced by Gibbons-Hawking-Horowitz-Perry [10]). We explain this here only for the linear case. The coupling between the boundary conditions for the fields ϕ and ψ in the non-linear case will be worked out in detail below. Mathematically, the chirality condition is explained in [16]. We consider the chirality operator $G = i\gamma_1\gamma_2$, and we can decompose the spinor bundle ΣM of M into the eigensubbundles of G for the eigenvalues ± 1. Restricting to the boundary, we have the decomposition $\mathbf{S} := \Sigma M|_{\partial M} = V^+ \oplus V^-$. With $\overrightarrow{n}$ being the outward unit normal vector field on ∂M, the orthogonal projection onto the eigensubbundle $V^\pm$:

$$\begin{aligned} \mathbf{B}^\pm : L^2(\mathbf{S}) &\to L^2(V^\pm) \\ \psi &\mapsto \frac{1}{2}\left(I \pm \overrightarrow{n} G\right)\psi, \end{aligned}$$

defines a local elliptic boundary condition for the Dirac operator $\partial\!\!\!/$ (see [16]). We say a spinor $\psi \in W^{1,4/3}(\Sigma M)$ satisfies the boundary condition $\mathbf{B}^\pm$ if

$$\mathbf{B}^\pm \, \psi|_{\partial M} = 0. \tag{1.9}$$

Our main analytical results then are concerned with weak solutions of the field equations for (1.8), that is, for weakly Dirac-harmonic maps (again, see the main text, e.g. Definition 2.1, for a precise definition) with such boundary conditions. We shall prove

Theorem 1.1. *Let M be a compact Riemann spin surface with boundary ∂M, N be any compact Riemannian manifold, and $\mathcal{S}$ be a closed submanifold of N. Let (ϕ, ψ) be a weakly Dirac-harmonic map from M to N with free boundary on $\mathcal{S}$. Then for any $\alpha \in (0, 1)$,*

$$\phi \in C^{0,\alpha}(M, N).$$

Theorem 1.2. *Let M be a compact Riemann spin surface with boundary ∂M, N be any compact Riemannian manifold, and $\mathcal{S}$ be a closed, totally geodesic submanifold of N. Let (ϕ, ψ) be a weakly Dirac-harmonic map from M to N with free boundary on $\mathcal{S}$ and suppose that $\phi \in C^{0,\alpha}(M, N)$ for any $\alpha \in (0, 1)$. Then there exists some $\beta \in (0, 1)$ such that*

$$\phi \in C^{1,\beta}(M, N), \quad \psi \in C^{1,\beta}(\Sigma M \otimes \phi^{-1}TN).$$

In fact, we shall start by showing the regularity of weakly Dirac-harmonic maps in the interior of M. This was shown independently by Wang-Xu [28] by a different method inspired by [24, 25]. Our methods will also utilize the general strategy of Rivière [24] who had achieved an important generalization of the earlier results of Wente [27] and Hélein [14, 15]. Rivière's approach has been adapted to Dirichlet boundary regularity by Müller-Schikorra [22], and this work will also be useful for our purposes.

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2. Interior regularity

Let M be a Riemann surface equipped with a conformal metric, which by conformal invariance of our functionals can then be assumed Euclidean, and with a fixed spin structure, ΣM the spinor bundle, let ϕ be a smooth map from M to another Riemannian manifold (N, g) of dimension $d \geq 2$. Denote $\phi^{-1}TN$ the pull-back bundle of TN by ϕ and consider the twisted bundle $\Sigma M \otimes \phi^{-1}TN$. On $\Sigma M \otimes \phi^{-1}TN$ there is a metric induced from the metrics on ΣM and $\phi^{-1}TN$. Also we have a natural connection $\widetilde{\nabla}$ on $\Sigma M \otimes \phi^{-1}TN$ induced from those on ΣM and $\phi^{-1}TN$. In local coordinates, the section ψ of $\Sigma M \otimes \phi^{-1}TN$ can be expressed by

$$\psi(x) = \sum_{j=1}^{d} \psi^j(x) \otimes \partial y^j(\phi(x)),$$

where ψ^j is a spinor and $\{\partial y^j\}$ is the natural local basis on N. $\widetilde{\nabla}$ can be expressed by

$$\widetilde{\nabla}\psi = \sum_{i=1}^{d} \nabla\psi^i(x) \otimes \partial y^i(\phi(x)) + \sum_{i,j,k=1}^{d} \Gamma^i_{jk}(\phi(x))\nabla\phi^j(x) \cdot \psi^k(x) \otimes \partial y^i(\phi(x)).$$

Now we define the *Dirac operator along the map* ϕ by

$$\begin{aligned} D\!\!\!/\psi &:= \gamma_\alpha \cdot \widetilde{\nabla}_{\gamma_\alpha}\psi \\ &= \sum_i \partial\!\!\!/\psi^i(x) \otimes \partial y^i(\phi(x)) + \sum_{i,j,k=1}^{d} \Gamma^i_{jk}(\phi(x))\nabla_{\gamma_\alpha}\phi^j(x)\gamma_\alpha \cdot \psi^k(x) \otimes \partial y^i(\phi(x)), \end{aligned}$$

where γ_1, γ_2 is the local orthonormal frame on M and $\partial\!\!\!/ := \sum_{\alpha=1}^2 \gamma_\alpha \cdot \nabla_{\gamma_\alpha}$ is the usual Dirac operator.

Set

$$\mathcal{X}(M, N) := \left\{(\phi, \psi) \,|\, \phi \in C^\infty(M, N) \text{ and } \psi \in C^\infty(\Sigma M \otimes \phi^{-1}TN)\right\}.$$

On $\mathcal{X}(M, N)$, we consider the following functional

$$\begin{aligned} L(\phi, \psi) &:= \int_M [|d\phi|^2 + (\psi, D\!\!\!/\psi)] \\ &= \int_M [g_{ij}(\phi)\frac{\partial\phi^i}{\partial x_\alpha}\frac{\partial\phi^j}{\partial x_\alpha} + g_{ij}(\phi)(\psi^i, D\!\!\!/\psi^j)]. \end{aligned}$$

(Recall that the domain metric can be taken as Euclidean.) The Euler-Lagrange equations of $L(\cdot, \cdot)$ are the following ones:

$$\tau^m(\phi) - \frac{1}{2}R^m_{lij}(\phi)(\psi^i, \nabla\phi^l \cdot \psi^j) = 0, \quad m = 1, 2, \ldots, d, \tag{2.10}$$

$$D\!\!\!/\psi^i = \partial\!\!\!/\psi^i + \Gamma^i_{jk}(\phi)\partial_\alpha\phi^j\gamma_\alpha \cdot \psi^k = 0, \quad i = 1, 2, \ldots, d, \tag{2.11}$$

where $\tau(\phi)$ is the tension field of the map ϕ. Solutions (ϕ, ψ) to (2.10) and (2.11) are called Dirac-harmonic maps from M to N.

Let (N', g') be another Riemannian manifold and $f : N \to N'$ a smooth map. For any $(\phi, \psi) \in \mathcal{X}(M, N)$ we set

$$\phi' = f \circ \phi \quad \text{and} \quad \psi' = f_* \psi.$$

It is clear that ψ' is a spinor along the map ϕ'. Let A be the second fundamental form of f, i.e., $A(X, Y) = (\nabla_X df)(Y)$ for any $X, Y \in \Gamma(TN)$. The tension fields of ϕ and ϕ' have the following relation

$$\tau'(\phi') = \sum_{\alpha=1}^{2} A(d\phi(\gamma_\alpha), d\phi(\gamma_\alpha)) + df(\tau(\phi)). \tag{2.12}$$

It is also easy to check that the Dirac operators $\not{D}$ and $\not{D}'$ corresponding to ϕ and ϕ' respectively are related by the following

$$\not{D}'\psi' = f_*(\not{D}\psi) + \mathcal{A}(d\phi(\gamma_\alpha), \gamma_\alpha \cdot \psi), \tag{2.13}$$

where$\mathcal{A}(d\phi(\gamma_\alpha), \gamma_\alpha \cdot \psi) := \phi^i_\alpha \gamma_\alpha \cdot \psi^j \otimes A(\partial y^i, \partial y^j)$. Furthermore, if $f : N \to N'$ is an isometric immersion, then $A(\cdot, \cdot)$ is the second fundamental form of the submanifold N in N', and

$$\nabla'_X \xi = -P(\xi; X) + \nabla^\perp_X \xi, \qquad \nabla'_X Y = \nabla_X Y + A(X, Y)$$

$\forall X, Y \in \Gamma(TN)$, $\xi \in \Gamma(T^\perp N)$, where $P(\xi; \cdot)$ denotes the shape operator. We can rewrite equations (2.10) (2.11) in terms of $\mathcal{A}$ and the geometric data of the ambient space N'.

Denote

$$R(\phi, \psi) := \frac{1}{2} R^m_{lij} \langle \psi^i, \nabla\phi^l \cdot \psi^j \rangle \otimes \partial y^m.$$

By the equation of Gauss, we have (see [6, 7, 19, 29])

$$R(\phi, \psi) = \mathrm{Re}\, \mathcal{P}(\mathcal{A}(d\phi(\gamma_\alpha), \gamma_\alpha \cdot \psi); \psi) + R'(\phi, \psi). \tag{2.14}$$

where $\mathcal{P}(\mathcal{A}(d\phi(\gamma_\alpha), \gamma_\alpha \cdot \psi); \psi) := P(A(\partial y^l, \partial y^j); \partial y^i)\langle \psi^i, \gamma_\alpha \cdot \psi^j \rangle \phi^l_\alpha$. Therefore, by using (2.12) and (2.13), and identifying ϕ with ϕ' and ψ with ψ', we can rewrite (2.10) and (2.11) as follows:

$$\tau'(\phi) = A(d\phi(\gamma_\alpha), d\phi(\gamma_\alpha)) + \mathrm{Re}\, \mathcal{P}(\mathcal{A}(d\phi(\gamma_\alpha), \gamma_\alpha \cdot \psi); \psi) + R'(\phi, \psi), \tag{2.15}$$

$$\not{D}'\psi = \mathcal{A}(d\phi(\gamma_\alpha), \gamma_\alpha \cdot \psi). \tag{2.16}$$

In order to introduce the notion of weak solutions of the Euler-Lagrange equations, we embed N isometrically into some $N' = \mathbb{R}^K$ via the Nash-Moser embedding theorem. Then the above equations become

$$-\Delta\phi = A(d\phi, d\phi) + \mathrm{Re}\, \mathcal{P}(\mathcal{A}(d\phi(\gamma_\alpha), \gamma_\alpha \cdot \psi); \psi), \tag{2.17}$$

$$\not{\partial}\psi = \mathcal{A}(d\phi(\gamma_\alpha), \gamma_\alpha \cdot \psi). \tag{2.18}$$

Denote

$$H^1(M, N) := \left\{ \phi \in H^1(M, \mathbb{R}^K) \mid \phi(x) \in N \ a.e.\ x \in M \right\};$$

$$W^{1,4/3}(\Sigma M \otimes \phi^{-1}TN) := \left\{ \psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN) \mid \int_M |\nabla\psi|^{4/3} < +\infty, \int_M |\psi|^4 < +\infty \right\}.$$

Here, $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$, the spinor field along the map ϕ, should be understood as a K-tuple of spinors $(\psi^1, \psi^2, ..., \psi^K)$ satisfying

$$\sum_i \nu_i \psi^i = 0, \quad \text{for any normal vector } \nu = \sum_{i=1}^{K} \nu_i E_i \text{ at } \phi(x),$$

where $\{E_i, i = 1, 2, ..., K\}$ is the standard basis of $\mathbb{R}^K$. Denote

$$\mathcal{X}^{1,2}_{1,4/3}(M, N) := \left\{ (\phi, \psi) \in H^1(M, N) \times W^{1,4/3}(\Sigma M \otimes \phi^{-1}TN) \right\}.$$

Critical points $(\phi, \psi) \in \mathcal{X}^{1,2}_{1,4/3}(M, N)$ of the functional $L(\cdot, \cdot)$ are called weakly Dirac-harmonic maps from M to N (see [7]), equivalently,

Definition 2.1. *We call $(\phi,\psi)\in\mathcal{X}^{1,2}_{1,4/3}(M,N)$ a weakly Dirac-harmonic map from M to N if*

$$\int_M [\langle d\phi, d\eta\rangle + \langle A(d\phi,d\phi) + \mathrm{Re}\mathcal{P}(\mathcal{A}(d\phi(\gamma_\alpha),\gamma_\alpha\cdot\psi);\psi),\eta\rangle] = 0, \tag{2.19}$$

$$\int_M [\langle \psi, \partial\!\!\!/\xi\rangle - \langle \mathcal{A}(d\phi(\gamma_\alpha),\gamma_\alpha\cdot\psi),\xi\rangle] = 0, \tag{2.20}$$

for all $\eta\in H^1_0\cap L^\infty(M,\mathbb{R}^K)$ and $\xi\in W^{1,4/3}_0\cap L^\infty(\Sigma M\otimes\mathbb{R}^K)$.

Let us recall the following regularity result in two dimensional conformally invariant variational problems by Rivière [24]. Denote $B_1 := \left\{(x_1,x_2)\in\mathbb{R}^2 | x_1^2+x_2^2\le 1\right\}$ the unit disk in $\mathbb{R}^2$ and write $z = x_1 + ix_2$.

Theorem A. *Let $u\in H^1(B_1,\mathbb{R}^K)$ be a weak solution of*

$$-\Delta u = \Omega\cdot\nabla u. \tag{2.21}$$

where $\Omega = (\Omega^i_j)_{1\le i,j\le K}\in L^2(B_1, so(K)\otimes\mathbb{R}^2)$. Then u is continuous.

To prove the smoothness of weakly Dirac-harmonic maps, it is sufficient to show the continuity of the map (see [7]).

Theorem B. *Let $(\phi,\psi): B_1\to N$ be a weakly Dirac-harmonic map, if ϕ is continuous, then (ϕ,ψ) is smooth.*

When $N=\mathbb{S}^d$, the continuity of weakly Dirac-harmonic maps was proved by Chen-Jost-Li-Wang in [7], using Wente's Lemma [27]. Zhu extended this result to the case that N is a compact hypersurface in the Euclidean space $\mathbb{R}^{d+1}$ [29]. The case of a general target N was shown independently by Wang-Xu [28], where Hélein's technique of moving frame [14, 15] and the Coulomb gauge construction, due to Rivière [24] and Rivière-Struwe [25], are combined.

Here, following the notations in [29], we show that the extrinsic equations (2.17) in the case of a general compact target can also be written in the same form as (2.21) and hence can be used to prove the continuity of weakly Dirac-harmonic maps.

Theorem 2.1. *Let M be a Riemann spin surface, N be any compact Riemannian manifold, (ϕ,ψ) a weakly Dirac-harmonic map from M to N, then ϕ is continuous in the interior of M and consequently, (ϕ,ψ) is smooth.*

Proof. We follow the approach in [29]. We assume W.L.O.G that $M=B_1$ and take the orthonormal basis $\gamma_1=\partial_{x_1},\gamma_2=\partial_{x_2}$. Fix a canonical coordinate $(y^1,y^2,...,y^K)$ of $\mathbb{R}^K$. Let ν_l, $l=d+1,...,K$ be an orthonormal frame field for the normal bundle $T^\perp N$ to N (the target N considered is always assumed to be oriented). Denote by ν_l the corresponding unit normal vector field along the map ϕ. We write

$$\phi = \phi^i\partial y^i, \quad \psi = \psi^j\otimes\partial y^j,$$

and denote $\phi_\alpha := \phi_*(\gamma_\alpha) = \phi_{x_\alpha}, \alpha = 1,2$. Then, we proceed as in [29] to write (2.17) and (2.18) in the following extrinsic form in terms of the orthonormal frame field ν_l, $l=d+1,...,K$, for $T^\perp N$

$$-\Delta\phi^m = \phi^i_\alpha\left(\phi^j_\alpha\frac{\partial\nu^i_l}{\partial y^j}\nu^m_l - \phi^j_\alpha\frac{\partial\nu^m_l}{\partial y^j}\nu^i_l\right) + \phi^i_\alpha\langle\psi^k,\gamma_\alpha\cdot\psi^j\rangle\left(\frac{\partial\nu^i_l}{\partial y^j}(\frac{\partial\nu_l}{\partial y^k})^{\top,m} - \frac{\partial\nu^i_l}{\partial y^k}(\frac{\partial\nu_l}{\partial y^j})^{\top,m}\right), \tag{2.22}$$

$$\partial\!\!\!/\psi^m = \frac{\partial\nu^i_l}{\partial y^j}\nu^m_l\phi^i_\alpha\gamma_\alpha\cdot\psi^j. \tag{2.23}$$

Here $\top$ denotes the orthogonal projection : $\mathbb{R}^K \to T_yN$ and $(\cdot)^i$ denotes the i-th component of a vector of $\mathbb{R}^K$. Note that $\phi_\alpha \in TN$ and $(\frac{\partial \nu_l}{\partial y^j})^\perp \in T^\perp N$, hence, we have

$$\sum_i \phi^i_\alpha (\frac{\partial \nu_l}{\partial y^j})^{\perp,i} = 0, \quad \forall \alpha, l, j. \tag{2.24}$$

where $\perp$ denotes the orthogonal projection : $\mathbb{R}^K \to T^\perp_y N$. Decomposing the vector $\frac{\partial \nu_l}{\partial y^j}$ into tangent part and normal part and then applying (2.24), we get

$$\frac{\partial \nu^i_l}{\partial y^j}\phi^i_\alpha = (\frac{\partial \nu_l}{\partial y^j})^i \phi^i_\alpha = \left((\frac{\partial \nu_l}{\partial y^j})^{\top,i} + (\frac{\partial \nu_l}{\partial y^j})^{\perp,i} \right)\phi^i_\alpha = (\frac{\partial \nu_l}{\partial y^j})^{\top,i}\phi^i_\alpha. \tag{2.25}$$

Thus, the equations (2.22) and (2.23) become

$$-\Delta\phi^m = \phi^i_\alpha \left(\phi^j_\alpha \frac{\partial \nu^i_l}{\partial y^j}\nu^m_l - \phi^j_\alpha \frac{\partial \nu^m_l}{\partial y^j}\nu^i_l \right) + \phi^i_\alpha \langle \psi^k, \gamma_\alpha \cdot \psi^j \rangle \left((\frac{\partial \nu_l}{\partial y^j})^{\top,i} (\frac{\partial \nu_l}{\partial y^k})^{\top,m} - (\frac{\partial \nu_l}{\partial y^k})^{\top,i} (\frac{\partial \nu_l}{\partial y^j})^{\top,m} \right), \tag{2.26}$$

$$\partial\!\!\!/ \psi^m = \frac{\partial \nu^i_l}{\partial y^j}\nu^m_l \phi^i_\alpha \gamma_\alpha \cdot \psi^j. \tag{2.27}$$

Denote

$$\Omega^m_i := \begin{pmatrix} \lambda^m_i \\ \mu^m_i \end{pmatrix}, \quad i, m = 1, 2, ..., K,$$

where

$$\lambda^m_i := (\frac{\partial \nu^i_l}{\partial y^j}\nu^m_l - \frac{\partial \nu^m_l}{\partial y^j}\nu^i_l)\phi^j_1 + \left((\frac{\partial \nu_l}{\partial y^j})^{\top,i} (\frac{\partial \nu_l}{\partial y^k})^{\top,m} - (\frac{\partial \nu_l}{\partial y^k})^{\top,i} (\frac{\partial \nu_l}{\partial y^j})^{\top,m} \right) \langle \psi^k, \gamma_1 \cdot \psi^j \rangle,$$

$$\mu^m_i := (\frac{\partial \nu^i_l}{\partial y^j}\nu^m_l - \frac{\partial \nu^m_l}{\partial y^j}\nu^i_l)\phi^j_2 + \left((\frac{\partial \nu_l}{\partial y^j})^{\top,i} (\frac{\partial \nu_l}{\partial y^k})^{\top,m} - (\frac{\partial \nu_l}{\partial y^k})^{\top,i} (\frac{\partial \nu_l}{\partial y^j})^{\top,m} \right) \langle \psi^k, \gamma_2 \cdot \psi^j \rangle.$$

Then we can write (2.26) in the following form

$$-\Delta\phi^m = \Omega^m_i \cdot \nabla\phi^i.$$

It is easy to verify that $\Omega = (\Omega^m_i)_{1\le i,m\le K} \in L^2(B_1, so(K) \otimes \mathbb{R}^2)$. By Theorem A, we have $\phi \in C^0(B_1, N)$ and consequently, (ϕ, ψ) is smooth. □

3. Free boundary problem for Dirac-harmonic maps

In this section, we shall study the free boundary problem for Dirac-harmonic maps.

First, we impose the free boundary condition for the map in the classical sense, namely, the boundary of the domain is mapped freely into a submanifold of the target. Next, motivated by the supersymmetric sigma model with boundaries (see Albertsson- Lindström-Zabzine [1, 2]), we impose the boundary condition for the spinor field using a chirality operator.

To begin with, let us recall the chirality boundary conditions for the usual Dirac operator $\partial\!\!\!/$ (see [16]).

Chirality boundary conditions for the Dirac operator $\partial\!\!\!/$

Let M be a compact Riemannian spin surface with boundary $\partial M \neq \emptyset$. Then M admits a chirality operator $G = \gamma(\omega_2)$, the Clifford multiplication by the complex volume form $\omega_2 = i\gamma_1\gamma_2$. G is an endomorphism of the spinor bundle ΣM satisfying:

$$G^2 = I, \quad \langle G\psi, G\varphi\rangle = \langle \psi, \varphi\rangle, \tag{3.28}$$

$$\nabla_X(G\psi) = G\nabla_X\psi, \quad X\cdot G\psi = -G(X\cdot\psi). \tag{3.29}$$

$\forall X \in \Gamma(TM), \psi, \varphi \in \Gamma(\Sigma M)$. Here I denotes the identity endomorphism of ΣM.

Denote

$$\mathbf{S} := \Sigma M|_{\partial M}$$

the restricted spinor bundle with induced Hermitian product.

Let $\overrightarrow{n}$ be the outward unit normal vector field on ∂M. One can verify that $\overrightarrow{n}G : \Gamma(\mathbf{S}) \to \Gamma(\mathbf{S})$ is a self-adjoint endomorphism whose square is the identity, namely

$$\langle \overrightarrow{n}G\psi, \varphi\rangle = \langle \psi, \overrightarrow{n}G\varphi\rangle \tag{3.30}$$

$$(\overrightarrow{n}G)^2 = I. \tag{3.31}$$

Hence, we can decompose $\mathbf{S} = V^+ \oplus V^-$, where $V^\pm$ is the eigensubbundle corresponding to the eigenvalue ±1. One verifies that the orthogonal projection onto the eigensubbundle $V^\pm$:

$$\begin{aligned} \mathbf{B}^\pm : L^2(\mathbf{S}) &\to L^2(V^\pm) \\ \psi &\mapsto \frac{1}{2}\left(I \pm \overrightarrow{n}G\right)\psi, \end{aligned}$$

defines a local elliptic boundary condition for the Dirac operator $\partial\!\!\!/$ (see [16]). We say a spinor $\psi \in W^{1,4/3}(\Sigma M)$ satisfies the boundary condition $\mathbf{B}^\pm$ if

$$\mathbf{B}^\pm \psi|_{\partial M} = 0. \tag{3.32}$$

The following proposition was shown in [16]. For the sake of completeness, we present the proof here using our notations.

Proposition 3.1. *If $\varphi, \psi \in W^{1,4/3}(\Sigma M)$ satisfy the boundary condition $\mathbf{B}^\pm$ then*

$$\langle \overrightarrow{n}\cdot\psi, \varphi\rangle = 0, \quad \text{on } \partial M. \tag{3.33}$$

In particular,

$$\int_{\partial M} \langle \overrightarrow{n}\cdot\psi, \varphi\rangle = 0. \tag{3.34}$$

Proof. Let $\varphi, \psi \in W^{1,4/3}(\Sigma M)$ satisfy the boundary condition $\mathbf{B}^\pm$, namely, $\mathbf{B}^\pm \psi|_{\partial M} = \mathbf{B}^\pm \varphi|_{\partial M} = 0$. Then

$$\overrightarrow{n}G\psi = \mp\psi, \quad \overrightarrow{n}G\varphi = \mp\varphi.$$

Hence, applying the properties (3.28) - (3.31) of G, we get

$$\langle \overrightarrow{n}\cdot\psi, \varphi\rangle = \langle G\overrightarrow{n}\cdot\psi, G\varphi\rangle = \langle -\overrightarrow{n}G\psi, -\overrightarrow{n}\,\overrightarrow{n}G\varphi\rangle = (-1)^2(\mp1)^2\langle\psi, \overrightarrow{n}\varphi\rangle = -\langle \overrightarrow{n}\cdot\psi, \varphi\rangle.$$

(3.33) and (3.34) follow immediately. □

Let M be the upper-half Euclidean space $\mathbb{R}^2_+$. We identify the Clifford multiplication by the orthonormal frame $\partial x_1, \partial x_2$ with the following matrices:

$$\gamma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then we can take the chirality operator $G := i\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that $\overrightarrow{n} = -\partial x_2 = -\gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and hence we can calculate $\mathbf{B}^\pm = \frac{1}{2}\left(I \pm \overrightarrow{n}\cdot G\right) = \frac{1}{2}\begin{pmatrix} 1 & \pm1 \\ \pm1 & 1 \end{pmatrix}$.

By the standard chirality decomposition, we can write $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, then the boundary condition (3.32) becomes

$$\psi_+ = \mp\psi_- \quad \text{on } \partial M.$$

Next, we will extend the chirality boundary condition to the Dirac operator along a map.

Chirality boundary condition for the Dirac operator $\not{D}$ along a map ϕ

When $\partial M \neq \emptyset$, the Dirac operator $\not{D}$ along a map ϕ is in general not formally self-adjoint. In fact, we have the following property analogous to the usual Dirac operator $\not{\partial}$.

Proposition 3.2.

$$\int_M \langle\psi, \not{D}\varphi\rangle = \int_M \langle\not{D}\psi, \varphi\rangle - \int_{\partial M} \langle\vec{n}\cdot\psi, \varphi\rangle,$$

for all $\psi, \varphi \in C^\infty(\Sigma M \otimes \phi^{-1}TN)$, where $\langle\psi, \varphi\rangle := g_{ij}(\phi)\langle\psi^i, \varphi^j\rangle$.

Proof. Choose a local orthonormal frame $\{\gamma_\alpha\}_{\alpha=1}^2$ on M. Given $\psi, \varphi \in C^\infty(\Sigma M \otimes \phi^{-1}TN)$, define

$$f := \langle\gamma_\alpha \cdot \psi, \varphi\rangle\gamma_\alpha,$$

then f is independent of the choice of such a frame γ_α and hence is globally defined. We calculate

$$\begin{aligned}
\int_M \langle\psi, \not{D}\varphi\rangle &= \int_M \langle\not{D}\psi, \varphi\rangle - \int_M \gamma_\alpha\langle\gamma_\alpha\cdot\psi, \varphi\rangle \\
&= \int_M \langle\not{D}\psi, \varphi\rangle - \int_M \mathrm{div} f \\
&= \int_M \langle\not{D}\psi, \varphi\rangle - \int_{\partial M} f\cdot\vec{n} \\
&= \int_M \langle\not{D}\psi, \varphi\rangle - \int_{\partial M} \langle\gamma_\alpha\cdot\psi, \varphi\rangle\langle\gamma_\alpha, \vec{n}\rangle. \\
&= \int_M \langle\not{D}\psi, \varphi\rangle - \int_{\partial M} \langle\vec{n}\cdot\psi, \varphi\rangle.
\end{aligned}$$

Here in the last step we have used the fact that $\vec{n} = \langle\gamma_\alpha, \vec{n}\rangle\gamma_\alpha$. □

To extend the chirality boundary condition to the Dirac operator $\not{D}$ along a map from M to N, we need some geometric structure on the target N.

Given a submanifold $\mathcal{S}$ of N. We assume that there is an endomorphism $R(y) : T_yN \to T_yN, \forall y \in \mathcal{S}$. The $(1,1)$ tensor R is called *compatible* if it preserves the metric on TN, namely,

$$\langle R(y)V, R(y)W\rangle = \langle V, W\rangle, \quad \forall V, W \in T_yN, \quad \forall y \in \mathcal{S}$$

and it squares to the identity, more precisely,

$$R(y)R(y)V = V, \quad \forall V \in T_yN, \quad \forall y \in \mathcal{S}.$$

Such compatible $(1,1)$ tensors on $\mathcal{S}$ always exist. For instance, we can take $R \equiv \pm\, id$, where

$$id : T_yN \to T_yN, \quad \forall y \in \mathcal{S}$$

denotes the identity endomorphism.

Let $\mathcal{S}$ be a closed submanifold of N with a compatible $(1,1)$ tensor R and consider a map $\phi \in C^\infty(M,N)$ satisfying the free boundary condition in the classical sense, namely, $\phi(\partial M) \subset \mathcal{S}$. We denote by

$$\mathbf{S}_\phi := \left(\Sigma M \otimes \phi^{-1}TN\right)|_{\partial M}$$

the restricted (twisted) spinor bundle with the induced metric.

Let $\psi \in C^\infty(\mathbf{S}_\phi)$. Given $x \in \partial M$, then $\phi(x) \in \mathcal{S}$. Choose a local orthonormal frame $\{V_i\}$ on a neighborhood of $\phi(x)$ (still denote by $\{V_i\}$ the corresponding orthonormal frame along the map ϕ). Locally, we can write

$$\psi = \sum_i \psi^i \otimes V_i.$$

Denote by Id the identity endomorphism acting on $C^\infty(\phi^{-1}TN|_{\partial M})$. Then, one can verify that the endomorphism $\overrightarrow{n}G \otimes R : C^\infty(\mathbf{S}_\phi) \to C^\infty(\mathbf{S}_\phi)$ defined by

$$\left(\overrightarrow{n}G \otimes R\right)\psi := \sum_i \overrightarrow{n}G\psi^i \otimes RV_i, \quad \forall \psi = \sum_i \psi^i \otimes V_i \in C^\infty(\mathbf{S}_\phi). \tag{3.35}$$

is self-adjoint and its square is the identity, namely

$$\left\langle (\overrightarrow{n}G \otimes R)\psi, \varphi \right\rangle = \left\langle \psi, (\overrightarrow{n}G \otimes R)\varphi \right\rangle, \quad \forall \psi, \varphi \in C^\infty(\mathbf{S}_\phi) \tag{3.36}$$

$$\left(\overrightarrow{n}G \otimes R\right)^2 = I \otimes Id. \tag{3.37}$$

Hence, we can decompose the twisted bundle $\mathbf{S}_\phi = V_\phi^+ \oplus V_\phi^-$, where $V_\phi^\pm$ is the eigensubbundle corresponding to the eigenvalue ± 1. One verifies that the orthogonal projection onto the eigensubbundle $V_\phi^\pm$:

$$\begin{aligned} \mathbf{B}_\phi{}^\pm : C^\infty(\mathbf{S}_\phi) &\to C^\infty(V_\phi^\pm) \\ \psi &\mapsto \frac{1}{2}\left(I \otimes Id \pm \overrightarrow{n}G \otimes R\right)\psi, \end{aligned}$$

defines an elliptic boundary condition for the Dirac operator $\not{D}$ along the map ϕ. We say a spinor field $\psi \in C^\infty(\Sigma M \otimes \phi^{-1}TN)$ along a map ϕ satisfies the boundary condition $\mathbf{B}_\phi^\pm$ if

$$\mathbf{B}_\phi^\pm \, \psi|_{\partial M} = 0. \tag{3.38}$$

The following proposition generalizes the results of Proposition 3.1 to the case of spinor fields along a map:

Proposition 3.3. *If $\varphi, \psi \in C^\infty(\Sigma M \otimes \phi^{-1}TN)$ satisfy the chirality boundary condition $\mathbf{B}_\phi^\pm$, then*

$$\langle \overrightarrow{n} \cdot \psi, \varphi \rangle = 0, \quad \text{on } \partial M. \tag{3.39}$$

In particular, we have

$$\int_{\partial M} \langle \overrightarrow{n} \cdot \psi, \varphi \rangle = 0. \tag{3.40}$$

Proof. Let $\psi, \varphi \in C^\infty(\Sigma M \otimes \phi^{-1}TN)$ satisfying the chirality boundary condition $\mathbf{B}_\phi^\pm$, namely, $\mathbf{B}_\phi^\pm \psi|_{\partial M} = \mathbf{B}_\phi^\pm \varphi|_{\partial M} = 0$. Choosing a local orthonormal frame $\{V_i\}$ on a neighborhood of $\phi(x)$ for $x \in \partial M$, we can write

$$\psi = \sum_i \psi^i \otimes V_i, \quad \varphi = \sum_j \varphi^j \otimes V_j.$$

Then the chirality boundary conditions $\mathbf{B}_\phi^\pm$ for ψ and φ read:

$$\psi = \sum_i \psi^i \otimes V_i = \mp \sum_i \overrightarrow{n}G\psi^i \otimes RV_i, \quad \varphi = \sum_j \varphi^j \otimes V_j = \mp \sum_j \overrightarrow{n}G\varphi^j \otimes RV_j.$$

At the point x, we can calculate

$$\begin{aligned}\langle \vec{n}\cdot\psi,\varphi\rangle &= (\mp 1)^2\sum_{i,j}\langle \vec{n}\,\vec{n}G\psi^i\otimes RV_i, \vec{n}G\varphi^j\otimes RV_j\rangle\\ &= \sum_{i,j}\langle \vec{n}\,\vec{n}G\psi^i, \vec{n}G\varphi^j\rangle\langle RV_i, RV_j\rangle\\ &= \sum_{i,j}\langle -\vec{n}\psi^i,\varphi^j\rangle\langle V_i, V_j\rangle\\ &= \sum_{i,j} -\langle \vec{n}\psi^i\otimes V_i,\varphi^j\otimes V_j\rangle\\ &= \sum_{i,j} -\langle \vec{n}\cdot\psi,\varphi\rangle.\end{aligned}$$

Since the point $x\in\partial M$ is arbitrary, we obtain (3.39) and (3.40). □

Free boundary conditions for Dirac-harmonic maps

Let $\mathcal{S}$ be a closed p-dimensional submanifold of N. It turns out that one can associate to it a natural $(1,1)$ tensor R that is compatible.

To see this, we consider a tubular neighborhood $\mathbf{U}_\delta := \left\{z\in N \mid \mathrm{dist}^N(z,\mathcal{S})<\delta\right\}$ of $\mathcal{S}$ in N, where $\delta>0$ is a constant small enough such that for any $z\in\mathbf{U}_\delta$, there exists a unique minimal geodesic γ_z connecting z and $z'\in\mathcal{S}$ which attains the distance from z to the submanifold $\mathcal{S}$.

On $\mathbf{U}_\delta$, we can define the geodesic reflection σ as follows:

$$\sigma:\mathbf{U}_\delta\to\mathbf{U}_\delta,\quad z:=\exp_{z'}v\mapsto\sigma(z):=\exp_{z'}(-v),$$

where $v\in T_{z'}N$ is uniquely determined by z. Clearly, $\sigma^2=id:\mathbf{U}_\delta\to\mathbf{U}_\delta$, and for δ small enough, the map σ is a diffeomorphism. Associated to this σ, there is a $(1,1)$ tensor R on $\mathcal{S}$ defined by

$$R(z):=D\sigma(z),\quad \forall z\in\mathcal{S}.$$

The $(1,1)$ tensor R is well defined on $\mathcal{S}$, since $\sigma|_{\mathcal{S}}=id$ and hence $R(z):T_zN\to T_zN$ is an endomorphism for $z\in\mathcal{S}$. To show the compatibility of R, it is most convenient to take the *adapted* coordinates $\{y^i\}_{i=1,2,\cdots,d}$ in some neighborhood $\mathrm{U}\subset\mathbf{U}_\delta$ of a given point $P\in\mathcal{S}$, such that $\{y^a\}_{a=1,2,\cdots,p}$ are coordinates in $\mathcal{S}$, $\{y^\lambda\}_{\lambda=p+1,\cdots,d}$ are the directions normal to $\mathcal{S}$ and

$$\mathcal{S}\cap\mathrm{U}=\{\mathrm{y}\in\mathrm{U}|\mathrm{y}^{\mathrm{p}+1}=...=\mathrm{y}^{\mathrm{d}}=0\}.$$

In the sequel, the index ranges are:

$$1\le a,b,\cdots\le p,\qquad p+1\le\lambda,\mu,\cdots\le d,\qquad 1\le i,j,k,\cdots\le d.$$

Note that the adapted coordinates $\{y^i\}_{i=1,2,\cdots,d}$ are exactly the geodesic parallel coordinates for the submanifold $\mathcal{S}$. These coordinates also go under the name of Fermi coordinates in the literature. We refer to [12] for more details. In such coordinates, the diffeomorphism $\sigma|_{\mathrm{U}}:\mathrm{U}\to\mathrm{U}$ is given by

$$\sigma:(y^1,...,y^p,y^{p+1},...,y^d)\to(y^1,...,y^p,-y^{p+1},...,-y^d)$$

Consequently, we have

$$\begin{aligned}D\sigma(\partial y^k) &= \partial y^k,\quad k=1,...,p\\ D\sigma(\partial y^m) &= -\partial y^m,\quad m=p+1,...,d.\end{aligned}$$

The tensor R and the metric g take the following forms

$$R=\begin{pmatrix}\delta^a_b & 0\\ 0 & -\delta^\lambda_\mu\end{pmatrix},\quad g=\begin{pmatrix}g_{ab} & 0\\ 0 & g_{\lambda\mu}\end{pmatrix}.$$

It is easy to verify that R is compatible. Moreover, R satisfies the following additional property:

$$R(z)|_{T_z\mathcal{S}} = id, \quad R(z)|_{T_z^\perp\mathcal{S}} = -\, id, \quad \forall z \in \mathcal{S}$$

where id denotes the identity endomorphism and $T_z^\perp\mathcal{S}$ is the subspace of T_zN that is normal to $T_z\mathcal{S}$.

Given a closed p-dimensional submanifold $\mathcal{S}$ of N. In the sequel, we will always associate to it the compatible $(1,1)$ tensor R constructed via the geodesic reflection σ for $\mathcal{S}$. It turns out that this tensor is the most natural one from a geometrical and analytical point of view.

Let $\phi \in C^\infty(M,N)$ satisfying the boundary condition that $\phi(\partial M) \subset \mathcal{S}$ and let $\psi \in C^\infty(\Sigma M \otimes \phi^{-1}TN)$. We impose the free boundary condition for ψ as the chirality boundary condition corresponding to $\mathcal{S}$, namely,

$$\mathbf{B}^\pm_\phi\, \psi|_{\partial M} = 0.$$

or in a local form

$$\psi^i = \mp R^i_j \overrightarrow{n} G\psi^j, \quad i = 1,2,\cdots,d, \quad \text{on } \partial M.$$

When $M = \mathbb{R}^2_+$, we identify the Clifford multiplication by $\partial x_1, \partial x_2$ with the matrices γ_1, γ_2, take the chirality operator $G := i\gamma_1\gamma_2$ and decompose $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$. Then, the chirality boundary condition $\mathbf{B}^\pm_\phi$ corresponding to $\mathcal{S}$ becomes:

$$\psi^i_+ = \mp R^i_j\, \psi^j_-, \quad i = 1,2,\cdots,d, \quad \text{on } \partial M. \tag{3.41}$$

Remark 3.1. *In the physics literature (see* [1]*), the above coordinate system* $\{y^i\}_{i=1,2,\cdots,d}$ *is said to be adapted to the brane* $\mathcal{S}$*. And* (3.41) *is the fermionic boundary condition considered in* [1]*, where it is a priori assumed that there exists some compatible (1,1) tensor R defined on some region including* $\mathcal{S}$*.*

Set

$$\mathcal{X}(M,N;\mathcal{S}) := \left\{(\phi,\psi)|\phi \in C^\infty(M,N), \phi(\partial M) \subset \mathcal{S}; \psi \in C^\infty(\Sigma M \otimes \phi^{-1}TN), \mathbf{B}^\pm_\phi\, \psi|_{\partial M} = 0\right\}.$$

Definition 3.1. $(\phi,\psi) \in \mathcal{X}(M,N;\mathcal{S})$ *is called a Dirac-harmonic map from M to N with free boundary on* $\mathcal{S}$ *if it is a critical point of* $L(\cdot,\cdot)$ *in* $\mathcal{X}(M,N;\mathcal{S})$*.*

Let (ϕ,ψ) be a Dirac-harmonic map from M to N with a free boundary on $\mathcal{S} \subset N$.

First, we consider a family of $(\phi_t,\psi_t) \in \mathcal{X}(M,N;\mathcal{S})$ with $\phi_t \equiv \phi$ and $\frac{d\psi_t}{dt}|_{t=0} = \xi$. Then we calculate

$$\begin{aligned}
\frac{dL(\phi_t,\psi_t)}{dt}|_{t=0} &= \int_M \frac{d}{dt}\langle\psi_t, \not{D}\psi_t\rangle|_{t=0} \\
&= \int_M \langle\xi, \not{D}\psi\rangle + \int_M \langle\psi, \not{D}\xi\rangle \\
&= 2\int_M \mathrm{Re}\langle\xi, \not{D}\psi\rangle - \int_{\partial M} \langle\overrightarrow{n}\cdot\psi, \xi\rangle.
\end{aligned}$$

Note that ψ, ξ satisfy the boundary condition $\mathbf{B}^\pm_\phi$, hence, it follows from Proposition 3.3 that $\int_{\partial M}\langle\overrightarrow{n}\cdot\psi,\xi\rangle = 0$.

Next, we consider a family of $(\phi_t,\psi_t) \in \mathcal{X}(M,N;\mathcal{S})$ with $\frac{d\phi_t}{dt}|_{t=0} = \eta$ and $\psi_t = \psi^i_t \otimes \partial y^i(\phi_t)$, $\psi^i_t \equiv \psi^i$. Then we have

$$\begin{aligned}
\frac{dL(\phi_t,\psi_t)}{dt}|_{t=0} &= \int_M 2\langle d\phi, d\eta\rangle + \int_M \langle\psi, \frac{d}{dt}\not{D}\psi_t\rangle|_{t=0} \\
&= \int_M 2\langle d\phi, d\eta\rangle + \int_M 2\langle R(\phi,\psi),\eta\rangle + \int_M \langle\psi, \not{D}(\psi^i \otimes \nabla_{\partial t}\partial y^i)\rangle \\
&= \int_M 2\langle -\tau(\phi),\eta\rangle + \int_M 2\langle R(\phi,\psi),\eta\rangle + \int_M \langle\not{D}\psi, \psi^i \otimes \nabla_{\partial t}\partial y^i\rangle \\
&\quad + \int_{\partial M} 2\langle\phi_{\overrightarrow{n}},\eta\rangle - \int_{\partial M}\langle\overrightarrow{n}\cdot\psi, \psi^i \otimes \nabla_{\partial t}\partial y^i\rangle.
\end{aligned}$$

Here $\phi_{\vec{n}} = \frac{\partial \phi}{\partial \vec{n}}$. Note that, for simplicity, we used the local expression of ψ, namely, $\psi = \psi^i \otimes \partial y^i$, where y^i is a local coordinate of N. By using the expression $\psi^i \otimes \nabla_{\partial t} \partial y^i = \eta^j \Gamma^k_{ji} \psi^i \otimes \partial y^k$ and requiring the vanishing of the boundary integral, we have

$$0 = \int_{\partial M} 2\langle \phi_{\vec{n}}, \eta \rangle - \int_{\partial M} \langle \vec{n} \cdot \psi, \psi^i \otimes \nabla_{\partial t} \partial y^i \rangle = \int_{\partial M} g_{mj} \left(2\phi^m_{\vec{n}} - g^{nm} \langle \vec{n} \cdot \psi^l, \psi^i \rangle \Gamma^k_{in} g_{kl} \right) \eta^j.$$

Since $\eta = \frac{d\phi_t}{dt}|_{t=0}$ is arbitrary, it follows that

$$\left(2\phi^m_{\vec{n}} - g^{nm} \langle \vec{n} \cdot \psi^l, \psi^i \rangle \Gamma^k_{in} g_{kl} \right) \partial_m \perp \mathcal{S}, \tag{3.42}$$

here and in the sequel, for simplicity, we write $\partial_i := \frac{\partial}{\partial y^i}$, $\partial_\lambda := \frac{\partial}{\partial y^\lambda}$, and $\partial_a := \frac{\partial}{\partial y^a}$ etc..

From the free boundary conditions for the spinor fields:

$$\psi^i = \mp R^i_j \vec{n} G \psi^j, \quad \text{on } \partial M$$

where $R = (R^i_j) = \begin{pmatrix} \delta^a_b & 0 \\ 0 & -\delta^\lambda_\mu \end{pmatrix}$, one easily verifies that

$$\langle \vec{n} \cdot \psi^a, \psi^b \rangle = 0, \quad \langle \vec{n} \cdot \psi^\lambda, \psi^\mu \rangle = 0, \quad \text{on } \partial M$$

for $a, b = 1, 2, \cdots, p$ and $\lambda, \mu = p+1, \cdots, d$.

Let us continue to consider (3.42). We note that

$$\begin{aligned} g^{mn} g_{kl} \Gamma^k_{in} \langle \vec{n} \cdot \psi^l, \psi^i \rangle &= g^{mn} g_{\lambda l} \Gamma^\lambda_{in} \langle \vec{n} \cdot \psi^l, \psi^i \rangle + g^{mn} g_{al} \Gamma^a_{in} \langle \vec{n} \cdot \psi^l, \psi^i \rangle \\ &= g^{mn} g_{\lambda\mu} \Gamma^\lambda_{in} \langle \vec{n} \cdot \psi^\mu, \psi^i \rangle + g^{mn} g_{ab} \Gamma^a_{in} \langle \vec{n} \cdot \psi^b, \psi^i \rangle \\ &= g^{mn} g_{\lambda\mu} \Gamma^\lambda_{an} \langle \vec{n} \cdot \psi^\mu, \psi^a \rangle + g^{mn} g_{ab} \Gamma^a_{\lambda n} \langle \vec{n} \cdot \psi^b, \psi^\lambda \rangle \\ &= g^{mn} g_{\lambda\mu} \Gamma^\lambda_{an} \langle \vec{n} \cdot \psi^\mu, \psi^a \rangle + g^{mn} g_{\lambda\mu} \Gamma^\mu_{bn} \langle \psi^b, \vec{n} \cdot \psi^\lambda \rangle, \end{aligned}$$

namely,

$$g^{mn} g_{kl} \Gamma^k_{in} \langle \vec{n} \cdot \psi^l, \psi^i \rangle = 2 g^{mn} g_{\lambda\mu} \Gamma^\lambda_{an} \langle \vec{n} \cdot \psi^\mu, \psi^a \rangle, \qquad m = 1, 2, \cdots, d.$$

Using this we have

$$\begin{aligned} \left(2\phi^m_{\vec{n}} - g^{nm} \langle \vec{n} \cdot \psi^l, \psi^i \rangle \Gamma^k_{in} g_{kl} \right) \partial_m \perp \mathcal{S} \quad &\Leftrightarrow \quad \left(2\phi^c_{\vec{n}} - g^{nc} \langle \vec{n} \cdot \psi^l, \psi^i \rangle \Gamma^k_{in} g_{kl} \right) \partial_c = 0 \\ &\Leftrightarrow \quad 2\phi^c_{\vec{n}} \partial_c - g^{dc} \langle \vec{n} \cdot \psi^l, \psi^i \rangle \Gamma^k_{id} g_{kl} \partial_c = 0 \\ &\Leftrightarrow \quad (\frac{\partial \phi}{\partial \vec{n}})^\top - g^{cd} \Gamma^\lambda_{ad} g_{\lambda\mu} \langle \vec{n} \cdot \psi^\mu, \psi^a \rangle \partial_c = 0. \end{aligned}$$

On the other hand, for the second fundamental form $A_{\mathcal{S}}(\cdot, \cdot)$ of $\mathcal{S}$ in N, it holds that $A_{\mathcal{S}}(\partial_a, \partial_d) = (\nabla_{\partial_a} \partial_d)^\perp = \Gamma^\mu_{ad} \partial_\mu$, using this in (3.42), we obtain

$$\begin{aligned} \left(2\phi^m_{\vec{n}} - g^{nm} \langle \vec{n} \cdot \psi^l, \psi^i \rangle \Gamma^k_{in} g_{kl} \right) \partial_m \perp \mathcal{S} \Leftrightarrow \left(\frac{\partial \phi}{\partial \vec{n}} \right)^\top &= g^{cd} \langle A_{\mathcal{S}}(\partial_a, \partial_d), \partial_\mu \rangle \langle \vec{n} \cdot \psi^\mu, \psi^a \rangle \partial_c \\ &= g^{cd} \langle A_{\mathcal{S}}(\psi^\top, \partial_d), \vec{n} \cdot \psi^\perp \rangle \partial_c \\ &= g^{cd} \langle P_{\mathcal{S}}(\vec{n} \cdot \psi^\perp; \psi^\top), \partial_d \rangle \partial_c \\ &= P_{\mathcal{S}}(\vec{n} \cdot \psi^\perp; \psi^\top). \end{aligned}$$

Here $P_{\mathcal{S}}(\cdot; \cdot)$ is the shape operator of $\mathcal{S}$ in N. Therefore, we have

Proposition 3.4. *The condition* (3.42) *is equivalent to*

$$\left(\frac{\partial \phi}{\partial \vec{n}} \right)^\top = P_{\mathcal{S}}(\vec{n} \cdot \psi^\perp; \psi^\top),$$

in particular, if $\mathcal{S}$ is a totally geodesic submanifold in N, this reads

$$\frac{\partial \phi}{\partial \overrightarrow{n}} \perp \mathcal{S}.$$

Remark 3.2. *The condition $\frac{\partial \phi}{\partial \overrightarrow{n}} \perp \mathcal{S}$ is exactly the orthogonality condition in the theory of minimal surfaces with free boundaries (see the survey paper by Hildebrandt* [17] *and the references therein). In the case of Dirac-harmonic maps with free boundaries, the orthogonality condition appears when the supporting submanifold $\mathcal{S}$ is totally geodesic or the spinor field vanishes, namely $\psi \equiv 0$.*

The above discussions lead to the following equivalent definition of Dirac-harmonic maps with a free boundary on $\mathcal{S}$.

Definition 3.2. $(\phi, \psi) \in \mathcal{X}(M, N; \mathcal{S})$ *is called a Dirac-harmonic map from M to N with free boundary $\mathcal{S} \subset N$ if (ϕ, ψ) is Dirac-harmonic in M, namely,*

$$\tau(\phi) = R(\phi, \psi),$$
$$\not{D}\psi = 0,$$

and satisfies the following free boundary conditions:

i)

$$\left(\frac{\partial \phi}{\partial \overrightarrow{n}}\right)^{\top} = P_{\mathcal{S}}(\overrightarrow{n} \cdot \psi^{\perp}; \psi^{\top}), \quad \text{on } \partial M$$

ii)

$$\mathbf{B}^{\pm}_{\phi}\, \psi|_{\partial M} = 0.$$

Weakly Dirac-harmonic maps with free boundary on $\mathcal{S}$

In order to define the free boundary conditions for weakly Dirac-harmonic maps, we shall use the isometric embedding $N \hookrightarrow \mathbb{R}^K$. Using the orthogonal decomposition $\mathbb{R}^K_y = T_yN \oplus T^{\perp}_y N$, for any $y \in N$, we can consider the bundles $\Sigma M \otimes \phi^{-1}TN$ and $\mathbf{S}_\phi = \left(\Sigma M \otimes \phi^{-1}TN\right)|_{\partial M}$ as subbundles of $\Sigma M \otimes \phi^{-1}\mathbb{R}^K$ and $\left(\Sigma M \otimes \phi^{-1}\mathbb{R}^K\right)|_{\partial M}$, respectively. Moreover, we denote

$$L^2(\mathbf{S}_\phi) := \left\{\psi|_{\partial M} \mid \psi \in W^{1,4/3}(\Sigma M \otimes \phi^{-1}TN)\right\}.$$

Let $V_\delta N$ be a tubular neighborhood of N in $\mathbb{R}^K$ with a projection $P : V_\delta N \to N$ (see [15]), we define

$$\widetilde{R}(y) := D(\sigma \circ P)(y), \quad y \in \mathcal{S}.$$

For $y \in \mathcal{S}$, since $R(y) = D\sigma(y)$, we have $\widetilde{R}(y) = D(\sigma \circ P)(y) = D\sigma(y) \circ (DP)(y) = R(y) \circ (DP)(y)$. Moreover, for all $V, W \in T_yN$ and $y \in \mathcal{S}$, there holds $DP(y)V = V$ and hence

$$\left\langle \widetilde{R}(y)V, \widetilde{R}(y)W\right\rangle_{\mathbb{R}^K_y} = \langle R(y)[(DP)(y)V], R(y)[(DP)(y)W]\rangle_{\mathbb{R}^K_y} = \langle R(y)V, R(y)W\rangle_{T_yN} = \langle V, W\rangle_{T_yN} = \langle V, W\rangle_{\mathbb{R}^K_y}.$$

On the other hand, since $(\sigma \circ P) \circ (\sigma \circ P) = (\sigma \circ \sigma \circ P) = P = id$ on $\mathbf{U}_\delta \subset N$, we get

$$\widetilde{R}(y)\widetilde{R}(y)V = V, \quad \forall V \in T_yN, \forall y \in \mathcal{S}.$$

Therefore, we can define, in analogy to the case of smooth sections, an endomorphism

$$\overrightarrow{n}G \otimes \widetilde{R} : L^2(\mathbf{S}_\phi) \to L^2(\mathbf{S}_\phi),$$

which is self-adjoint and squares to the identity. Also, we can decompose $\mathbf{S}_\phi = V^+_\phi \oplus V^-_\phi$ and define an elliptic boundary condition

$$\widetilde{\mathbf{B}}^{\pm}_{\phi} : L^2(\mathbf{S}_\phi) \to L^2(V^{\pm}_\phi)$$

for $\not{D}$. For convenience of notation, we still denote $\widetilde{\mathbf{B}}^{\pm}_{\phi}$ by $\mathbf{B}^{\pm}_{\phi}$.

One easily verifies that the results in Proposition 3.3 hold for $W^{1,4/3}$ sections of the bundle $\Sigma M \otimes \phi^{-1}TN$ with $\phi \in H^1(M,N)$. More precisely, we have

Proposition 3.5. *If $\varphi, \psi \in W^{1,4/3}(\Sigma M \otimes \phi^{-1}TN)$ satisfy the chirality boundary condition $\mathbf{B}^{\pm}_{\phi}$, then*

$$\langle \overrightarrow{n} \cdot \psi, \varphi \rangle = 0, \quad \text{a.e. on } \partial M.$$

In particular, we have

$$\int_{\partial M} \langle \overrightarrow{n} \cdot \psi, \varphi \rangle = 0.$$

Now we introduce the class $\mathcal{X}^{1,2}_{1,4/3}(M,N;\mathcal{S})$ of admissible fields (ϕ,ψ) with free boundary on the supporting submanifold $\mathcal{S} \subset N$ as follows:

$$\mathcal{X}^{1,2}_{1,4/3}(M,N;\mathcal{S}) := \left\{(\phi,\psi) | \phi \in H^1(M,N), \phi(\partial M) \subset \mathcal{S}; \psi \in W^{1,4/3}(\Sigma M \otimes \phi^{-1}TN), \mathbf{B}^{\pm}_{\phi}\, \psi|_{\partial M} = 0\right\}$$

where "$\phi(\partial M) \subset \mathcal{S}$" means that the L^2-trace $\phi|_{\partial M}$ of ϕ maps $\mathcal{H}^1$-almost all of ∂M into $\mathcal{S}$ and "$\mathbf{B}^{\pm}_{\phi}\, \psi|_{\partial M} = 0$" means that the L^2-traces $\mathbf{B}^{\pm}_{\phi}\, \psi|_{\partial M}$ vanish on $\mathcal{H}^1$-almost all of ∂M.

Definition 3.3. *$(\phi,\psi) \in \mathcal{X}^{1,2}_{1,4/3}(M,N;\mathcal{S})$ is called a weakly Dirac-harmonic map with free boundary $\mathcal{S}$ if it is a critical point of the action functional $L(\cdot,\cdot)$ in $\mathcal{X}^{1,2}_{1,4/3}(M,N;\mathcal{S})$.*

One verifies, similarly to Wang-Xu [28], that a Dirac-harmonic map with free boundary on $\mathcal{S}$ is invariant under a totally geodesic, isometric embedding of the target. Therefore, adapting Hélein's enlargement argument (see [14, 15]), we assume W.L.O.G. that there exists a global orthonormal frame $\{\widehat{V}_i\}_{i=1}^d$ on N. Set $V_i(x) = \widehat{V}_i(\phi(x))$, $i = 1,2,...,d$, then $\{V_i\}$ is an orthonormal frame along the map ϕ. The spinor field ψ along ϕ can be written as

$$\psi = \sum_{i=1}^{d} \psi^i \otimes V_i,$$

Using the frame $\{\widehat{V}_i\}_{i=1}^d$, it is not difficult to derive (similarly to the calculations in [6, 28]) the following two propositions (proofs omitted):

Proposition 3.6. *Let $(\phi,\psi) \in \mathcal{X}^{1,2}_{1,4/3}(M,N)$ be a weakly Dirac-harmonic map. Then*

$$\int_M d\phi \cdot \nabla V + \int_M \left\langle \psi^i, \gamma_\alpha \cdot \psi^j \right\rangle \left\langle V_i, R(\phi)\,(V, \phi_*(\gamma_\alpha))\, V_j \right\rangle = 0,$$

$$\int_M \langle \psi, \not{D}\xi \rangle = 0,$$

for all compactly supported $V \in H^1 \cap L^\infty(M,\phi^{-1}TN)$ and for all compactly supported $\xi \in W^{1,4/3} \cap L^\infty(\Sigma M \otimes \phi^{-1}TN)$.

Proposition 3.7. *Let $(\phi,\psi) \in \mathcal{X}^{1,2}_{1,4/3}(M,N;\mathcal{S})$ be a weakly Dirac-harmonic map with free boundary on $\mathcal{S}$. Then*

$$\int_M d\phi \cdot \nabla V + \int_M \left\langle \psi^i, \gamma_\alpha \cdot \psi^j \right\rangle \left\langle V_i, R(\phi)\,(V, \phi_*(\gamma_\alpha))\, V_j \right\rangle = 0,$$

$$\int_M \langle \psi, \not{D}\xi \rangle = 0,$$

for all $V \in H^1 \cap L^\infty(M,\phi^{-1}TN)$ such that $V(x) \in T_{\phi(x)}\mathcal{S}$ for a.e.$x \in \partial M$ and for all $\xi \in W^{1,4/3} \cap L^\infty(\Sigma M \otimes \phi^{-1}TN)$ such that $\mathbf{B}^{\pm}_{\phi}\, \xi|_{\partial M} = 0$.

The rest of this section will be devoted to studying the regularity of weakly Dirac-harmonic maps with free boundary on $\mathcal{S}$. For simplicity, we will locate our problem in a small neighborhood of a boundary point. To this end, we consider the case that the domain M is $B_1^+ := \left\{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 \le 1, x_2 \ge 0\right\}$ and the free boundary portion is $I := \left\{(x_1, 0) \in \mathbb{R}^2 | -1 \le x_1 \le 1\right\}$. Moreover, we identify ∂x_α with γ_α, $\alpha = 1, 2$.

The reflection principle

The following Lemma, analogous to Lemma 3.1 in [26], shows that the image of ϕ over a sufficiently small neighborhood of a boundary point is contained in a tubular neighborhood of the supporting submanifold $\mathcal{S}$. Therefore, we can use the geodesic reflection σ to reflect the two fields (ϕ, ψ) across $\mathcal{S}$ when restricted to a sufficiently small domain.

Lemma 3.1. *Let N be a compact Riemannian manifold, isometrically embedded in $\mathbb{R}^K$ and $\mathcal{S}$ a closed submanifold in N. Then there is an $\epsilon_0 = \epsilon_0(N) > 0$ such that for all weakly Dirac-harmonic maps $(\phi, \psi) \in \mathcal{X}^{1,2}_{1,4/3}(B_1^+, N; \mathcal{S})$ with a free boundary on $\mathcal{S}$ and*

$$\int_{B_1^+} \left(|d\phi|^2 + |\psi|^4\right) \le \epsilon_0, \tag{3.43}$$

it holds $\text{dist}(\phi(x), \mathcal{S}) \le C\epsilon_0^{1/2}$ *for all $x \in B_{1/4}^+$ with a constant $C = C(N)$. Moreover, there is a $Q \in \mathcal{S}$ such that $\phi(x) \in B_{C\epsilon_0^{1/2}}(Q)$ for all $x \in B_{1/4}^+$ with a constant $C = C(N)$.*

Proof. To prove this Lemma, it is sufficient to prove an interior estimate for Dirac-harmonic maps on surfaces. More precisely, let $x_0 \in B_{1/4}^+ \backslash \partial\mathbb{R}^2_+$ be an arbitrary point and set $R := \frac{1}{3}\text{dist}(x_0, \partial\mathbb{R}^2_+)$. Given $x \in B_{2R}(x_0)$, one can verify that $B_R(x) \subset B_1^+$. Define

$$\widetilde{\phi}(z) := \phi(x + Rz), \quad \widetilde{\psi}(z) := R^{\frac{1}{2}}\psi(x + Rz), \quad z \in B_1.$$

Then by assumption (3.43), we have

$$\int_{B_1} \left(|d\widetilde{\phi}|^2 + |\widetilde{\psi}|^4\right) = \int_{B_R(x)} \left(|d\phi|^2 + |\psi|^4\right) \le \int_{B_1^+} \left(|d\phi|^2 + |\psi|^4\right) \le \epsilon_0.$$

Provided that ϵ_0 is sufficiently small, then we can apply the ϵ-regularity for Dirac-harmonic maps from surfaces (see Theorem 3.2 in [7] or Theorem 4.3 in [6]) to get

$$\| d\widetilde{\phi} \|_{L^\infty(B_{1/2})} \le C \| d\widetilde{\phi} \|_{L^2(B_1)} \le C\sqrt{\epsilon_0}.$$

where $C > 0$ is a constant depending only on the geometry of N. Note that $d\widetilde{\phi}(0) = R \cdot d\phi(x)$. Hence,

$$|d\phi(x)| = \frac{|d\widetilde{\phi}(0)|}{R} \le \frac{C\sqrt{\epsilon_0}}{R}.$$

for all $x \in B_{2R}(x_0)$.

The rest of the proof can use the same arguments as in the proof of Lemma 3.1 in [26]. Therefore we obtain

$$|\phi(x_0) - \overline{\phi}| \le C\sqrt{\epsilon_0}$$

and

$$\text{dist}(\overline{\phi}, \mathcal{S}) \le C[R^{2-n} \int_{B_{5R}^+(x_1)} |d\phi|^2]^{1/2} \le C\sqrt{\epsilon_0},$$

where $\overline{\phi} := \fint_{B_{5R}^+(x_1)} \phi$.

Furthermore, since $\mathcal{S}$ is compact, then there is a point $Q \in \mathcal{S}$ such that $\text{dist}(\overline{\phi}, \mathcal{S}) = \text{dist}(\overline{\phi}, Q)$. Hence we have

$$\text{dist}(\phi(x_0), Q) \le |\phi(x_0) - \overline{\phi}| + \text{dist}(\overline{\phi}, Q) \le C\sqrt{\epsilon_0}.$$

This completes the proof. □

The above lemma shows that

$$\phi(B^+_{1/4}) \subset \mathbf{U}_\delta := \{z \in N : \mathrm{dist}^N(z, \mathcal{S}) < \delta\}$$

for some $\delta > 0$, provided that the energy of ϕ over the half disk is sufficiently small.

Let $(\phi, \psi) \in \mathcal{X}^{1,2}_{1,4/3}(B^+_1, N; \mathcal{S})$ be a weakly Dirac-harmonic map with free boundary on $\mathcal{S}$. By the conformal invariance of weakly Dirac-harmonic maps from surfaces, we can W.L.O.G assume that $\phi(B^+_1) \subset \mathbf{U}_\delta$.

Denote

$$\Sigma(x) := D\sigma(\phi(x)), \quad x \in B^+_1$$

Define a morphism $\mathbf{T}^\pm_\phi : W^{1,4/3}(\Sigma B^+_1 \otimes \phi^{-1}TN) \to W^{1,4/3}(\Sigma B^+_1 \otimes (\sigma \circ \phi)^{-1}TN)$ by

$$\mathbf{T}^\pm_\phi := \pm i\gamma_1 \otimes \Sigma.$$

Here $\mathbf{T}^\pm_\phi$ corresponds to $\mathbf{B}^\pm_\phi$. In the sequel, we will only consider the case of $(\mathbf{B}^+_\phi, \mathbf{T}^+_\phi)$ and omit the symbol " + ", because the case of $(\mathbf{B}^-_\phi, \mathbf{T}^-_\phi)$ is analogous.

For $x = (x_1, x_2)$, denote $x^* := (x_1, -x_2)$. Then, we extend the two fields (ϕ, ψ) to the lower half disc $B^-_1 := \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 < 1, x_2 \le 0\}$ as follows (and still denote them by (ϕ, ψ)):

$$\begin{aligned} \phi(x^*) &:= \sigma(\phi(x)), \quad x^* \in B^-_1, \\ \psi(x^*) &:= \mathbf{T}_\phi(x)\psi(x), \quad x^* \in B^-_1, \end{aligned}$$

The extension for (ϕ, ψ) is well defined. To see this, we verify that for a.e. $x \in I$ the following hold:

$$\phi(x) = \sigma(\phi(x)), \quad \psi(x) = \left(-\overrightarrow{n} G \otimes R(x)\right)\psi(x) = (i\gamma_1 \otimes \Sigma(x))\,\psi(x) = \mathbf{T}_\phi(x)\psi(x).$$

Using the extended map ϕ, we can extend $\Sigma(x)$ to B_1. Since $\sigma = \sigma^{-1}$, one verifies that (see also [26])

$$\Sigma^{-1}(x) = D\sigma(\phi(x))^{-1} = D\sigma(\sigma(\phi(x))) = D\sigma(\phi(x^*)) = \Sigma(x^*), \tag{3.44}$$

namely, $\Sigma(x)\Sigma(x^*) = Id(\phi(x))$. Moreover, we can extend $\mathbf{T}_\phi$ to some morphism (still denoted by $\mathbf{T}_\phi$): $W^{1,4/3}(\Sigma B_1 \otimes \phi^{-1}TN) \to W^{1,4/3}(\Sigma B_1 \otimes (\sigma \circ \phi)^{-1}TN)$. Note that for $\psi \in W^{1,4/3}(\Sigma B_1 \otimes \phi^{-1}TN)$, if we write $\psi(x) = \psi^i(x) \otimes V_i(x), x \in B_1$, then

$$\psi(x^*) = \mathbf{T}_\phi(x)\psi(x) = i\gamma_1\psi^i(x) \otimes \Sigma(x)V_i(x), \quad x^* \in B_1$$

One checks that $\mathbf{T}_\phi(x)\mathbf{T}_\phi(x^*)\psi(x^*) = \psi(x^*)$ for any $\psi \in W^{1,4/3}(\Sigma B_1 \otimes \phi^{-1}TN)$.

Remark 3.3. *We note that our reflection for Dirac-harmonic maps is a natural generalization of the one for harmonic maps considered by Gulliver-Jost* [13] *and Scheven* [26].

Using the geodesic reflection σ, we are able to extend the metric on the bundle $\phi^{-1}TN \to B^+_1$ to some metric h on the bundle $\phi^{-1}TN \to B_1$ with the extended map ϕ as follows:

$$\langle V(x), W(x)\rangle_h := \begin{cases} \langle V(x), W(x)\rangle, & x \in B^+_1, \\ \langle \Sigma(x)V(x), \Sigma(x)W(x)\rangle, & x \in B^-_1, \end{cases}$$

where $V, W \in \Gamma(B_1, \phi^{-1}TN)$. Consequently, the induced metrics on $\Sigma B^+_1 \otimes \phi^{-1}TN$, $TB^+_1 \otimes \phi^{-1}TN$ and $T^*B^+_1 \otimes \phi^{-1}TN$ extend to metrics (with respect to h) on $\Sigma B_1 \otimes \phi^{-1}TN$, $TB_1 \otimes \phi^{-1}TN$ and $T^*B_1 \otimes \phi^{-1}TN$.

Lemma 3.2. *For $\psi, \varphi \in W^{1,4/3}(\Sigma B_1 \otimes \phi^{-1}TN)$, there holds*

$$\langle \psi(x), \varphi(x)\rangle_h = \left\langle \mathbf{T}_\phi(x)\psi(x), \mathbf{T}_\phi(x)\varphi(x)\right\rangle, \quad \forall x \in B^-_1.$$

Proof. Given $\psi, \varphi \in W^{1,4/3}(\Sigma B_1 \otimes \phi^{-1}TN)$, we write $\psi(x) = \psi^i(x) \otimes V_i(x)$ and $\varphi(x) = \varphi^j(x) \otimes V_j(x)$. Then for $x \in B_1^-$,

$$\begin{aligned}
\langle \psi(x), \varphi(x) \rangle_h &= \left\langle \psi^i(x), \varphi^j(x) \right\rangle \left\langle \Sigma(x)V_i(x), \Sigma(x)V_j(x) \right\rangle \\
&= \left\langle i\gamma_1 \psi^i(x), i\gamma_1 \varphi^j(x) \right\rangle \left\langle \Sigma(x)V_i(x), \Sigma(x)V_j(x) \right\rangle \\
&= \left\langle \mathbf{T}_\phi(x)\psi(x), \mathbf{T}_\phi(x)\varphi(x) \right\rangle .
\end{aligned}$$

Thus, we have proved the lemma. □

Note that given a vector field $V(x) \in T_{\phi(x)}N$, $x \in B_1$, there holds $\Sigma(x)V(x) = D\sigma(\phi(x))V(x) \in T_{\sigma\circ\phi(x)}N$. We define the covariant derivative ∇^h with respect to h as follows (see also [26])

$$\nabla^h_{X(x)} V(x) := \begin{cases} \nabla_{\phi_*(X(x))} V(x), & x \in B_1^+, \\ \Sigma(x^*)\nabla_{(\sigma\circ\phi)_*(X(x))}(\Sigma(x)V(x)) = \Sigma(x^*)\nabla_{\Sigma(x)\phi_*(X(x))}(\Sigma(x)V(x)), & x \in B_1^-. \end{cases}$$

where $X \in \Gamma(TB_1)$, $V \in \Gamma(B_1, \phi^{-1}TN)$ and ∇ is the Levi-Civita connection on N (also denote the induced connection for $\phi^{-1}TN$ by ∇). One easily verifies that ∇^h is compatible with h, namely,

$$d\left(\langle V(x), W(x) \rangle_h\right) = \left\langle \nabla^h V(x), W(x) \right\rangle_h + \left\langle V(x), \nabla^h W(x) \right\rangle_h, \quad x \in B_1. \tag{3.45}$$

Moreover, we define the tensor $R^h(\phi)$ (with symmetries similar to the Riemann curvature tensor $R(\phi)$):

$$R^h(\phi)(V(x), W(x))U(x) := \begin{cases} R(\phi)(V(x), W(x))U(x), & x \in B_1^+, \\ \Sigma(x^*)R(\phi)\left(\Sigma(x)V(x), \Sigma(x)W(x)\right)(\Sigma(x)U(x)), & x \in B_1^-. \end{cases}$$

Recall that the Dirac operator along the map ϕ can be written as:

$$\not{D} = \not{\partial} \otimes Id + \gamma_\alpha \otimes \nabla_{\phi_*(\gamma_\alpha)}.$$

Now we define the Dirac operator along the extended map ϕ with respect to the extended metric h as follows:

$$\not{D}^h := \not{\partial} \otimes Id + \gamma_\alpha \otimes \nabla^h_{\gamma_\alpha}.$$

The following lemma gives a relation between $\not{D}^h$ and $\not{D}$:

Lemma 3.3. *For any $\xi \in W^{1,4/3}(\Sigma B_1 \otimes \phi^{-1}TN)$, denote $\xi^*(x) := \mathbf{T}_\phi(x^*)\xi(x^*)$, $\forall x \in B_1$, then there holds*

$$\not{D}^h_{x^*}\xi(x^*) = \mathbf{T}_\phi(x)\not{D}_x\xi^*(x), \quad \forall x \in B_1.$$

Proof. Write $\xi = \xi^i \otimes V_i$. Then, $\forall x \in B_1$, we calculate

$$\begin{aligned}
\mathbf{T}_\phi(x)\not{D}_x\xi^*(x) &= \mathbf{T}_\phi(x)\not{D}_x\left(i\gamma_1\xi^i(x^*) \otimes \Sigma(x^*)V_i(x^*)\right) \\
&= \mathbf{T}_\phi(x)\left\{\not{\partial}_x(i\gamma_1\xi^i(x^*)) \otimes \Sigma(x^*)V_i(x^*) + \gamma_\alpha(i\gamma_1)\xi^i(x^*) \otimes \nabla_{\phi_*(\partial x_\alpha)}(\Sigma(x^*)V_i(x^*))\right\} \\
&= (i\gamma_1)\not{\partial}_x(i\gamma_1\xi^i(x^*)) \otimes \Sigma(x)\Sigma(x^*)V_i(x^*) + (i\gamma_1)\gamma_\alpha(i\gamma_1)\xi^i(x^*) \otimes \Sigma(x)\nabla_{\phi_*(\partial x_\alpha)}(\Sigma(x^*)V_i(x^*)) \\
&= \not{\partial}_{x^*}\xi^i(x^*) \otimes V_i(x^*) + \gamma_\alpha\xi^i(x^*) \otimes \Sigma(x)\nabla_{\Sigma(x^*)\phi_*(\partial x_\alpha^*)}(\Sigma(x^*)V_i(x^*)).
\end{aligned} \tag{3.46}$$

Here, we have used the fact that

$$(i\gamma_1)\not{\partial}_x(i\gamma_1\xi^i(x^*)) = \not{\partial}_{x^*}\xi^i(x^*)$$

and the following identities (which can be verified using $\phi(x) = \sigma(\phi(x^*))$):

$$\phi_*(\partial x_1) = \Sigma(x^*)\phi_*(\partial x_1^*), \quad \phi_*(\partial x_2) = -\Sigma(x^*)\phi_*(\partial x_2^*),$$

On the other hand, by definition of $\not{D}^h$, we have that $\forall x^* \in B_1$,

$$\begin{aligned}
\not{D}^h_{x^*}\xi(x^*) &= \not{\partial}_{x^*}\xi^i(x^*) \otimes V_i(x^*) + \gamma_\alpha\xi^i(x^*) \otimes \nabla^h_{\partial x_\alpha^*}(V_i(x^*)) \\
&= \not{\partial}_{x^*}\xi^i(x^*) \otimes V_i(x^*) + \gamma_\alpha\xi^i(x^*) \otimes \Sigma(x)\nabla_{\Sigma(x^*)\phi_*(\partial x_\alpha^*)}(\Sigma(x^*)V_i(x^*)).
\end{aligned} \tag{3.47}$$

Combining (3.46) and (3.47) proves the lemma. □

Theorem 3.1. *Let $(\phi, \psi) \in \mathcal{X}^{1,2}_{1,4/3}(B_1^+, N; \mathcal{S})$ be a weakly Dirac-harmonic map with free boundary on $\mathcal{S}$. We extend the two fields (ϕ, ψ) to the whole disk B_1 as before. Then*

$$\int_{B_1} d\phi \cdot_h \nabla^h V + \int_{B_1} \left\langle \psi^i, \gamma_\alpha \cdot \psi^j \right\rangle \left\langle V_i, R^h(\phi)\,(V, \phi_*(\partial x_\alpha))\, V_j \right\rangle_h = 0,$$

$$\int_{B_1} \left\langle \psi, \not{D}^h \xi \right\rangle_h = 0,$$

for all compactly supported $V \in H^1 \cap L^\infty(B_1, \phi^{-1}TN)$ and all compactly supported $\xi \in W^{1,4/3} \cap L^\infty(\Sigma B_1 \otimes \phi^{-1}TN)$.

Proof. First, given a compactly supported vector field $V \in H^1 \cap L^\infty(B_1, \phi^{-1}TN)$. We proceed as in [26] to decompose the vector field V into the equivariant and the antiequivariant part with respect to the diffeomorphism σ, namely, $V = V_e + V_a$, where for $x \in B_1$

$$V_e(x) := \frac{1}{2}[V(x) + \Sigma(x^*)V(x^*)], \quad V_a(x) := \frac{1}{2}[V(x) - \Sigma(x^*)V(x^*)]$$

Since $\Sigma(x)\Sigma(x^*) = Id(\phi(x))$, one checks

$$V_e(x^*) = \Sigma(x)V(x), \quad V_a(x^*) = -\Sigma(x)V_a(x)$$

By (3.44), we have for $x_0 \in I$

$$V_e(x_0) \;=\; \frac{1}{2}[V(x_0) + \Sigma(x_0)V(x_0)] \in T_{\phi(x_0)}\mathcal{S}.$$

Hence, $V_e|_{B_1^+}$ is an admissible variation vector field for ϕ with respect to the free boundary condition $\phi(I) \subset \mathcal{S}$. It follows from Proposition 3.7 that

$$\int_{B_1^+} d\phi \cdot \nabla V_e + \int_{B_1^+} \left\langle \psi^i, \gamma_\alpha \cdot \psi^j \right\rangle \left\langle V_i, R(\phi)\,(V_e, \phi_*(\partial x_\alpha))\, V_j \right\rangle = 0. \tag{3.48}$$

Applying the equivariance of V_e and the symmetry properties of ∇^h (see its definition), one verifies

$$\int_{B_1^-} d\phi \cdot_h \nabla^h V_e = \int_{B_1^+} d\phi \cdot \nabla V_e. \tag{3.49}$$

In view of the antiequivariance of V_a, we calculate analogously and obtain

$$\int_{B_1^-} d\phi \cdot_h \nabla^h V_a = -\int_{B_1^+} d\phi \cdot \nabla V_a. \tag{3.50}$$

Recall that $\psi(x^*) = (i\gamma_\alpha)\psi^i(x) \otimes \Sigma(x)V_i(x)$. We claim that the following two identities hold:

$$\begin{aligned} &\int_{x^* \in B_1^-} \left\langle (i\gamma_1)\psi^i(x), \gamma_\alpha \cdot (i\gamma_1)\psi^j(x) \right\rangle \left\langle \Sigma(x)V_i(x), R^h(\phi)\,(V_e, \phi_*(\partial x^*_\alpha))\, \Sigma(x)V_j(x) \right\rangle_h \\ &= \int_{x \in B_1^+} \left\langle \psi^i(x), \gamma_\alpha \cdot \psi^j(x) \right\rangle \left\langle V_i(x), R(\phi)\,(V_e(x), \phi_*(\partial x_\alpha))\, V_j(x) \right\rangle, \end{aligned} \tag{3.51}$$

$$\begin{aligned} &\int_{x^* \in B_1^-} \left\langle (i\gamma_1)\psi^i(x), \gamma_\alpha \cdot (i\gamma_1)\psi^j(x) \right\rangle \left\langle \Sigma(x)V_i(x), R^h(\phi)\,(V_a, \phi_*(\partial x^*_\alpha))\, \Sigma(x)V_j(x) \right\rangle_h \\ &= -\int_{x \in B_1^+} \left\langle \psi^i(x), \gamma_\alpha \cdot \psi^j(x) \right\rangle \left\langle V_i(x), R(\phi)\,(V_a, \phi_*(\partial x_\alpha))\, V_j(x) \right\rangle. \end{aligned} \tag{3.52}$$

If the claim is true, then combining (3.48) - (3.52) gives

$$\int_{B_1} d\phi \cdot_h \nabla^h V + \int_{B_1} \left\langle \psi^i, \gamma_\alpha \cdot \psi^j \right\rangle \left\langle V_i, R^h(\phi)\,(V, \phi_*(\partial x_\alpha))\, V_j \right\rangle_h = 0.$$

Now it is sufficient to prove the claim. Let $x = (x_1, x_2) \in B_1^+$, then $x^* = (x_1, -x_2) \in B_1^-$. Since $\phi(x^*) = \sigma(\phi(x))$, we have

$$\phi_*(\partial x_1^*) = \Sigma(x)\phi_*(\partial x_1), \quad \phi_*(\partial x_2^*) = -\Sigma(x)\phi_*(\partial x_2).$$

Hence, we calculate

$$\begin{aligned}
& \left\langle (i\gamma_1)\psi^i(x), \gamma_\alpha \cdot (i\gamma_1)\psi^j(x) \right\rangle \left\langle \Sigma(x)V_i(x), R^h(\phi)\left(V_e(x^*), \phi_*(\partial x_\alpha^*)\right)\Sigma(x)V_j(x) \right\rangle_h \\
= & \left\langle (i\gamma_1)\psi^i(x), \gamma_1 \cdot (i\gamma_1)\psi^j(x) \right\rangle \left\langle \Sigma(x)V_i(x), R^h(\phi)\left(\Sigma(x)V_e(x), \Sigma(x)\phi_*(\partial x_1)\right)\Sigma(x)V_j(x) \right\rangle_h \\
& + \left\langle (i\gamma_1)\psi^i(x), \gamma_2 \cdot (i\gamma_1)\psi^j(x) \right\rangle \left\langle \Sigma(x)V_i(x), R^h(\phi)\left(\Sigma(x)V_e(x), -\Sigma(x)\phi_*(\partial x_2)\right)\Sigma(x)V_j(x) \right\rangle_h \\
= & \left\langle \psi^i(x), \gamma_\alpha \cdot \psi^j(x) \right\rangle \left\langle \Sigma(x)V_i(x), R^h(\phi)\left(\Sigma(x)V_e(x), \Sigma(x)\phi_*(\partial x_\alpha)\right)\Sigma(x)V_j(x) \right\rangle_h \\
= & \left\langle \psi^i(x), \gamma_\alpha \cdot \psi^j(x) \right\rangle \left\langle \Sigma(x)V_i(x), \Sigma(x)R(\phi)\left(V_e(x), \phi_*(\partial x_\alpha)\right)V_j(x) \right\rangle_h \\
= & \left\langle \psi^i(x), \gamma_\alpha \cdot \psi^j(x) \right\rangle \left\langle V_i(x), R(\phi)\left(V_e(x), \phi_*(\partial x_\alpha)\right)V_j(x) \right\rangle.
\end{aligned}$$

Integrating the above identity for $x^* \in B_1^-$ and changing variables $x^* \to x$, we have (3.51). Similarly, using the fact that $V_a(x^*) = -\Sigma(x)V_a(x)$, one checks (3.52).

Next, given a compactly supported $\xi \in W^{1,4/3} \cap L^\infty(\Sigma B_1 \otimes \phi^{-1}TN)$. We have (recall that $\overrightarrow{n} = -\gamma_2$)

$$\int_{B_1^+} \langle \psi, \not{D}^h \xi \rangle_h = \int_{B_1^+} \langle \not{D}\psi, \xi \rangle - \int_I \langle (-\gamma_2) \cdot \psi, \xi \rangle.$$

By Lemma 3.2 and Lemma 3.3, we calculate

$$\begin{aligned}
\int_{x^* \in B_1^-} \left\langle \psi(x^*), \not{D}^h_{x^*}\xi(x^*) \right\rangle_h &= \int_{x^* \in B_1^-} \left\langle \mathbf{T}_\phi(x^*)\psi(x^*), \mathbf{T}_\phi(x^*)\not{D}^h_{x^*}\xi(x^*) \right\rangle \\
&= \int_{x \in B_1^+} \langle \psi(x), \not{D}_x \xi^*(x) \rangle \\
&= \int_{x \in B_1^+} \langle \not{D}\psi(x), \xi^*(x) \rangle - \int_I \langle (-\gamma_2) \cdot \psi(x), \xi^*(x) \rangle.
\end{aligned}$$

Hence,

$$\int_{B_1} \langle \psi, \not{D}^h \xi \rangle_h = \int_{B_1^+} \langle \not{D}\psi, \xi + \xi^* \rangle - \int_I \langle (-\gamma_2) \cdot \psi, \xi + \xi^* \rangle. \tag{3.53}$$

For $x \in I$, one verifies that

$$\begin{aligned}
\mathbf{B}_\phi(\xi + \xi^*)(x) &= \frac{1}{2}\left(I \otimes Id - i\gamma_1 \otimes \Sigma\right)(\xi + \xi^*) \\
&= \frac{1}{2}\left(I \otimes Id - i\gamma_1 \otimes \Sigma\right)(\xi + i\gamma_1 \otimes \Sigma\xi) \\
&= \frac{1}{2}\left[(I \otimes Id)\xi - \left((i\gamma_1)^2 \otimes \Sigma^2\right)\xi\right] \\
&= 0.
\end{aligned}$$

Therefore, $\xi + \xi^*$ satisfies the following chirality boundary condition on I:

$$\mathbf{B}_\phi(\xi + \xi^*)|_I = 0.$$

Recall that, by assumption, ψ satisfies the same chirality boundary condition. Hence, by Proposition 3.5,

$$\int_I \langle \overrightarrow{n} \cdot \psi, \xi + \xi^* \rangle = \int_I \langle (-\gamma_2) \cdot \psi, \xi + \xi^* \rangle = 0.$$

Note that $\not{D}\psi = 0$ in B_1^+, we get from (3.53) that

$$\int_{B_1} \langle \psi, \not{D}^h \xi \rangle_h = 0.$$

This completes the proof. □

Continuity of weakly Dirac-harmonic maps at a free boundary

Starting with the global orthonormal frame $V_i(x) = \widehat{V_i}(\phi(x)), i = 1, 2, ..., d$ on $\phi^{-1}TN$, we can apply the orthonormalization procedure by Gram-Schmidt to construct an H^1-tangent frame $e_i(x) \in T_{\phi(x)}N$ that is orthonormal with respect to h (see [26]). This construction gives the following estimate

$$\sup_{1 \le i \le d} |\nabla e_i(x)| \le C |d\phi(x)|, x \in B_1. \tag{3.54}$$

where $C = C(\mathcal{S}, N)$ is a constant.

Define

$$\Re_{lm} := \sum_{i,j,\alpha} \left\langle \psi^i(x), \gamma_\alpha \cdot \psi^j(x) \right\rangle \left\langle e_i, R^h(\phi)\,(e_l, e_m)\, e_j \right\rangle_h dx_\alpha,$$

then, by the symmetry properties of $R^h(\phi)$, one can verify (similarly to [28]) that $\Re_{lm} = -\Re_{ml}$ and $\overline{\Re_{lm}} = \Re_{lm}$, for $1 \le l, m \le d$. Moreover, we get

Proposition 3.8.

$$\Re = (\Re_{lm}) \in L^2(B_1, so(d) \otimes \wedge^1 \mathbb{R}^2).$$

Using $\Re_{lm}$, we can write

$$\begin{aligned} \left\langle \psi^i(x), \gamma_\alpha \cdot \psi^j(x) \right\rangle \left\langle e_i, R^h(\phi)\,(e_l, \phi_*(\partial x_\alpha))\, e_j \right\rangle_h &= \left\langle \psi^i(x), \gamma_\alpha \cdot \psi^j(x) \right\rangle \left\langle e_i, R^h(\phi)\,(e_l, (\phi_*(\partial x_\alpha) \cdot_h e_m) e_m)\, e_j \right\rangle_h \\ &= (\phi_*(\partial x_\alpha) \cdot_h e_m) \left\langle \psi^i(x), \gamma_\alpha \cdot \psi^j(x) \right\rangle \left\langle e_i, R^h(\phi)\,(e_l, e_m)\, e_j \right\rangle_h \\ &= \Re_{lm} \cdot (d\phi \cdot_h e_m). \end{aligned}$$

Note that here $d\phi = \phi_*(\partial x_\alpha) dx_\alpha$ and $d\phi \cdot_h e_m = (\phi_*(\partial x_\alpha) \cdot_h e_m) dx_\alpha$.

Given any $\varphi \in C_0^\infty(B_1)$. Fix $1 \le i \le d$ and take $V = \varphi e_i$ in Theorem 3.1, we get

$$\begin{aligned} 0 &= \int_{B_1} d\phi \cdot_h \nabla^h(\varphi e_i) + \int_{B_1} \left\langle \psi^l(x), \gamma_\alpha \cdot \psi^m(x) \right\rangle \left\langle e_l, R^h(\phi)\,((\varphi e_i), \phi_*(\partial x_\alpha))\, e_m \right\rangle_h \\ &= \int_{B_1} (d\phi \cdot_h e_i) d\varphi + \int_{B_1} (\nabla^h e_i \cdot_h e_j)(d\phi \cdot_h e_i)\varphi + \int_{B_1} \Re_{ij} \cdot (d\phi \cdot_h e_j)\varphi, \end{aligned}$$

Since $\varphi \in C_0^\infty(B_1)$ is arbitrary, we have

$$d^*(d\phi \cdot_h e_i) = \left((\nabla^h e_i \cdot_h e_j) + \Re_{ij} \right) (d\phi \cdot_h e_j). \tag{3.55}$$

Note that $e_i(x) \in T_{\phi(x)}N$ is an H^1-tangent frame that is orthonormal with respect to h and ∇^h is compatible with h, one verifies that $(\nabla^h e_i \cdot_h e_j)$ is antisymmetric with respect to the indices i and j. Moreover, we have

Proposition 3.9.

$$(\nabla^h e_i \cdot_h e_j)_{i,j} \in L^2(B_1, so(d) \otimes \wedge^1 \mathbb{R}^2).$$

To proceed, let us recall the Coulomb gauge construction theorem due to Rivière [24] and Rivière-Struwe [25] (we only need to consider the case that the domain is two dimensional and hence we use the norm L^2 instead of $M^{2,2}$).

Lemma 3.4. *There exist* $\epsilon_1 > 0$ *and* $C > 0$ *such that if* $\Omega \in L^2(B_1, so(d) \otimes \wedge^1\mathbb{R}^2)$ *satisfies*

$$\| \Omega \|_{L^2(B_1)} \leq \epsilon_1,$$

then there exist $P \in H^1(B_1, SO(d))$ *and* $\zeta \in H^1(B_1, so(d) \otimes \wedge^2\mathbb{R}^2)$ *such that*

$$\begin{aligned} P^{-1}dP + P^{-1}\Omega P &= d^*\zeta \quad \text{in } B_1, \\ d\zeta &= 0 \quad \text{in } B_1, \\ \zeta &= 0 \quad \text{on } \partial B_1. \end{aligned}$$

Moreover, ∇P *and* $\nabla\zeta$ *belong to* $L^2(B_1)$ *with*

$$\| \nabla P \|_{L^2(B_1)} + \| \nabla\zeta \|_{L^2(B_1)} \leq C \| \Omega \|_{L^2(B_1)} \leq C\epsilon_1.$$

The above lemma can be applied to study the regularity of weakly Dirac-harmonic maps with free boundary when the two fields are extended to the whole disc.

Lemma 3.5. *There exists* $\epsilon_2 > 0$ *such that if* $(\phi, \psi) \in \mathcal{X}^{1,2}_{1,4/3}(B_1^+, N; \mathcal{S})$ *is a weakly Dirac-harmonic map with free boundary on* $\mathcal{S}$ *satisfying*

$$\| d\phi \|^2_{L^2(B_1^+)} + \| \psi \|^4_{L^4(B_1^+)} \leq \epsilon_2^2,$$

then $\phi \in C^{0,\alpha}(B^+_{1/2}, N)$*, for any* $\alpha \in (0, 1)$*. Moreover, we have*

$$[\phi]_{C^{0,\alpha}(B^+_{1/2})} \leq C \| d\phi \|_{L^2(B_1^+)} .$$

Remark 3.4. *The scheme of proof will be similar to the ones of* [25, 28]*, however we need to present the details here in order to set up our framework for the extended metric h.*

Proof. First we extend the two fields (ϕ, ψ) to the whole disk B_1 as before. Then, combing Proposition 3.8 and Proposition 3.9 gives $\Omega = (\Omega_{ij}) := \left((\nabla^h e_i \cdot_h e_j) + \Re_{ij}\right) \in L^2(B_1, so(d) \otimes \wedge^1\mathbb{R}^2)$. Moreover, (3.54) gives

$$\| \Omega \|_{L^2(B_1)} \leq C[\| d\phi \|_{L^2(B_1^+)} + \| \psi \|^2_{L^4(B_1^+)}] \leq C\epsilon_2 \leq \epsilon_1,$$

where $\epsilon_1 > 0$ is the same constant as in Lemma 3.3 and $\epsilon_2 > 0$ is chosen to be sufficiently small. Hence, it follows from Lemma 3.3 that there are $P \in H^1(B_1, SO(d))$ and $\zeta \in H^1(B_1, so(d) \otimes \wedge^2\mathbb{R}^2)$ satisfying

$$\begin{aligned} P^{-1}dP + P^{-1}\Omega P &= d^*\zeta \quad \text{in } B_1, \\ d\zeta &= 0 \quad \text{in } B_1, \\ \zeta &= 0 \quad \text{on } \partial B_1. \end{aligned} \tag{3.56}$$

and

$$\| \nabla P \|_{L^2(B_1)} + \| \nabla\zeta \|_{L^2(B_1)} \leq C \| \Omega \|_{L^2(B_1)} \leq C\epsilon_2.$$

We write $P = (P_{ij}), P^{-1} = (P_{ji})$, and $\zeta = (\zeta_{ij})$. Since $P \in H^1(B_1, SO(d))$ and hence $P^{-1}P = P^TP = I_d$, we have $dP^{-1} = -P^{-1}dPP^{-1}$. Using (3.55) and (3.56), we calculate

$$d^*\left[P^{-1}\begin{pmatrix} d\phi \cdot_h e_1 \\ \vdots \\ d\phi \cdot_h e_d \end{pmatrix}\right] = \left(dP^{-1}P + P^{-1}\Omega P\right) \cdot P^{-1}\begin{pmatrix} d\phi \cdot_h e_1 \\ \vdots \\ d\phi \cdot_h e_d \end{pmatrix} = -d^*\zeta \cdot P^{-1}\begin{pmatrix} d\phi \cdot_h e_1 \\ \vdots \\ d\phi \cdot_h e_d \end{pmatrix}$$

Equivalently, we have

$$-d^*(P_{ji}(d\phi \cdot_h e_j)) = d^*\zeta_{il} \cdot (P_{ml}(d\phi \cdot_h e_m)), \quad i = 1, 2, ..., d, \quad \text{in } B_1. \tag{3.57}$$

For any $0 < R \leq 1/4$, let $B_R \subset B_{1/2}$ be an arbitrary disc of radius R and $\tau \in C_0^\infty(B_{1/2})$ satisfying $0 \leq \tau \leq 1, \tau \equiv 1$ in B_R, $\tau \equiv 0$ outside B_{2R}, and $|\nabla\tau| \leq 4/R$. Denote $\widetilde{\phi} := \tau(\phi - \overline{\phi}_R)$, where $\overline{\phi}_R := \fint_{B_R} \phi$.

For each $1 \le i \le d$, the 1-form $\Sigma_{j=1}^{d} P_{ji}(d\widetilde{\phi} \cdot_h e_j) \in L^2(\mathbb{R}^2, \wedge^1\mathbb{R}^2)$, extended by 0 outside of B_{2R}, admits a Hodge-de Rham decomposition of the following form

$$\Sigma_{j=1}^{d} P_{ji}(d\widetilde{\phi} \cdot_h e_j) = df_i + d^* g_i + h_i, \tag{3.58}$$

where $f_i \in H_0^1(B_R)$, $g_i \in H_0^1(B_R, \wedge^2\mathbb{R}^2)$ is a closed 2-form, namely, $dg_i = 0$ in B_R, and $h_i \in L^2(B_R, \wedge^1\mathbb{R}^2)$ is a harmonic 1-form (we refer to Iwaniec-Martin [18] for more details of the Hodge decomposition of forms in Sobolev spaces).

Taking first d^* and then d of both sides of (3.58) and applying (3.57) gives for $1 \le i \le d$,

$$\begin{aligned} -\Delta f_i &= d^* \zeta_{il}(P_{jl}(d\phi \cdot_h e_j)) \quad \text{in } B_R, \\ \Delta g_i &= dP_{ji} \wedge (d\phi \cdot_h e_j) + P_{ji} d\phi \wedge_h de_j \quad \text{in } B_R. \end{aligned}$$

For $1 < p < 2$, let $q = p/(p-1)$ be the conjugate exponent. By the duality characterization of $\| \nabla f \|_{L^p(B_R)}$ for $f \in W_0^{1,p}(B_R)$, we get

$$\| \nabla f \|_{L^p(B_R)} \le C \sup \left\{ \int_{B_R} \nabla f \cdot \nabla \varphi dx : \varphi \in W_0^{1,q}(B_R), \| \nabla \varphi \|_{L^q(B_R)} \le 1 \right\}. \tag{3.59}$$

Since $q > 2$, by the Sobolev embedding theorem, we have $W_0^{1,q}(B_R) \hookrightarrow C^{0,1-2/q}(B_R)$ and for $\varphi \in W_0^{1,q}(B_R)$ with $\| \nabla \varphi \|_{L^q(B_R)} \le 1$ the following estimate holds

$$\| \varphi \|_{L^\infty(B_R)} \le CR^{1-2/q}, \quad \| \nabla \varphi \|_{L^2(B_R)} \le CR^{1-2/q}. \tag{3.60}$$

For any such φ, we can estimate f_i (similarly to Rivière-Struwe [25] and Wang-Xu [28]) as follows:

$$\begin{aligned} \int_{B_R} df_i \cdot d\varphi = -\int_{B_R} \Delta f_i \cdot \varphi &= \int_{B_R} d^* \zeta_{il} \cdot (P_{jl}(d\phi \cdot_h e_j)) \cdot \varphi \\ &= \int_{B_R} d^* \zeta_{il} \cdot (P_{jl}(d\phi \cdot \widehat{e_j})) \cdot \varphi \\ &= -\int_{B_R} d^* \zeta_{il} \cdot d(P_{jl} \widehat{e_j} \varphi) \widetilde{\phi} \\ &\le C \| d^* \zeta_{il} \cdot d(P_{jl} \widehat{e_j} \varphi) \|_{\mathcal{H}^1(\mathbb{R}^2)} [\phi]_{\mathrm{BMO}(B_R)} \\ &\le C \| \nabla \zeta \|_{L^2(B_R)} \left(\| \nabla P \|_{L^2(B_R)} + \sum_j \| \nabla \widehat{e_j} \|_{L^2(B_R)} \right) \| \varphi \|_{L^\infty(B_R)} [\phi]_{\mathrm{BMO}(B_R)} \\ &\quad + C \| \nabla \zeta \|_{L^2(B_R)} \| \nabla \varphi \|_{L^2(B_R)} [\phi]_{\mathrm{BMO}(B_R)} \\ &\le C \| \nabla \zeta \|_{L^2(B_R)} \left(\| \nabla P \|_{L^2(B_R)} + \| d\phi \|_{L^2(B_R)} \right) \| \varphi \|_{L^\infty(B_R)} [\phi]_{\mathrm{BMO}(B_R)} \\ &\quad + C \| \nabla \zeta \|_{L^2(B_R)} \| \nabla \varphi \|_{L^2(B_R)} [\phi]_{\mathrm{BMO}(B_R)} \\ &\le C\epsilon_2 \left[\| \varphi \|_{L^\infty(B_R)} + \| \nabla \varphi \|_{L^2(B_R)} \right] [\phi]_{\mathrm{BMO}(B_R)} \\ &\le C\epsilon_2 R^{2/p-1} [\phi]_{\mathrm{BMO}(B_R)}, \end{aligned}$$

where we have used the notations $d\phi \cdot_h e_j = d\phi \cdot (h_{jl} e_l)$, $\widehat{e_j} := h_{jl} e_l$, (3.54) and the following estimates:

$$\sum_j |\widehat{e_j}| \le C \sum_j |e_j|, \quad \sum_j \| \nabla \widehat{e_j} \|_{L^2(B_R)} \le C \| d\phi \|_{L^2(B_R)}.$$

By (3.59), we get

$$\left(R^{p-2} \int_{B_R} |\nabla f_i|^p \right)^{1/p} \le C\epsilon_2 [\phi]_{\mathrm{BMO}(B_R)}. \tag{3.61}$$

Similarly, for any $\varphi \in W_0^{1,q}(B_R)$ satisfying (3.60), we can estimate g_i as follows

$$\begin{aligned}
\int_{B_R} dg_i \cdot d\varphi &= -\int_{B_R} \Delta g_i \cdot \varphi \\
&= -\int_{B_R} \Big[dP_{ji} \wedge (d\phi \cdot_h e_j) + P_{ji} d\phi \wedge_h de_j\Big]\varphi \\
&= -\int_{B_R} \Big[dP_{ji} \wedge (d\phi \cdot \widehat{e_j}) + P_{ji} d\phi \wedge (h_{jl} de_l)\Big]\varphi \\
&= \int_{B_R} \Big[dP_{ji} \wedge d(\varphi\widehat{e_j}) + d(P_{ji}h_{jl}\varphi) \wedge de_l\Big]\widetilde{\phi} \\
&\leq C\Big[\| dP_{ji} \wedge d(\varphi\widehat{e_j}) \|_{\mathcal{H}^1(\mathbb{R}^2)} + \| d(P_{ji}h_{jl}\varphi) \wedge de_l \|_{\mathcal{H}^1(\mathbb{R}^2)}\Big][\phi]_{\mathrm{BMO}(B_R)} \\
&\leq C \| \nabla P \|_{L^2(B_R)} \left(\| \nabla\varphi \|_{L^2(B_R)} + \sum_j \| \nabla\widehat{e_j} \|_{L^2(B_R)} \| \varphi \|_{L^\infty(B_R)}\right)[\phi]_{\mathrm{BMO}(B_R)} \\
&\quad + C\left(\sum_l \| \nabla e_l \|_{L^2(B_R)}\right)\Big(\| \nabla\varphi \|_{L^2(B_R)} + (\| \nabla P \|_{L^2(B_R)} + \| \nabla h \|_{L^2(B_R)}) \| \varphi \|_{L^\infty(B_R)}\Big)[\phi]_{\mathrm{BMO}(B_R)} \\
&\leq C \| \nabla P \|_{L^2(B_R)} \Big(\| \nabla\varphi \|_{L^2(B_R)} + \| d\phi \|_{L^2(B_R)} \| \varphi \|_{L^\infty(B_R)}\Big)[\phi]_{\mathrm{BMO}(B_R)} \\
&\quad + C \| d\phi \|_{L^2(B_R)} \Big(\| \nabla\varphi \|_{L^2(B_R)} + \| \nabla P \|_{L^2(B_R)} \| \varphi \|_{L^\infty(B_R)} + \| \varphi \|_{L^\infty(B_R)}\Big)[\phi]_{\mathrm{BMO}(B_R)} \\
&\leq C\epsilon_2 R^{2/p-1}[\phi]_{\mathrm{BMO}(B_R)}.
\end{aligned}$$

Again, using (3.59), we have

$$\left(R^{p-2}\int_{B_R} |\nabla g_i|^p\right)^{1/p} \leq C\epsilon_2[\phi]_{\mathrm{BMO}(B_R)}. \tag{3.62}$$

To estimate the harmonic 1-form h_i, we apply the classical Campanato estimates for harmonic functions (see Giaquinta [11]) , (3.61) and (3.62) to get that for any $0 < r < R$,

$$\begin{aligned}
r^{p-2}\int_{B_r} |h_i|^p &\leq C\left(\frac{r}{R}\right)^p \left(R^{p-2}\int_{B_R} |h_i|^p\right) \\
&\leq C\left(\frac{r}{R}\right)^p \left(R^{p-2}\int_{B_R} |P_{ji}(d\widetilde{\phi} \cdot_h e_j) - df_i - d^* g_i|^p\right) \\
&\leq C\left(\frac{r}{R}\right)^p \left(R^{p-2}\int_{B_R} |d\phi|^p + |\nabla f_i|^p + |\nabla g_i|^p\right) \\
&\leq C\left(\frac{r}{R}\right)^p \left(R^{p-2}\int_{B_R} |d\phi|^p + \epsilon_2^p[\phi]^p_{\mathrm{BMO}(B_R)}\right)
\end{aligned}$$

To proceed, we note that by the definition of the extended metric h, there holds (we may need to take $\delta > 0$ small enough so that the tubular neighborhood $\mathbf{U}_\delta$ of $\mathcal{S}$ is sufficiently close to $\mathcal{S}$)

$$|d\phi| \leq C(N, \mathcal{S}) \sum_i |d\phi \cdot_h e_i|,$$

Then using $\widetilde{d\phi}\cdot_h e_j = P_{ij}(df_i + d^*g_i + h_i)$ and $P \in H^1(B_1, SO(d))$, we can estimate

$$\begin{aligned} r^{p-2}\int_{B_r}|d\phi|^p &\leq Cr^{p-2}\int_{B_R}(|\nabla f_i|^p + |\nabla g_i|^p) + Cr^{p-2}\int_{B_r}|h_i|^p \\ &\leq Cr^{p-2}\int_{B_R}(|\nabla f_i|^p + |\nabla g_i|^p) + C\left(\frac{r}{R}\right)^p\left(R^{p-2}\int_{B_R}|d\phi|^p + \epsilon_2^p[\phi]^p_{\mathrm{BMO}(B_R)}\right) \\ &\leq C\left(\frac{r}{R}\right)^p\left\{R^{p-2}\int_{B_R}|d\phi|^p + \left(\frac{r}{R}\right)^{-2}\epsilon_2^p\,[\phi]^p_{\mathrm{BMO}(B_R)}\right\}. \end{aligned}$$

An iteration argument (see [25, 28] for more details), combined with Morrey's decay lemma (see [11]), implies that $\phi \in C^{0,\alpha}(B_{1/2})$, for any $\alpha \in (0,1)$ and $[\phi]_{C^{0,\alpha}(B_{1/2})} \leq C \parallel d\phi \parallel_{L^2(B_1)}$. Since ϕ is extended to B_1 by reflection, it follows that $[\phi]_{C^{0,\alpha}(B^+_{1/2})} \leq C \parallel d\phi \parallel_{L^2(B_1)} \leq C \parallel d\phi \parallel_{L^2(B_1^+)}$. Thus, we have completed the proof. □

Theorem 3.2. *Let M be a compact Riemann spin surface with boundary ∂M, N be any compact Riemannian manifold, and $\mathcal{S}$ be a closed submanifold of N. Let (ϕ,ψ) be a weakly Dirac-harmonic map from M to N with free boundary on $\mathcal{S}$. Then for any $\alpha \in (0,1)$,*

$$\phi \in C^{0,\alpha}(M,N).$$

Proof. Applying Lemma 3.5 and rescaling the two fields (ϕ,ψ) if necessary. □

Higher regularity of continuous weakly Dirac-harmonic maps at a free boundary

Let (ϕ,ψ) be a weakly Dirac-harmonic map from M to N with free boundary on $\mathcal{S} \subset N$ and suppose that $\phi \in C^{0,\alpha}(M,N)$ for any $\alpha \in (0,1)$. For simplicity, we assume that $M = B_1^+$ and consider the higher regularity of ϕ at the boundary point $0 \in I$. As before, we take the *adapted* coordinates $\{y^i\}_{i=1,2,\cdots,d}$ in some neighborhood $\mathrm{U} \subset \mathbf{U}_\delta$ of the point $\phi(0) \in \mathcal{S}$. By conformal invariance and continuity of ϕ, we assume W.L.O.G. $\phi(B_1^+) \subset \mathrm{U} \subset \mathbf{U}_\delta$. Denote

$$\eta_i := \begin{cases} 1, & i = 1,...,p, \\ -1, & i = p+1,...,d. \end{cases}$$

Then the two extended fields (ϕ,ψ) can be written as follows for $k = 1,2,...,d$:

$$\phi^k(x) = \begin{cases} \phi^k(x), & x \in B_1^+, \\ \eta_k\phi^k(x^*), & x \in B_1^-. \end{cases}$$

and

$$\psi_+^k(x) = \begin{cases} \psi_+^k(x), & x \in B_1^+, \\ \eta_k\psi_-^k(x^*), & x \in B_1^-. \end{cases}$$

$$\psi_-^k(x) = \begin{cases} \psi_-^k(x), & x \in B_1^+, \\ \eta_k\psi_+^k(x^*), & x \in B_1^-. \end{cases}$$

One can verify that

$$\partial y^k(\phi(x^*)) \;=\; \eta_k\, D\sigma(\phi(x))\,\partial y^k(\phi(x)) = \eta_k\,\Sigma(x)\,\partial y^k(\phi(x)), \quad x \in B_1, \quad k = 1,2,\cdots,d.$$

For convenience of notation, we shall henceforth also denote the extended metric h by $\widetilde{g}$.

Now we define some geometric data associated to the extended metric $\widetilde{g}$ as follows for $x \in B_1$:

$$\begin{aligned}
\widetilde{g}_{ij}(\phi(x)) &:= \left\langle \partial y^i(\phi(x)), \partial y^j(\phi(x)) \right\rangle_{\widetilde{g}} \\
\widetilde{\Gamma}^k_{ij}(\phi(x)) &:= \begin{cases} \widetilde{g}^{kl}(\phi(x)) \left\langle \nabla_{\partial y^i(\phi(x))} \partial y^j(\phi(x)), \partial y^l(\phi(x)) \right\rangle_{\widetilde{g}}, & x \in B_1^+ \\ \widetilde{g}^{kl}(\phi(x)) \left\langle \Sigma(x^*) \nabla_{\Sigma(x)\partial y^i(\phi(x))} \Sigma(x) \partial y^j(\phi(x)), \partial y^l(\phi(x)) \right\rangle_{\widetilde{g}}, & x \in B_1^- \end{cases} \\
\widetilde{R}_{mlij}(\phi(x)) &:= \left\langle \partial y^j(\phi(x)), R^h(\phi)(\partial y^m(\phi(x)), \partial y^l(\phi(x))) \partial y^i(\phi(x)) \right\rangle_{\widetilde{g}} \\
\widetilde{R}^m_{lij}(\phi(x)) &:= \widetilde{g}^{mk}(\phi(x)) \widetilde{R}_{ijlk}(\phi(x)),
\end{aligned}$$

where $\left(\widetilde{g}^{mk}(\phi(x))\right)_{mk}$ is the inverse matrix of $(\widetilde{g}_{mk}(\phi(x)))_{mk}$. Then, we have

Lemma 3.6.

$$\widetilde{g}_{ij}(\phi(x)) = \begin{cases} g_{ij}(\phi(x)), & x \in B_1^+, \\ \eta_i \eta_j g_{ij}(\phi(x^*)), & x \in B_1^-. \end{cases}$$

$$\widetilde{\Gamma}^k_{ij}(\phi(x)) = \begin{cases} \Gamma^k_{ij}(\phi(x)), & x \in B_1^+, \\ \eta_i \eta_j \eta_k \Gamma^k_{ij}(\phi(x^*)), & x \in B_1^-. \end{cases}$$

$$\widetilde{R}_{mlij}(\phi(x)) = \begin{cases} R_{mlij}(\phi(x)), & x \in B_1^+, \\ \eta_i \eta_j \eta_l \eta_m R_{mlij}(\phi(x^*)), & x \in B_1^-. \end{cases}$$

$$\widetilde{R}^m_{lij}(\phi(x)) = \begin{cases} R^m_{lij}(\phi(x)), & x \in B_1^+, \\ \eta_i \eta_j \eta_l \eta_m \widetilde{R}^m_{lij}(\phi(x^*)), & x \in B_1^-. \end{cases}$$

Proof. By definition of h and $R^h(\phi)$, it is sufficient to consider the case of $x \in B_1^-$. For this, we calculate

$$\begin{aligned}
\widetilde{g}_{ij}(\phi(x)) = \left\langle \partial y^i(\phi(x)), \partial y^j(\phi(x)) \right\rangle_{\widetilde{g}} &= \left\langle \Sigma(x) \partial y^i(\phi(x)), \Sigma(x) \partial y^j(\phi(x)) \right\rangle \\
&= \left\langle \eta_i \partial y^i(\phi(x^*)), \eta_j \partial y^j(\phi(x)) \right\rangle \\
&= \eta_i \eta_j g_{ij}(\phi(x^*)).
\end{aligned}$$

It is easy to verify that $\widetilde{g}^{ij}(\phi(x)) = \eta_i \eta_j g^{ij}(\phi(x^*))$. Moreover, we have

$$\begin{aligned}
\widetilde{\Gamma}^k_{ij}(\phi(x)) &= \widetilde{g}^{kl}(\phi(x)) \left\langle \Sigma(x) \Sigma(x^*) \nabla_{\Sigma(x)\partial y^i(\phi(x))} \Sigma(x) \partial y^j(\phi(x)), \Sigma(x) \partial y^l(\phi(x)) \right\rangle \\
&= \eta_k \eta_l g^{kl}(\phi(x^*)) \eta_i \eta_j \eta_l \left\langle \nabla_{\partial y^i(\phi(x^*))} \partial y^j(\phi(x^*)), \partial y^l(\phi(x^*)) \right\rangle \\
&= \eta_i \eta_j \eta_k \Gamma^k_{ij}(\phi(x^*)).
\end{aligned}$$

$$\begin{aligned}
\widetilde{R}_{mlij}(\phi(x)) &= \left\langle \partial y^j(\phi(x)), R^h(\phi)(\partial y^m(\phi(x)), \partial y^l(\phi(x))) \partial y^i(\phi(x)) \right\rangle_{\widetilde{g}} \\
&= \left\langle \Sigma(x) \partial y^j(\phi(x)), \Sigma(x) \Sigma(x^*) R(\phi)(\Sigma(x) \partial y^m(\phi(x)), \Sigma(x) \partial y^l(\phi(x))) \Sigma(x) \partial y^i(\phi(x)) \right\rangle \\
&= \left\langle \eta_j \partial y^j(\phi(x^*)), R(\phi)(\eta_m \partial y^m(\phi(x^*)), \eta_l \partial y^l(\phi(x^*))) \eta_i \partial y^i(\phi(x^*)) \right\rangle \\
&= \eta_i \eta_j \eta_l \eta_m R_{mlij}(\phi(x^*)).
\end{aligned}$$

and

$$\widetilde{R}^m_{lij}(\phi(x)) = \widetilde{g}^{mk}(\phi(x)) \widetilde{R}_{ijlk}(\phi(x)) = \eta_m \eta_k \eta_k \eta_l \eta_i \eta_j g^{mk}(\phi(x^*)) R_{ijlk}(\phi(x^*)) = \eta_i \eta_j \eta_l \eta_m R^m_{lij}(\phi(x^*)).$$

□

Remark 3.5. *In the adapted coordinates $\{y^i\}$, we have $g_{ij}(y) = 0$, for $y \in$ U, $i \in \{1, 2, ..., p\}$, $j \in \{p+1, ..., d\}$. Hence, both $\widetilde{g}_{ij}(\phi)$ and $\widetilde{g}^{ij}(\phi)$ are continuous (they are in fact Lipschitz, see also* [13]*).*

Now we can write the equations of the extended fields (ϕ, ψ) in terms of the data $\widetilde{g}_{ij}$, $\widetilde{\Gamma}^k_{ij}$ and $\widetilde{R}^m{}_{lij}$.

Proposition 3.10. *Assumptions and notations as before. The extended fields (ϕ, ψ) satisfy in B_1*

$$\Delta\phi^m + \widetilde{\Gamma}^m_{ij}(\phi)\phi^i_\alpha\phi^j_\alpha - \frac{1}{2}\widetilde{R}^m{}_{lij}(\phi)\langle\psi^i, \nabla\phi^l \cdot \psi^j\rangle = 0, \quad m = 1, 2, \cdots, d,$$

$$\partial\!\!\!/\psi^i + \widetilde{\Gamma}^i_{jk}(\phi)\partial_\alpha\phi^j\gamma_\alpha \cdot \psi^k = 0, \quad i = 1, 2, \cdots, d,$$

Proof. Note that $\phi(B_1^+) \subset$ U, the proposition follows from applying Lemma 3.6 and Theorem 3.1 with $V_i(x) = \partial y^i(\phi(x))$, $V(x) = \widetilde{g}^{mj}(\phi(x))\eta_j(x)\otimes\partial y^m(\phi(x))$, $\xi = \widetilde{g}^{mk}(\phi(x))\xi_k(x)\otimes\partial y^m(\phi(x))$, where $\eta_j \in H^1_0 \cap L^\infty(B_1)$ and $\xi_k \in W^{1,4/3}_0 \cap L^\infty(\Sigma B_1)$ are arbitrarily chosen. □

Proposition 3.11. *Assumptions and notations as before. If in addition we assume that $\mathcal{S}$ is totally geodesic, then for all $m, i, j \in \{1, 2, ..., d\}$ and any $\gamma \in (0, 1)$,*

$$\widetilde{\Gamma}^m_{ij}(\phi) \in C^{0,\gamma}(B_1).$$

Proof. By definition, we have $\widetilde{\Gamma}^m_{ij}(\phi(x)) = \eta_i\eta_j\eta_m\Gamma^m_{ij}(\phi(x^*))$, for $x \in B_1^+$. Note that both Γ^m_{ij} and ϕ are continuous, hence, to prove the continuity of $\widetilde{\Gamma}^m_{ij}(\phi)$, it is sufficient to show that the following terms

$$\Gamma^\perp_{\top\top}, \quad \Gamma^\top_{\top\perp}, \quad \Gamma^\perp_{\perp\perp} \tag{3.63}$$

vanish on $\mathcal{S}$. Here and in the sequel, $\top$ denotes the tangential index $\{1, 2, ..., p\}$ and $\perp$ denotes the normal index $\{p + 1, p + 2, ..., d\}$. To verify this, firstly we note that (see [12])

$$\begin{aligned} g_{\perp\perp} &\equiv 1, \quad \text{on U} \\ g_{\top\perp} &\equiv 0, \quad \text{on U} \end{aligned} \tag{3.64}$$

It follows that

$$g_{\perp\perp,\perp} = g_{\perp\perp,\top} = g_{\top\perp,\perp} = g_{\top\perp,\top} \equiv 0, \quad \text{on U}.$$

Next, we calculate

$$\begin{aligned} \Gamma^\perp_{\top\top} &= \frac{1}{2}g^{\perp\perp}(g_{\perp\top,\top} + g_{\perp\top,\top} - g_{\top\top,\perp}) = -\frac{1}{2}g^{\perp\perp}g_{\top\top,\perp} \quad \text{on U} \\ \Gamma^\top_{\top\perp} &= \frac{1}{2}g^{\top\top}(g_{\top\perp,\top} + g_{\top\top,\perp} - g_{\top\perp,\top}) = \frac{1}{2}g^{\top\top}g_{\top\top,\perp} \quad \text{on U} \\ \Gamma^\perp_{\perp\perp} &= \frac{1}{2}g^{\perp\perp}g_{\perp\perp,\perp} = 0, \quad \text{on U} \end{aligned}$$

Since $\mathcal{S}$ is totally geodesic, we have $\Gamma^\perp_{\top\top} = 0$ on $\mathcal{S}$. Therefore,

$$\Gamma^\perp_{\top\top} = -\frac{1}{2}g^{\perp\perp}g_{\top\top,\perp} = 0, \quad \text{on } \mathcal{S}.$$

By (3.64), it follows that

$$\Gamma^\top_{\top\perp} = \frac{1}{2}g^{\top\top}g_{\top\top,\perp} = 0, \quad \text{on } \mathcal{S}$$

Now we have verified that all the terms in (3.63) vanish on $\mathcal{S}$ and hence $\widetilde{\Gamma}^m_{ij}(\phi) \in C^0$. Moreover, we can write

$$\widetilde{\Gamma}^m_{ij}(\phi(x)) = \begin{cases} \Gamma^m_{ij}(\phi(x)), & x \in B_1^+, \\ \Gamma^m_{ij}(\phi(x^*)), & x \in B_1^-. \end{cases}$$

Note that $\phi(B_1^+) \subset$ U, $\Gamma^m_{ij} \in C^1(\overline{\text{U}})$ and $\phi \in C^{0,\gamma}(B_1^+)$ for any $\gamma \in (0, 1)$. Therefore, for any $\gamma \in (0, 1)$, we have $\| \widetilde{\Gamma}^m_{ij}(\phi) \|_{C^{0,\gamma}(B_1)} \leq 2 \| \Gamma^m_{ij}(\phi) \|_{C^{0,\gamma}(B_1^+)} < +\infty$. This completes the proof. □

Theorem 3.3. *Let M be a compact Riemann spin surface with boundary ∂M, N be any compact Riemannian manifold, and $\mathcal{S}$ be a closed, totally geodesic submanifold of N. Let (ϕ, ψ) be a weakly Dirac-harmonic map from M to N with free boundary on $\mathcal{S}$ and suppose that $\phi \in C^{0,\alpha}(M, N)$ for any $\alpha \in (0, 1)$. Then there exists some $\beta \in (0, 1)$ such that*

$$\phi \in C^{1,\beta}(M, N), \quad \psi \in C^{1,\beta}(\Sigma M \otimes \phi^{-1}TN).$$

Proof. Combining Lemma 3.6, Proposition 3.10, Proposition 3.11 and applying similar arguments as in the proof of Theorem 2.3. [7], we get $\phi \in C^{1,\beta}(M, N)$ and $\psi \in C^{1,\beta}(\Sigma M \otimes \phi^{-1}TN)$ for some $\beta \in (0, 1)$. □

Remark 3.6. *Following the same strategy as in the proof of Theorem 2.3.* [7], *we take $G = (G^1, G^2, ...G^d)$, where*

$$G^m(x, \phi, d\phi) := \widetilde{\Gamma}^m_{ij}(\phi)\phi^i_\alpha\phi^j_\alpha - \frac{1}{2}\widetilde{R}^m_{lij}(\phi)\langle\psi^i, \nabla\phi^l \cdot \psi^j\rangle,$$

then using the formulas in Lemma 3.6, we have the following pointwise estimate (used in (2.41), page 70, [7]*)*

$$|\nabla G| \leq C(N, \mathcal{S})\left(|d\phi|^3 + |\psi||\nabla\psi||d\phi| + |\psi|^2|d\phi|^2 + |\nabla^2\phi||d\phi| + |\nabla^2\phi||\psi|^2\right), \text{a.e. in } B_1.$$

4. Dirichlet boundary problem for Dirac-harmonic maps

In this section, we shall study Dirichlet boundary problem for weakly Dirac-harmonic maps.

To proceed, we recall that the regularity up to the boundary for weak solutions satisfying (2.21) with continuous boundary trace was established by Müller-Schikorra [22]. More precisely, they proved that

Theorem C. *Let $D \subset \mathbb{R}^2$ be a simply connected domain with C^2 boundary ∂D. Let $u \in H^1(D, \mathbb{R}^K)$, $f \in L^s(D, \mathbb{R}^K)$, $s > 1$ satisfying*

$$-\Delta u = \Omega \cdot \nabla u + f, \quad u|_{\partial D} \in C^0,$$

where $\Omega = (\Omega^i_j)_{1\leq i,j\leq K} \in L^2(D, so(K) \otimes \mathbb{R}^2)$, then u is continuous up to the boundary.

In view of the extrinsic equation (2.26) in the proof of Theorem 2.1, we can apply Theorem C to obtain the following Dirichlet boundary regularity for weakly Dirac-harmonic maps:

Theorem 4.1. *Let (ϕ, ψ) be a weakly Dirac-harmonic map from B_1 to a compact Riemannian manifold N. If ϕ satisfies the following Dirichlet boundary value condition:*

$$\phi|_{\partial B_1} \in C^0,$$

then ϕ is continuous up to the boundary ∂B_1.

Proof. We proceed as in the proof of Theorem 2.1. Recall that the equations for the map ϕ can be written in the following form

$$-\Delta\phi^m = \Omega^m_i \cdot \nabla\phi^i.$$

with some $\Omega = (\Omega^m_i)_{1\leq i,m\leq K} \in L^2(B_1, so(K) \otimes \mathbb{R}^2)$. Applying Theorem C gives that ϕ is continuous up to the boundary ∂B_1. □

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