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Rigorous derivation of a homogenized bending-torsion theory for inextensible rods from 3d elasticity

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RIGOROUS DERIVATION OF A HOMOGENIZED BENDING-TORSION THEORY FOR INEXTENSIBLE RODS FROM 3D ELASTICITY

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ABSTRACT. We present a rigorous derivation of a homogenized, bending-torsion theory for inextensible rods from three-dimensional nonlinear elasticity in the spirit of Γ-convergence. We start with the elastic energy functional associated to a nonlinear composite material. In a stress-free reference configuration it occupies a thin cylindrical domain with thickness $h \ll 1$. We consider composite materials that feature a periodic microstructure with period $\varepsilon \ll 1$. We study the behavior as ε and h simultaneously converge to zero and prove that the energy (scaled by h^{-4}) Γ-converges towards a non-convex, singular energy functional. The energy is only finite for configurations that correspond to pure bending and twisting of the rod. In this case, the energy is quadratic in curvature and torsion.

Our derivation leads to a new relaxation formula that uniquely determines the homogenized coefficients. It turns out that their precise structure additionally depends on the ratio h/ε and, in particular, different relaxation formulas arise for $h \ll \varepsilon$, $\varepsilon \sim h$ and $\varepsilon \ll h$. Although, the initial elastic energy functional and the limiting functional are non-convex, our analysis leads to a relaxation formula that is quadratic and involves only relaxation over a single cell. Moreover, we derive an explicit formula for isotropic materials in the cases $h \ll \varepsilon$ and $h \gg \varepsilon$, and prove that the Γ-limits associated to homogenization and dimension reduction in general do not commute.

Keywords: nonlinear elasticity, dimension reduction, homogenization, rod theory, two-scale convergence.

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CONTENTS

1. INTRODUCTION

A three-dimensional, elastic rod is an elastic body that occupies in a reference configuration a slender, cylindrical domain $\Omega_h := \omega \times (hS) \subset \mathbb{R}^3$. Here $\omega := (0,1)$ is the center-line, $S \subset \mathbb{R}^2$

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the cross section and h a small positive parameter that we refer to as the thickness of the rod. We study rods made of composite materials that feature a periodic microstructure with small period ε in the lateral direction. The associated elastic energy per unit volume is given by the nonconvex integral functional

$$\mathcal{E}^{\varepsilon,h}(v) := \frac{1}{|\Omega_h|}\int_{\Omega_h} W_{\varepsilon,h}(x, \nabla v(x))\,\mathrm{d}x, \tag{1}$$

where $v : \Omega_h \to \mathbb{R}^3$ is a sufficiently smooth deformation, and $W_{\varepsilon,h}(x, F)$ is a frame indifferent, possibly unbounded, stored energy function. Since we are interested in microstructured composites the stored energy function $W_{\varepsilon,h}$ is supposed to oscillate on scale ε in the lateral component of x, and to vary on a scale comparable to h in cross-sectional directions.

The goal of this paper is to understand the effective behavior of the energy $\mathcal{E}^{\varepsilon,h}$ for small parameters ε and h by rigorous mathematical analysis. For the *homogenization* limit, $\varepsilon \to 0$, we expect that the oscillating-in-space energy density W_ε can be replaced by a non-oscillating stored energy function given by a multi-cell homogenization formula. The zero-thickness limit, $h \to 0$, is associated to *dimension reduction*. Both limits ($h \to 0$ with $\varepsilon > 0$ fixed and $\varepsilon \to 0$ with $h > 0$ fixed) are understood reasonably well. In the present paper we analyze the behavior of (1) as both parameters ε and h *simultaneously* tend to zero. Therefore, we assume that the period of the composite is given as a function of the thickness, i.e. $\varepsilon = \varepsilon(h)$. As a main result we prove in Theorems 3.1 and 3.3 that the scaled energy functional $h^{-2}\mathcal{E}^{\varepsilon(h),h}$ and the associated minimization problem converge to a homogenized, nonlinear bending-torsion theory for inextensible rods in the sense of Γ-convergence. The limiting theory is a nonconvex, singular functional defined for *rod configurations*; these are pairs (v, R) where $v : \omega \to \mathbb{R}^3$ is an isometric space curve and $R : \omega \to \mathrm{SO}(3)$ is a rotation field adapted to v (i.e. at every point the tangent of v is given by the first column of R). Physically, the pair (v, R) describes the bending of the rod's center-line and the torsion of the rod's cross-section. For a rod configuration the matrix field $R^tR_{,1}$ is skew-symmetric and its entries are associated to curvature and torsion of the deformed rod. The limiting energy is quadratic in curvature and torsion and takes the form

$$(v, R) \mapsto \int_\omega Q\big(\, x_1, R^tR_{,1}(x_1)\,\big)\,\mathrm{d}x_1, \tag{2}$$

where the quadratic energy density Q encodes the effective behavior of the rod and is given by a relaxation formula (see Definition 2.12 below).

Let us point out that the functional (2) is non-convex: its domain of definition, namely the class of rod configurations, is a non-convex set and additionally the mapping $(v, R) \mapsto R^tR_{,1}$ is nonlinear. Despite the severe non-convexity of (1) and (2), our result reveals that Q is given by a *convex, single-cell* relaxation formula. This is remarkable: as it is well-known (see e.g. [Mül87, Bra85, GMT93]), in general the homogenization of nonconvex integral functionals leads to non-convex relaxation formulas where an infinite number of periodicity cells has to be taken into account.

As it is natural to expect, the quadratic energy density Q depends on the properties of W_ε and the geometry of the rod's cross section. Interestingly, we find that Q additionally depends on the asymptotics of the ratio $h/\varepsilon(h)$, since the relaxation mechanisms associated to homogenization and dimension reduction couple in a nontrivial way. In particular, the cases $h \ll \varepsilon$, $h \sim \varepsilon$ and $h \gg \varepsilon$ have to be distinguished. In the cases $h \ll \varepsilon$ and $h \gg \varepsilon$ the two fine-scales separate. We prove in Section 5 that in these two cases the effective behavior is similar to that obtained by passing to the limits associated to homogenization and dimension reduction separately. As a side result we prove that both limits do not commute in general and derive an explicit formula for isotropic materials.

A brief survey of the literature. The derivation of lower dimensional models (e.g. for rods and plates) from the three-dimensional theory is a fundamental problem in nonlinear elasticity. Classical approaches are often based on extra kinematic assumptions. In this paper we follow a different philosophy, and approach the problem in the framework of Γ-convergence. Within this framework the relevant patterns and constraints in the kinematics of the deformations naturally emerge in the limiting process from the elastic energy itself. Hence, suitable compactness results have to be established. The main challenge in this context is due to the peculiarity that slender, non-linearly elastic bodies can undergo large deformations at low energy. For example, the scaling (w. r. t. thickness) of the elastic energy per volume associated to a stretched rod is of order 1, while bending of the rod leads to energies that scale as the square of the thickness. For this reason different theories emerge for different scalings of the elastic energy. First results in variational dimension reduction are due to Acerbi, Buttazo & Percivale [ABP91]. They prove that (1) (with $\varepsilon > 0$ fixed) Γ-converges to a one-dimensional elastic string theory which shows no resistance to compression. LeDret & Raoult established in [LDR95] a similar result for two-dimensional membranes. In the present contribution we scale the energy by h^{-2}. For this so called *bending regime* Mora & Müller [MM03] proved Γ-convergence to a nonlinear bending-torsion theory for inextensible rods. At the heart of their analysis is a quantitative geometric rigidity estimate (see Theorem 4.1) that has been developed by Friesecke, James & Müller in the seminal work [FJM02], where a nonlinear bending theory for plates was derived. Based on the geometric rigidity estimate various lower dimensional models have been derived (see for instance [FJM06] and the references therein, see [MM03], [Sca06, Sca09] for results related to rod theory). Let us also mention that Mielke [A.88] rigorously derived the fully nonlinear rod equations by appealing to the method of center- manifolds.

The first rigorously treated homogenization problems in elasticity theory were concerned with linearly elastic, periodic composites. Various methods have been developed in this context, see for instance [Spa68, SP74, Tar77, BLP78, SP80, OSI84, MT97, ZKO94, CD99]. In a nutshell, for linearly elastic composites the effective elastic behavior can be derived by solving a linear corrector problem; which is related to the observation that in the linear (or convex) case, solutions of the elasticity system essentially oscillate on the scale given by the microstructure of the composite. For non-linearly elastic composites this is not the case and considerable additional difficulties arise: As stated in [GMT93] “the main difference with the linear case is that the macroscopic behavior of the nonlinear composite can be of a nature completely different from the microscopic behavior of its constituents.” For instance, homogenized nonlinear materials might lose strong ellipticity due to buckling phenomena (see [TM85] and [GMT93]). The first rigorous homogenization results relevant in nonlinear elasticity are due to Müller [Mül87] and Braides [Bra85] and stated in the language of Γ-convergence (see [DGF75, DGDM83, DM93]). They study nonconvex energy functionals with oscillating energy densities, and prove that the limiting energy is again an integral functional with a homogeneous-in-space energy density W_{hom} determined by a multi-cell homogenization formula. For their approach the energy density is required to satisfy a p-growth condition from above. In nonlinear elasticity it is desired to consider materials with the physical property $W(x, F) = \infty$ for all $F \in \mathbb{M}^3$ with $\det F \leq 0$. The homogenization of such materials is still an open problem. Let us remark that the desired behavior of the material for $\det F$ close to 0, does not play a role in our context. Indeed, our result applies to unbounded energy densities, since our argument only relies on the behavior of the material close to $\mathrm{SO}(3)$.

Regarding homogenization, in this paper we appeal to a combination of direct methods from the calculus of variations, and two-scale convergence as introduced by Nguetseng [Ngu89] and further developed by Allaire [All92]. More precisely, in Section 6 we introduce a modified version of two-scale convergence tailor made for the analysis on thinning domains – the content of this chapter is of independent interest.

While the literature for homogenization and dimension reduction is vast, only few results are known for the combination of both in the nonlinear situation. As far as we know, our result is the first rigorous one for simultaneous homogenization and dimension reduction in the bending regime. Results associated to the membrane regime are treated for instance by Braides, Fonseca & Francfort [BFF00], Shu [Shu00], Babadijan & Baìa [BB06]. Recently Velcic [Vela, Velb] considered the dimension reduction problem for rapidly precurved rods and plates in the von Karman regime which corresponds to a scaling of $\mathcal{E}^{\varepsilon(h),h}$ by h^{-4} and leads to a linear limit.

Let us mention that the main results of our paper were announced in a simplified version in the author's thesis, see [Neu10]. There we also study the case of elastic plates and present lower- and upper bounds for the Γ-limit (in the bending regime) that, although being non-optimal, already show a dependence on the ratio γ. The derivation of a complete Γ-convergence result for nonlinearly elastic plates with oscillating coefficients in the bending as well as in the von Karman regime is work in progress.

A motivation of our approach. Starting point of our derivation is the functional (1). To fix ideas, let $W_\varepsilon(x,F)=W(x_1/\varepsilon,F)$ where W is $[0,1)=:Y$-periodic in its spatial component. We assume that W is frame-indifferent, non-degenerate in the sense that $W(y,F)\gtrsim \operatorname{dist}^2(F,\mathrm{SO}(3))$, and minimal at the identity with $W(y,I)=0$. Moreover, we suppose that $W(y,\cdot)$ admits a quadratic expansion at the identity. Since by minimality of I the first two terms in the expansion drop out, we get $W(y,I+G)=Q(y,G)+o(|G|^2)$ where $Q(y,F)$ is a non-negative quadratic form.

We are interested in the derivation of a bending-torsion model. Therefore, we consider deformations v^h with $\mathcal{E}^{\varepsilon(h),h}(v^h)=O(h^2)$. The non-degeneracy of W directly implies

$$\limsup_{h\to 0}\frac{1}{h^2|\Omega_h|}\int_{\Omega_h}\operatorname{dist}^2(\nabla v^h(x),\mathrm{SO}(3))\,\mathrm{d}x<\infty. \tag{3}$$

Let $E^h=\frac{1}{h}\left(\sqrt{(\nabla v^h)^t\nabla v^h}-I\right)$ denote the *scaled nonlinear strain*, which by (3) and the elementary estimate $\operatorname{dist}^2(F,\mathrm{SO}(3))\geq|\sqrt{F^tF}-I|^2$ is equi-bounded in the sense that

$$\limsup_{h\to 0}\frac{1}{|\Omega_h|}\int_{\Omega_h}|E^h|^2\,\mathrm{d}x<\infty. \tag{4}$$

The nonlinear strain monitors the deviation of the right Cauchy-Green deformation tensor from the identity. For $h\ll 1$ the elastic energy depends on the nonlinear strain in a quadratic way: indeed, by frame-indifference, polar factorization, and the expansion of W at identity, a formal computation yields

$$\begin{aligned}\frac{1}{h^2}\mathcal{E}^{\varepsilon,h}(v^h) &\approx \frac{1}{h^2|\Omega_h|}\int_{\Omega_h}W\left(x_1/\varepsilon,\sqrt{(\nabla v^h)^t\nabla v^h}\right)\mathrm{d}x\\ &\approx \frac{1}{h^2|\Omega_h|}\int_{\Omega_h}W\left(x_1/\varepsilon,I+hE^h\right)\mathrm{d}x\\ &\approx \frac{1}{|\Omega_h|}\int_{\Omega_h}Q\left(x_1/\varepsilon,E^h\right)\mathrm{d}x+\text{higher order terms.}\end{aligned}$$

Clearly, this calculation would hold rigorously if E^h was bounded uniformly in h and x – which of course is not the case. However, to fix ideas let us take the expansion for granted. The crucial information that can be extracted from the expansion is the observation that for $h\ll 1$ the behavior of $\mathcal{E}^{\varepsilon(h),h}$ is dominated by the functional

$$v^h\mapsto\frac{1}{|\Omega_h|}\int_{\Omega_h}Q(x_1/\varepsilon(h),E^h)\,\mathrm{d}x. \tag{5}$$

We note that (5) is a nonconvex integral functional, for the mapping $v^h \mapsto E^h$ being nonlinear. However, seen as a function of the nonlinear strain, (5) is quadratic and, from this perspective, the passage $(\varepsilon(h), h) \to 0$ can be treated by convex homogenization methods.

Indeed, we pass to the limit in (5) by appealing to a version of two-scale convergence which is adapted to dimension reduction problems and introduced in Section 6. In particular, we prove that

$$\liminf_{h\to 0} \frac{1}{h^2}\mathcal{E}^{\varepsilon(h),h}(v^h) \geq \inf \iint_{\Omega\times Y} Q(y,E)\,\mathrm{d}y\,\mathrm{d}x$$

where the infimum on the right-hand side is taken over all weak two-scale cluster points E of the family E^h as $h \to 0$. In a second step, see Theorem 3.5, we identify the general structure of E and put it in relation to the limits of v^h and ∇v^h. By virtue of the non-linearity of $v^h \mapsto E^h$ and the presence of oscillations induced by the material microstructure, this is a delicate task. Let us comment on this. In [MM03] Mora and Müller derive a nonlinear bending-torsion theory for rods as a Γ-limit from 3d-elasticity. As a central ingredient, they prove, based on the geometric rigidity estimate Theorem 4.1, a compactness result that can be stated as follows: if (3) is satisfied, then, up to a subsequence (and modulo the subtraction of a constant from v^h), the cross sectional averages $\langle v^h\rangle_{hS} := \fint_{hS} v^h$ and $\langle \nabla v^h\rangle_{hS} := \fint_{hS} \nabla v^h$ converge in L^2 to a rod configuration (v,R). In Theorem 3.5 we prove that any weak two-scale limit of (E^h) takes roughly the form

$$\operatorname{sym}\left[(R^t R_{,1})\begin{pmatrix}0\\x_2\\x_3\end{pmatrix}\otimes \mathbf{e}_1\right] + G \tag{6}$$

where G belongs to a (precisely described) subspace of $L^2(\Omega\times\mathcal{Y},\mathbb{M}^3_{\mathrm{sym}})$. The first term in (6) is determined by the limit (v,R), and is associated to the macroscopic curvature and torsion of the rod configuration. The second term, G, is a relaxation field and captures properties associated to the scaled stretch, shear and cross-sectional deformation, as well as oscillations along the lateral direction. Interestingly, it turns out that the admissible relaxation fields satisfy a set of kinematic constraints that couple the behavior in the cross sectional directions with the oscillatory behavior in the lateral direction. In particular, the precise structure of the constraints depend in a subtle way on the scaling of the fine-scale ratio $h/\varepsilon(h)$. In the second part of Theorem 3.5 we prove that the identification is optimal in the sense that any rod configuration (v,R) and any admissible relaxation field G can be obtained as the limit of a sequence of deformations v^h and the associated scaled nonlinear strains E^h, respectively. Hence, it is not surprising that the limit of $h^{-2}\mathcal{E}^{\varepsilon(h),h}$ is given by a functional that roughly takes the form

$$\inf_{\substack{G\text{ admissible}\\\text{relaxation field}}} \iint_{\omega\times S\times Y} Q\left(y, \operatorname{sym}\left[(R^t R_{,1})\begin{pmatrix}0\\x_2\\x_3\end{pmatrix}\otimes \mathbf{e}_1\right] + G\right)\mathrm{d}y\,\mathrm{d}x.$$

In Theorem 3.1 and Theorem 3.3 we present the precise convergence result and limiting functional. The precise relaxation formula (in a localized form) is defined in Definition 2.12.

Outline of the paper. The article is structured as follows: In Section 2 we introduce the general framework, basic concepts and notation. The main results of this paper are presented in Section 3. In particular, in Theorem 3.1 we study the asymptotic behavior of the minimization problem associated to the energy (1) extended by a loading term and subject to one-sided boundary conditions. In Theorem 3.3 we state the general Γ-convergence result and Theorem 3.5 contains the two-scale identification and compactness result for the nonlinear strain. The proofs are contained in Section 4. In Section 5 we analyze the limiting model in the regimes $h \ll \varepsilon$ and $h \gg \varepsilon$. In particular, we compare the effective coefficients obtained by the simultaneous limit with those obtained by consecutively passing to the limits $h \to 0$ and $\varepsilon \to 0$, and demonstrate that in general homogenization and dimension reduction do not commute. In Section 6 we introduce a version of two-scale convergence adapted to dimension reduction problems. In particular, in Theorem 6.11 we precisely identify the structure of two-scale limits that emerge from scaled gradients, and in Proposition 6.12 we prove a new Korn-type inequality for the emerging limiting function space which consists of functions in $H^1(S\times Y,\mathbb{R}^3)$ that are periodic in the third component. Both results are needed in the proof of Theorem 3.5.

Basic notation.

- $\mathbb{R}^+ := [0,\infty)$ is the set of non-negative real numbers;
- $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ denotes the standard basis in $\mathbb{R}^3$;
- we decompose $x\in\mathbb{R}^3$ as $x=(x_1,x_2,x_3)=(x_1,\bar{x})$ with $x_i\in\mathbb{R}$ and $\bar{x}\in\mathbb{R}^2$;
- $a\otimes b$ denotes the outer product of vectors a and b; in particular, $\mathbf{e}_i\otimes\mathbf{e}_j$ is the unique 3×3-matrix which is 1 in the ith row and jth column, and zero elsewhere;
- $\left(\,a\,|\,A\,\right)$ denotes the 3×3 matrix whose first column is given by $a\in\mathbb{R}^3$ and whose remaining columns are given by $A\in\mathbb{R}^{3\times 2}$;
- $\left(\,a_1\,|\,a_2\,|\,a_3\,\right)$ denotes the 3×3 matrix whose columns are given by $a_1,a_2,a_3\in\mathbb{R}^3$;
- $\mathbb{M}^d$ denotes the space of $d\times d$ matrices;
- for $F\in\mathbb{M}^d$ we write F^t, $\operatorname{trace}F$, $\operatorname{sym}F=1/2(F+F^t)$ and $\operatorname{skw}F=F-\operatorname{sym}F$ for the transposition of F, the symmetric-, skew-symmetric part of F and the trace of F;
- $\mathrm{SO}(d):=\{\,R\in\mathbb{M}^d\,:\,R^tR=I,\ \det R=1\,\}$ is the set of rotations of $\mathbb{R}^d$;
- $\mathrm{Skew}(d)$ is the set of skew-symmetric $d\times d$ matrices;
- $\mathbb{T}^d_{\text{sym}}$ denotes the space of symmetric fourth order tensors on $\mathbb{R}^d$;
- $Y:=[0,1)$ is the unit cell of periodicity;
- $\partial_i u$ and $u_{,i}$ denote the partial derivative of u in direction $\mathbf{e}_i$. We set $\bar{\nabla}u:=\left(\,\partial_2 u\,|\,\partial_3 u\,\right)$ and define the scaled deformation gradient as $\nabla_h u:=\left(\,\partial_1 u\,|\,\frac{1}{h}\bar{\nabla}u\,\right)$.
- $⨍_\omega := \frac{1}{|\omega|}\int_\omega$ denotes the integral mean;
- ε and h denote generic elements of vanishing sequences of positive numbers $\{\varepsilon\}$ and $\{h\}$, respectively;
- the notation $\varepsilon\overset{\gamma}{\sim}h$ indicates that ε is coupled to h with ratio γ, see Definition 2.10;
- $A\lesssim B$ means that the inequality holds up to a multiplicative constant that only depends on the domains ω and S, and the constants of ellipticity α, β. We write $A\ll B$ if the constant is much smaller than 1.
- $L^p(D,\mathbb{R}^d)$, $H^1(D,\mathbb{R}^d)$, $W^{1,p}(D,\mathbb{R}^d)$, $H^1_0(D,\mathbb{R}^d)$, and $W^{1,p}_0(D,\mathbb{R}^d)$ denote the standard Lebesgue, Hilbert and Sobolev spaces; if no confusion occurs, we tacitly write $L^p(D), H^1(D),\ldots$ or even simply $L^p, H^1,\ldots$.
- In this paper we frequently encounter function spaces of periodic functions. Following [Vis06] we write $C(\mathcal{Y})$ to denote the space of continuous functions $f:\mathbb{R}\to\mathbb{R}$ satisfying $f(y+1)=f(y)$ for all $y\in\mathbb{R}$. Clearly, $C(\mathcal{Y})$ endowed with the norm $||f||_\infty:=\sup_{y\in Y}|f(y)|$ is a Banach space. Moreover, we set $C^k(\mathcal{Y}):=C^k(\mathbb{R})\cap C(\mathcal{Y})$ and denote by $L^2(\mathcal{Y})$, $H^1(\mathcal{Y})$ and $H^1(S\times\mathcal{Y})$ ($S\subset\mathbb{R}^2$ Lipschitz domain) the closure of $C^\infty(\mathcal{Y})$ and $C^\infty(\bar{S},C^\infty(\mathcal{Y}))$ w. r. t. the norm in $L^2(Y)$, $H^1(Y)$ and $H^1(S\times Y)$, respectively. Obviously, they are Banach spaces.
- For $A\subset\mathbb{R}^d$ measurable and X a Banach space, $L^2(A,X)$ is understood in the sense of Bochner. We tacitly identify the spaces $L^2(A,L^2(B))$ and $L^2(A\times B)$; since whenever $f\in L^2(A\times B)$, then there exists a function $\tilde{f}\in L^2(A,L^2(B))$ with $f=\tilde{f}$ almost everywhere in $A\times B$.

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2. General framework

Kinematics. Throughout this paper Ω_h denotes the cylindrical domain $\omega \times (hS)$, where $\omega := (0,1)$, $S \subset \mathbb{R}^2$ is an open, bounded, connected domain with Lipschitz boundary and h a small positive parameter. Physically, we think of Ω_h, ω, S and h as the *reference domain*, *center-line*, *(up)scaled cross section* and *thickness* of a three-dimensional rod.

Deformations of the rod are described by sufficiently smooth mappings $v^h : \Omega_h \to \mathbb{R}^3$. Since we are interested in the behavior $h \to 0$, it is convenient to work on the *scaled reference domain* $\Omega := \omega \times S$. Let $\pi^h : \mathbb{R}^3 \to \mathbb{R}^3$ denote the change of coordinates $\pi^h(x_1, \bar{x}) := (x_1, h\bar{x})$. Here and below, we refer to the components of $x \in \mathbb{R}^3$ by $x = (x_1, x_2, x_3) = (x_1, \bar{x})$. Clearly, for sufficiently smooth maps $v^h : \Omega_h \to \mathbb{R}^3$, the composition $u^h := v^h \circ \pi^h$ is a map from $\Omega \to \mathbb{R}^3$ and $\nabla_h u^h = (\nabla v^h) \circ \pi^h$ where $\nabla_h := \left(\partial_1 \,|\, \frac{1}{h}\bar{\nabla} \right)$ denotes the *scaled deformation gradient*. As explained in the introduction a central role in our analysis is played by the *scaled nonlinear strain* which is defined for $u \in H^1(\Omega, \mathbb{R}^3)$ by

$$\mathrm{E}^h(u) := \frac{\sqrt{(\nabla_h u)^{\mathrm{t}} \nabla_h u} - I}{h}.$$

We frequently associate to functions v^h and u^h defined on Ω_h and Ω their *cross sectional averages* $\left\langle v^h \right\rangle_{hS} := \fint_{hS} v^h(\cdot, \bar{x})\,\mathrm{d}\bar{x}$ and $\left\langle u^h \right\rangle_S := \fint_S u^h(\cdot, \bar{x})\,\mathrm{d}\bar{x}$, respectively.

Next, we introduce one-sided boundary conditions for three-dimensional deformations of the rod.

Definition 2.1 (boundary conditions)**.** Define the vector field

$$\mathbf{d}_S : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \mathbf{d}_S(x_2, x_3) := \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} - c_S \tag{7}$$

where $c_S \in \mathbb{R}^3$ is chosen such that $\langle \mathbf{d}_S \rangle_S = 0$. Let $v^0 \in \mathbb{R}^3$ and $R^0 \in \mathrm{SO}(3)$. We say $u \in H^1(\Omega, \mathbb{R}^3)$ and $v \in H^1(\Omega_h, \mathbb{R}^3)$ satisfy the one-sided boundary condition associated to (v^0, R^0), if

$$\begin{aligned} & u(0,\bar{x}) = v^0 + hR^0 \mathbf{d}_S(\bar{x}) \quad \text{almost everywhere in } S, \\ \text{and} \quad & v(0,\bar{x}) = v^0 + R^0 \mathbf{d}_S(\bar{x}) \quad \text{almost everywhere in } hS, \end{aligned} \tag{8}$$

respectively. We write

$$\begin{aligned} \mathcal{A}^h_{(v^0,R^0)}(\Omega) &:= \left\{ u \in H^1(\Omega, \mathbb{R}^3) \,:\, u \text{ satisfies (8)} \right\}, \\ \mathcal{A}_{(v^0,R^0)}(\Omega_h) &:= \left\{ v \in H^1(\Omega_h, \mathbb{R}^3) \,:\, v \text{ satisfies (8)} \right\} \end{aligned}$$

for the associated set of admissible deformations.

Next, we introduce some notation and basic properties for the one-dimensional rod theory.

Definition 2.2 (rod configuration)**.** A *rod configuration* consists of a pair (v, R) where $v \in H^2(\omega, \mathbb{R}^3)$, $R \in H^1(\omega, \mathbb{M}^3)$, and $R(x_1) \in \mathrm{SO}(3)$, $v_{,1}(x_1) = R(x_1)\mathbf{e}_1$ for almost every $x_1 \in \omega$. We write $\mathcal{A}^{\mathrm{rod}}$ and $\mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)}$ for the set of arbitrary rod configurations and the set of those that additionally satisfy $v(0) = v^0$ and $R(0) = R^0$. With slight abuse of notation we write $v \in \mathcal{A}^{\mathrm{rod}}$ if $(v, R) \in \mathcal{A}^{\mathrm{rod}}$ for some R, and similarly $R \in \mathcal{A}^{\mathrm{rod}}$, $v \in \mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)}$ and $R \in \mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)}$.

A central quantity in rod theory is the *rod strain* $K := R^{\mathrm{t}} R_{,1}$ associated to a rod configuration $(v, R) \in \mathcal{A}^{\mathrm{rod}}$. By construction we have

$$K = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \tau \\ -\kappa_2 & -\tau & 0 \end{pmatrix}$$

where the scalar fields κ_1, κ_2 and τ are associated to the curvature of v and the torsion of the frame, respectively. A rod configuration (v, R) is uniquely determined by K and its one-sided boundary value $(v(0), R(0))$:

Lemma 2.3 (approximation of rod configurations)**.** *Let $K \in L^2(\omega, \mathrm{Skew}(3))$, $v^0 \in \mathbb{R}^3$ and $R^0 \in \mathrm{SO}(3)$ be given. There exists a unique rod configuration $(v, R) \in \mathcal{A}^{rod}_{(v^0,R^0)}$ with $K = R^t R_{,1}$ almost everywhere. Additionally, for all $\delta > 0$ there exists a rod configuration $(v^{(\delta)}, R^{(\delta)}) \in \mathcal{A}^{rod}_{(v^0,R^0)}$ with $v^{(\delta)}$ and $R^{(\delta)}$ analytic such that*

$$\int_\omega |v^{(\delta)} - v|^2 + |R^{(\delta)} - R|^2 + |(R^{(\delta)})^t R^{(\delta)}_{,1} - K|^2 \, \mathrm{d}x_1 \le \delta. \tag{9}$$

Proof. The proof of the lemma is standard. For analytic K it relies on the Picard-Lindelöf theorem, which yields uniqueness and existence of an analytic solution $R : [0, 1] \to \mathbb{M}^3$ to the initial value problem $R(0) = R^0$ and $R_{,1} = RK$ in $[0, 1]$. Since K is skew-symmetric and $R^0 \in \mathrm{SO}(3)$, R is generically $\mathrm{SO}(3)$-valued. Thus, by setting $v(x_1) := v^0 + \int_0^{x_1} R(s)\mathbf{e}_1 \, \mathrm{d}s$ the pair (v, R) is an analytic rod configuration in $\mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)}$. The general case with $K \in L^2(\omega, \mathrm{Skew}(3))$ follows by an approximation argument, as it is demonstrated in [FMP10, Lemma 4.4]). □

The proof of the following corollary is easy and left to the reader.

Corollary 2.4. *Let $(v^h, R^h) \in \mathcal{A}^{rod}$ be a sequence of rod configurations. The following properties are equivalent:*

(a) There exists $(v, R) \in \mathcal{A}^{rod}$ such that

$$v^h \rightharpoonup v \quad \text{weakly in } H^2(\omega, \mathbb{R}^3) \qquad \text{and} \qquad R^h \rightharpoonup R \quad \text{weakly in } H^1(\omega, \mathrm{Skew}(3))$$

(b) There exists $(v^0, R^0) \in \mathbb{R}^3 \times \mathrm{SO}(3)$ and $K \in L^2(\omega, \mathrm{Skew}(3))$ such that

$$v^h(0) \to v^0, \quad R^h(0) \to R^0, \qquad (R^h)^t R^h_{,1} \rightharpoonup K \quad \text{weakly in } L^2(\omega, \mathrm{Skew}(3)).$$

Moreover, if (b) holds for (v^0, R^0, K), then (a) holds for the unique rod configuration (v, R) with $R(0) = R^0$, $R_{,1} = RK$ in ω and $v(x_1) = v^0 + \int_0^{x_1} R(s)\mathbf{e}_1 \, \mathrm{d}s$.

Constitutive relation. The elastic energy associated to a three-dimensional hyperelastic rod is given by an integral of the form $\int_{\Omega_h} W(x, \nabla v(x)) \, \mathrm{d}x$ where $W(x, \cdot)$ encodes the elastic properties of the material at the material point x. We consider materials of the following type:

Definition 2.5 (nonlinear material law)**.** Let $0 < \alpha \le \beta$ and $\rho > 0$. The class $\mathcal{W}(\alpha, \beta, \rho)$ consists of all measurable functions $W : \mathbb{M}^3 \to [0, \infty]$ that satisfy the following properties:

(W1) W is frame indifferent, i.e.

$$W(RF) = W(F) \quad \text{for all } F \in \mathbb{M}^3,\ R \in \mathrm{SO}(3);$$

(W2) W is non degenerate, i.e.

$$W(F) \ge \alpha \operatorname{dist}^2(F, \mathrm{SO}(3)) \quad \text{for all } F \in \mathbb{M}^3;$$

$$W(F) \le \beta \operatorname{dist}^2(F, \mathrm{SO}(3)) \quad \text{for all } F \in \mathbb{M}^3 \text{ with } \operatorname{dist}^2(F, \mathrm{SO}(3)) \le \rho;$$

(W3) W is minimal at I, i.e.

$$W(I) = 0$$

(W4) W admits a quadratic expansion at I, i.e.

$$W(I + G) = Q(G) + o(|G|^2) \qquad \text{for all } G \in \mathbb{M}^3$$

where $Q : \mathbb{M}^3 \to \mathbb{R}$ is a quadratic form.

An energy density of class $\mathcal{W}(\alpha,\beta,\rho)$ corresponds to a (spatially homogeneous) material with a single, quadratic energy well at $\mathrm{SO}(3)$. For such materials Hooke's law holds in the following generalized sense: for infinitesimally small strains, the stress-strain relation is linear and determined by the quadratic term Q in the expansion of W at identity (see [MN11] and [DMNP02] where this is made rigorous in the language of Γ-convergence). By construction, Q satisfies certain conditions that are common in linear elasticity, namely

$$\text{(Q1)} \qquad \alpha|\operatorname{sym} G| \leq Q(G) \leq \beta|\operatorname{sym} G|^2 \qquad \text{for all } G \in \mathbb{M}^3.$$

Definition 2.6. We say a quadratic form $Q : \mathbb{M}^3 \to \mathbb{R}$ belongs to the class $\mathcal{Q}(\alpha,\beta)$, if (Q1) is satisfied.

Lemma 2.7. *Let $W \in \mathcal{W}(\alpha,\beta,\rho)$ and let Q be the quadratic form associated to W through the expansion (W4). Then $Q \in \mathcal{Q}(\alpha,\beta)$ and there exists a monotone function $r : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{\infty\}$ that can be chosen only depending on the parameters α,β and ρ, such that $r(\delta) \to 0$ as $\delta \to 0$ and*

$$\text{(10)} \qquad \forall G \in \mathbb{M}^3 \ : \ |W(I+G) - Q(G)| \leq |G|^2 r(|G|).$$

Proof. The statement easily follows from the properties of W and the expansion

$$\operatorname{dist}^2(I+hG,\mathrm{SO}(3)) = |\sqrt{(I+hG)^t(I+hG)} - I|^2 = h^2|\operatorname{sym} G|^2 + o(h^2)$$

which holds for all $G \in \mathbb{M}^3$ and $h \ll 1$.

□

For heterogeneous materials the energy density additionally depends on the spatial position. In order to guarantee the measurability of the composition $W(\cdot,\nabla v(\cdot))$ we need the following definition:

Definition 2.8. We say a function $W : \Omega \times \mathbb{M}^3 \to \mathbb{R} \cup \{\infty\}$ is *sup-measurable*, if the superposition $x \mapsto W(x,F(x))$ is measurable for all measurable maps $F : \Omega \to \mathbb{M}^3$.

Now, we are in position to state the assumptions on the composite under investigation. We suppose that the stored energy function in (1) can be written in the form $W_{\varepsilon,h}(x,F) = W_\varepsilon(x_1,\frac{1}{h}\bar{x},F)$ which excludes oscillations (on scales that are small compared to the thickness) of the material in the cross-sectional directions. We assume that W_ε satisfies the following conditions.

Assumption 2.9 (conditions on the composite)**.** Let $W_\varepsilon : \Omega \to \mathbb{R} \cup \{\infty\}$ be a sequence of energy densities. We assume that there exist parameters (α,β,ρ) and a quadratic energy density $Q : \Omega \times \mathbb{R} \times \mathbb{M}^3 \to \mathbb{R}^+$ such that the following properties hold:

(A1) $W_\varepsilon : \Omega \to \mathbb{R}\cup\{\infty\}$ is sup-measurable and $W_\varepsilon(x,\cdot) \in \mathcal{W}(\alpha,\beta,\rho)$ for almost every $x \in \Omega$.
(A2) The quadratic term Q_ε in the expansion of W_ε at identity satisfies

$$\limsup_{\varepsilon\to 0} \operatorname*{ess\,sup}_{x\in\Omega} \max_{\substack{G\in\mathbb{M}^3\\|G|=1}} |Q_\varepsilon(x,G) - Q(x,x_1/\varepsilon,G)| = 0.$$

(A3) For all $G \in \mathbb{M}^3$ the function

$$\Omega \times \mathbb{R} \ni (x,y) \mapsto Q(x,y,G) \in \mathbb{R}^+$$

is a Y-periodic Carathéodory function in the sense that $(x,G) \mapsto Q(x,y,G)$ is continuous and $y \mapsto Q(x,y,G)$ is measurable and Y-periodic.

Remarks. 1. Let W_ε satisfy Assumption 2.9 and let Q denote the associated quadratic form. Then $Q(x,y,\cdot)$ automatically belongs to class $\mathcal{Q}(\alpha,\beta)$ for almost every $(x,y)\in\Omega\times\mathbb{R}$. Furthermore, there exists a tensor field $\mathbb{C}\in C(\overline{\Omega},L^\infty(\mathcal{Y},\mathbb{T}^3_{\rm sym}))$ with

$$\mathbb{C}(x,y)A:B=\frac{Q(x,y,A+B)-Q(x,y,A)-Q(x,y,B)}{2}\qquad\text{for all } A,B\in\mathbb{M}^3$$

almost everywhere in $\Omega\times\mathbb{R}$.

2. Obviously, conditions (A2) and (A3) impose additional assumptions on the microstructure and the measurability of W_ε. As we are going to see, our analysis basically depends on the behavior of W_ε close to $\mathrm{SO}(3)$. Since this behavior is captured by the quadratic form Q, it is convenient to put the needed assumptions regarding measurability and periodicity of the microstructure rather on Q than on the stored energy function W_ε.
3. Let $W_0:\mathbb{R}\times\mathbb{M}^3\to[0,\infty]$ be a Borel-function, $[0,1)$-periodic in its first component and $W_0(y,\cdot)\in\mathcal{W}(\alpha,\beta,\rho)$ for almost every $y\in\mathbb{R}$. Then the family $W_\varepsilon(x,F):=W_0(x_1/\varepsilon,F)$ satisfies Assumption 2.9. The family models a composite that is constant in cross sectional direction and periodic in the lateral direction. An explicit example is given by the following model functional for isotropic materials of St. Venant-Kirchhoff type

$$W_\varepsilon(x,F):=\begin{cases}2\mu(x_1/\varepsilon)|\sqrt{F^tF}-I|^2+\lambda(x_1/\varepsilon)|\operatorname{trace}(\sqrt{F^tF}-I)|^2 & \text{for } \det F>0\\ \infty & \text{else,}\end{cases}$$

where $\mu,\lambda\in L^\infty(\mathcal{Y})$ satisfy $2\mu\geq\alpha$, $\lambda\geq 0$.

Relaxation formula. The effective material law for the homogenized rod with reduced dimension can be computed by means of a *relaxation formula* where "inner degrees of freedom", represented by *relaxation fields*, are minimized out. The structure of these relaxation fields depends on the limit of the ratio $h/\varepsilon(h)$ as $h\to 0$. For the precise definition of the relaxation formula we need the following vocabulary.

Definition 2.10 (coupling of the fine-scales)**.** We say ε and h are coupled with ratio $\gamma\in[0,\infty]$, and briefly write $\varepsilon\overset{\gamma}{\sim}h$, if $\varepsilon=\varepsilon(h)$ is a monotone function from $(0,\infty)$ to $(0,\infty)$ with the following properties: as $h\to 0$ we have $\varepsilon(h)\to 0$ and $h/\varepsilon(h)\to\gamma$ for some *fine-scale ratio* $\gamma\in[0,\infty]$.

Definition 2.11 (space of relaxation fields)**.** Let $\gamma\in[0,\infty]$. We denote by $\mathcal{G}_\gamma(S{\times}\mathcal{Y})$ the following subspace of $L^2(S{\times}\mathcal{Y},\mathrm{Sym}(3))$

$$\begin{aligned}\mathcal{G}_\gamma(S{\times}\mathcal{Y}):=\Big\{\,G(\bar x,y)=\operatorname{sym}\Big[\partial_y\Psi(y)\big(\mathbf{d}_S(\bar x)\otimes\mathbf{e}_1\big)+a(\mathbf{e}_1\otimes\mathbf{e}_1)+F(\bar x,y)\Big]\ :\\ a\in\mathbb{R},\qquad F\in\mathcal{F}_\gamma(S{\times}\mathcal{Y}),\\ \Psi\in H^1(\mathcal{Y},\mathrm{Skew}(3))\text{ for }\gamma=0\text{ and }\Psi=0\text{ for }\gamma>0\,\Big\}.\end{aligned}$$

Above, $\mathbf{d}_S:\mathbb{R}^2\to\mathbb{R}^3$ is defined in (7) and the space $\mathcal{F}_\gamma(S\times\mathcal{Y})$ is given by Definition 6.10.

We are now in position to state the relaxation formula that (as we are going to prove in Section 3) determines the effective behavior of the one-dimensional rod.

Definition 2.12 (relaxation formula)**.** Let Q be as in Assumption 2.9. For $\gamma\in[0,\infty]$ define $Q_\gamma:\omega\times\mathrm{Skew}(3)\to\mathbb{R}^+$ by

$$Q_\gamma(x_1,K):=\inf\left\{\iint_{S\times Y}Q\Big(x_1,\bar x,y,\ K(\mathbf{d}_S(\bar x){\otimes}\mathbf{e}_1)+G(\bar x,y)\Big)\,\mathrm{d}y\,\mathrm{d}\bar x\ :\ G\in\mathcal{G}_\gamma(S{\times}\mathcal{Y})\right\}.$$

Remark 1. If $Q(x, y, G)$ is independent on the fast variable y, then it is easy to check that the relaxation formula simplifies to

$$Q_\gamma(x_1, K) := \inf_{\substack{a\in\mathbb{R}\\ \phi\in H^1(S,\mathbb{R}^3)}} \int_S Q\Big(x_1, \bar{x},\, K(\mathbf{d}_S(\bar{x})\otimes\mathbf{e}_1) + a(\mathbf{e}_1\otimes\mathbf{e}_1) + \big(\,0\,|\,\bar{\nabla}\phi(\bar{x})\,\big)\Big)\,\mathrm{d}\bar{x}.$$

In particular, we recover the result for non-oscillatory materials derived in [MM03].

Proposition 2.13. *In the situation above the following properties hold:*

(a) $\mathrm{Skew}(3) \ni K \mapsto Q_\gamma(x_1, K)$ *is quadratic and convex for all* $x_1 \in \omega$.
(b) $\omega \ni x_1 \mapsto Q_\gamma(x_1, K)$ *is continuous for all* $K \in \mathrm{Skew}(3)$.
(c) There exists a linear, bounded operator

$$L^2(\omega, \mathrm{Skew}(3)) \ni K \mapsto G_K \in L^2(\omega, \mathcal{G}_\gamma(S\times\mathcal{Y}))$$

such that

$$\begin{aligned}\int_\omega Q_\gamma(x_1, K(x_1))\,\mathrm{d}x_1 &= \iint_{\Omega\times Y} Q(x, y, K(\mathbf{d}_S\otimes\mathbf{e}_1) + G_K)\,\mathrm{d}y\,\mathrm{d}x\\ &= \inf_{G\in L^2(\omega;\mathcal{G}_\gamma(S\times\mathcal{Y}))}\iint_{\Omega\times Y} Q(x, y, K(\mathbf{d}_S\otimes\mathbf{e}_1) + G)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

and $||G_K||_{L^2(\Omega\times Y,\mathbb{M}^3)} \lesssim ||K||_{L^2(\omega,\mathbb{M}^3)}$.
(d) For all $K \in \mathrm{Skew}(3)$ *we have*

$$|K|^2 \lesssim \min_{x_1\in\omega} Q_\gamma(x_1, K) \le \max_{x_1\in\omega} Q_\gamma(x_1, K) \lesssim |K|^2.$$

(For the proof see Subsection 4.6).

3. Main Results

In this section we present our main results. The corresponding proofs are presented in Section 4. Unless otherwise stated we assume that the following properties are fulfilled throughout this section:

- ε and h are coupled with ratio $\gamma \in [0, \infty]$ in the sense of Definition 2.10;
- Ω and Ω_h are cylindrical domains as described in Subsection 2;
- W_ε satisfies Assumption 2.9 and Q denotes the associated quadratic form;
- Q_γ is defined by the relaxation formula in Definition 2.12.

Our first result identifies the asymptotic behavior of minimizers and minima of the elastic energy (1) extended by a loading term and subject to one-sided boundary conditions. The precise setup is the following. Let boundary data $v^0 \in \mathbb{R}^3$, $R^0 \in \mathrm{SO}(3)$, a sequence of applied loads $f^h \in L^2(\Omega_h, \mathbb{R}^3)$ and a limiting load $f \in L^2(\omega, \mathbb{R}^3)$ be given and from now on fixed. We suppose that

$$\text{(11)}\qquad \frac{1}{h^2}\left\langle f^h\right\rangle_{hS} \rightharpoonup f \quad \text{weakly in } L^2(\omega,\mathbb{R}^3) \text{ and } \limsup_{h\to 0}\frac{1}{h^2}\fint_{\Omega_h}|f^h - \left\langle f^h\right\rangle_{hS}|^2\,\mathrm{d}x = 0.$$

We consider the total energy functional $\mathcal{E}^{\varepsilon,h}[\cdot; f^h] : H^1(\Omega_h, \mathbb{R}^3) \to \mathbb{R}\cup\{\infty\}$ given by

$$\text{(12)}\qquad \mathcal{E}^{\varepsilon,h}[v; f^h] := \frac{1}{|\Omega_h|}\int_{\Omega_h} W_\varepsilon(x_1, \tfrac{1}{h}\bar{x}, \nabla v(x)) - v(x)\cdot f^h(x)\,\mathrm{d}x.$$

Let $v^h \in \mathcal{A}_{(v^0,R^0)}(\Omega_h)$ be a *sequence of almost minimizers* of (12) in the sense that

$$\text{(13)}\qquad \limsup_{h\to 0}\frac{1}{h^2}\left(\mathcal{E}^{\varepsilon(h),h}[v^h; f^h] - \mathcal{E}^{\varepsilon(h),h}_{\inf}[f^h]\right) = 0$$

where $\mathcal{E}^{\varepsilon,h}_{\inf}[f^h]$ denotes the infimum of $\mathcal{E}^{\varepsilon,h}[\cdot;f^h]$ among all deformations in $\mathcal{A}_{(v^0,R^0)}(\Omega_h)$.

As a main result of this paper we prove that the limiting behavior of (12) is given by the functional

$$\mathcal{E}_\gamma[\cdot,\cdot;f]\,:\,\mathcal{A}^{\text{rod}}\to\mathbb{R}\cup\{\infty\},\qquad \mathcal{E}_\gamma[v,R;f]:=\frac{1}{|\Omega|}\int_\omega Q_\gamma(x_1,R^{\mathrm{t}}R_{,1})-v\cdot f\,\mathrm{d}x_1.$$

Theorem 3.1. *In the situation above the following properties hold:*

(a) (Convergence of almost minimizers). The sequences $\left\langle v^h\right\rangle_{hS}$ and $\left\langle \nabla v^h\right\rangle_{hS}$ are norm bounded in $H^1(\omega,\mathbb{R}^3)$ and $L^2(\omega,\mathbb{M}^3)$, respectively. Moreover, up to a subsequence,

$$\begin{aligned}\left\langle v^h\right\rangle_{hS}&\rightharpoonup v^* &&\textit{weakly in } H^1(\omega,\mathbb{R}^3),\\ \left\langle \nabla v^h\right\rangle_{hS}&\to R^* &&\textit{strongly in } L^2(\omega,\mathbb{M}^3),\end{aligned}$$

where $(v^*,R^*)\in\mathcal{A}^{rod}_{(v^0,R^0)}$ minimizes $\mathcal{E}_\gamma[\cdot,\cdot,f]$ among all rod configurations in $\mathcal{A}^{rod}_{(v^0,R^0)}$.

(b) (Convergence of minima).

$$\begin{aligned}\lim_{h\to0}\frac{1}{h^2}\mathcal{E}^{\varepsilon(h),h}[v^h;f^h]&=\lim_{h\to0}\frac{1}{h^2}\mathcal{E}^{\varepsilon(h),h}_{\inf}[f^h]\\ &=\min\left\{\mathcal{E}_\gamma[v,R;f]\;:\;(v,R)\in\mathcal{A}^{rod}_{(v^0,R^0)}\right\}\end{aligned}$$

(c) (Approximation of minimizers). Let $(v^*,R^*)\in\mathcal{A}^{rod}_{(v^0,R^0)}$ be a minimizer of $\mathcal{E}_\gamma[\cdot,\cdot,f]$. Then there exists a sequence $v^h_*\in\mathcal{A}^{rod}_{(v^0,R^0)}(\Omega_h)$ such that

$$\left\langle v^h_*\right\rangle_{hS}\rightharpoonup v^*\quad\textit{weakly in } H^1(\omega,\mathbb{R}^3),\qquad \left\langle \nabla v^h_*\right\rangle_{hS}\to R^*\quad\textit{strongly in } L^2(\omega,\mathbb{M}^3),$$
$$\lim_{h\to0}\frac{1}{h^2}\mathcal{E}^{\varepsilon(h),h}[v^h_*;f^h]=\mathcal{E}_\gamma[v^*,R^*;f].$$

Remark 2. If $\mathcal{E}_\gamma[\cdot,\cdot;f]$ has a unique minimizer, say (v^*,R^*), then every sequence of almost minimizers converges to (v^*,R^*). However, since $\mathcal{E}_\gamma[\cdot,\cdot;f]$ is a non-convex functional, it does not exhibit a unique minimizer in general.

The previous theorem follows from a general Γ-convergence statement that we present next. For convenience we shall work on the fixed domain Ω; therefore, we introduce the scaled functional

$$\mathcal{I}^{\varepsilon,h}\,:L^2(\Omega,\mathbb{R}^3)\to[0,\infty],\qquad \mathcal{I}^{\varepsilon,h}(u):=\begin{cases}\dfrac{1}{h^2}\displaystyle\int_\Omega W_\varepsilon(x,\nabla_h u(x))\,\mathrm{d}x & \text{for } u\in H^1(\Omega,\mathbb{R}^3),\\ \infty & \text{else.}\end{cases}$$

For $\gamma\in[0,\infty]$ we introduce the functional $\mathcal{I}_\gamma\,:L^2(\Omega,\mathbb{R}^3)\times L^2(\Omega,\mathbb{M}^3)\to[0,\infty]$ by

$$\mathcal{I}_\gamma(v,R):=\begin{cases}\displaystyle\int_\omega Q_\gamma(x_1,R^{\mathrm{t}}R_{,1})\,\mathrm{d}x_1 & \text{for } (v,R)\in\mathcal{A}^{\text{rod}}\\ \infty & \text{else.}\end{cases}$$

Here and below, we tacitly identify functions defined on Ω that are constant in the cross-sectional directions with their cross-sectional average. In particular, we shall write $(v,R)\in\mathcal{A}^{\text{rod}}$ for functions $(v,R)\in L^2(\Omega,\mathbb{R}^3)\times L^2(\Omega,\mathbb{M}^3)$ provided $v=\langle v\rangle_S$, $R=\langle R\rangle_S$ and $(\langle v\rangle_S,\langle R\rangle_S)\in\mathcal{A}^{\text{rod}}$.

We prove that $\mathcal{I}^{\varepsilon(h),h}$ converges to $\mathcal{I}_\gamma$ in the sense of Γ-convergence w. r. t. strong convergence of $(u^h,\nabla_h u^h)$ in L^2. The choice of this notion of convergence is natural, as can be seen by the following compactness result.

Proposition 3.2 (compactness and stability of boundary conditions). *Let $u^h \in L^2(\Omega, \mathbb{R}^3)$ be a sequence satisfying*

$$\limsup_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(u^h) < \infty. \tag{14}$$

Then there exist $(v, R) \in \mathcal{A}^{rod}$ and $c^h \in \mathbb{R}^3$ such that, up to a subsequence,

$$\begin{aligned} u^h - c^h &\to v && \textit{strongly in } L^2(\Omega, \mathbb{R}^3),\\ \nabla_h u^h &\to R && \textit{strongly in } L^2(\Omega, \mathbb{M}^3). \end{aligned} \tag{15}$$

Additionally, if $u^h \in \mathcal{A}^h_{(u^0,R^0)}(\Omega)$ then the convergence holds for $c^h = 0$ and we have $(v, R) \in \mathcal{A}^{rod}_{(u^0,R^0)}$.

The convergence statement reads as follows:

Theorem 3.3 (convergence of the scaled energy).

(a) (Lower bound). Let $u^h \in L^2(\Omega, \mathbb{R}^3)$ be a sequence such that $(u^h, \nabla_h u) \to (v, R)$ strongly in L^2. Then

$$\liminf_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(u^h) \geq \mathcal{I}_\gamma(v, R).$$

(b) (Recovery sequence). Let $(v, R) \in \mathcal{A}^{rod}$. Then there exists a sequence $u^h \in \mathcal{A}^h_{(v(0),R(0))}(\Omega)$ such that $(u^h, \nabla_h u^h) \to (v, R)$ strongly in L^2 and

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(u^h) = \mathcal{I}_\gamma(v, R).$$

As explained in the introduction, for the proof of Theorem 3.3 a precise understanding of the oscillatory behavior in the nonlinear strain along sequences with equi-bounded energy is mandatory. In the next result we precisely identify the class of nonlinear limiting strains that can emerge as two-scale limits of sequences with finite bending energy. For the statement we need the following version of two-scale convergence that is adapted to dimension reduction problems and only captures oscillations in x_1-direction.

Definition 3.4 (two-scale convergence). Let ε and h be coupled with ratio $\gamma \in [0, \infty]$. Let $\Omega := \omega \times S$ with $\omega \subset \mathbb{R}$ a (possibly unbounded) open interval and $S \subset \mathbb{R}^2$ a Lipschitz domain. We say a sequence $g^h \in L^p(\Omega)$, $p \in [1, \infty)$, weakly two-scale converges in L^p to the function $g \in L^p(\omega, L^2(S\times\mathcal{Y}))$ as $h \to 0$, if the sequence g^h is bounded in $L^p(\Omega)$ and

$$\lim_{h\to 0} \int_\Omega g^h(x)\, \psi(x, \tfrac{x_1}{\varepsilon(h)})\, \mathrm{d}x = \iint_{\Omega\times Y} g(x, y)\, \psi(x, y)\, \mathrm{d}y\, \mathrm{d}x$$

for all $\psi \in C^\infty_c(\Omega, C(\mathcal{Y}))$. We say g_h strongly two-scale converges to g if additionally

$$\lim_{h\to 0} ||g^h||_{L^p(\Omega)} = ||g||_{L^p(\Omega\times Y)}.$$

If the meaning is clear from the context, we shall simply write $g^h \xrightarrow{2,\gamma} g$ in L^p (resp. $g^h \xrightarrow{2,\gamma} g$ in L^p) for weak (resp. strong) two-scale convergence in L^p. Moreover, for technical reasons we tacitly identify a two-scale limit g defined on $\omega \times S\times\mathcal{Y}$ with its extension by zero to $\mathbb{R} \times S\times\mathcal{Y}$. In Section 6 we give a detailed discussion of this type of convergence.

Theorem 3.5 (two-scale identification of the nonlinear limiting strain). *Let $u^h \in H^1(\Omega, \mathbb{R}^3)$ be a sequence satisfying*

$$\limsup_{h\to 0} \frac{1}{h^2} \int_\Omega \operatorname{dist}^2(\nabla_h u^h, \mathrm{SO}(3))\, \mathrm{d}x < \infty. \tag{16}$$

Recall the definition of $\mathcal{G}_\gamma(S \times \mathcal{Y})$ in Definition 2.11. The following properties hold.

(a) *(Compactness and identification). There exist* $R \in \mathcal{A}^{rod}$ *and* $G \in L^2(\omega, \mathcal{G}_\gamma(S\times\mathcal{Y}))$ *such that, up to a subsequence,*

$$\mathrm{E}_h(u_h) \xrightarrow{2,\gamma} E \qquad \text{weakly two-scale in } L^2$$

where

$$E(x_1, \bar{x}, y) = \mathrm{sym}\left[(R^t(x_1)R_{,1}(x_1))(\mathbf{d}_S(\bar{x}) \otimes \mathbf{e}_1)\right] + G(x_1, y, \bar{x}) \quad \text{almost everywhere.}$$

(b) *(Approximation). For all* $(v, R) \in \mathcal{A}^{rod}$ *and* $G \in L^2(\omega, \mathcal{G}_\gamma(S\times\mathcal{Y}))$ *there exists a sequence* $u^h \in A^h_{(v(0),R(0))}(\Omega)$ *such that*

$$(u^h, \nabla_h u^h) \to (u, R) \qquad \text{strongly in } L^2$$

$$\mathrm{E}_h(u_h) \xrightarrow{2,\gamma} \mathrm{sym}\left[(R^t R_{,1})(\mathbf{d}_S \otimes \mathbf{e}_1)\right] + G \quad \text{strongly two-scale in } L^2,$$

and

$$\limsup_{h\to 0} \operatorname*{ess\,sup}_{x\in\Omega} \left(h^{-1} \operatorname{dist}^2(\nabla_h u^h(x), \mathrm{SO}(3)) + h|\,\mathrm{E}_h(u_h)|^2 \right) = 0.$$

A central ingredient in the proof of Theorem 3.5 is the following decomposition.

Proposition 3.6. *Let* $0 < h \ll 1$. *For every deformation* $u \in H^1(\Omega, \mathbb{R}^3)$ *there exist a rod configuration* $(v^h, R^h) \in \mathcal{A}^{rod}$ *and a corrector field* $\phi^h \in H^1(\Omega, \mathbb{R}^3)$ *such that the following properties hold.*

(a) *The representation*

$$u(x) = v^h(x_1) + hR^h(x_1)\mathbf{d}_S(\bar{x}) + h\phi^h(x)$$

holds almost everywhere in Ω.

(b) *The function* v^h *satisfies the boundary condition* $v^h(0) = \langle u(0,\cdot)\rangle_S$. *Additionally, if* $u \in \mathcal{A}^h_{(u^0,R^0)}(\Omega)$ *then* $(v^h, R^h) \in \mathcal{A}^{rod}_{(u^0,R^0)}$ *and* $\phi^h(0,\cdot) = 0$ *almost everywhere in* S.

(c) *The function* $K^h := (R^h)^t R^h_{,1}$ *is a piecewise constant map from* ω *to* $\mathrm{Skew}(3)$ *and its jumpset is contained in* $h\mathbb{Z}$.

(d) *The estimates*

$$\|h^{-1}(\nabla_h u - R^h)\|^2_{L^2(\Omega)} + \|R^h_{,1}\|^2_{L^2(\omega)} + \|\phi^h\|^2_{L^2(\Omega)|} + \|\nabla_h \phi^h\|^2_{L^2(\Omega)} \lesssim \frac{1}{h^2}\int_\Omega \operatorname{dist}^2(\nabla_h u, \mathrm{SO}(3))\,\mathrm{d}x$$

$$\|R^h\|^2_{H^1(\omega)} + \|v^h\|^2_{H^2(\omega)} \lesssim 1 + |\,\langle u(0,\cdot)\rangle_S\,|^2 + \frac{1}{h^2}\int_\Omega \operatorname{dist}^2(\nabla_h u, \mathrm{SO}(3))\,\mathrm{d}x$$

hold up to a multiplicative constant that only depends on the cross section S.

Remark 3. Let u^h be a sequence of (scaled) deformations in $H^1(\Omega, \mathbb{R}^3)$ with $\frac{1}{h^2}\int_\Omega \operatorname{dist}^2(\nabla_h u^h(x), \mathrm{SO}(3))\,\mathrm{d}x \lesssim 1$ small. The previous proposition shows that for $h \ll 1$ the three-dimensional deformation u^h is close (in L^2) to a deformation of the form

$$u^h_{\text{cosserat}}(x_1, \bar{x}) := v^h(x_1) + hR^h(x_1)\mathbf{d}_S(\bar{x}) \tag{17}$$

where (v^h, R^h) is a suitable rod-configuration. The deformation u^h_{cosserat} is classic: it is the extension of (v^h, R^h) to a three-dimensional deformation based on the standard Cosserat-ansatz. In particular, the deformation u^h_{cosserat} purely bends the center-line $\omega \times \{0\}$ according to v^h. The fibers in the normal directions remain orthogonal to the center-line in the deformed configuration. It is well-known that the Cosserat-ansatz is not optimal in the sense that the error $\|\nabla_h u^h - \nabla_h u^h_{\text{cosserat}}\|_{L^2(\Omega)}$ has order one. The reason for this is that u^h_{cosserat} does not take Poisson's effect into account: in curved parts of the rod the material away from the rod's center-line is squeezed or stretched. Therefore, it prefers to relax in a material-specific direction that is non-tangential to the center-line. This relaxation phenomena is captured by the corrector field ϕ^h.

4. Proof of the main results

In this section we present the proofs of our main results in the following ordering: Propositions 3.6, 3.2, Theorems 3.5, 3.3, 3.1 and Proposition 2.12.

4.1. **Proof of Proposition 3.6.** The proof crucially relies on the following geometric rigidity estimate developed by Friesecke, James and Müller:

Theorem 4.1 (see [FJM02, FJM06]). *Let U be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. There exists a constant $C(U)$ with the following property: for each $v \in H^1(U, \mathbb{R}^d)$ there is an associated rotation $R \in \mathrm{SO}(d)$ such that*

$$\int_U |\nabla v(x) - R|^2 \,\mathrm{d}x \leq C(U) \int_\Omega \mathrm{dist}^2(\nabla v(x), \mathrm{SO}(d)) \,\mathrm{d}x.$$

Moreover, the constant $C(U)$ is invariant under uniform scaling of U.

Based on this theorem in [FJM02] a compactness result for deformations with finite bending energy has been developed. It is the main ingredient in the rigorous derivation of the nonlinear plate and nonlinear rod theory from three-dimensional elasticity (see [FJM02] and [MM03]). Our proof of Proposition 3.6 is inspired by these results. However, as we seek for a precise two-scale identification of the nonlinear limiting strain, a slightly finer construction is needed to rule out certain oscillations in quantities associated to curvature and torsion.

The general idea is the following. In a first step we approximate $\nabla_h u^h$ by a piecewise constant rotation field:

Lemma 4.2. *In the situation of Proposition 3.6 there exists a piecewise constant rotation field $R^h_{pc} : \mathbb{R} \to \mathrm{SO}(3)$ satisfying the following properties: R^h_{pc} is continuous from the right, its jumpset is contained in $h\mathbb{Z}$ and*

$$\int_\Omega |\nabla_h u^h - R^h_{pc}|^2 \,\mathrm{d}x + h \sum_{\xi \in h\mathbb{Z}} |[[R^h_{pc}(\xi)]]|^2 \lesssim \int_\Omega \mathrm{dist}^2(\nabla_h u^h(x), \mathrm{SO}(3)) \,\mathrm{d}x$$

where $[[R^h_{pc}(\xi)]] := \lim_{\delta \downarrow 0} \left(R^h_{pc}(\xi) - R^h_{pc}(\xi - \delta)\right)$ denotes the jump of R^h_{pc} at ξ. The constant in the estimate only depends on the geometry of S. For $u^h \in \mathcal{A}^h_{(u^0,R^0)}(\Omega)$ the rotation field additionally satisfies $R^h_{pc} = R^0$ on $[0, h)$.

The proof of the statement follows [FJM02, Theorem 4.1]. Since some additional arguments are needed to treat the boundary condition and to guarantee the asserted structure of the jumpset, we prove Lemma 4.2 in the appendix.

In a second step (see Lemma 4.3 below) we regularize the rotation field R^h_{pc} and get an approximation $R^h : \omega \to \mathrm{SO}(3)$ of class H^1. Finally, we draw the conclusion by comparing u^h to a "hand made" deformation that is based on R^h and constructed via a standard Cosserat ansatz for inextensible rods. Let us point out that the construction is made in such a way that $K^h := (R^h)^t R^h_{,1}$ is a piecewise constant function with jumpset contained in $h\mathbb{Z}$. This is an important feature, since for $\gamma = \infty$ this allows us to rule out oscillations in curvature and torsion on scale $\varepsilon(h)$.

The regularization step is based on the following lemma the proof of which is easy. For the reader's convenience we present it in the appendix.

Lemma 4.3. *Let $s_1 < s_2$ and $R_1, R_2 \in \mathrm{SO}(3)$. There exists a map $R \in C^\infty([s_1, s_2], \mathrm{SO}(3))$ and a skew symmetric matrix K such that*

$$R(s_1) = R_1, \qquad R(s_2) = R_2, \qquad R(s)^{\mathrm{t}} R_{,1}(s) = K \quad \textit{for all } s \in [s_1, s_2]$$

and

$$\sup_{s\in[s_1,s_2]} |R_{,1}(s)|^2 \lesssim \frac{|R_2 - R_1|^2}{(s_2 - s_1)^2}$$

up to a universal multiplicative constant.

Proof of Proposition 3.6. The proof will be divided into three steps.

Step 1. Proof of (a), (b) and (c).

We assert that there exists $R^h : \omega \to \mathrm{SO}(3)$ of class H^1 such that

$$\int_\Omega |\nabla_h u - R^h|^2 + h^2 |R^h_{,1}|^2 \,\mathrm{d}x_1 \lesssim \int_\Omega \mathrm{dist}^2(\nabla_h u^h, \mathrm{SO}(3)) \,\mathrm{d}x, \tag{18}$$

$$K^h := (R^h)^{\mathrm{t}} R^h_{,1} \quad \text{is a piecewise constant map from } \omega \text{ to } \mathrm{Skew}(3) \text{ and its jumpset is contained in } h\mathbb{Z}. \tag{19}$$

Indeed, let $R^h_{pc} : \mathbb{R} \to \mathrm{SO}(3)$ denote the approximation from Lemma 4.2. R^h_{pc} is constant on each interval $[\xi, \xi + h)$, $\xi \in h\mathbb{Z}$. For $\xi \in h\mathbb{Z}$, let $R^h_\xi : [\xi, \xi + h] \to \mathrm{SO}(3)$ denote the smooth function from Lemma 4.3 applied with $s_1 = \xi$, $s_2 = \xi + h$, $R_1 = R^h_{pc}(\xi)$ and $R_2 = R^h_{pc}(\xi + h)$. For $s \in \mathbb{R}$ set $R^h(s) := R^h_\xi(s)$ where $\xi \in h\mathbb{Z}$ is determined by $s \in [\xi, \xi + h)$. By construction $R^h : \mathbb{R} \to \mathrm{SO}(3)$ is a function of class H^1_{loc} and (after restriction to ω) satisfies (19). It remains to show that R^h satisfies (18). We start with the estimate for the second term in (18):

$$\begin{aligned} h^2 \int_\omega |R^h_{,1}|^2 \,\mathrm{d}x_1 \quad &\le \sum_{\substack{\xi \in h\mathbb{Z} \\ [\xi,\xi+h)\cap\omega\neq\emptyset}} \int_\xi^{\xi+h} h^2 |(R^h_\xi)_{,1}|^2 \,\mathrm{d}x_1 \overset{\text{Lemma 4.3}}{\lesssim} h \sum_{\xi\in h\mathbb{Z}} |[[R^h_{pc}(\xi + h)]]|^2 \\ &\overset{\text{Lemma 4.2}}{\lesssim} \int_\Omega \mathrm{dist}^2(\nabla_h u, \mathrm{SO}(3)) \,\mathrm{d}x. \end{aligned}$$

The first term in (18) is estimated by comparison to R^h_{pc}:

$$\int_\Omega |\nabla_h u - R^h|^2 \,\mathrm{d}x \lesssim \int_\Omega |\nabla_h u - R^h_{pc}|^2 \,\mathrm{d}x + \int_\omega |R^h_{pc} - R^h|^2 \,\mathrm{d}x_1. \tag{20}$$

By Lemma 4.2, the first term on the right-hand side is controlled by the right-hand side in (18). The second term is bounded from above by $\sum_\xi \int_\xi^{\xi+h} |R^h_{pc}(\xi) - R^h(x_1)|^2 \,\mathrm{d}x_1$ where the sum runs over all $\xi \in h\mathbb{Z}$ with $[\xi, \xi + h) \cap \omega \neq \emptyset$. By construction we have for $x_1 \in [\xi, \xi + h)$

$$|R^h_{pc}(\xi) - R^h(x_1)|^2 = |\int_\xi^{x_1} R^h_{,1}(s) \,\mathrm{d}s|^2 \le h^2 \sup_{s\in[\xi,\xi+h)} |(R^h_\xi)_{,1}|^2 \overset{\text{Lemma 4.3}}{\lesssim} |[[R^h_{pc}(\xi + h)]]|^2.$$

By integration over $[\xi, \xi + h)$ and summation in ξ we get

$$\int_\omega |R^h_{pc} - R^h|^2 \,\mathrm{d}x_1 \lesssim h \sum_{\xi\in h\mathbb{Z}} |[[R^h_{pc}(\xi)]]|^2 \overset{\text{Lemma 4.2}}{\lesssim} \int_\Omega \mathrm{dist}^2(\nabla_h u, \mathrm{SO}(3)) \,\mathrm{d}x.$$

This concludes the proof of (18).

Next, we introduce the functions

$$v^h(x_1) := \langle u(0,\cdot)\rangle_S + \int_0^{x_1} R^h(s)\mathbf{e}_1\,\mathrm{d}s,$$
$$\phi^h(x) := \frac{u(x) - \big(v^h(x_1) + hR^h(x_1)\mathbf{d}_S(\bar{x})\big)}{h}.$$

With (18) and (19) at hand, it is straightforward to check that (a), (b) and (c) are satisfied by (v^h, R^h) and ϕ^h. The second part of (b) follows from the fact that $R^h(0) = R^0$ if $u^h \in \mathcal{A}^h_{(v^0,R^0)}(\Omega)$ (see Lemma 4.2).

Step 2. Proof of the first estimate in (d).

The estimate of the first two terms on the left-hand side in the first inequality in (d) directly follows from (18). Moreover, differentiation of the identity in (a) and (18) lead to

$$||\nabla_h\phi^h||^2_{L^2(\Omega)} \lesssim \frac{1}{h^2}\int_\Omega \mathrm{dist}^2(\nabla_h u(x), \mathrm{SO}(3))\,\mathrm{d}x. \tag{21}$$

Hence, it remains to prove that the L^2-norm of ϕ^h is controlled by the right-hand side in (21). Since $\int_S \phi^h - \langle\phi^h\rangle_S\,\mathrm{d}\bar{x} = 0$ a.e. in ω, the Poincaré inequality yields

$$\int_\Omega |\phi^h|^2\,\mathrm{d}x \lesssim \int_\Omega |\phi^h - \left\langle\phi^h\right\rangle_S|^2\,\mathrm{d}x + \int_\omega |\left\langle\phi^h\right\rangle_S|^2\,\mathrm{d}x_1 \lesssim h^2\int_\Omega |\nabla_h\phi^h|^2\,\mathrm{d}x + \int_\omega |\left\langle\phi^h\right\rangle_S|^2\,\mathrm{d}x_1.$$

By (21) it suffices to estimate the second term on the right-hand side. By construction we have $\left\langle\phi^h\right\rangle_S \in H^1(\omega,\mathbb{R}^3)$ and by (b) we have $\left\langle\phi^h(0,\cdot)\right\rangle_S = 0$. Hence, Poincaré's inequality yields

$$\int_\omega |\left\langle\phi^h\right\rangle_S|^2\,\mathrm{d}x_1 \lesssim \int_\omega |\left\langle\phi^h_{,1}\right\rangle_S|^2\,\mathrm{d}x_1 \le \int_\Omega |\nabla_h\phi^h|^2\,\mathrm{d}x \overset{(21)}{\lesssim} \frac{1}{h^2}\int_\Omega \mathrm{dist}^2(\nabla_h u, \mathrm{SO}(3))\,\mathrm{d}x.$$

Step 3. Proof of the second estimate in (d).

Since $(v^h, R^h) \in \mathcal{A}^{\mathrm{rod}}$, we have $|v^h_{,1}|^2 = 1$ and $|R^h|^2 = |I|^2 = 3$ almost everywhere in ω. Hence, it suffices to estimate the L^2-norm of v^h. By Poincaré's inequality we have $\int_\omega |v^h|^2\,\mathrm{d}x_1 \lesssim |v^h(0)|^2 + \int_\omega |v^h_{,1}|^2\,\mathrm{d}x_1$. By (b) and the constrained $|v^h_{,1}| = 1$, the right-hand side equals $|\langle u(0,\cdot)\rangle_S|^2 + |\omega|$. Combined with the first estimate in (d), this proves the assertion. ☐

4.2. **Proof of Proposition 3.2.** The proof is divided into two steps. In Step 1 we prove (15). In the second step we treat the case with prescribed boundary data.

Step 1. Proof of (15).

Without loss of generality we may assume that $\int_\Omega u^h\,\mathrm{d}x = 0$; otherwise set $c^h := \fint_\Omega u^h\,\mathrm{d}x$ and consider the sequence $u^h - c^h$. Our proof starts with the decomposition introduced in Proposition 3.6: there exist $(v^h, R^h) \in \mathcal{A}^{\mathrm{rod}}$ and $\phi^h \in H^1(\Omega,\mathbb{R}^3)$ such that

$$u^h = \big(v^h + hR^h\mathbf{d}_S\big) + h\phi^h. \tag{22}$$

By Proposition 3.6 (d), the non-degeneracy condition (W2) and assumption (14), we find that

$$\limsup_{h\to 0}\Big(||h^{-1}(\nabla_h u^h - R^h)||^2_{L^2(\Omega)} + ||\phi^h||^2_{L^2(\Omega)} + ||\nabla_h\phi^h||^2_{L^2(\Omega)}\Big) \lesssim \limsup_{h\to 0}\frac{1}{h^2}\int_\Omega \mathrm{dist}^2(\nabla_h u^h, \mathrm{SO}(3))\,\mathrm{d}x < \infty. \tag{23a}$$

Since $\int u^h\,\mathrm{d}x = 0$ by assumption, the previous estimate directly implies that $\limsup_{h\to 0}\|u^h\|_{H^1} < \infty$. Hence, by the continuity of the trace operator we have $\limsup_{h\to 0}|\left\langle u^h(0,\cdot)\right\rangle_S| < \infty$, and Proposition 3.6 (d) yields

$$\limsup_{h\to 0}\left(\|v^h\|^2_{H^2(\omega)} + \|R^h\|^2_{H^1(\omega)}\right) < \infty. \tag{23b}$$

We assert that, up to a subsequence,

$$u^h - v^h \to 0 \quad \text{strongly in } L^2(\Omega), \tag{24}$$

$$v^h \rightharpoonup v \quad \text{weakly in } H^2(\omega), \tag{25}$$

$$R^h \rightharpoonup R \quad \text{weakly in } H^1(\omega). \tag{26}$$

In view of identity (22) it is evident that (24) – (26) imply the convergence of $(u^h, \nabla_h u^h)$ to (v, R) as asserted in (15). We prove (24) – (26). Obviously, (24) directly follows from (22), (23a) and (25). We prove (25) and (26): by (23b) there obviously exist $v \in H^2(\omega,\mathbb{R}^3)$ and $R \in H^1(\omega,\mathbb{M}^3)$ such that, up to a subsequence, $v^h \rightharpoonup v$ weakly in H^2 and $R^h \rightharpoonup R$ weakly in H^1. The embedding $H^1(\omega) \subset L^2(\omega)$ is compact. Since strong convergence in L^2 implies pointwise convergence almost everywhere (for a subsequence), we have $R(x_1) \in \mathrm{SO}(3)$ almost everywhere; thus, $R \in \mathcal{A}^{\mathrm{rod}}$. It remains to argue that $v_{,1} = R\mathbf{e}_1$. But this is obvious, since $R^h\mathbf{e}_1 = v^h_{,1} \to v_{,1}$ and $R^h\mathbf{e}_1 \to R\mathbf{e}_1$ in L^2.

Step 2. Boundary conditions.

Suppose that $u^h \in \mathcal{A}_{(v^0,R^0)}(\Omega)$. Then Proposition 3.6 implies that identity (22) and estimates (23a), (23b) hold for $(v^h, R^h) \in \mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)}$ and $\phi^h \in H^1(\Omega,\mathbb{R}^3)$ with $\phi^h(0,\cdot) = 0$ almost everywhere. From (22) and the identity $\int_S \mathbf{d}_S\,\mathrm{d}\bar{x} = 0$ we infer that $c^h := \fint_\Omega u^h\,\mathrm{d}x = \fint_\omega v^h\,\mathrm{d}x_1 + h\fint_\Omega \phi^h\,\mathrm{d}x$. The right-hand side is bounded as can be seen by appealing to Proposition 3.6 (d). Hence, the sequence c^h is compact in $\mathbb{R}^3$ and arguments similar to those in Step 1 show that (24), (25) and (26) hold for a subsequence. Since (v^h, R^h) belongs to $\mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)}$, the limit (v, R) satisfies the same one-sided boundary condition and the proof is complete. $\square$

4.3. **Proof of Theorem 3.5.** For the proof of Theorem 3.5 we need some auxiliary results regarding two-scale convergence. We refer to Section 6 for a brief introduction to two-scale convergence.

Lemma 4.4. *Let ε and h be coupled with ratio $\gamma \in [0,\infty]$. Let $f^h \in L^2(\Omega)$ be a weakly two-scale convergent sequence with limit $f \in L^2(\Omega\times\mathcal{Y})$. Let χ^h be a sequence of measurable functions from Ω to $\mathbb{R}$ and suppose that*

$$\limsup_{h\to 0}\operatorname*{ess\,sup}_{x\in\Omega}|\chi^h(x)| < \infty, \qquad \chi^h \to 1 \quad \textit{strongly in } L^1(\Omega).$$

Then $\chi^h f^h \xrightharpoonup{2,\gamma} f$ weakly two-scale in L^2. The result remains valid for vector-valued functions.

Proof. First, we pass to a subsequence (not relabeled) such that $\chi^h \to 1$ strongly in $L^2(\Omega)$. By appealing to Lemma 6.6 we infer that the product $g^h := \chi^h f^h$ weakly two-scale converges to f in L^1. On the other hand, the sequence g^h is bounded in $L^2(\Omega)$. Thus, by Lemma 6.5 we get $g^h \xrightharpoonup{2,\gamma} g$ weakly two-scale in L^2, and the uniqueness of the two-scale limit yields $g = f$. $\square$

Lemma 4.5. *Let ε and h be coupled with ratio $\gamma \in [0,\infty]$. Consider sequences $R^h \in L^2(\Omega,\mathrm{SO}(3))$, $G^h \in L^2(\Omega,\mathbb{M}^3)$ and define*

$$E^h := \frac{\sqrt{(D^h)^t D^h} - I}{h}, \qquad D^h := R^h\left(I + hG^h\right).$$

Suppose that $G^h \xrightharpoonup{2,\gamma} G$ *weakly two-scale in* L^2 *where* $G \in L^2(\Omega \times Y, \mathbb{M}^3)$*. Then*

$$E^h \rightharpoonup \operatorname{sym} G \qquad \textit{weakly two-scale in } L^2.$$

Moreover, if $G^h \xrightarrow{2,\gamma} G$ *strongly two-scale in* L^2 *and*

$$\limsup_{h\to 0} \operatorname*{ess\,sup}_{x\in\Omega} h|G^h(x)| = 0, \tag{27}$$

then $E^h \xrightarrow{2,\gamma} \operatorname{sym} G$ *strongly two-scale in* L^2*.*

Proof. For convenience set

$$\Lambda : \mathbb{M}^3 \to \mathbb{M}^3, \qquad F \mapsto \sqrt{(I+F)^t(I+F)} - I.$$

We claim that every sequence $\widetilde{G}^h \in L^2(\Omega, M^3)$, that fulfills (27), satisfies

$$\lim_{h\to 0} \int_\Omega \left| \frac{\Lambda(h\widetilde{G}^h)}{h} - \operatorname{sym}(\widetilde{G}^h) \right|^2 \, \mathrm{d}x = 0. \tag{28}$$

Indeed, due to (27) the sequence $h\widetilde{G}^h$ converges to 0 in L^∞ and (28) follows from the expansion $\Lambda(F) = \operatorname{sym} F + o(|F|)$.

Let us first prove the second part of the lemma: Suppose that G^h fulfills (28) and $G^h \xrightarrow{2,\gamma} G$ in L^2. Since $E^h = \frac{\Lambda(hG^h)}{h}$, (28) directly implies that $E^h \xrightarrow{2,\gamma} \operatorname{sym} G$ strongly two-scale in L^2.

Next, we prove the first part of the lemma: Let $G^h \xrightharpoonup{2,\gamma} G$ in L^2. Since $E^h = \frac{\Lambda(hG^h)}{h}$, the chain of inequalities

$$\forall F \in \mathbb{M}^3 : \qquad |\Lambda(F)|^2 \le \operatorname{dist}^2(I+F, \mathrm{SO}(3)) \le |F|^2$$

implies that the sequence E^h is bounded in L^2. Consequently, by Lemma 6.5 we have $E^h \xrightharpoonup{2,\gamma} E$ weakly two-scale in L^2 for a subsequence and $E \in L^2(\Omega \times \mathcal{Y}, \mathbb{M}^3)$. We have to show that

$$E = \operatorname{sym} G. \tag{29}$$

For the argument, introduce the truncation

$$\chi^h(x) := \begin{cases} 1 & \text{if } |G^h(x)| \le h^{-1/2}, \\ 0 & \text{else.} \end{cases}$$

Since the sequence E_h is bounded in L^2, Chebyshev's inequality yields $\chi^h \to 1$ strongly in L^1. Hence, we can apply Lemma 4.4 and get

$$\begin{aligned} \chi^h E^h &\xrightharpoonup{2,\gamma} E && \text{weakly two-scale in } L^2, \\ \chi^h G^h &\xrightharpoonup{2,\gamma} G && \text{weakly two-scale in } L^2. \end{aligned} \tag{30}$$

On the other hand, $\widetilde{G}^h := \chi^h G^h$ satisfies (27). Hence, (29) follows from (28) and (30). □

Lemma 4.6. *Let* ε *and* h *be coupled with ratio* $\gamma = \infty$*. Let* $f^h \in L^2(\omega)$ *be a weakly two-scale convergent sequence with limit* $f \in L^2(\omega \times \mathcal{Y})$*. Suppose that for all* $h \ll 1$ *the function* f^h *is piecewise constant and its jumpset is contained in* $h\mathbb{Z}$*. Then* $f = \int_Y f \, \mathrm{d}y$ *almost everywhere. The result remains valid for vector-valued functions.*

Proof. Set $\Lambda^h := \{\, \xi \in \varepsilon(h)\mathbb{Z} \,:\, (\xi, \xi+\varepsilon(h)) \subset \omega \cap [\eta, \eta+h] \text{ for some } \eta \in h\mathbb{Z} \,\}$ and define the indicator function

$$\chi^h(x_1) := \begin{cases} 1 & \text{if } x_1 \in [\xi, \xi+\varepsilon(h)] \text{ for some } \xi \in \Lambda^h, \\ 0 & \text{else.} \end{cases}$$

We claim that

$$\chi^h \to 1 \qquad \text{strongly in } L^2(\omega) \tag{31}$$

$$\chi^h f^h \overset{2,\gamma}{\rightharpoonup} \int_Y f \,\mathrm{d}y \qquad \text{weakly two-scale in } L^1. \tag{32}$$

Clearly, if this is the case, then application of Lemma 4.4 yields $f = \int_Y f \,\mathrm{d}y$ as desired.

We prove (31). As it is easy to check, we have

$$\{\chi^h = 0\} \subset \{\, x_1 \in \omega \,:\, \operatorname{dist}(x_1, \partial\omega) \le 2h \,\} \cup \{\, x_1 \in \omega \,:\, \operatorname{dist}(x_1, h\mathbb{Z}) \le 2\varepsilon(h) \,\}.$$

Since $\varepsilon(h) \ll h$ by assumption, the measure of the set on the right-hand side vanishes for $h \to 0$, and thus (31) follows.

Next, we prove (32). By assumption the jumpset of f^h is contained in $h\mathbb{Z}$. Hence, f^h is constant on each of the intervals $(\xi, \xi+\varepsilon(h))$, $\xi \in \Lambda^h$. By definition of the unfolding operator (see Definition 6.1) a direct computation yields

$$\mathcal{T}_{\varepsilon(h)}(\chi^h f^h) = (\mathcal{T}_{\varepsilon(h)}\chi^h) \int_Y \mathcal{T}_{\varepsilon(h)} f^h \,\mathrm{d}y.$$

On the right-hand side we can pass to the limit, since $\mathcal{T}_{\varepsilon(h)}\chi^h$ strongly converges in L^2 to the indicator function of $\omega \times Y$, while $\int_Y \mathcal{T}_{\varepsilon(h)} f^h \,\mathrm{d}y \rightharpoonup \int_Y f \,\mathrm{d}y$ weakly in L^2. Thus, (32) follows from Proposition 6.3. □

Proof of part (a) of Theorem 3.5. Without loss of generality assume that $\int_\Omega u^h \,\mathrm{d}x = 0$. In the proof we tacitly pass to suitable subsequences if necessary. The proof relies on the decomposition

$$u^h = \left(v^h + hR^h \mathbf{d}_S \right) + h\phi^h \tag{33}$$

where $(v^h, R^h) \in \mathcal{A}^{\mathrm{rod}}$ and $\phi^h \in H^1(\Omega, \mathbb{R}^3)$ are constructed according to Proposition 3.6. Set $K^h := (R^h)^{\mathrm{t}} R^h_{,1}$. As a direct consequence of Proposition 3.6 (d) and assumption (16) we get the estimate

$$\limsup_{h\to 0} \Big(||K^h||_{L^2(\omega)} + ||h^{-1}(\nabla_h u^h - R^h)||_{L^2(\Omega)} + ||\phi^h||_{L^2(\Omega)} + ||\nabla_h \phi^h||_{L^2(\Omega)} + ||R^h||_{H^1(\omega)} \Big) < \infty. \tag{34}$$

After these preparations we proceed in three steps.

Step 1. General compactness result.

We assert that there exist

$$R \in \mathcal{A}^{\mathrm{rod}}, \qquad K \in L^2(\omega \times \mathcal{Y}, \mathrm{Skew}(3)), \qquad \phi \in H^1(\omega, \mathbb{R}^3), \qquad F \in L^2(\omega, \mathcal{F}_\gamma(S \times \mathcal{Y}))$$

with $\phi(0) = 0$ such that

(35) $$R^h \to R \qquad \text{strongly in } L^2(\omega),$$

(36) $$\nabla_h u^h \to R \qquad \text{strongly in } L^2(\Omega),$$

(37) $$K^h \xrightarrow{2,\gamma} K \qquad \text{weakly two-scale in } L^2(\omega)$$

(38) $$\phi^h \to \phi \qquad \text{strongly in } L^2(\Omega),$$

(39) $$\nabla_h \phi^h \xrightarrow{2,\gamma} \partial_1 \phi \otimes \mathbf{e}_1 + F \qquad \text{weakly two-scale in } L^2.$$

The proof of (35) is similar to that of Proposition 3.2. (36) follows by combining (34) and (35). Convergence (37) directly follows from (34) and Lemma 6.5. Application of the two-scale compactness and identification result for scaled gradients (see Theorem 6.11) yields (38) and (39).

Step 2. Identification of K.

We assert that

$$K = \begin{cases} R^t R_{,1} + \partial_y \Psi & \text{for } \gamma \in [0,\infty) \text{ and some } \Psi \in L^2(\omega, H^1(\mathcal{Y}, \mathrm{Skew}(3))), \\ R^t R_{,1} & \text{for } \gamma = \infty. \end{cases}$$

Indeed, since $R^h \to R$ strongly and $R^h_{,1} \rightharpoonup R_{,1}$ weakly, we deduce that

$$K^h = (R^h)^t R^h_{,1} \rightharpoonup \bar{K} := R^t R_{,1} \qquad \text{weakly in } L^2(\omega, \mathrm{Skew}(3)).$$

By Lemma 6.8 (c) we have $\bar{K} = \int_Y K(x_1, y)\,\mathrm{d}y$. Thus, $\Psi(x_1, y) := \int_0^y K(x_1, s) - \bar{K}(x_1)\,\mathrm{d}s$ defines a map in $L^2(\omega, H^1(\mathcal{Y}, \mathrm{Skew}(3)))$ with $K(x_1, y) = \bar{K}(x_1) + \partial_y \Psi(x_1, y)$. This completes the argument for $\gamma \in [0, \infty)$. For $\gamma = \infty$ we argue as follows: by construction (see Proposition 3.6) the function K^h is piecewise constant and its jumpset is contained in $h\mathbb{Z}$. Hence, Lemma 4.6 (applied with $f^h = K^h$) yields $K = \int_Y K\,\mathrm{d}y$ and we conclude $\partial_y \Psi = 0$.

Step 3.

Let $\Psi \in H^1(\mathcal{Y}, \mathrm{Skew}(3))$ and $\gamma \in (0, \infty)$. We assert that there exists a function $\psi \in H^1(S \times \mathcal{Y}, \mathbb{R}^3)$ with

(40) $$\mathrm{sym}[\partial_y \Psi (\mathbf{d}_S \otimes \mathbf{e}_1)] = \mathrm{sym}\left(\, \partial_y \psi \,|\, \tfrac{1}{\gamma} \bar{\nabla} \psi \,\right).$$

The proof consists in an explicit construction. For $i, j \in \{1, 2, 3\}$ set $\Psi_{ij} := \mathbf{e}_i \cdot (\Psi \mathbf{e}_j)$ and $c_{ij} := \int_Y \Psi_{ij}(y)\,\mathrm{d}y$. We define ψ as follows

$$\psi(\bar{x}, y) := \begin{pmatrix} \Psi_{12}(y)\mathbf{d}_S(\bar{x}) \cdot \mathbf{e}_2 + \Psi_{13}(y)\mathbf{d}_S(\bar{x}) \cdot \mathbf{e}_3 - c_{12}x_2 - c_{13}x_3 \\ \Psi_{23}(y)\mathbf{d}_S(\bar{x}) \cdot \mathbf{e}_3 - \frac{1}{\gamma}\int_0^y \Psi_{12}(s) - c_{12}\,\mathrm{d}s \\ -\Psi_{23}(y)\mathbf{d}_S(\bar{x}) \cdot \mathbf{e}_2 - \frac{1}{\gamma}\int_0^y \Psi_{13}(s) - c_{13}\,\mathrm{d}s \end{pmatrix}.$$

By construction we have $\psi \in H^1(S \times \mathcal{Y}, \mathbb{R}^3)$ and (40) follows by a tedious but straightforward calculation that we leave to the reader.

Step 4. Conclusion.

We only have to identify the limit of $\mathrm{E}_h(u^h)$. Differentiation of (33) yields

$$\nabla_h u^h = R^h \left(I + h G^h \right) \qquad \text{where} \qquad G^h := K^h (\mathbf{d}_S \otimes \mathbf{e}_1) + (R^h)^t \nabla_h \phi^h.$$

By (35), (37), (39) and Step 2, we have

(41) $$G^h \xrightarrow{2,\gamma} R^t R_{,1}(\mathbf{d}_S \otimes \mathbf{e}_1) + G \qquad \text{weakly two-scale in } L^2$$

where $G := \partial_y \Psi (\mathbf{d}_S \otimes \mathbf{e}_1) + R^t(\partial_1 \phi \otimes \mathbf{e}_1 + F)$. Hence, Lemma 4.5 yields

(42) $$\mathrm{E}_h(\nabla_h u^h) \xrightarrow{2,\gamma} \mathrm{sym}\left[R^t R_{,1} (\mathbf{d}_S \otimes \mathbf{e}_1) \right] + \mathrm{sym}\, G.$$

What is left is to show that $\operatorname{sym} G\in\mathcal{G}:=L^2(\omega,\mathcal{G}_\gamma(S\times\mathcal{Y}))$. By linearity it suffices to argue that the three matrix fields

$$G_1:=\operatorname{sym}[\partial_y\Psi(\mathbf{d}_S\otimes\mathbf{e}_1)],\qquad G_2:=\operatorname{sym}[R^{\mathrm{t}}(\partial_1\phi\otimes\mathbf{e}_1)]\quad\text{and}\quad G_3:=\operatorname{sym}[R^{\mathrm{t}}F]$$

belong to $\mathcal{G}$. Let us argue that $G_1\in\mathcal{G}$. For $\gamma=0$ this directly follows from the definition of $\mathcal{G}$. For $\gamma=\infty$ we have $G_1=0$ by Step 2 and the statement is trivial. For $\gamma\in(0,\infty)$ the statement follows from (40). We argue that $G_2\in\mathcal{G}$. To this end set $a_i:=(R^{\mathrm{t}}\partial_1\phi)\cdot\mathbf{e}_i$ for $i\in\{1,2,3\}$ and $\varphi(x_1,\bar{x}):=(a_2(x_1)x_2+a_3(x_1)x_3)\mathbf{e}_1$. We note that $G_2=a_1(\mathbf{e}_1\otimes\mathbf{e}_1)+\operatorname{sym}\left(0\,|\,\bar\nabla\varphi\right)$. Since $\left(0\,|\,\bar\nabla\varphi\right)\in L^2(\omega,\mathcal{F}_\gamma(S\times\mathcal{Y}))$, the assertion follows. Eventually we note that $G_3\in\mathcal{G}$, since $R^{\mathrm{t}}F\in L^2(\omega,\mathcal{F}_\gamma(S\times\mathcal{Y}))$. $\square$

Proof of part (b) of Theorem 3.5. We have to prove the following statement:

Fix $v^0\in\mathbb{R}^3$, $R^0\in\mathrm{SO}(3)$, $K\in L^2(\omega,\mathrm{Skew}(3))$, $\Psi\in L^2(\omega,H^1(\mathcal{Y},\mathrm{Skew}(3)))$, $a\in L^2(\omega)$ and $F\in L^2(\omega,\mathcal{F}_\gamma(S\times\mathcal{Y}))$. Let $(v,R)\in\mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)}$ denote the unique rod configuration satisfying $R^{\mathrm{t}}R_{,1}=K$. We assert that there there exist sequences of smooth mappings v^h,R^h,a^h and ϕ^h with

$$(v^h,R^h)\in\mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)},\qquad a^h\in C^\infty_c(\omega),\qquad \phi^h\in C^\infty_c(\omega,C^\infty(\mathcal{Y},\mathbb{R}^3))$$

such that the three-dimensional deformation u^h defined by

$$u^h(x):=v^h(x_1)+hR^h(x_1)\mathbf{d}_S(\bar{x})+h\phi^h(x)+h\int_0^{x_1}(R^h(s)\mathbf{e}_1)a^h(s)\,\mathrm{d}s\tag{43}$$

satisfies the following convergence properties:

$$u^h\to v\qquad\text{strongly in }L^2(\Omega,\mathbb{R}^3),\tag{44a}$$

$$\nabla_h u^h\to R\qquad\text{strongly in }L^2(\Omega,\mathbb{M}^3),\tag{44b}$$

$$\mathrm{E}^h(u^h)\xrightarrow{2,\gamma}(K+\partial_y\Psi)\mathbf{d}_S\otimes\mathbf{e}_1+a(\mathbf{e}_1\otimes\mathbf{e}_1)+\operatorname{sym}F\qquad\text{strongly two-scale in }L^2,\tag{44c}$$

$$\limsup_{h\to0}\operatorname*{ess\,sup}_{\Omega}\left(h^{-1}\operatorname{dist}^2(\nabla_h u^h,\mathrm{SO}(3))+h|\,\mathrm{E}^h(u^h))|\right)=0.\tag{44d}$$

The proof of this assertion is divided into three steps.

Step 2. Approximation of a, F and $K+\partial_y\Psi$.

We claim that there exist sequences $K^h\in C^\infty_c(\omega,\mathrm{Skew}(3))$, $\phi^h\in C^\infty_c(\omega,C^\infty(\bar{S},\mathbb{R}^3))$ and $a^h\in C^\infty_c(\omega)$ such that

$$\begin{cases}K^h\xrightarrow{2,\gamma}K+\partial_y\Psi\quad\text{strongly two-scale in }L^2\\ \quad\text{and }\limsup_{h\to0}\sqrt{h}||K^h||_{L^\infty(\omega)}=0,\end{cases}\tag{45a}$$

$$\begin{cases}\nabla_h\phi^h\xrightarrow{2,\gamma}RF\quad\text{strongly two-scale in }L^2,\quad\phi^h\rightharpoonup0\quad\text{weakly two-scale in }H^1\\ \quad\text{and }\limsup_{h\to0}\sqrt{h}\left(||\phi^h||_{L^\infty}+||\nabla_h\phi^h||_{L^\infty}\right)=0,\end{cases}\tag{45b}$$

$$\limsup_{h\to0}\left(\int_\omega|a^h-a|^2\,\mathrm{d}x_1+\sqrt{h}||a^h||_{L^\infty(\omega)}\right)=0.\tag{45c}$$

The sequence K^h can be constructed as follows. For $\delta>0$ let $\widetilde{K}^\delta\in C^\infty_c(\omega,C^\infty(\mathcal{Y},\mathrm{Skew}(3)))$ satisfy $||\widetilde{K}^\delta-(K+\partial_y\Psi)||_{L^2(\omega\times Y)}\leq\delta$. Set $K^{h,\delta}(x_1):=\widetilde{K}^\delta(x_1,x_1/\varepsilon(h))$. We prove that a suitable diagonal sequence of $\widetilde{K}^{h,\delta}$ satisfies (45a). To this end we introduce the quantity $\rho_{h,\delta}:=||\mathcal{T}_{\varepsilon(h)}K^{h,\delta}-(K+\partial_y\Psi)||_{L^2(\mathbb{R}\times Y)}+\sqrt{h}||K^{h,\delta}||_{L^\infty(\omega)}$ where $\mathcal{T}_{\varepsilon(h)}$ denotes the periodic unfolding operator introduced in Definition 6.1 and $(K + \partial_y\Psi)$ is extended by zero to the domain $\mathbb{R} \times Y$. By construction we have $K^{h,\delta} \xrightarrow{2,\gamma} \widetilde{K}^\delta$ strongly two-scale in L^2 for every $\delta > 0$ fixed. Therefore, it is easy to check that $\lim_{\delta\to 0}\lim_{h\to 0}\rho_{h,\delta} = 0$. Hence, by Lemma 4.7 (see below) there exists a mapping $h \mapsto \delta(h)$ such that $\lim_{h\to 0}\delta(h) = 0$ and $\lim_{h\to 0}\rho_{h,\delta(h)} = 0$. But this implies that $K^h := K^{h,\delta(h)}$ satisfies (45a).

(45c) follows by a similar construction. Since $RF \in L^2(\omega, \mathcal{F}_\gamma(S \times \mathcal{Y}))$, (45b) follows from Theorem 6.11 (b).

Step 2. Construction of (v^h, R^h); proof of (44a) and (44b).

The construction of the rod configuration (v^h, R^h) relies on Lemma 2.3: there exists a unique and analytic configuration $(v^h, R^h) \in \mathcal{A}^{\text{rod}}_{(v^0,R^0)}$ with $(R^h)^{\mathrm{t}} R^h_{,1} = K^h$ where K^h is the sequence constructed in Step 1. Due to convergence (45a), Lemma 6.8 and the identity $\int_Y K + \partial_y\Psi \,\mathrm{d}y = K$, we have $K^h \rightharpoonup K$ weakly in L^2. Hence, $R^h \rightharpoonup R$ weakly in H^1 and $v^h \rightharpoonup v$ weakly in H^2, as it is made precise in Corollary 2.4. In combination with (45a) – (45c) this implies (44a) and (44b) as can be checked by a direct calculation.

Step 3. Conclusion; proof of (44c) and (44d).

Define u^h by (43) and note that

$$\begin{aligned}\nabla_h u^h &= R^h(\,I + hG^h\,)\\ \text{where}\quad G^h &:= K^h(\mathbf{d}_S \otimes \mathbf{e}_1) + (R^h)^{\mathrm{t}}\nabla_h\phi^h + a^h(\mathbf{e}_1 \otimes \mathbf{e}_1).\end{aligned} \tag{46}$$

We assert that

$$G^h \xrightarrow{2,\gamma} (K + \partial_y\Psi)(\mathbf{d}_S \otimes \mathbf{e}_1) + F + a(\mathbf{e}_1 \otimes \mathbf{e}_1). \tag{47}$$

Indeed, the convergence of the first term in the definition of G^h follows from (45a). By (45a), we have $R^h \to R$ strongly in L^2. Hence, with (45b) we get $(R^h)^{\mathrm{t}}\nabla_h\phi^h \xrightarrow{2,\gamma} R^{\mathrm{t}}RF = F$ strongly two-scale in L^2. The convergence of the remaining term follows from (45c). With (47) at hand we can easily prove (44c): since $\limsup_{h\to 0}\sqrt{h}||G^h||_{L^\infty(\Omega)} = 0$ by construction, we can apply Lemma 4.5 and (44c) follows from (47). Condition (44d) holds by construction. $\square$

In the proof of the previous theorem we used the following lemma:

Lemma 4.7. *(diagonalization lemma, see* [Att84, Corollary 1.16] *and* [Mül87, Lemma 2.2]*). Assume $\varepsilon, \delta > 0$ and let $f : [0,\infty) \times [0,\infty) \to \mathbb{R} \cup \{\infty\}$. Then there is a mapping $\varepsilon \mapsto \delta(\varepsilon)$ such that $\varepsilon \to 0$ implies $\delta(\varepsilon) \to 0$ and*

$$\limsup_{\varepsilon\to 0} f(\delta(\varepsilon), \varepsilon) \le \limsup_{\delta\to 0}\limsup_{\varepsilon\to 0} f(\delta, \varepsilon).$$

4.4. **Proof of Theorem 3.3.** We need the following auxiliary lemma:

Lemma 4.8 (convex homogenization)**.** *Let Q_ε and Q be as in Assumption 2.9, and let $\widetilde{E}_h$ be a sequence in $L^2(\Omega, \mathbb{M}^3)$.*

(a) If $\widetilde{E}^h \xrightharpoonup{2,\gamma} \widetilde{E}$ weakly two-scale in L^2, then

$$\liminf_{h\to 0}\int_\Omega Q_{\varepsilon(h)}(x, \widetilde{E}^h(x))\,\mathrm{d}x \ge \int_{\Omega\times Y} Q(x, y, \widetilde{E}(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

(b) If $\widetilde{E}^h \xrightarrow{2,\gamma} \widetilde{E}$ strongly two-scale in L^2, then

$$\lim_{h\to 0}\int_\Omega Q_{\varepsilon(h)}(x, \widetilde{E}^h(x))\,\mathrm{d}x = \int_{\Omega\times Y} Q(x, y, \widetilde{E}(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

Proof. Since the sequence $\widetilde{E}^h$ is bounded in $L^2(\Omega)$, Assumption 2.9 (A2) implies that

$$\limsup_{h\to 0}\left|\int_\Omega Q_{\varepsilon(h)}(x,\widetilde{E}^h(x))\,\mathrm{d}x-\mathcal{Q}^h(\widetilde{E}^h)\right|=0.$$

where the functional $\mathcal{Q}^h\,:\,L^2(\Omega,\mathbb{M}^3)\to\mathbb{R}^+$ is given by $\mathcal{Q}^h(E):=\int_\Omega Q(x,x_1/\varepsilon(h),E(x))\,\mathrm{d}x$. As a consequence, it suffices to prove the following (lower semi-)continuity properties of $\mathcal{Q}^h(\widetilde{E}^h)$:

$$\begin{aligned}
&\widetilde{E}^h \xrightharpoonup{2,\gamma} \widetilde{E} \qquad \text{weakly two-scale in } L^2\\
&\qquad\Rightarrow\qquad \liminf_{h\to 0}\mathcal{Q}^h(\widetilde{E}^h)\geq\iint_{\Omega\times Y}Q(x,y,\widetilde{E}(x,y))\,\mathrm{d}y\,\mathrm{d}x.\\
&\widetilde{E}^h \xrightarrow{2,\gamma} \widetilde{E} \qquad \text{strongly two-scale in } L^2\\
&\qquad\Rightarrow\qquad \lim_{h\to 0}\mathcal{Q}^h(\widetilde{E}^h)=\iint_{\Omega\times Y}Q(x,y,\widetilde{E}(x,y))\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}\tag{48}$$

By Assumption 2.9 the quadratic energy density $Q(x,y,F)$ is continuous in x, measurable and periodic in y, convex and continuous in F and satisfies quadratic growth conditions (uniformly in x,y). Therefore, (48) follows by [Vis07, Proposition 1.3]. For the reader's convenient we briefly sketch the argument: by the integral identity of periodic unfolding, see Lemma 6.2, we have

$$\mathcal{Q}^h(\widetilde{E}^h)=\iint_{\mathbb{R}\times S\times Y}Q(S_{\varepsilon(h)}(x,y),y,\mathcal{T}_{\varepsilon(h)}\widetilde{E}^h(x))\,\mathrm{d}y\,\mathrm{d}x\tag{49}$$

where $S_{\varepsilon(h)}(x,y):=(\varepsilon(h)\lfloor x_1/\varepsilon(h)\rfloor+\varepsilon(h)y,\bar{x})$. We note that $S_{\varepsilon(h)}(x,y)\to x$ uniformly and $\mathcal{T}_{\varepsilon(h)}\widetilde{E}^h\rightharpoonup\widetilde{E}$ weakly in L^2 by Proposition 6.3. Hence, (48) basically follows from the continuity (resp. lower-semi-continuity) of the integral functionals on the right-hand side of (49) w. r. t. strong convergence (resp. weak convergence) in $L^2(\mathbb{R}\times S\times Y,\mathbb{M}^3)$, and the property that the support of the limit $\widetilde{E}$ is contained in $\Omega\times Y$. Let us remark that for the precise argument the continuity of $x_1\mapsto Q(x_1,\bar{x},y,F)$ is needed. □

Proof of Theorem 3.3 (Lower bound). Without loss of generality we assume that

$$\liminf_{h\to 0}\mathcal{I}^{\varepsilon(h),h}(u^h)=\limsup_{h\to 0}\mathcal{I}^{\varepsilon(h),h}(u^h)<\infty.$$

Hence, assumption (W2) implies that the sequence u^h satisfies

$$\limsup_{h\to 0}\frac{1}{h^2}\int_\Omega\operatorname{dist}^2(\nabla_h u^h(x),\mathrm{SO}(3))\,\mathrm{d}x\lesssim 1.\tag{50}$$

To shorten the notation we set $E^h:=\mathrm{E}^h(u^h)$. In a first step we prove the abstract lower bound

$$\liminf_{h\to 0}\mathcal{I}^{\varepsilon(h),h}(u^h)\geq\inf_{E\in\mathfrak{E}}\iint_{\Omega\times Y}Q(x,y,E(x,y))\,\mathrm{d}y\,\mathrm{d}x\tag{51}$$

where $\mathfrak{E}\subset L^2(\Omega\times\mathcal{Y},\mathbb{M}^3)$ denotes the set of weak two-scale cluster points associated to the sequence of nonlinear strains E^h :

$$\mathfrak{E}:=\left\{\,E\in L^2(\Omega\times\mathcal{Y},\mathbb{M}^3)\,:\,E^h\xrightharpoonup{2,\gamma}E\quad\text{weakly two-scale in } L^2 \text{ for a subsequence}\,\right\}.$$

In the second step we identify $\mathfrak{E}$ by appealing to Theorem 3.5.

Step 1. Abstract lower bound.

We prove (51) by combining a truncation argument used in [FJM02] with convex homogenization techniques. It suffices to show the following: Let $u^h \in H^1(\Omega, \mathbb{R}^3)$ be an arbitrary sequence that satisfies (50) and suppose that $E^h \xrightarrow{2,\gamma} E$ weakly two-scale in L^2. Then

$$\liminf_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(u^h) \geq \iint_{\Omega\times Y} Q(x, y, E(x, y))\,\mathrm{d}y\,\mathrm{d}x. \tag{52}$$

In a nutshell the argument relies on the following three ingredients:

– by polar factorization there exists $R^h : \Omega \to \mathrm{SO}(3)$ measurable such that

$$\nabla_h u^h(x) = R^h(x)(I + hE^h(x)) \tag{53}$$

holds almost everywhere – except for an exceptional set with small measure;

– by assumption (W4) and Lemma 2.7 the estimate

$$W_\varepsilon(x, I + G) \geq Q_\varepsilon(x; G) - |G|^2\, r(|G|) \tag{54}$$

holds, where $r : \mathbb{R}^+ \to [0, \infty]$ is increasing, satisfies $\lim_{\delta\to 0} r(\delta) = 0$ and only depends on the constants of ellipticity α, β;

– the lower semi-continuity of convex, oscillating integral functionals as stated in Lemma 4.8.

We want to apply estimate (54) with $G = hE^h$. However, since E^h is only bounded in L^2 and not pointwise, the direct application is not possible. Furthermore, identity (53) only holds for $x \in \Omega$ with $\det \nabla_h u^h(x) > 0$. For both reasons, we introduce the truncation

$$\chi^h(x) := \begin{cases} 1 & \text{if } \operatorname{dist}^2(\nabla_h u^h(x), \mathrm{SO}(3)) \leq h \text{ and } |E^h(x)| \leq h^{-1/2} \\ 0 & \text{else.} \end{cases}$$

By construction we have $\chi^h |hE^h|^2 \leq h$. Furthermore, for $h \ll 1$ and $x \in \{\chi^h = 1\}$ the matrix $\nabla_h u^h(x)$ is close to $\mathrm{SO}(3)$ and therefore its determinant is positive. Thus, by the polar factorization, identity (53) holds for $x \in \{\chi^h = 1\}$. By the non-negativity of $W_{\varepsilon(h)}$, its minimality at I, see (W3), and frame-indifference, see (W1), we have

$$\begin{aligned} \frac{1}{h^2} W_{\varepsilon(h)}(x, \nabla_h u^h(x)) \quad &\geq \quad \frac{\chi^h(x)}{h^2} W_{\varepsilon(h)}(x, \nabla_h u^h(x)) \\ &\overset{(53),(W1),(W3)}{=} \frac{1}{h^2} W_{\varepsilon(h)}(x, I + h\chi^h(x)E^h(x)) \\ &\overset{(54)}{\geq} \quad Q_{\varepsilon(h)}(x, \chi^h(x)E^h(x)) - |E^h(x)|^2 r(\sqrt{h}). \end{aligned}$$

In the last line we used that $h|\chi^h E^h| \leq \sqrt{h}$ by definition of the truncation. Since E^h is bounded in L^2 and $r(\sqrt{h}) \to 0$, we arrive at

$$\liminf_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(u^h) \geq \liminf_{h\to 0} \int_\Omega Q_{\varepsilon(h)}(x, \chi^h(x)E^h(x))\,\mathrm{d}x. \tag{55}$$

Next, we argue that

$$\chi^h E^h \xrightarrow{2,\gamma} E \qquad \text{weakly two-scale in } L^2. \tag{56}$$

Provided that $\chi^h \to 1$ strongly in L^1, this follows from Lemma 4.4 and $E^h \xrightarrow{2,\gamma} E$ in L^2 (which holds by assumption). Indeed, since for all $x \in \Omega$ with $\chi^h(x) = 0$ we have $1 \leq h^{-1} \operatorname{dist}^2(\nabla_h u^h(x), \mathrm{SO}(3)) + h|E^h(x)|^2$, we get

$$\int_\Omega |1 - \chi^h|\,\mathrm{d}x \leq h \left(\frac{1}{h^2} \int_\Omega \operatorname{dist}^2(\nabla_h u^h(x), \mathrm{SO}(3))\,\mathrm{d}x + \int_\Omega |E^h|^2\,\mathrm{d}x \right) \to 0$$

by appealing to (50) and the boundedness of E^h in L^2. Eventually, the desired lower bound (52) follows from (55), Lemma 4.8 applied with $\widetilde{E}^h = \chi^h E^h$ and (56).

Step 2. Identification of $\mathfrak{E}$ and conclusion.

By assumption we have $(u^h, \nabla_h u^h) \to (v, R)$ in L^2. Because of (50), Proposition 3.2 applies and we deduce that (u, R) is a rod configuration. Moreover, by Theorem 3.5 (a) we have $\mathfrak{E} \subset \{\operatorname{sym} E \,:\, E \in \mathfrak{E}_0\,\}$ where

$$\mathfrak{E}_0 := \left\{ E(x_1, \bar{x}, y) = (R^t(x_1)R_{,1}(x_1))(\mathbf{d}_S(\bar{x}) \otimes \mathbf{e}_1) + G(x_1, y, \bar{x}) \,:\, G \in L^2(\omega, \mathcal{G}_\gamma(S \times \mathcal{Y})) \right\}.$$

Hence, with (52) we get

$$\begin{aligned}
&\liminf_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(u^h)\\
&\qquad\geq \inf_{E\in\mathfrak{E}_0} \iint_{\Omega\times Y} Q(x, y, \operatorname{sym} E(x, y))\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad\geq \int_\omega \inf_{G\in\mathcal{G}_\gamma(S\times\mathcal{Y})} \iint_{S\times Y} Q\big(\,x_1, \bar{x}, y,\ (R^t(x_1)R_{,1}(x_1))(\mathbf{d}_S(\bar{x}) \otimes \mathbf{e}_1) + G(\bar{x}, y)\,\big)\,\mathrm{d}y\,\mathrm{d}\bar{x}\,\mathrm{d}x_1\\
&\qquad\overset{\text{Def. 2.12}}{\geq} \int_\omega Q_\gamma\big(\,x_1,\ R^t(x_1)R_{,1}(x_1)\,\big)\,\mathrm{d}x_1.
\end{aligned}$$

The second estimates follows from the definition of $\mathfrak{E}_0$ and the fact that Q vanishes for skew-symmetric matrices. □

Proof of Theorem 3.3 (Recovery sequence). For the argument set $K := R^t R_{,1}$. Since $R \in \mathcal{A}^{\mathrm{rod}}$ by assumption, we have $K \in L^2(\omega, \operatorname{Skew}(3))$. By Proposition 2.13 there exists a relaxation field $G \in L^2(\omega, \mathcal{G}_\gamma(S \times \mathcal{Y}))$ such that

$$\mathcal{I}_\gamma(u, R) = \iint_{\Omega\times Y} Q(x, y, K(x_1)(\mathbf{d}_S(\bar{x}) \otimes \mathbf{e}_1) + G(x, y))\,\mathrm{d}y\,\mathrm{d}x.$$

By Theorem 3.5 there exists a sequence $u^h \in H^1(\Omega, \mathbb{R}^3)$ satisfying the one-sided boundary condition associated to $(v(0), R(0))$ and

$$(u^h, \nabla_h u^h) \to (v, R) \qquad \text{strongly in } L^2 \tag{57}$$

$$\mathrm{E}_h(u_h) \xrightarrow{2,\gamma} \operatorname{sym}\big[(R^t R_{,1})(\mathbf{d}_S \otimes \mathbf{e}_1)\big] + G \qquad \text{strongly two-scale in } L^2, \tag{58}$$

$$\limsup_{h\to 0} \operatorname*{ess\,sup}_{x\in\Omega} \Big(h^{-1} \operatorname{dist}^2(\nabla_h u^h(x), \mathrm{SO}(3)) + h|\,\mathrm{E}_h(u_h)|^2\Big) = 0. \tag{59}$$

To complete the proof it suffices to show that

$$\lim_{h\to 0} \frac{1}{h^2}\mathcal{I}^{\varepsilon(h),h}(u^h) = \iint_{\Omega\times Y} Q(x, y, K(x_1)(\mathbf{d}_S(\bar{x}) \otimes \mathbf{e}_1) + G(x, y))\,\mathrm{d}y\,\mathrm{d}x. \tag{60}$$

For convenience we set $E^h := \mathrm{E}^h(\nabla_h u^h)$. By (59) we have $\det \nabla_h u^h > 0$ for $h \ll 1$. Hence, by polar factorization, the frame-indifference of $W_{\varepsilon(h)}$ and the expansion of $W_{\varepsilon(h)}$ at identity (see assumption (W4)), we get for almost every $x \in \Omega$

$$\frac{1}{h^2}W_{\varepsilon(h)}(x, \nabla_h u^h(x)) = \frac{1}{h^2}W_{\varepsilon(h)}(x, I + hE^h(x)) = Q_{\varepsilon(h)}(x, E^h(x)) + q^h(x),$$
$$\text{where } |q^h(x)| \overset{(10)}{\leq} r(h|E^h(x)|^2)|E^h(x)|^2$$

By virtue of (59) the remainder term q^h vanishes uniformly as $h \to 0$. Hence, (60) follows from (58) and Lemma 4.8 (b).

□

4.5. **Proof of Theorem 3.1.** The proof will be divided into 5 steps. In Step 1 we rewrite $\mathcal{E}^{\varepsilon(h),h}[v^h;f^h]$ in terms of the scaled deformation

$$u^h:\Omega\to\mathbb{R}^3,\qquad u^h(x):=v^h(x_1,h\bar{x}) \tag{61}$$

and the scaled functional $\mathcal{I}^{\varepsilon(h),h}(u^h)$. In Steps 2 and 3 we prove that u^h has equi-bounded energy, that is

$$\limsup_{h\to0}\mathcal{I}^{\varepsilon(h),h}(u^h)<\infty. \tag{62}$$

Finally, in Steps 4 and 5 we prove the theorem by appealing to Proposition 3.2, Theorem 3.3 and standard arguments from the theory of Γ-convergence.

Step 1. Scaled formulation.

Let u^h be defined by (61). Introduce scaled loads g^h, $g_1^h\in L^2(\omega,\mathbb{R}^3)$ and $g_2^h\in L^2(\Omega,\mathbb{R}^3)$ by

$$g^h(x):=f^h(x_1,h\bar{x}),\qquad g_1^h:=\frac{1}{h^2}\left\langle g^h\right\rangle_S\quad\text{and}\quad g_2^h:=\frac{g^h-\langle g^h\rangle_S}{h}.$$

We assert that

$$\begin{aligned}\mathcal{I}^{\varepsilon(h),h}(u^h)-\frac{|\Omega|}{h^2}\mathcal{E}^{\varepsilon(h),h}[v^h;f^h] &= \frac{1}{h^2}\int_\Omega u^h\cdot g^h\,\mathrm{d}x\\ &= \int_\omega\left\langle u^h\right\rangle_S\cdot g_1^h\,\mathrm{d}x_1-\int_\Omega\frac{u^h-\langle u^h\rangle_S}{h}\cdot g_2^h.\end{aligned} \tag{63}$$

Indeed, by the identity $\nabla_h u^h(x_1,\bar{x})=\nabla v^h(x_1,h\bar{x})$ and a change of coordinates we get

$$[\text{l. h. s. of (63)}]=\frac{|\Omega|}{h^2}\fint_{\Omega_h}v^h\cdot f^h\,\mathrm{d}x=\frac{1}{h^2}\int_\Omega u^h\cdot g^h\,\mathrm{d}x.$$

The second identity in (63) follows by the definition of g_1^h, g_2^h and orthogonality in $L^2(\Omega,\mathbb{R}^3)$.

Step 2. Estimate of the loading term.

We assert that

$$\left|\frac{1}{h^2}\int_\Omega u^h\cdot g^h\,\mathrm{d}x\right|\lesssim\max\{\|g_1^h\|_{L^2},\|g_2^h\|_{L^2}\}\left(1+h\sqrt{\mathcal{I}^{\varepsilon(h),h}(u^h)}+|v^0|\right).$$

Here comes the argument: By the second identity in (63) and Cauchy-Schwarz inequality we have

$$\begin{aligned}\left|\frac{1}{h^2}\int_\Omega u^h\cdot g^h\,\mathrm{d}x\right| &\le \left\|\left\langle u^h\right\rangle_S\right\|_{L^2(\omega)}\|g_1^h\|_{L^2(\omega)}+\left\|h^{-1}(u^h-\left\langle u^h\right\rangle_S)\right\|_{L^2(\Omega)}\|g_2^h\|_{L^2(\Omega)}\\ &\lesssim c^h\left(\left\|\left\langle u^h\right\rangle_S\right\|_{L^2(\omega)}+\left\|h^{-1}(u^h-\left\langle u^h\right\rangle_S)\right\|_{L^2(\Omega)}\right)\end{aligned}$$

where $c^h:=\max\{\|g_1^h\|_{L^2},\|g_2^h\|_{L^2}\}$. Recall that $\langle u^h(0,\cdot)\rangle_S=\langle v^h(0,\cdot)\rangle_{hS}=v^0$ by assumption. Hence, by Poincaré's inequality for functions with prescribed boundary data applied to $\left\langle u^h\right\rangle_S$, and by Poincaré's inequality for functions with zero integral mean applied to $h^{-1}(u^h-\left\langle u^h\right\rangle_S)$, we get

$$\left\|\left\langle u^h\right\rangle_S\right\|_{L^2(\omega)}+\left\|h^{-1}(u^h-\left\langle u^h\right\rangle_S)\right\|_{L^2(\Omega)}\lesssim|v_0|+\|\nabla_h u^h\|_{L^2(\Omega)}.$$

Hence, it remains to prove that

$$\|\nabla_h u^h\|_{L^2(\Omega)}\lesssim1+h\sqrt{\mathcal{I}^{\varepsilon(h),h}(u^h)}. \tag{64}$$

But this follows from assumption (W2) and the elementary estimate $|F|^2\lesssim1+\operatorname{dist}^2(F,\mathrm{SO}(3))$.

Step 3. Proof of (62).

From identity (63) and the estimate in Step 2 we get

$$\mathcal{I}^{\varepsilon(h),h}(u^h) \lesssim (1+\max\{||g_1^h||_{L^2}, ||g_2^h||_{L^2}\}) \left(1 + h\sqrt{\mathcal{I}^{\varepsilon(h),h}(u^h)} + |v^0| + \frac{|\Omega|}{h^2}\mathcal{E}^{\varepsilon(h),h}[v^h; f^h]\right).$$

By assumption (11) we have $\limsup_{h\to 0}\max\{||g_1^h||_{L^2}, ||g_2^h||_{L^2}\} < \infty$. Hence, for (62) we only need to show that $\limsup_{h\to 0}\frac{|\Omega|}{h^2}\mathcal{E}^{\varepsilon(h),h}[v^h; f^h] < \infty$. Since v^h is a sequence of almost minimizers, see (13), it suffices to prove that

$$\limsup_{h\to 0}\frac{1}{h^2}\mathcal{E}_{\inf}^{\varepsilon(h),h}[f^h] < \infty. \tag{65}$$

In order to get this we consider the deformation $\widetilde{v}^h : \Omega_h \to \mathbb{R}^3$, $\widetilde{v}^h(x) := v^0 + R^0 x$. By construction $\widetilde{v}^h$ satisfies the one-sided boundary condition, i.e. $\widetilde{v}^h \in \mathcal{A}_{(v^0,R^0)}(\Omega_h)$. Consequently,

$$\frac{1}{h^2}\mathcal{E}_{\inf}^{\varepsilon(h),h}[f^h] \le \frac{1}{h^2}\mathcal{E}^{\varepsilon(h),h}[\widetilde{v}^h; f^h].$$

Since $\widetilde{v}^h$ is a rigid body motion, the right-hand side reduces to $-\frac{1}{h^2}\fint_{\Omega_h}\widetilde{v}^h\cdot f^h\,\mathrm{d}x$, the absolute value of which is bounded by a constant independent of h, as can be easily checked by appealing to assumption (11). Hence, the assertion is proved.

Step 4. Proof Theorem 3.1 (c).

Let $(v^*, R^*) \in \mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)}$ be a minimizer of the functional $\mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)} \ni (v,R) \mapsto \mathcal{E}_\gamma[v,R;f]$. The existence of such a minimizer follows by the direct method of the calculus of variations. Let $u_*^h \in \mathcal{A}^h_{(v^0,R^0)}(\Omega)$ denote a recovery sequence satisfying $(u_*^h, \nabla_h u_*^h) \to (v^*, R^*)$ in L^2 and

$$\mathcal{I}^{\varepsilon(h),h}(u_*^h) \to \mathcal{I}_\gamma(v^*, R^*). \tag{66}$$

The existence of such a sequence is guaranteed by Theorem 3.3 (b). By appealing to assumption (11) it is easy to check that

$$\lim_{h\to 0}\frac{1}{h^2}\int_\Omega u^h\cdot g^h\,\mathrm{d}x = \int_\omega v^*\cdot f\,\mathrm{d}x_1. \tag{67}$$

Now, consider the rescaled deformation $v_*^h \in H^1(\Omega_h, \mathbb{R}^3)$, $v_*^h(x) := u_*^h(x_1, \frac{1}{h}\bar{x})$. By construction v_*^h satisfies the desired one-sided boundary conditions and converges to (v^*, R^*) as asserted in the theorem. Furthermore, by (66), (67) and identity (63) we find that

$$\lim_{h\to 0}\frac{1}{h^2}\mathcal{E}^{\varepsilon(h),h}[v_*^h; f] = \frac{1}{|\Omega|}\left(\mathcal{I}_\gamma(v^*, R^*) - \int_\omega v^*\cdot f\,\mathrm{d}x_1\right) = \mathcal{E}_\gamma[v^*, R^*; f].$$

Step 5. Proof of Theorem 3.1 (a) and (b).

We first argue that the sequences $\langle v^h\rangle_{hS}$ and $\langle \nabla v^h\rangle_{hS}$ are bounded in H^1 and L^2, respectively. Since $\langle v^h(0,\cdot)\rangle_{hS} = v^0$, Poincaré's inequality applies. Therefore, we only need to show that the sequence $\langle \nabla v^h\rangle_{hS}$ is bounded in L^2. But this follows from the identity $\langle \nabla v^h\rangle_{hS} = \langle \nabla_h u^h\rangle_S$, (62) and (64).

For the proof of the remaining assertions we have to pass to a suitable subsequence. To this end set $e_* := \liminf_{h\to 0}\frac{1}{h^2}\mathcal{E}_{\inf}^{\varepsilon(h),h}[f^h]$. By virtue of identity (63) and the estimate in Step 2 e_* is a finite number. Hence, we can pass to a subsequence (that we do not relabel) such that $\mathcal{E}_{\inf}^{\varepsilon(h),h}[f^h] \to e_*$. Since v^h is a sequence of almost minimizers, we get $\mathcal{E}^{\varepsilon(h),h}[v^h; f^h] \to e_*$. After these preparations we appeal to Proposition 3.2: since (62) is satisfied, there exists a rod configuration $(v^*, R^*) \in \mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)}$ such that, up to a further subsequence,

$$(u^h, \nabla_h u^h) \to (v^*, R^*) \qquad \text{strongly in } L^2. \tag{68}$$

Obviously, this already implies that v^h converges to (v^*, R^*) as asserted in the theorem.

Hence, we only need to show the convergence of the energy and to prove that (v^*, R^*) is a minimizer. To this end, we first claim that (68) implies

$$\lim_{h\to 0} \frac{1}{h^2}\int_\Omega u^h \cdot g^h \,\mathrm{d}x = \int_\omega v^* \cdot f \,\mathrm{d}x_1, \tag{69}$$

$$\liminf_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(u^h) \geq \mathcal{I}_\gamma(v^*, R^*). \tag{70}$$

The lower bound (70) is a direct consequence of Theorem 3.3 (a). Convergence (69) follows from the second identity in (63), convergence (68) and the assumptions on the loading, see (11).

The combination of (69), (70) and identity (63) yields

$$e_* = \lim_{h\to 0} \mathcal{E}^{\varepsilon(h),h}[v^h; f^h] \geq \mathcal{E}_\gamma[v^*, R^*; f]$$

The right-hand side can be bounded from below by passing to its minimum; we get

$$e_* \geq \min\{\, \mathcal{E}_\gamma[v, R; f] \,:\, (v, R) \in \mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)} \,\}.$$

The proof is completed by showing the opposite inequality. To this end let v^h_* denote the sequence constructed in Step 4. We have

$$e_* \leq \limsup_{h\to 0} \mathcal{E}^{\varepsilon(h),h}_{\mathrm{inf}}[f^h] \leq \limsup_{h\to 0} \mathcal{E}^{\varepsilon(h),h}[v^h_*; f^h] = \min\{\, \mathcal{E}_\gamma[v, R; f] \,:\, (v, R) \in \mathcal{A}^{\mathrm{rod}}_{(v^0,R^0)} \,\}$$

which completes the proof of statement (a). In particular, the previous inequality shows that $\mathcal{E}^{\varepsilon(h),h}_{\mathrm{inf}}[f^h]$ converges to e_* for the entire sequence and statement (b) is proved. □

4.6. **Proof of Proposition 2.13.** The proof of the statements (a) - (c) is easy. Therefore, we only give the main idea. Let $x_1 \in \omega$ and $K \in \mathcal{K} := \{\, \mathrm{skw}(\mathbf{e}_1 \otimes \mathbf{e}_2), \mathrm{skw}(\mathbf{e}_1 \otimes \mathbf{e}_3), \mathrm{skw}(\mathbf{e}_2 \otimes \mathbf{e}_3) \,\}$ be fixed. By the direct method of the calculus of variations one shows that there exists $G_{K,x_1} \in \mathcal{G}_\gamma(S \times \mathcal{Y})$ that minimizes the functional

$$\mathcal{G}_\gamma(S \times \mathcal{Y}) \ni G \mapsto \iint_{S\times Y} Q\big(\, x_1, \bar{x}, y, K(\mathbf{d}_S \otimes \mathbf{e}_1) + G \,\big) \,\mathrm{d}y \,\mathrm{d}\bar{x}. \tag{71}$$

We note that the minimum is precisely $Q_\gamma(x_1, K)$. Since Q satisfies (Q1) and is continuous in x_1, we can find solutions G_{K,x_1} that continuously depend on x_1 and that satisfy $||G_{K,x_1}||_{L^2(S\times Y)} \lesssim 1$. We note that $\mathcal{K}$ is a basis of $\mathrm{Skew}(3)$. Hence, we can construct a linear operator L that associates to each matrix field $K \in C(\overline{\omega}, \mathrm{Skew}(3))$ a continuous mapping $G_K : \omega \to \mathcal{G}_\gamma(S \times \mathcal{Y})$ with $||G_K||_{L^2(\Omega\times Y)} \lesssim ||K||_{L^2(\omega)}$ such that $G_K(x_1)$ minimizes (71) with K replaced by $K(x_1)$ for all $x_1 \in \omega$. Since L is linear, statement (a) follows. Since L applied to constant K yields a mapping that continuously depends on x_1, statement (b) follows. Obviously, L can be extended to a continuous operator with domain $L^2(\omega, \mathrm{Skew}(3))$. This proves (c).

Next, we prove (d). The upper bound directly follows from the definition of Q_γ, (Q1) and statement (c). We prove the lower estimate in detail. By Proposition 2.13 (c), it suffices to show that

$$\iint_{S\times Y} Q(x_1, \bar{x}, y, \mathrm{sym}[K(\mathbf{d}_S \otimes \mathbf{e}_1)] + G) \,\mathrm{d}y \,\mathrm{d}\bar{x} \gtrsim |K|^2. \tag{72}$$

Since the following argument and the estimates below hold uniformly in x_1, we omit the latter in our notation. For our purpose it is convenient to introduce the vector field

$$\mathbf{d}_S^\perp(x_2, x_3) := \begin{pmatrix} 0 \\ x_3 \\ -x_2 \end{pmatrix} - c_S$$

where $c_S \in \mathbb{R}^3$ is chosen such that $\langle \mathbf{d}_S^\perp \rangle_S = 0$.

Step 1. A basic coercivity estimate.

We assert that

$$\inf_{F\in X_\gamma}\iint_{S\times Y}|\operatorname{sym}\big(\,\mathbf{d}_S^\perp\,|\,0\,|\,0\,\big)+F|^2\,\mathrm{d}y\,\mathrm{d}\bar x\gtrsim 1 \tag{73}$$

where

$$\begin{aligned}
X_0 &:=\Big\{\operatorname{sym}\big(\,\partial_y\alpha\,\mathbf{d}_S^\perp\,|\,\partial_2\varphi\,\mathbf{e}_1\,|\,\partial_3\varphi\,\mathbf{e}_1\,\big)\;:\;\alpha\in H^1(\mathcal{Y}),\;\varphi\in L^2(\mathcal{Y},H^1(S))\,\Big\},\\
X_\infty &:=\Big\{\operatorname{sym}\big(\,0\,|\,\partial_2\varphi\,\mathbf{e}_1\,|\,\partial_3\varphi\,\mathbf{e}_1\,\big)\;:\;\varphi\in H^1(S)\,\Big\},\\
X_\gamma &:=\Big\{\operatorname{sym}\big(\,\partial_y\phi\,|\,\tfrac{1}{\gamma}\bar\nabla\phi\,\big)\;:\;\phi\in H^1(S\times\mathcal{Y},\mathbb{R}^3)\,\Big\}\quad\text{for}\quad\gamma\in(0,\infty).
\end{aligned}\tag{74}$$

The proof for $\gamma\in\{0,\infty\}$ is easy: since the vector field $S\ni\bar x\mapsto(x_3,-x_2)^{\mathrm t}\in\mathbb{R}^2$ is not a gradient-field, we have

$$\inf_{\varphi\in H^1(S)}\int_S|\operatorname{sym}\big(\,\mathbf{d}_S^\perp\,|\,\partial_2\varphi\,\mathbf{e}_1\,|\,\partial_3\varphi\,\mathbf{e}_1\,\big)|^2\,\mathrm{d}\bar x\gtrsim 1. \tag{75}$$

This already proves the assertion for $\gamma=\infty$. For $\gamma=0$, we note that for all $\alpha\in H^1(\mathcal{Y})$ and $\varphi\in L^2(\mathcal{Y},H^1(S))$ we have by orthogonality

$$\begin{aligned}
&\iint_{S\times Y}|\operatorname{sym}\big(\,(1+\partial_y\alpha)\,\mathbf{d}_S^\perp\,|\,\partial_2\varphi\,\mathbf{e}_1\,|\,\partial_3\varphi\,\mathbf{e}_1\,\big)|^2\,\mathrm{d}y\,\mathrm{d}\bar x\\
&\gtrsim\quad\left(\int_Y(1+\partial_y\alpha)^2\,\mathrm{d}y\right)\inf_{\hat\varphi\in H^1(S)}\int_S|\operatorname{sym}\big(\,\mathbf{d}_S^\perp\,|\,\partial_2\hat\varphi\,\mathbf{e}_1\,|\,\partial_3\hat\varphi\,\mathbf{e}_1\,\big)|^2\,\mathrm{d}\bar x\\
&\overset{(75)}{\gtrsim}\quad 1.
\end{aligned}$$

It remains to treat the case $\gamma\in(0,\infty)$. For convenience, we shall write ∇_γ for the scaled gradient $\big(\,\partial_y\,|\,\frac{1}{\gamma}\bar\nabla\,\big)$ in the sequel. Moreover, let $\mathcal{R}$ be defined as in Proposition 6.12 and let $\mathcal{H}$ denote the orthogonal complement of $\mathcal{R}$ in $H^1(S\times\mathcal{Y},\mathbb{R}^3)$. We note that $\mathcal{H}$ is a Hilbert space. By the Korn inequality (see Proposition 6.12) we have for $\phi\in\mathcal{H}$

$$\|\phi\|_{H^1(S\times Y,\mathbb{R}^3)}\lesssim\|\operatorname{sym}\nabla_\gamma\phi\|_{L^2(S\times Y,\mathbb{R}^3)} \tag{76}$$

up to a constant that depends on S and γ. Thus, the right-hand side of (76) defines an equivalent norm on $\mathcal{H}$. Because $H^1(S\times\mathcal{Y},\mathbb{R}^3)$ is the direct sum of $\mathcal{R}$ and $\mathcal{H}$, and since for all $\zeta\in\mathcal{R}$ the matrix field $\nabla_\gamma\zeta$ is skew-symmetric, it suffices to show that

$$\inf_{\phi\in\mathcal{H}}\iint_{S\times Y}|\operatorname{sym}\big(\,\mathbf{d}_S^\perp\,|\,0\,|\,0\,\big)+\operatorname{sym}\nabla_\gamma\phi|^2\,\mathrm{d}y\,\mathrm{d}\bar x\gtrsim 1.$$

For the argument, we note that the minimum on the left-hand side is attained for a unique function $\phi\in\mathcal{H}$, which is independent of the y-variable; indeed, by convexity and due to estimate (76), the associated integral functional is lower semi-continuous and coercive. Hence, by the direct method of the calculus of variations there exists a unique minimizer. Moreover, if ϕ is a minimizer, then for all translations $t\in\mathbb{R}$ the function $(\bar x,y)\mapsto\phi(\bar x,y+t)$ is a minimizer as well. Thus, by uniqueness we infer that ϕ does not depend on y. In conclusion, we have

$$\begin{aligned}
&\inf_{\phi\in\mathcal{H}}\iint_{S\times Y}\left|\,\mathrm{sym}\left(\,\mathbf{d}_S^\perp\,|\,0\,|\,0\,\right)+\mathrm{sym}\,\nabla_\gamma\phi\right|^2\,\mathrm{d}y\,\mathrm{d}\bar{x}\\
&\quad=\inf_{\substack{\phi\in\mathcal{H}\\ \phi\text{ indep of }y}}\iint_{S\times Y}\left|\,\mathrm{sym}\left(\,\mathbf{d}_S^\perp\,|\,\bar\nabla\phi\,\right)\right|^2\,\mathrm{d}y\,\mathrm{d}\bar{x}\\
&\quad=\inf_{\hat\varphi\in H^1(S)}\int_S\left|\,\mathrm{sym}\left(\,\mathbf{d}_S^\perp\,|\,\partial_2\hat\varphi\,\mathbf{e}_1\,|\,\partial_3\hat\varphi\,\mathbf{e}_1\,\right)\right|^2\,\mathrm{d}\bar{x}\\
&\quad\overset{(75)}{\gtrsim}\ 1.
\end{aligned}$$

Above, the second identity holds by orthogonality: the lower right 2×2-submatrix of $\mathrm{sym}\left(\,0\,|\,\bar\nabla\phi\,\right)$ is orthogonal to 3×3-matrices that are nonzero only in the first column and row.

Step 2. Decomposition of G

For this and the following step it is convenient to decompose $\mathrm{Skew}(3)$ into the space spanned by the matrix $E_{23} := (\mathbf{e}_2\otimes\mathbf{e}_3-\mathbf{e}_3\otimes\mathbf{e}_2)$ and its orthogonal complement. Set $\tau := \frac{1}{2}\,\mathrm{trace}(K^{\mathrm{t}}E_{23})$. Then $K_1 := \tau E_{23}$ and $K_2 := K-K_1$ is an orthogonal decomposition of K and

$$K(\mathbf{d}_S\otimes\mathbf{e}_1)=\tau(\mathbf{d}_S^\perp\otimes\mathbf{e}_1)+K_2(\mathbf{d}_S\otimes\mathbf{e}_1).$$

We claim that G can be written as a sum G_1+G_2 with $G_1\in X_\gamma$ such that the matrix fields G_2, $G_1+\tau\,\mathrm{sym}(\mathbf{d}_S^\perp\otimes\mathbf{e}_1)$ and $K_2(\mathbf{d}_S\otimes\mathbf{e}_1)$ are pairwise orthogonal w. r. t. the inner product in $L^2(S\times Y,\mathbb{M}^3)$.

Construction for $\gamma=0$: By definition there exist $a\in\mathbb{R}$, $\Psi\in H^1(\mathcal{Y},\mathrm{Skew}(3))$, $\hat\phi\in H^1(\mathcal{Y},\mathbb{R}^3)$ and $\bar\phi\in L^2(\mathcal{Y},H^1(S,\mathbb{R}^3))$ such that $G=\mathrm{sym}[a(\mathbf{e}_1\times\mathbf{e}_1)+\partial_y\Psi(\mathbf{d}_S\otimes\mathbf{e}_1)+\left(\,\partial_y\hat\phi\,|\,\bar\nabla\bar\phi\,\right)]$. Set

$$\begin{aligned}
\alpha &:= \tfrac{1}{2}\,\mathrm{trace}(\Psi^{\mathrm{t}}E_{23}), &\qquad \varphi &:= \bar\phi\cdot\mathbf{e}_1+\partial_y\hat\phi\cdot\mathbf{d}_S\\
G_1 &:= \mathrm{sym}\left(\,\partial_y\alpha\,\mathbf{d}_S^\perp\,|\,\partial_2\varphi\,\mathbf{e}_1\,|\,\partial_3\varphi\,\mathbf{e}_1\,\right), &\qquad G_2 &:= G-G_1.
\end{aligned}$$

Then $\alpha\in H^1(\mathcal{Y})$, $\varphi\in L^2(\mathcal{Y},H^1(S))$ and a simple calculation shows that the decomposition $G=G_1+G_2$ satisfies the claimed properties.

Construction for $\gamma=\infty$: by definition there exist $a\in\mathbb{R}$ and $\hat\phi\in L^2(S,H^1(\mathcal{Y},\mathbb{R}^3))$ and $\bar\phi\in H^1(S,\mathbb{R}^3)$ such that $G=\mathrm{sym}[a(\mathbf{e}_1\otimes\mathbf{e}_1)+\left(\,\partial_y\hat\phi\,|\,\bar\nabla\bar\phi\,\right)]$. Set

$$\varphi := \bar\phi\cdot\mathbf{e}_1\qquad G_1 := \mathrm{sym}\left(\,0\,|\,\partial_2\varphi\,\mathbf{e}_1\,|\,\partial_2\varphi\,\mathbf{e}_1\,\right),\qquad G_2 := G-G_1.$$

Then $\varphi\in H^1(S)$ and a simple calculation shows that the decomposition $G=G_1+G_2$ satisfies the claimed properties.

Construction for $\gamma\in(0,\infty)$: by definition there exist $a\in\mathbb{R}$ and $\phi\in H^1(S\times\mathcal{Y})$ such that $G=\mathrm{sym}[a(\mathbf{e}_1\times\mathbf{e}_1)+\left(\,\partial_y\phi\,|\,\frac{1}{\gamma}\bar\nabla\phi\,\right)]$. The claim follows with $G_1 := \mathrm{sym}\left(\,\partial_y\phi\,|\,\frac{1}{\gamma}\bar\nabla\phi\,\right)$ and $G_2 := a(\mathbf{e}_1\otimes\mathbf{e}_1)$.

Step 3. Conclusion.

We decompose G as detailed in Step 2. Because Q is positive definite for symmetric matrices, see property (Q1), we get

$$[\text{l. h. s. of (72)}]\gtrsim\iint_{S\times Y}\left|\,\mathrm{sym}[K_2(\mathbf{d}_S\otimes\mathbf{e}_1)+\tau(\mathbf{d}_S^\perp\otimes\mathbf{e}_1)]+G_1+G_2\right|^2\,\mathrm{d}y\,\mathrm{d}\bar{x}.$$

By construction the matrix fields $\operatorname{sym}[K_2(\mathbf{d}_S\otimes\mathbf{e}_1)]$, G_2 and $G_1+\tau\operatorname{sym}(\mathbf{d}_S^\perp\otimes\mathbf{e}_1)$ are pairwise orthogonal. Thus, by Pythagoras' theorem and Step 1 we get

$$\begin{aligned}
[\text{l. h. s. of (72)}] &\gtrsim |K_2|^2+\iint_{S\times Y}|\tau\operatorname{sym}(\mathbf{d}_S^\perp\otimes\mathbf{e}_1)+G_1|^2\,\mathrm{d}y\,\mathrm{d}\bar x\\
&\overset{G_1\in X_\gamma}{\gtrsim} |K_2|^2+\inf_{F\in X_\gamma}\iint_{S\times Y}|\tau\operatorname{sym}(\mathbf{d}_S^\perp\otimes\mathbf{e}_1)+F|^2\,\mathrm{d}y\,\mathrm{d}\bar x\\
&= |K_2|^2+\tau^2\inf_{F\in X_\gamma}\iint_{S\times Y}|\operatorname{sym}(\mathbf{d}_S^\perp\otimes\mathbf{e}_1)+F|^2\,\mathrm{d}y\,\mathrm{d}\bar x\\
&\overset{(73)}{\gtrsim} |K_2|^2+\tau^2\gtrsim|K|^2.
\end{aligned}$$

$\square$

5. The effective behavior for $\gamma=0$ and $\gamma=\infty$

In this section we study the limiting behavior for laterally layered composites in the three-scale regimes $\gamma=0$ and $\gamma=\infty$. As we are going to see, these cases correspond to an iteration of the relaxation steps associated to dimension reduction and homogenization, respectively. As a side result, we prove that the limits associated to dimension reduction and homogenization in general do not commute.

Throughout this section we suppose that the composite is constant in the cross-sectional direction and periodic in the lateral direction; that is, we assume that $W_\varepsilon(x_1,F):=W(x_1/\varepsilon,F)$ where

(B1) $W:\mathbb{R}\times\mathbb{M}^3\to[0,\infty]$ is a Carathéodory function, periodic and measurable in its first variable and continuous in its second variable;
(B2) $W(y,\cdot)$ is of class $\mathcal{W}(\alpha,\beta,\rho)$ for almost every y.

Evidently, in this case Assumption 2.9 is fulfilled. The associated quadratic energy density takes the form $Q(y,G)$ and is Y-periodic in y. We associate to $Q(y,G)$ four relaxed energy densities that correspond to dimension reduction and homogenization. For $y\in Y$, $K\in\operatorname{Skew}(3)$ and $G\in\mathbb{M}^3$ define

$$\begin{aligned}
Q_{\text{rod}}(y,K)&:=\inf_{\bar\phi\in H^1(S,\mathbb{R}^3)}\int_S Q\left(y,K(\mathbf{d}_S\otimes\mathbf{e}_1)+\left(0\,|\,\bar\nabla\bar\phi\right)\right)\,\mathrm{d}\bar x,\\
Q_{\text{hom}}(G)&:=\inf_{\hat\phi\in H^1(\mathcal{Y},\mathbb{R}^3)}\int_Y Q\left(y,G+\partial_y\hat\phi\otimes\mathbf{e}_1\right)\,\mathrm{d}y,\\
Q_{\text{rod}\circ\text{hom}}(K)&:=\inf_{\bar\phi\in H^1(S,\mathbb{R}^3)}\int_S Q_{\text{hom}}\left(K\mathbf{d}_S\otimes\mathbf{e}_1+\left(0\,|\,\bar\nabla\bar\phi\right)\right)\,\mathrm{d}\bar x,\\
Q_{\text{hom}\circ\text{rod}}(K)&:=\inf_{\Psi\in H^1(\mathcal{Y},\text{Skew}(3))}\int_Y Q\left(y,K+\partial_y\Psi\right)\,\mathrm{d}y.
\end{aligned}\tag{77}$$

The following result shows that Q_γ for $\gamma\in\{0,\infty\}$ can be computed by two consecutive relaxation steps.

Lemma 5.1. *Let (B1) and (B2) be satisfied. Then for $\gamma\in\{0,\infty\}$ we have*

$$Q_\gamma=\begin{cases}Q_{\textit{hom}\circ\textit{rod}} & \textit{for } \gamma=0,\\ Q_{\textit{rod}\circ\textit{hom}} & \textit{for } \gamma=\infty.\end{cases}\tag{78}$$

Proof. We only give the proof for $\gamma = 0$; the case $\gamma = \infty$ is similar and left to the reader. Fix $K \in \mathrm{Skew}(3)$. We first prove the inequality

$$Q_0(K) \geq Q_{\text{hom}\,\circ\,\text{rod}}(K). \tag{79}$$

Motivated by the relaxation formula for Q_0, see Definition 2.12, and the definition of $\mathcal{G}_0(S \times \mathcal{Y})$, see Definition 2.11, we call functions $a, \Psi, \hat{\phi}$ and $\bar{\phi}$ *admissible* if $\Psi \in H^1(\mathcal{Y}, \mathrm{Skew}(3))$, $\hat{\phi} \in H^1(\mathcal{Y}, \mathbb{R}^3)$, $\bar{\phi} \in L^2(\mathcal{Y}, H^1(S, \mathbb{R}^3))$ and $a \in \mathbb{R}$. We show that for almost every $y \in Y$ and all admissible functions $\Psi, \hat{\phi}, \bar{\phi}, a$ we have

$$\int_S Q\Big(y, (K + \partial_y \Psi(y))(\mathbf{d}_S(\bar{x}) \otimes \mathbf{e}_1) + a(\mathbf{e}_1 \otimes \mathbf{e}_1) + \big(\,\partial_y \hat{\phi}(y)\,|\,\bar{\nabla}\bar{\phi}(\bar{x}, y)\,\big)\Big)\,\mathrm{d}\bar{x} \geq Q_{\text{rod}}(y, K + \partial_y \Psi(y)) \tag{80}$$

Clearly, integration of this inequality in y and passing to the infimum over $\Psi \in H^1(\mathcal{Y}, \mathrm{Skew}(3))$ yields (79). For the proof of (80) fix $y \in Y$. Set

$$\widetilde{a} := a + \partial_y \hat{\phi}(y) \cdot \mathbf{e}_1, \quad \widetilde{\phi}(\bar{x}) := \bar{\phi}(\bar{x}, y) + \begin{pmatrix} \partial_y \hat{\phi}(y) \cdot \mathbf{d}_S(\bar{x}) \\ 0 \\ 0 \end{pmatrix} - \mathbf{c}_2 x_2 - \mathbf{c}_3 x_3$$

where $\mathbf{c}_2, \mathbf{c}_3 \in \mathbb{R}^3$ are chosen such that $\int_S \bar{\nabla}\widetilde{\phi}\,\mathrm{d}\bar{x} = 0$. By construction we have

$$a(\mathbf{e}_1 \otimes \mathbf{e}_1) + \mathrm{sym}\,\big(\,\partial_y \hat{\phi}\,|\,\bar{\nabla}\bar{\phi}\,\big) = \widetilde{a}(\mathbf{e}_1 \otimes \mathbf{e}_1) + \mathrm{sym}\,\big(\,0\,|\,\mathbf{c}_2\,|\,\mathbf{c}_3\,\big) + \mathrm{sym}\,\big(\,0\,|\,\bar{\nabla}\widetilde{\phi}\,\big).$$

Since $Q(y, F)$ only depends on the symmetric part of F, the previous identity shows that the left-hand side in (80) is equal to

$$\int_S Q\Big(y, (K + \partial_y \Psi)(\mathbf{d}_S \otimes \mathbf{e}_1) + \widetilde{a}(\mathbf{e}_1 \otimes \mathbf{e}_1) + \big(\,0\,|\,\mathbf{c}_1\,|\,\mathbf{c}_2\,\big) + \big(\,0\,|\,\bar{\nabla}\widetilde{\phi}\,\big)\Big)\,\mathrm{d}\bar{x}.$$

Since $\int_S \mathbf{d}_S\,\mathrm{d}\bar{x} = 0$ and $\int_S \bar{\nabla}\widetilde{\phi}\,\mathrm{d}\bar{x} = 0$, by orthogonality the previous line turns into

$$\int_S Q\Big(y, (K + \partial_y \Psi)(\mathbf{d}_S \otimes \mathbf{e}_1) + \big(\,0\,|\,\bar{\nabla}\widetilde{\phi}\,\big)\Big)\,\mathrm{d}\bar{x} + |S| Q\Big(y, \widetilde{a}(\mathbf{e}_1 \otimes \mathbf{e}_1) + \big(\,0\,|\,\mathbf{c}_1\,|\,\mathbf{c}_2\,\big)\Big).$$

The first term is bounded from below by $Q_{\text{rod}}(y, K + \partial_y \Psi)$. Since the second term is non-negative, (80) follows.

It remains to prove the opposite inequality. We only sketch the argument; the details are left to the reader. Fix $K \in \mathrm{Skew}(3)$ and let $\Psi_K \in H^1(\mathcal{Y}, \mathrm{Skew}(3))$ denote a function with $Q_{\text{hom}\,\circ\,\text{rod}}(K) = \int_Y Q_{\text{rod}}(y, K + \partial_y \Psi_K(y))\,\mathrm{d}y$. Now, choose $\bar{\phi}_K \in L^2(Y, H^1(S, \mathbb{R}^3))$ such that for almost every y we have

$$Q_{\text{rod}}(y, K + \partial_y \Psi_K(y)) = \int_S Q\left(y, (K + \partial_y \Psi_K(y))(\mathbf{d}_S(\bar{x}) \otimes \mathbf{e}_1) + \big(\,0\,|\,\bar{\nabla}\bar{\phi}_K(\bar{x}, y)\,\big)\right)\,\mathrm{d}\bar{x}.$$

By integration in y, the left-hand side turns into $Q_{\text{hom}\,\circ\,\text{rod}}(K)$ while the right-hand side is clearly bounded from below by $Q_0(K)$, since $\Psi_K, \bar{\phi}_K$ are admissible functions. This proves the inequality opposite to (79). $\square$

The energy densities defined above naturally appear when passing to the homogenization and zero-thickness limit in the scaled elastic energy functional consecutively. In the following we make this precise. For this purpose we introduce the functionals

$$\begin{aligned}
&\mathcal{I}^{\varepsilon,h}: H^1(\Omega,\mathbb{R}^3)\to[0,\infty], && \mathcal{I}^{\varepsilon,h}(u):=\frac{1}{h^2}\int_\Omega W(x_1/\varepsilon,\nabla_h u)\,\mathrm{d}x,\\
&\mathcal{I}^h_{\text{hom}}: H^1(\Omega,\mathbb{R}^3)\to[0,\infty], && \mathcal{I}^h_{\text{hom}}(u):=\frac{1}{h^2}\int_\Omega W_{\text{hom}}(\nabla_h u)\,\mathrm{d}x,\\
&\mathcal{I}^\varepsilon_{\text{rod}}: \mathcal{A}^{\text{rod}}\to[0,\infty], && \mathcal{I}^\varepsilon_{\text{rod}}(u,R):=\int_\omega Q_{\text{rod}}(x_1/\varepsilon,R^{\mathrm{t}}R_{,1})\,\mathrm{d}x_1,\\
&\mathcal{I}_\gamma: \mathcal{A}^{\text{rod}}\to[0,\infty], && \mathcal{I}_\gamma(u,R):=\int_\omega Q_\gamma(R^{\mathrm{t}}R_{,1})\,\mathrm{d}x_1,\qquad \text{for }\gamma\in\{0,\infty\}.
\end{aligned}$$

We extend these functionals by ∞ from their domains $H^1(\Omega,\mathbb{R}^3)$ and $\mathcal{A}^{\text{rod}}$ to $L^2(\Omega,\mathbb{R}^3)$ and $L^2(\Omega,\mathbb{R}^3)\times L^2(\Omega,\mathbb{M}^3)$, respectively. Above, the homogenized energy density W_{hom} is defined by the well-known multi-cell homogenization formula that we recall below. By Lemma 5.1 we have for $(v,R)\in\mathcal{A}^{\text{rod}}$

$$\mathcal{I}_\gamma(v,R)=\begin{cases}\int_\omega Q_{\text{rod}\circ\text{hom}}(R^{\mathrm{t}}R_{,1})\,\mathrm{d}x_1 & \text{for }\gamma=0,\\ \int_\omega Q_{\text{hom}\circ\text{rod}}(R^{\mathrm{t}}R_{,1})\,\mathrm{d}x_1 & \text{for }\gamma=\infty.\end{cases}\tag{81}$$

To avoid technicalities in the non-convex homogenization step, we suppose that besides (B1) and (B2) the energy density additionally satisfies a quadratic growth condition at infinity:

(B3) there exists a constant $\hat\beta>0$ such that
$$W(y,F)\le\hat\beta(1+|F|^2)$$
for all $F\in\mathbb{M}^3$ and almost every y.

The passage to the zero-thickness limit in the functionals $\mathcal{I}^{\varepsilon,h}$ and $\mathcal{I}^h_{\text{hom}}$ can be treated by appealing to Theorem 3.3. In contrast, for the homogenization limits we need some additional considerations. Let us discuss the homogenization step associated to $\gamma=\infty$, which corresponds to "dimension reduction after homogenization". It is well-known (see [Mül87, Bra85]) that the homogenization of the non-convex integrand W leads to the multi-cell homogenization formula

$$W_{\text{hom}}(F):=\inf_{k\in\mathbb{N}}\inf_{\hat\phi\in H^1_\#((0,k)^3,\mathbb{R}^3)}\fint_{(0,k)^3}W(z_1,F+\nabla\hat\phi(z))\,\mathrm{d}z$$

where $H^1_\#((0,k)^3,\mathbb{R}^3)$ denotes the space of $(0,k)^3$-periodic functions in $H^1_{\text{loc}}(\mathbb{R}^3,\mathbb{R}^3)$ and z_1 stands for the first component of $z\in\mathbb{R}^3$. A priori, it is not clear at all that W_{hom} satisfies the properties needed for a subsequent dimension reduction step. We have to check whether the class $\mathcal{W}(\alpha,\beta,\rho)$ is stable under homogenization. In [MN11, GN11] we prove that homogenization and linearization do commute for materials of type $\mathcal{W}(\alpha,\beta,\rho)$. The following observation is a direct consequence of this commutativity result.

Lemma 5.2. *Let W satisfy assumptions (B1) – (B3) and let $Q(y,G)$ denote the quadratic term in the expansion of W at identity. Then $W_{hom}:\mathbb{M}^3\to\mathbb{R}^+$ is a continuous, quasiconvex energy density of class $\mathcal{W}(\alpha',\beta',\rho')$ (for some admissible parameters α',β' and ρ'). Moreover, for all $G\in\mathbb{M}^3$ the expansion $W_{hom}(I+G)=Q_{hom}(G)+o(|G|^2)$ holds where Q_{hom} is given by (77).*

Proof. The property of W_{hom} to be quasiconvex, continuous and of class $\mathcal{W}(\alpha',\beta',\rho')$ directly follows from [MN11, Lemma 1]. Furthermore, [MN11, Theorem 1] states that W_{hom} admits a quadratic expansion at identity with quadratic term given by

$$\widehat{Q}(F):=\inf_{\hat\phi\in H^1(\mathcal{Z},\mathbb{R}^3)}\int_{(0,1)^3}Q(z_1,F+\nabla\hat\phi(z))\,\mathrm{d}z.$$

It remains to check that $\widehat{Q} = Q_{\text{hom}}$. But this easily follows from the observation that for all $F \in \mathbb{M}^3$ the minimum problem above admits a minimizer $\phi \in H^1_\#((0,1)^3,\mathbb{R}^3)$ that is independent of the second and third component of $z \in \mathbb{R}^3$. $\square$

We are now ready to prove the following result:

Proposition 5.3. *Let (B1) – (B3) be satisfied. Then*

$$\mathcal{I}^{\varepsilon,h} \xrightarrow{h} \mathcal{I}^{\varepsilon}_{rod} \xrightarrow{\varepsilon} \mathcal{I}_0 \qquad \textit{and} \qquad \mathcal{I}^{\varepsilon,h} \xrightarrow{\varepsilon} \mathcal{I}^{h}_{hom} \xrightarrow{h} \mathcal{I}_\infty$$

where $\stackrel{h}{\to}$ *and* $\stackrel{\varepsilon}{\to}$ *denote* $\Gamma(L^2)$*-convergence as* $\varepsilon \to 0$ *and* $h \to 0$*, respectively.*

Proof. **Step 1.** The case $\varepsilon \ll h$.

The convergence $\mathcal{I}^{\varepsilon,h} \to \mathcal{I}^h_{\text{hom}}$ for $h > 0$ fixed and $\varepsilon \to 0$, is proved in [Mül87] and [Bra85]. The convergence $\mathcal{I}^h_{\text{hom}} \to \mathcal{I}_\infty$ as $h \to 0$ directly follows from Theorem 3.3, Lemma 5.2 and (81).

Step 2. The case $h \ll \varepsilon$.

The convergence $\mathcal{I}^{\varepsilon,h} \to \mathcal{I}^{\varepsilon}_{\text{rod}}$ for $\varepsilon > 0$ fixed and $h \to 0$ directly follows from Theorem 3.3. It remains to show that $\mathcal{I}^{\varepsilon}_{\text{rod}}$ $\Gamma(L^2)$-converges to $\mathcal{I}_0$. For the argument, we first note that for all $x_1 \in \omega$, $\varepsilon > 0$ and $K \in \text{Skew}(3)$ we have

$$Q_{\text{rod}}(x_1/\varepsilon, K) \gtrsim |K|^2. \tag{82}$$

Indeed, this directly follows from Proposition 2.13 (c). For the Γ-convergence statement, we have to show the following two properties from the sequential characterization of Γ-convergence in L^2:

– (Lower bound). For all sequences $(u^\varepsilon, R^\varepsilon) \in L^2(\Omega,\mathbb{R}^3) \times L^2(\Omega,\mathbb{M}^3)$ converging in L^2 to some (v, R) it is
$$\liminf_{\varepsilon\to 0} \mathcal{I}^{\varepsilon}_{\text{rod}}(u^\varepsilon, R^\varepsilon) \geq \mathcal{I}_0(v,R).$$
– (Recovery sequence). For all $(v,R) \in \mathcal{A}^{\text{rod}}$ there exists a sequence of rod configurations $(u^\varepsilon, R^\varepsilon) \subset \mathcal{A}^{\text{rod}}$ such that $(u^\varepsilon, R^\varepsilon) \to (v,R)$ in L^2 and
$$\lim_{\varepsilon\to 0} \mathcal{I}^{\varepsilon}_{\text{rod}}(u^\varepsilon, R^\varepsilon) = \mathcal{I}_0(v,R).$$

We only prove the lower bound statement. The construction of recovery sequences is easy and left to the reader. It suffices to consider the case where $(u^\varepsilon, R^\varepsilon)$ is a sequence of rod configurations and $\mathcal{I}^{\varepsilon}_{\text{rod}}(u^\varepsilon, R^\varepsilon)$ converges to a finite number. By (82) we can pass to a subsequence (not relabeled) such that $K^\varepsilon := (R^\varepsilon)^{\text{t}} R^\varepsilon_{,1}$ weakly two-scale converges in L^2 to a field $\widehat{K} \in L^2(\omega \times Y, \text{Skew}(3))$. Set $K(x_1) := \int_Y \widehat{K}(x_1, y)\,\mathrm{d}y$. By construction there exists a mapping $\Psi \in L^2(\omega, H^1(\mathcal{Y}, \text{Skew}(3)))$ with $\widehat{K} = K + \partial_y \Psi$. Thus, by the lower semi-continuity of convex integral functionals w. r. t. weak two-scale convergence we get

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \mathcal{I}^{\varepsilon}_{\text{rod}}(u^\varepsilon, R^\varepsilon) &= \liminf_{\varepsilon\to 0} \int_\omega Q_{\text{rod}}(x_1/\varepsilon, K^\varepsilon)\,\mathrm{d}x_1 \geq \iint_{\omega\times Y} Q_{\text{rod}}(y, K + \partial_y\Psi)\,\mathrm{d}y\,\mathrm{d}x_1 \\
&\geq \int_\omega Q_{\text{hom}\circ\text{rod}}(K)\,\mathrm{d}y = \int_\omega Q_0(K)\,\mathrm{d}y.
\end{aligned}$$

Since (v, R) is a rod configuration and $K = R^{\text{t}} R_{,1}$, the right-hand side is equal to $\mathcal{I}_0(v,R)$ which completes the argument. $\square$

5.1. **An explicit formula for isotropic materials.** Assume that (B1) and (B2) are satisfied. For isotropic materials the energy density Q takes the form

$$Q(y,G) = 2\mu(y)|\operatorname{sym} G|^2 + \lambda(y)(\operatorname{trace} G)^2 \tag{83}$$

where μ and λ are non-negative, measurable, Y-periodic functions with $\operatorname{ess\,inf}_{\mathbb{R}}(2\mu+\lambda) > 0$. In the following we prove an explicit formula for Q_γ in the regimes $\gamma \in \{0,\infty\}$. For convenience we introduce some notation. We set $m := 2\mu + \lambda$ and for scalar fields $\rho : Y \to \mathbb{R}^+$ we write $\langle \rho \rangle := \int_Y \rho \, \mathrm{d}y$ and $\langle \rho \rangle_{\text{hom}} := \left\langle \rho^{-1} \right\rangle^{-1}$ to denote the average and the harmonic mean, respectively. Moreover, we introduce two quadratic functionals that only depend on the geometry of S: for $\kappa_2, \kappa_3 \in \mathbb{R}$ and $\tau \in \mathbb{R}$ define

$$\begin{aligned} q_{S,1}(\tau) &:= \left(\inf_{\varphi \in H^1(S)} |\big(\, \partial_2\varphi \,|\, \partial_3\varphi \,\big) + \big(\, \mathbf{d}_S \cdot \mathbf{e}_3 \,|\, -\mathbf{d}_S \cdot \mathbf{e}_2 \,\big)|^2 \, \mathrm{d}\bar{x} \right) \tau^2 \\ q_{S,2}(\kappa_2,\kappa_3) &:= \int_S |\, (\kappa_2 \mathbf{e}_2 + \kappa_3 \mathbf{e}_3) \cdot \mathbf{d}_S(\bar{x}) \,|^2 \, \mathrm{d}\bar{x} \end{aligned}$$

Lemma 5.4. *Let Q be given by (83). Then for $\gamma \in \{0,\infty\}$ we have*

$$Q_\gamma(K) = \langle \mu \rangle_{hom} \, q_{S,1}(\tau) + c_\gamma(\mu,\lambda) q_{S,2}(k_2,k_3), \qquad K = \begin{pmatrix} 0 & \kappa_2 & \kappa_3 \\ -\kappa_2 & 0 & \tau \\ -\kappa_3 & -\tau & 0 \end{pmatrix}$$

where

$$c_\gamma(\mu,\lambda) := \begin{cases} \left\langle \dfrac{\mu(2\mu+3\lambda)}{\mu+\lambda} \right\rangle_{hom} & \text{for } \gamma = 0, \\[2ex] \langle m \rangle_{hom} \dfrac{\langle \mu \rangle + \langle \lambda \rangle - \left\langle \lambda^2/m \right\rangle}{\langle \mu \rangle + \langle \lambda \rangle + \langle m \rangle_{hom} \left\langle \lambda/m \right\rangle^2 - \left\langle \lambda^2/m \right\rangle} & \text{for } \gamma = \infty. \end{cases}$$

As a direct consequence of Proposition 5.3 and Lemma 5.4 we get the following:

Corollary 5.5. *There exists an isotropic composite material for which dimension reduction and homogenization do not commute in the sense that $\mathcal{I}_{hom \circ rod} \neq \mathcal{I}_{rod \circ hom}$.*

Proof of Lemma 5.4. We introduce the following notation for matrices $G \in \mathbb{M}^3$:

$$G = \begin{pmatrix} G_{11} & G_{12} \quad G_{13} \\ G_{21} & \\ G_{31} & \widehat{G} \end{pmatrix}, \qquad \widehat{G} \in \mathbb{M}^2.$$

Let $a_1, \ldots, a_5$ be in $L^2(\mathcal{Y})$. For $y \in Y$ and $G \in \mathbb{M}^3$ define

$$\begin{aligned} \widehat{Q}(y,G) \;=\; & a_1(y) G_{11}^2 + a_2(y) \left((G_{12}+G_{21})^2 + (G_{13}+G_{31})^2 \right) \\ & + a_3(y) |\operatorname{sym} \widehat{G}|^2 + a_4(y) (\operatorname{trace} \widehat{G})^2 + a_5(y) G_{11} (\operatorname{trace} \widehat{G}). \end{aligned} \tag{84}$$

Step 1. A one-dimensional homogenization formula.

Let $\nu_1, \nu_2 \in L^2(\mathcal{Y})$, $c_1, c_2 \in \mathbb{R}$ and suppose that $\nu_1 > 0$. Then

$$\begin{aligned} &\inf_{\varphi \in H^1(\mathcal{Y})} \int_Y \nu_1 (c_2 + \partial_y \varphi)^2 + \nu_2 c_1 (c_2 + \partial_y \varphi) \, \mathrm{d}y \\ &\qquad = \langle \nu_1 \rangle_{\text{hom}} c_2^2 + \langle \nu_1 \rangle_{\text{hom}} \langle \nu_2/\nu_1 \rangle c_1 c_2 + \frac{1}{4} \left(\langle \nu_1 \rangle_{\text{hom}} \langle \nu_2/\nu_1 \rangle^2 - \left\langle (\nu_2)^2/\nu_1 \right\rangle \right) c_1^2. \end{aligned}$$

The argument is standard. First, we note that the associated Euler-Lagrange equation reads $\int_Y (2\nu_1(c_2+\partial_y\varphi)+\nu_2 c_1)\,\partial_y\eta\,\mathrm{d}y=0$ for all $\eta\in H^1(\mathcal{Y})$. We deduce that

$$2\nu_1(c_2+\partial_y\varphi)+\nu_2 c_1 = C$$

for a constant $C\in\mathbb{R}$. In order to identify C, we divide both sides by ν_1, integrate over Y, use the identity $\int_Y \partial_y\varphi\,\mathrm{d}y=0$ and get $\langle\nu_1\rangle_{\text{hom}}^{-1} C = 2c_2+\langle\nu_2/\nu_1\rangle\,c_1$. Now, a short computation yields the asserted identity.

Step 2. Homogenization.

Let $\widehat{Q}$ be defined as in (84) and suppose that $a_1>0$ and $a_2>0$. Then

$$\begin{aligned}\widehat{Q}_{\text{hom}}(G) &= \inf_{\hat\phi\in H^1(\mathcal{Y},\mathbb{R}^3)} \widehat{Q}(y,G+\partial_y\hat\phi\otimes\mathbf{e}_1)\,\mathrm{d}y\\ &= b_1G_{11}^2+b_2\left((G_{12}+G_{21})^2+(G_{13}+G_{31})^2\right)\\ &\quad +b_3|\operatorname{sym}\widehat{G}|^2+b_4(\operatorname{trace}\widehat{G})^2+b_5G_{11}(\operatorname{trace}\widehat{G})\end{aligned}$$

where

$$\begin{gathered} b_1=\langle a_1\rangle_{\text{hom}}\,,\qquad b_2=\langle a_2\rangle_{\text{hom}}\,,\qquad b_3=\langle a_3\rangle\,,\\ b_4=\langle a_4\rangle+\frac14\left(\langle a_1\rangle_{\text{hom}}\langle a_5/a_1\rangle^2-\langle a_5^2/a_1\rangle\right),\qquad b_5=\langle a_1\rangle_{\text{hom}}\langle a_5/a_1\rangle\,.\end{gathered}$$

For the argument we use the notation $\hat\phi=(\hat\phi_1,\hat\phi_2,\hat\phi_3)$ to denote the components of $\hat\phi\in H^1(\mathcal{Y},\mathbb{R}^3)$. By definition we have

$$\begin{aligned}&\int_Y \widehat{Q}(y,G+\partial_y\hat\phi\otimes\mathbf{e}_1)\,\mathrm{d}y\\ &=\int_Y a_2(G_{12}+G_{21}+\partial_y\hat\phi_2)^2\,\mathrm{d}y+\int_Y a_2(G_{13}+G_{31}+\partial_y\hat\phi_3)^2\,\mathrm{d}y\\ &\quad+\langle a_3\rangle\,|\operatorname{sym}\widehat{G}|^2+\langle a_4\rangle\,(\operatorname{trace}\widehat{G})^2\\ &\quad+\int_Y a_1(G_{11}+\partial_y\hat\phi_1)^2+a_5(G_{11}+\partial_y\hat\phi_1)(\operatorname{trace}\widehat{G})\,\mathrm{d}y.\end{aligned}$$

The infimum over $\hat\phi\in H^1(\mathcal{Y},\mathbb{R}^3)$ can be computed by appealing to Step 1: indeed, for the first two integrals on the right-hand side we apply Step 1 with $\nu_1=a_2$, $\nu_2=0$, $c_1=0$, $c_2=G_{12}+G_{21}$ and $c_2=G_{13}+G_{31}$, respectively. For the third integral on the right-hand side we apply Step 1 with $\nu_1=a_1$, $\nu_2=a_5$, $c_1=\operatorname{trace}\widehat{G}$ and $c_2=G_{11}$.

Step 3. Relaxation associated to reduction of the dimension.

Let $\widehat{Q}$ be defined as in (84). Suppose that $a_2>0$ and $a_3+2a_4>0$. Then for almost every y

$$\begin{aligned}\widehat{Q}_{\text{rod}}(y,K) &= \inf_{\bar\phi\in H^1(S,\mathbb{R}^3)}\int_S Q\left(y,K(\mathbf{d}_S\otimes\mathbf{e}_1)+\left(0\,|\,\bar\nabla\bar\phi\right)\right)\,\mathrm{d}\bar x\\ &=\left(a_1-\frac{a_5^2}{2(a_3+2a_4)}\right)q_{S,2}(\kappa_2,\kappa_3)+a_2q_{S,1}(\tau).\end{aligned}$$

For the proof we follow the computation presented in [MM03]. Since the argument holds uniformly in y, we drop the dependency on y in our notation. To denote the components of $\bar\phi\in$

$H^1(S,\mathbb{R}^3)$ we use the notation $\bar\phi = (\bar\phi_1, \bar\phi_{23})$ with $\bar\phi_1 \in H^1(S)$ and $\bar\phi_{23} \in H^1(S,\mathbb{R}^2)$. Set $G = \operatorname{sym}[K(\mathbf{d}_S\otimes \mathbf{e}_1) + \left(\,0\,|\,\bar\nabla\bar\phi\,\right)]$. Then

$$G_{11} = (\kappa_2\mathbf{e}_2 + \kappa_3\mathbf{e}_3)\cdot \mathbf{d}_S, \qquad G_{12} = G_{21} = \frac{1}{2}(\partial_2\bar\phi_1 + \tau(\mathbf{d}_S\cdot\mathbf{e}_3)),$$
$$G_{13} = G_{31} = \frac{1}{2}(\partial_3\bar\phi_1 - \tau(\mathbf{d}_S\cdot\mathbf{e}_2)), \qquad \widehat{G} = \operatorname{sym}\bar\nabla\bar\phi_{23}.$$

By definition of $\widehat{Q}$ we have

$$\begin{aligned}
&\int_S Q\left(K(\mathbf{d}_S\otimes\mathbf{e}_1) + \left(\,0\,|\,\bar\nabla\bar\phi\,\right)\right)\,\mathrm{d}\bar x = \int_S Q(G)\,\mathrm{d}\bar x\\
&\quad= \quad a_1 q_{S,2}(\kappa_2,\kappa_3) + a_2\int_S |\big(\,\partial_2\bar\phi_1\,|\,\partial_3\bar\phi_1\,\big) + \tau\big(\,\mathbf{d}_S\cdot\mathbf{e}_3\,|\,-\mathbf{d}_S\cdot\mathbf{e}_2\,\big)|^2\,\mathrm{d}\bar x\\
&\qquad\quad + \int_S a_3|\operatorname{sym}\bar\nabla\bar\phi_{23}|^2 + a_4(\operatorname{trace}\bar\nabla\bar\phi_{23})^2 - a_5(y)(\kappa_2(\mathbf{d}_S\cdot\mathbf{e}_2) + \kappa_3(\mathbf{d}_S\cdot\mathbf{e}_3))(\operatorname{trace}\bar\nabla\bar\phi_{23})\,\mathrm{d}\bar x\\
&\quad\geq \quad a_1 q_{S,2}(\kappa_2,\kappa_3) + a_2 q_{S,1}(\tau) + q_{S,3}(\bar\phi_{23},\kappa_2,\kappa_3)
\end{aligned}$$

where

$$\begin{aligned}
q_{S,3}(\bar\phi_{23},\kappa_2,\kappa_3) := \int_S & a_3|\operatorname{sym}\bar\nabla\bar\phi_{23}|^2 + a_4(\operatorname{trace}\bar\nabla\bar\phi_{23})^2\\
&- a_5(y)(\kappa_2(\mathbf{d}_S\cdot\mathbf{e}_2) + \kappa_3(\mathbf{d}_S\cdot\mathbf{e}_3))(\operatorname{trace}\bar\nabla\bar\phi_{23})\,\mathrm{d}\bar x.
\end{aligned}$$

It remains to prove that $\inf\limits_{\phi\in H^1(S,\mathbb{R}^2)} q_{S,3}(\phi,\kappa_2,\kappa_3) = -\frac{a_5^2}{2(a_3+2a_4)} q_{S,2}(\kappa_2,\kappa_3)$. Indeed, this is true since the minimizer is explicitly given by the function

$$\phi = \frac{a_5}{2a_3+4a_4}\begin{pmatrix} \frac{1}{2}\kappa_2((\mathbf{d}_S\cdot\mathbf{e}_2)^2 - (\mathbf{d}_S\cdot\mathbf{e}_3)^2) + \kappa_3(\mathbf{d}_S\cdot\mathbf{e}_2)(\mathbf{d}_S\cdot\mathbf{e}_3)\\ \frac{1}{2}\kappa_3((\mathbf{d}_S\cdot\mathbf{e}_3)^2 - (\mathbf{d}_S\cdot\mathbf{e}_2)^2) + \kappa_2(\mathbf{d}_S\cdot\mathbf{e}_2)(\mathbf{d}_S\cdot\mathbf{e}_3)\end{pmatrix}$$

as can be easily checked by appealing to the associated Euler-Lagrange equation.

Step 4. Conclusion.

We note that $Q(y,G) = \widehat{Q}(y,G)$ with

$$a_1 = (2\mu+\lambda), \qquad a_2 = \mu, \qquad a_3 = 2\mu, \qquad a_4 = \lambda, \qquad a_5 = 2\lambda.$$

Hence, application of Step 3 yields

$$Q_{\text{rod}}(y,K) = \mu q_{S,1}(\tau) + \frac{\mu(2\mu+3\lambda)}{\mu+\lambda} q_{S,2}(\kappa_2,\kappa_3).$$

and the identity for $\gamma = 0$ easily follows from the one-dimensional homogenization formula in Step 1.

For $\gamma = \infty$, we apply Step 2 and get

$$\begin{aligned}
Q_{\text{hom}}(G) \quad = \quad & \langle m\rangle_{\text{hom}} G_{11}^2 + \langle\mu\rangle_{\text{hom}}\left((G_{12}+G_{21})^2 + (G_{13}+G_{31})^2\right)\\
&+2\langle\mu\rangle\,|\operatorname{sym}\widehat{G}|^2 + b_4(\operatorname{trace}\widehat{G})^2 + b_5 G_{11}(\operatorname{trace}\widehat{G})
\end{aligned}$$

with $b_4 = \langle\lambda\rangle + \langle m\rangle_{\text{hom}}\langle\lambda/m\rangle^2 - \left\langle\lambda^2/m\right\rangle$ and $b_5 = \langle m\rangle_{\text{hom}}\langle 2\lambda/m\rangle$. The asserted identity for $\gamma = \infty$ follows by Step 3 applied to Q_{hom}. □

6. Two-scale convergence methods for thin domains

In this section we introduce a modified version of two-scale convergence that is tailor made for the analysis of homogenization problems on thin domains. As a main result, we precisely identify the structure of two-scale limits that arise along sequences of scaled gradients. As we are going to see, the structure of the two-scale limits depends on the ratio between the fine-scales ε and h. The methodology that we develop in this section can be applied to various problems which combine homogenization and dimension reduction.

6.1. **Definition and basic properties.** Two-scale convergence was developed in [Ngu89], [All92], and can be viewed as an intermediate convergence between strong and weak convergence in L^p with the capability to capture oscillations on a prescribed microscale. In [ADH90, BLM96] it is noticed that two-scale convergence can be equivalently defined by appealing to a dilation operator which associates to each function a two-scale function that explicitly depends on an additional "fast" variable. In [CDG02] (see also [Vis06, MT07]) this approach has been investigated in a systematic way and led to the *periodic unfolding method*. For our purpose it is convenient to introduce a periodic unfolding operator that resolves only oscillations in x_1-direction.

Definition 6.1 (periodic unfolding)**.** Let $\omega \subset \mathbb{R}$ be (a possibly unbounded) open interval, $S \subset \mathbb{R}^2$ a Lipschitz domain and $\Omega := \omega \times S$. The periodic unfolding of a measurable function $f : \Omega \to \mathbb{R}$ is given by the function $\mathcal{T}_\varepsilon f : \mathbb{R}\times S\times\mathbb{R} \to \mathbb{R}$,

$$(\mathcal{T}_\varepsilon f)(x,y) := \begin{cases} f(\varepsilon\lfloor x_1/\varepsilon\rfloor + \varepsilon\{y\}, \bar{x}) & \text{if } \varepsilon\lfloor x_1/\varepsilon\rfloor + \varepsilon\{y\} \in \omega, \\ 0 & \text{else.} \end{cases}$$

Above, $\lfloor t\rfloor := \min\{\, k \geq t \,:\, k \in \mathbb{Z}^d \,\}$ and $\{t\} := t - \lfloor t\rfloor$ denote the integer and fractional part of $t \in \mathbb{R}$, respectively. The periodic unfolding of a vector valued map is defined componentwise.

Throughout this chapter we assume that ω, S and Ω are as in the definition above. The proofs of the following statements are similar up to minor modifications to those in the standard setting of periodic unfolding. Therefore, we omit the proofs and refer the reader to [Vis06, MT07, CDG02]. The central property of $\mathcal{T}_\varepsilon$ is the following integral identity.

Lemma 6.2 (integral identity)**.** *For every measurable function $f : \Omega \to \mathbb{R}$ the periodic unfolding $\mathcal{T}_\varepsilon f$ is measurable, and for all $f \in L^1(\Omega)$ and positive ε it is*

$$\int_\Omega f(x)\,\mathrm{d}x = \iint_{\mathbb{R}\times S\times Y} (\mathcal{T}_\varepsilon f)(x,y)\,\mathrm{d}y\,\mathrm{d}x. \tag{85}$$

In particular, for all $p \in [1,\infty]$ the operator

$$L^p(\Omega) \ni f \mapsto \mathcal{T}_\varepsilon f \in L^p(\mathbb{R}, L^p(S\times\mathcal{Y}))$$

is a (non-surjective) linear isometry.

With the operator $\mathcal{T}_\varepsilon$ at hand we can give an alternative characterization of two-scale convergence as introduced in Definition 3.4.

Proposition 6.3 (definition of two-scale convergence by periodic unfolding)**.** *Let $p \in [1,\infty)$ and $\varepsilon \overset{\gamma}{\sim} h$. Let $f \in L^p(\omega, L^p(S\times\mathcal{Y}))$ and let f^{ex} denote the extension of f to $\mathbb{R} \times S\times\mathbb{R}$ by zero. For a sequence $f^h \in L^p(\Omega)$ the following equivalences hold:*

$$f^h \xrightharpoonup{2,\gamma} f \quad \text{in } L^p \quad \Leftrightarrow \quad \mathcal{T}_{\varepsilon(h)} f^h \rightharpoonup f^{ex} \quad \text{weakly in } L^p(\mathbb{R}, L^p(S\times\mathcal{Y})),$$
$$f^h \xrightarrow{2,\gamma} f \quad \text{in } L^p \quad \Leftrightarrow \quad \mathcal{T}_{\varepsilon(h)} f^h \to f^{ex} \quad \text{strongly in } L^p(\mathbb{R}, L^p(S\times\mathcal{Y})).$$

In the following we gather basic properties of $\mathcal{T}_\varepsilon$.

Lemma 6.4. *Let $p \in [1,\infty)$, $\phi \in L^p(\Omega, C(\mathcal{Y}))$ and set $\phi^h(x) := \phi(x, x_1/\varepsilon(h))$. Then $\phi^h \xrightarrow{2,\gamma} \phi$ strongly two-scale in L^p.*

Lemma 6.5 (two-scale compactness)**.** *Every bounded sequence $f^h \in L^p(\Omega)$, $p \in (1,\infty)$, admits a weakly two-scale convergent subsequence in L^p.*

Lemma 6.6 (two-scale convergence of products)**.** *Let $p, q \in (1,\infty)$ be dual exponents. Consider sequences $f^h \in L^p(\Omega)$ and $g^h \in L^q(\Omega)$. If $f^h \xrightharpoonup{2,\gamma} f$ weakly two-scale in L^p and $g^h \xrightarrow{2,\gamma} g$ strongly two-scale in L^q, then $f^h g^h \xrightharpoonup{2,\gamma} fg$ weakly two-scale in L^1.*

Lemma 6.7 (two-scale compactness for functions in $H^1(\omega)$)**.** *Every bounded sequence $f^h \in H^1(\omega)$ admits a subsequence, such that*

$$\begin{aligned} f^h &\rightharpoonup f && \textit{weakly in } H^1(\omega),\\ \partial_1 f^h &\xrightharpoonup{2,\gamma} \partial_1 f + \partial_y \varphi && \textit{weakly two-scale in } L^2, \end{aligned}$$

where $f \in H^1(\omega)$ and $\varphi \in L^2(\omega, H^1(\mathcal{Y}))$.

Lemma 6.8. *Let $p \in [1,\infty)$, $f \in L^p(\Omega \times \mathcal{Y})$, $\bar f \in L^p(\Omega)$ and let f^h be a sequence in $L^p(\Omega)$. Then:*

(a) If $f^h \to \bar f$ strongly in $L^p(\Omega)$, then $f^h \xrightarrow{2,\gamma} \bar f$ strongly two-scale in L^p.
(b) If $f^h \xrightarrow{2,\gamma} f$ strongly two-scale in L^p, then $f^h \xrightharpoonup{2,\gamma} f$ weakly two-scale in L^p.
(c) If $f^h \xrightharpoonup{2,\gamma} f$ weakly two-scale in L^p, then $f^h \rightharpoonup \bar f := \int_Y f \,\mathrm{d}y$ weakly in $L^p(\Omega)$.
(d) If $f^h \xrightharpoonup{2,\gamma} f$ and $||f^h||_{L^p(\Omega)} \to ||f||_{L^p(\Omega\times Y)}$, then $f^h \xrightarrow{2,\gamma} f$ strongly two-scale in L^p, provided $p \in (1,\infty)$.

Lemma 6.9 (commutativity with superposition)**.** *Let $p \in [1,\infty]$, $m, n \geq 1$, $f : \Omega \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ be measurable functions. Then $\mathcal{T}_\varepsilon(g \circ f) = g \circ (\mathcal{T}_\varepsilon f)$.*

6.2. **Compactness for two-scale limits of scaled gradients.** We are interested in the structure of two-scale limits that arise from sequences of matrix fields of the form $\nabla_h u^h$ where $u^h \in H^1(\Omega, \mathbb{R}^3)$. On a formal level the structure can be easily understood by means of the following calculation. Consider the sequence

$$u^{\varepsilon,h}(x) := u(x, x_1/\varepsilon) + \varepsilon\hat\phi(x, x_1/\varepsilon) + h\bar\phi(x, x_1/\varepsilon).$$

where $u(x,y)$, $\hat\phi(x,y)$, $\bar\phi(x,y) \in C_c^\infty(\Omega, C^\infty(\mathcal{Y}, \mathbb{R}^3))$ are smooth functions periodic in y. Suppose that $\limsup_{h\to 0}\sup_{x\in\Omega} |\nabla_h u^{\varepsilon(h),h}(x)| < \infty$, then necessarily we have $u = u(x_1)$ and

$$\begin{aligned} &\hat\phi = \hat\phi(x_1, y), && \bar\phi = \bar\phi(x,y) && \text{for } \gamma = 0,\\ &\hat\phi = \hat\phi(x, y), && \bar\phi = \bar\phi(x,y) && \text{for } \gamma \in (0,\infty),\\ &\hat\phi = \hat\phi(x, y), && \bar\phi = \bar\phi(x) && \text{for } \gamma = \infty. \end{aligned}$$

Moreover, we find that (with $\phi := \bar\phi + \gamma\hat\phi$)

$$\nabla_h f^{\varepsilon(h),h} \xrightarrow{2,\gamma} \partial_1 u(x_1) \otimes \mathbf{e}_1 + \begin{cases} \big(\, \partial_y \hat\phi(x_1, y) \,|\, \bar\nabla \bar\phi(x,y) \,\big) & \text{for } \gamma = 0,\\ \big(\, \partial_y \phi(x, y) \,|\, \tfrac{1}{\gamma} \bar\nabla \phi(x,y) \,\big) & \text{for } \gamma \in (0,\infty),\\ \big(\, \partial_y \hat\phi(x, y) \,|\, \bar\nabla \bar\phi(x) \,\big) & \text{for } \gamma = \infty, \end{cases}$$

strongly two-scale in L^2. This formal observation can be made rigorous.

Definition 6.10. We denote by $\mathcal{F}_\gamma(S\times\mathcal{Y})$ the subspace of $L^2(S\times\mathcal{Y},\mathbb{M}^3)$ defined by

$$\Big\{\,F(\bar x,y)=\big(\,\partial_y\hat\phi(y)\,|\,\bar\nabla\bar\phi(\bar x,y)\,\big)\ :\ \hat\phi\in H^1(\mathcal{Y},\mathbb{R}^3),\ \bar\phi\in L^2(\mathcal{Y},H^1(S,\mathbb{R}^3))\,\Big\}\quad\text{for }\gamma=0,$$

$$\Big\{\,F(\bar x,y)=\big(\,\partial_y\phi(\bar x,y)\,|\,\tfrac{1}{\gamma}\bar\nabla\phi(\bar x,y)\,\big)\ :\ \phi\in H^1(S\times\mathcal{Y},\mathbb{R}^3)\,\Big\}\quad\text{for }\gamma\in(0,1),$$

$$\Big\{\,F(\bar x,y)=\big(\,\partial_y\hat\phi(\bar x,y)\,|\,\bar\nabla\bar\phi(\bar x)\,\big)\ :\ \hat\phi\in L^2(S,H^1(\mathcal{Y},\mathbb{R}^3)),\ \bar\phi\in H^1(S,\mathbb{R}^3)\,\Big\}\quad\text{for }\gamma=\infty.$$

Theorem 6.11 (two-scale compactness for scaled gradients). *Let $\gamma\in[0,\infty]$ and $\varepsilon\overset{\gamma}{\sim}h$. Let u^h be a sequence in $H^1(\Omega,\mathbb{R}^3)$ such that*

$$\limsup_{h\to 0}\int_\Omega |u^h|^2+|\nabla_h u^h|^2\,\mathrm{d}x<\infty. \tag{86}$$

Then:

(a) (compactness and identification). There exist $u\in H^1(\omega,\mathbb{R}^3)$ and $F\in L^2(\omega,\mathcal{F}_\gamma(S\times\mathcal{Y}))$ such that, up to a subsequence, $u^h\to u$ strongly in L^2 and

$$\nabla_h u^h \overset{2,\gamma}{\rightharpoonup} \partial_1 u\otimes \mathbf{e}_1+F\qquad\textit{weakly two-scale in } L^2.$$

(b) (approximation). For all $F\in L^2(\omega,\mathcal{F}_\gamma(S\times\mathcal{Y}))$ there exists a sequence $\phi^h\in C^\infty_c(\omega,C^\infty(\bar S,\mathbb{R}^3))$ such that

$$\phi^h\rightharpoonup 0\qquad\textit{weakly in } H^1,\qquad \nabla_h\phi^h\overset{2,\gamma}{\longrightarrow}F\qquad\textit{strongly two-scale in } L^2$$

and

$$\limsup_{h\to 0}\sup_{x\in\Omega}\sqrt{h}\,\Big(|\phi^h(x)|+|\nabla_h\phi^h(x)|\Big)=0.$$

For $\gamma\in(0,\infty)$ the space $\mathcal{F}_\gamma(S\times\mathcal{Y},\mathbb{R}^3)$ consists of scaled gradients of functions in $H^1(S\times\mathcal{Y},\mathbb{R}^3)$. In the latter space an inequality of Korn-type holds:

Proposition 6.12. *Let $\gamma\in(0,\infty)$ and $\phi\in H^1(S\times\mathcal{Y},\mathbb{R}^3)$. Set*

$$\mathcal{R}:=\left\{\,\zeta(\bar x,y):=\tau\begin{pmatrix}0\\ x_3\\ -x_2\end{pmatrix}+b\ :\ \tau\in\mathbb{R},\ b\in\mathbb{R}^3\,\right\}.$$

Then

$$\inf_{\zeta\in\mathcal{R}}\|\phi+\zeta\|_{H^1(S\times Y)}\lesssim\Big\|\mathrm{sym}\,\big(\,\partial_y\phi\,|\,\tfrac{1}{\gamma}\bar\nabla\phi\,\big)\Big\|_{L^2(S\times Y)}$$

where the constant only depends on S and γ.

Remark 4. Theorem 6.11 and Proposition 6.12 remain valid for general cylindrical domains $\Omega=\omega\times S$ with $\omega\subset\mathbb{R}^m$ and $S\subset\mathbb{R}^n$, see [Neu10].

Proof of Theorem 6.11 (a). The proof is divided into three steps. In Step 1 we reduce the problem to the case $\omega=\mathbb{R}$. In Step 2 we treat the contribution of u^h that is constant in cross-sectional directions. In Step 3 we treat the reminder.

Step 1. Reduction to the case $\omega=\mathbb{R}$.

If (a) holds for $\omega=\mathbb{R}$, then the statement also holds for every proper interval $\omega\subsetneq\mathbb{R}$. Indeed, this follows by an extension argument: without loss of generality let $\omega:=(0,1)$. We extend $u^h\in H^1(\Omega,\mathbb{R}^3)$ by reflection to $(-1,2)\times S$:

$$\tilde u^h(x):=\begin{cases}u^h(x) & \text{for } x\in\Omega\\ u^h(-x_1,\bar x) & \text{for } x\in(-1,0)\times S\\ u^h(2-x_1,\bar x) & \text{for } x\in(1,2)\times S.\end{cases}$$

Let ζ denote a $[0,1]$-valued cut-off function with $\zeta = 1$ in ω and support compactly contained in $[-1,2]$. Define $\hat{u}^h(x) := \zeta(x_1)\tilde{u}^h(x)$ for $x \in (-1,2)\times S$ and 0 otherwise. Then $\hat{u}^h \in H^1(\mathbb{R}\times S,\mathbb{R}^3)$ and

$$\limsup_{h\to 0}\int_{\mathbb{R}\times S}|\hat{u}^h|^2+|\nabla_h\hat{u}^h|^2\,\mathrm{d}x \lesssim \limsup_{h\to 0}\int_\Omega |u^h|^2+|\nabla_h u^h|^2\,\mathrm{d}x<\infty.$$

By assumption we can apply (a) to the extension $\hat{u}^h$. Hence, there exists $\hat{u}$ and $\hat{F}$ such that $\nabla_h\hat{u} \xrightharpoonup{2,\gamma} \partial_1\hat{u}\otimes\mathbf{e}_1+\hat{F}$ up to a subsequence. However, this directly implies weak two-scale convergence of the sequence $\nabla_h u$ to the limit $\partial_1 u\otimes \mathbf{e}_1+F$ with $u=\hat{u}|_\omega$ and $F=\hat{F}|_\omega$.

Step 2. Treatment of the cross-sectional average.

Let $\omega := \mathbb{R}$. Consider the splitting

$$u^h = \left\langle u^h\right\rangle_S + h\mathring{u}^h \quad\text{where}\quad \mathring{u}^h := \frac{u^h-\left\langle u^h\right\rangle_S}{h}. \tag{87}$$

We claim that statement (a) holds, provided that

$$h\nabla_h\mathring{u}^h \xrightharpoonup{2,\gamma} F \qquad \text{weakly two-scale in } L^2 \\ \text{for a subsequence and some } F\in L^2(\omega,\mathcal{F}_\gamma(S\times\mathcal{Y})). \tag{88}$$

Indeed, by assumption (86) the sequence $\left\langle u^h\right\rangle_S$ is bounded in $H^1(\omega,\mathbb{R}^3)$. Hence, by Lemma 6.7 there exists $u\in H^1(\omega,\mathbb{R}^3)$ and $\varphi\in L^2(\omega,H^1(\mathcal{Y},\mathbb{R}^3))$ such that (for a further subsequence) $\left\langle u^h\right\rangle_S \rightharpoonup u$ weakly in H^1 and $\partial_1\left\langle u^h\right\rangle_S \xrightharpoonup{2,\gamma} \partial_1 u+\partial_y\varphi$ weakly two-scale in L^2. Combined with (87) and (88) we find that $\nabla_h u^h \xrightharpoonup{2,\gamma} \partial_1 u\otimes\mathbf{e}_1+\partial_y\varphi\otimes\mathbf{e}_1+F$. Since $\partial_y\varphi\otimes\mathbf{e}_1\in L^2(\omega,\mathcal{F}_\gamma(S\times\mathcal{Y}))$, statement (a) follows.

Step 3. Conclusion.

It remains to prove (88) in the case $\omega=\mathbb{R}$. For the argument we introduce the mappings

$$U^h := \mathcal{T}_{\varepsilon(h)}\mathring{u}^h,\qquad V^h := \frac{h}{\varepsilon(h)}\left(U^h-\int_Y U^h\,\mathrm{d}y\right).$$

Since $\omega=\mathbb{R}$, we have $U^h,\,V^h\in L^2(\mathbb{R},H^1(S\times Y,\mathbb{R}^3))$. An elementary calculation shows that almost everywhere in $\Omega\times Y$ we have

$$\mathcal{T}_{\varepsilon(h)}(h\partial_1\mathring{u}^h)=\frac{h}{\varepsilon(h)}\partial_y\big(\mathcal{T}_{\varepsilon(h)}\mathring{u}^h\big)=\partial_y V^h \quad\text{and}\quad \mathcal{T}_{\varepsilon(h)}(\bar\nabla\mathring{u}^h)=\bar\nabla U^h. \tag{89}$$

Hence, it suffices to identify the limit of the sequences U^h and V^h. We proceed in three steps.

Step 3a. We claim that

$$\limsup_{h\to 0}\iint_{\Omega\times Y}|U^h|^2+|\bar\nabla U^h|^2\,\mathrm{d}y\,\mathrm{d}x<\infty, \tag{90a}$$

$$\limsup_{h\to 0}\iint_{\Omega\times Y}|V^h|^2+|\partial_y V^h|^2\,\mathrm{d}y\,\mathrm{d}x<\infty. \tag{90b}$$

This can be seen as follows: Since $\int_S \mathring{u}^h(x_1,\bar{x})\,\mathrm{d}\bar{x}=0$, Poincaré's inequality yields $||\mathring{u}^h||_{L^2(\Omega)}\lesssim ||\bar\nabla\mathring{u}^h||_{L^2(\Omega)}$. Thus, by appealing to the integral identity (85) and (89) we get

$$\iint_{\Omega\times Y}|U^h|^2+|\bar\nabla U^h|^2\,\mathrm{d}y\,\mathrm{d}x=\int_\Omega|\mathring{u}^h|^2+|\bar\nabla\mathring{u}^h|^2\,\mathrm{d}x\lesssim\int_\Omega|\bar\nabla\mathring{u}^h|^2\,\mathrm{d}x\le\int_\Omega|\nabla_h u^h|^2\,\mathrm{d}x.$$

Similarly, we have

$$\iint_{\Omega\times Y}|V^h|^2+|\partial_y V^h|^2\,\mathrm{d}y\,\mathrm{d}x \overset{\text{Poincaré}}{\lesssim} \iint_{\Omega\times Y}|\partial_y V^h|^2\,\mathrm{d}y\,\mathrm{d}x \overset{(89)}{=} \int_\Omega |h\partial_1\mathring{u}^h|^2\,\mathrm{d}x \overset{(87)}{\leq} 2\int_\Omega|\partial_1 u^h|^2\,\mathrm{d}x.$$

Thus, (90a) and (90b) follow from the previous two estimates and assumption (86).

Step 3b. We argue that

$$\text{(91)}\qquad \begin{aligned} U^h &\rightharpoonup U && \text{weakly in } L^2(\omega\times Y, H^1(S,\mathbb{R}^3)),\\ V^h &\rightharpoonup V && \text{weakly in } L^2(\Omega, H^1(Y,\mathbb{R}^3)). \end{aligned}$$

for a subsequence and

$$\text{(92)}\qquad \begin{aligned} U &\in \begin{cases} L^2(\omega\times Y, H^1(S,\mathbb{R}^3)) & \text{for } \gamma=0,\\ L^2(\omega, H^1(S{\times}\mathcal{Y},\mathbb{R}^3)) & \text{for } \gamma\in(0,\infty),\\ L^2(\omega, H^1(S,\mathbb{R}^3)) & \text{for } \gamma=\infty, \end{cases}\\ V &\in \begin{cases} L^2(\omega, H^1(\mathcal{Y},\mathbb{R}^3)) & \text{for } \gamma=0,\\ L^2(\omega, H^1(S{\times}\mathcal{Y},\mathbb{R}^3)) & \text{for } \gamma\in(0,\infty),\\ L^2(\omega, H^1(S{\times}\mathcal{Y},\mathbb{R}^3)) & \text{for } \gamma=\infty. \end{cases} \end{aligned}$$

Indeed, as a direct consequence of (90a), (90b) and the property that bounded sequences in $L^2(\omega\times Y, H^1(S,\mathbb{R}^3))$ (resp. $L^2(\omega, H^1(Y,\mathbb{R}^3))$) are precompact with respect to weak convergence, (91) holds for $U\in L^2(\omega{\times}Y, H^1(S,\mathbb{R}^3))$ and $V\in L^2(\Omega, H^1(S{\times}Y))$.

Proof of (92) for $\gamma=0$: We have $h/\varepsilon(h)\to 0$ for $\gamma=0$. Therefore the definition of V^h and the boundedness of U^h as a sequence in L^2 immediately imply that $V=0$. This completes the argument for $\gamma=0$.

Let us consider the case $\gamma>0$. We argue that $V\in L^2(\omega, H^1(S{\times}\mathcal{Y},\mathbb{R}^3))$, that is $V(x,y)$ is periodic in y. It suffices to show that

$$\text{(93)}\qquad \int_\Omega (V(x,1)-V(x,0))\cdot\psi(x)\,\mathrm{d}x=0 \qquad \text{for all } \psi\in C_c^\infty(\Omega,\mathbb{R}^3).$$

Indeed, we have

$$\begin{aligned}\int_\Omega (V(x,1)-V(x,0))\cdot\psi(x)\,\mathrm{d}x &= \int_\Omega\int_0^1 \partial_y V(x,y)\cdot\psi(x)\,\mathrm{d}y\,\mathrm{d}x\\ &= \lim_{h\to0}\iint_{\Omega\times Y}\partial_y V^h\cdot(\mathcal{T}_{\varepsilon(h)}\psi)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

The last line follows from (91) and $\mathcal{T}_{\varepsilon(h)}\psi\to\psi$ strongly in $L^2(\Omega\times Y,\mathbb{R}^3)$. On the other hand, by (89), Lemma 6.2 and integration by parts we get

$$\iint_{\Omega\times Y}\partial_y V^h\cdot(\mathcal{T}_{\varepsilon(h)}\psi)\,\mathrm{d}y\,\mathrm{d}x = h\int_\Omega \partial_1\mathring{u}^h\cdot\psi\,\mathrm{d}x = -h\int_\Omega \mathring{u}^h\cdot\partial_1\psi\,\mathrm{d}x.$$

Because U^h is the periodic unfolding of $\mathring{u}^h$, the sequence $\mathring{u}^h$ is bounded in $L^2(\Omega,\mathbb{R}^3)$ by (90a) and we deduce that the right-hand side of the previous identity vanishes for $h\to0$. This proves (93).

Proof of (92) for $\gamma\in(0,\infty)$: by appealing to the definition of V^h we deduce that $U=\frac{1}{\gamma}V+\int_Y U\,\mathrm{d}y$. As a consequence we get $U,V\in L^2(\omega,H^1(S{\times}\mathcal{Y},\mathbb{R}^3))$, which completes the proof for $\gamma\in(0,\infty)$.

Proof of (92) for $\gamma=\infty$: it remains to show that $U(x,y)$ is independent of y. Here comes the argument: Set $\bar U^h:=\int_Y U^h\,\mathrm{d}y$. By Poincaré's inequality we have

$$\begin{aligned}&\iint_{\Omega\times Y}|U^h(x,y)-\bar U^h(x)|^2\,\mathrm{d}y\,\mathrm{d}x\lesssim\iint_{\Omega\times Y}|\partial_y U^h(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x\\&=\frac{\varepsilon(h)^2}{h^2}\iint_{\Omega\times Y}|\tfrac{h}{\varepsilon(h)}\partial_y U^h|^2\,\mathrm{d}y\,\mathrm{d}x=\frac{\varepsilon(h)^2}{h^2}\iint_{\Omega\times Y}|\partial_y V^h|^2\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

The right-hand side vanishes as $h\to 0$ due to (90b) and $\frac{\varepsilon(h)}{h}\to 0$. Thus $U(x,\cdot)=\int_Y U(x,y)\,\mathrm{d}y$ and we deduce that $U(x,y)$ is independent of y.

Step 3c. We prove (88). By (89), (91) and Proposition 6.3 we find that, up to a subsequence,

$$h\nabla_h\mathring{u}^h\xrightarrow{2,\gamma}(\,\partial_y V\,|\,\bar\nabla U\,)$$

where U,V are chosen according to (92). It suffices to argue that the matrix field on the right-hand side belongs to $L^2(\omega,\mathcal{F}_\gamma(S\times\mathcal{Y}))$. Indeed, for $\gamma\in\{0,\infty\}$ this directly follows from (92), while for $\gamma\in(0,\infty)$ this is implied by (92) and the identity $U=\frac{1}{\gamma}V+\int_Y U\,\mathrm{d}y$. □

Proof of Theorem 6.11 (b). The proof is divided into two steps. In Step 1 we present a construction for smooth functions. In Step 2 we prove the statement by appealing to a diagonal-sequence construction.

Step 1. Smooth approximation.

Let

$$\begin{aligned}&\widehat\phi\in\begin{cases}C_c^\infty(\omega,C^\infty(\mathcal{Y},\mathbb{R}^3))&\text{for }\gamma=0,\\C_c^\infty(\omega,C^\infty(\bar S\times\mathcal{Y},\mathbb{R}^3))&\text{for }\gamma=\infty,\end{cases}\\&\bar\phi\in\begin{cases}C_c^\infty(\omega,C^\infty(\bar S\times\mathcal{Y},\mathbb{R}^3))&\text{for }\gamma=0,\\C_c^\infty(\omega,C^\infty(\bar S,\mathbb{R}^3))&\text{for }\gamma=\infty,\end{cases}\\&\phi\in\ C_c^\infty(\omega,C^\infty(\bar S\times\mathcal{Y},\mathbb{R}^3))\qquad\text{for }\gamma\in(0,\infty).\end{aligned}\tag{94}$$

Set

$$\phi^h(x):=\begin{cases}\varepsilon(h)\widehat\phi(x_1,x_1/\varepsilon(h))+h\bar\phi(x,x_1/\varepsilon(h))&\text{for }\gamma=0,\\\varepsilon(h)\phi(x,x_1/\varepsilon(h))&\text{for }\gamma\in(0,1),\\\varepsilon(h)\widehat\phi(x,x_1/\varepsilon(h))+h\bar\phi(x)&\text{for }\gamma=\infty.\end{cases}\tag{95}$$

It is straightforward to check that

$$\phi^h\rightharpoonup 0\qquad\text{weakly in }H^1(\Omega,\mathbb{R}^3),$$

$$\nabla_h\phi^h\xrightarrow{2,\gamma}\left\{\begin{matrix}(\,\partial_y\widehat\phi(x_1,y)\,|\,\bar\nabla\bar\phi(x,y)\,)&\text{for }\gamma=0\\(\,\partial_y\phi(x,y)\,|\,\frac{1}{\gamma}\bar\nabla\phi(x,y)\,)&\text{for }\gamma\in(0,\infty)\\(\,\partial_y\widehat\phi(x,y)\,|\,\bar\nabla\bar\phi(x)\,)&\text{for }\gamma=\infty\end{matrix}\right\}\quad\text{strongly two-scale in }L^2,$$

and

$$\limsup_{h\to0}\sup_{x\in\Omega}\sqrt{h}\Big(|\phi^h(x)|+|\nabla_h\phi^h(x)|\Big)=0.$$

Step 2. Conclusion.

Let $F\in L^2(\omega,\mathcal{F}_\gamma(S\times\mathcal{Y}))$ and choose $\widehat\phi$, $\bar\phi$ and ϕ according to Definition 6.10. If $\widehat\phi$, $\bar\phi$ and ϕ were smooth, we could directly conclude by appealing to Step 1. In the general case we proceed by approximation and extraction of a diagonal sequence. To this end let $\delta > 0$. By density we can find functions $\hat{\phi}^\delta$, $\bar{\phi}^\delta$ (resp. ϕ^δ) according to (94), such that

$$\delta \geq \begin{cases} \|\widehat{\phi}^\delta - \widehat{\phi}\|_{L^2(\omega,H^1(Y))} + \|\bar{\phi}^\delta - \bar{\phi}\|_{L^2(\omega\times Y,H^1(S))} & \text{for } \gamma = 0, \\ \|\phi^\delta - \phi\|_{L^2(\omega,H^1(S\times Y))} & \text{for } \gamma \in (0,\infty), \\ \|\widehat{\phi}^\delta - \widehat{\phi}\|_{L^2(\omega,H^1(Y))} + \|\bar{\phi}^\delta - \bar{\phi}\|_{L^2(\omega,H^1(S))} & \text{for } \gamma = \infty. \end{cases}$$

Set

$$F^\delta := \begin{cases} \left(\, \partial_y \hat{\phi}^\delta \,|\, \bar{\nabla}\bar{\phi}^\delta \,\right) & \text{for } \gamma \in \{0,\infty\}, \\ \left(\, \partial_y \phi^\delta \,|\, \frac{1}{\gamma}\bar{\nabla}\phi^\delta \,\right) & \text{for } \gamma \in (0,\infty), \end{cases}$$

and let $\phi^{\delta,h}$ be defined as in (95) with $\widehat{\phi}, \bar{\phi}, \phi$ replaced by $\widehat{\phi}^\delta, \bar{\phi}^\delta, \phi^\delta$. Set

$$\rho_{\delta,h} := \|\phi^{\delta,h}\|_{L^2(\Omega)} + \|\mathcal{T}_{\varepsilon(h)}\nabla_h\phi^{\delta,h} - F\|_{L^2(\mathbb{R}\times S\times Y)} + \sqrt{h}\left(\|\phi^h\|_{L^\infty(\Omega)} + \|\nabla_h\phi^{\delta,h}\|_{L^\infty(\Omega)}\right).$$

Then Step 1 implies that $\limsup_{h\to 0}\rho_{\delta,h} = \|F^\delta - F\|_{L^2(\mathbb{R}\times S\times Y)}$. Since $\|F^\delta - F\|_{L^2(\mathbb{R}\times S\times Y)} \leq \delta$ by construction, we deduce that $\limsup_{\delta\to 0}\limsup_{h\to 0}\rho_{\delta,h} = 0$. Thus, by appealing to Lemma 4.7 we can extract a diagonal sequence $\delta(h)$ such that $\rho_{\delta(h),h} \to 0$ as $h \to 0$. The latter directly implies that the sequence $\phi^h := \phi^{\delta(h),h}$ fulfills the claimed properties. □

Proof of Proposition 6.12. Let us first remark that it suffices to prove the statement for $\gamma = 1$. Indeed, for $\gamma \neq 1$ consider the scaled function $\phi_\gamma \in H^1(\gamma S\times\mathcal{Y},\mathbb{R}^3)$ defined by $\phi_\gamma(\bar{x},y) := \phi(\frac{1}{\gamma}\bar{x},y)$. Since $\partial_y\phi_\gamma(\gamma\bar{x},y) = \partial_y\phi(\bar{x},y)$ and $\bar{\nabla}\phi_\gamma(\gamma\bar{x},y) = \frac{1}{\gamma}\bar{\nabla}\phi(\bar{x},y)$, the desired estimate for ϕ follows from the statement applied with $\gamma = 1$ and a change of coordinates.

From now on let $\gamma = 1$. For convenience we use the notation $\widetilde{\nabla} := \left(\,\partial_y \,|\, \bar{\nabla}\,\right)$ and introduce the mapping

$$\mathbf{d}(\bar{x},y) := \begin{pmatrix} y \\ x_2 \\ x_3 \end{pmatrix} - \mathbf{c} \quad \text{where } \mathbf{c} \in \mathbb{R}^3 \text{ is chosen such that } \iint_{S\times Y} \mathbf{d}\,\mathrm{d}y\,\mathrm{d}\bar{x} = 0.$$

Our proof starts with the classical Korn inequality: up to a constant that only depends on the geometry of S we have

$$\min_{\zeta\in\mathcal{R}_0}\|\phi - \zeta\|_{H^1(S\times Y)} \lesssim \|\operatorname{sym}\widetilde{\nabla}\phi\|_{L^2(S\times Y)} \tag{96}$$

where $\mathcal{R}_0 := \left\{\,\zeta_0(\bar{x},y) = K\mathbf{d}(\bar{x},y) + b \text{ with } K \in \mathrm{Skew}(3),\ b \in \mathbb{R}^3\,\right\}$. Obviously, $\mathcal{R}_0$ endowed with the inner product of $H^1(S\times Y,\mathbb{R}^3)$ is a Hilbert space. Since $\mathcal{R}$ is a closed subspace of $\mathcal{R}_0$ we can represent $\mathcal{R}_0$ by the orthogonal decomposition

$$\mathcal{R}_0 = \mathcal{R} \oplus \mathcal{R}_1$$

where $\mathcal{R}_1$ denotes the orthogonal complement of $\mathcal{R}$ in $\mathcal{R}_0$. Let $P_\mathcal{R}, P_{\mathcal{R}_0}$ and $P_{\mathcal{R}_1}$ denote the orthogonal projections onto $\mathcal{R}, \mathcal{R}_0$ and $\mathcal{R}_1$, respectively. Our argument relies on the orthogonal decomposition

$$\phi = \psi + P_{\mathcal{R}_0}\phi = \psi + P_\mathcal{R}\phi + P_{\mathcal{R}_1}\phi. \tag{97}$$

By orthogonality of the terms in (97) and Pythagoras' Theorem we have

$$\begin{aligned} \min_{\zeta\in\mathcal{R}}\|\phi - \zeta\|^2_{H^1(S\times Y)} &\leq \|\phi - P_\mathcal{R}\phi\|^2_{H^1(S\times Y)} = \|\psi + P_{\mathcal{R}_1}\phi\|^2_{H^1(S\times Y)} \\ &= \|\psi\|^2_{H^1(S\times Y)} + \|P_{\mathcal{R}_1}\phi\|^2_{H^1(S\times Y)}. \end{aligned} \tag{98}$$

Hence, we only need to show that the right-hand side is controlled by $||\operatorname{sym}\widetilde{\nabla}\phi||^2_{L^2(S\times Y)}$. The estimate of ψ is easy: by orthogonality the left-hand side in (96) is minimized precisely by the orthogonal projection of ϕ onto $\mathcal{R}_0$. Hence,

$$(99)\quad ||\psi||^2_{H^1(S\times Y)} \overset{(97)}{=} ||\phi - P_{\mathcal{R}_0}\phi||^2_{H^1(S\times Y)} \overset{\text{orth.}}{=} \min_{\zeta\in\mathcal{R}_0} ||\phi+\zeta||^2_{H^1(S\times Y)} \overset{(96)}{\lesssim} ||\operatorname{sym}\widetilde{\nabla}\phi||^2_{L^2(S\times Y)}.$$

It remains to estimate $\zeta_1 := P_{\mathcal{R}_1}\phi$. In view of (99), it suffices to show that

$$(100)\qquad ||\zeta_1||^2_{H^1(S\times Y)} \lesssim ||\psi||^2_{H^1(S\times Y)}.$$

We prove this estimate in two steps: First, we show that

$$(101)\qquad ||\zeta_1||^2_{H^1(S\times Y)} \lesssim \int_S |\zeta_1(\bar x,1)-\zeta_1(\bar x,0)|^2\,\mathrm{d}\bar x$$

and secondly we argue that

$$(102)\qquad \int_S |\zeta_1(\bar x,1)-\zeta_1(\bar x,0)|^2\,\mathrm{d}\bar x = \int_S |\psi(\bar x,1)-\psi(\bar x,0)|^2\,\mathrm{d}\bar x.$$

Clearly, by combining both steps we find that $||\zeta_1||^2_{H^1(S\times Y)} \lesssim \int_S |\psi(\bar x,1)-\psi(\bar x,0)|^2\,\mathrm{d}\bar x \lesssim ||\psi||^2_{H^1(S\times Y)}$ and (100) follows.

We prove (101). It is easy to check that ζ_1 takes the form

$$\zeta_1(\bar x,y) = \big(\operatorname{skw}(\kappa_1\mathbf{e}_1\otimes\mathbf{e}_2+\kappa_2\mathbf{e}_1\otimes\mathbf{e}_3)\big)\,\mathbf{d}(\bar x,y)$$

for some constants $\kappa_1,\kappa_2\in\mathbb{R}$. In particular the integral mean of ζ_1 vanishes. Hence,

$$(103)\qquad ||\zeta_1||^2_{H^1(S\times Y)} \lesssim \iint_{S\times Y} |\tilde\nabla\zeta_1|^2\,\mathrm{d}y\,\mathrm{d}\bar x \lesssim |S\times Y|(\kappa_1^2+\kappa_2^2).$$

On the other hand, a direct calculation shows that $\zeta_1(\bar x,1)-\zeta_1(\bar x,0) = -\frac{1}{2}(\kappa_1\mathbf{e}_2+\kappa_2\mathbf{e}_3)$. Thus, $\int_S |\zeta_1(\bar x,1)-\zeta_1(\bar x,0)|^2\,\mathrm{d}\bar x = \frac{|S|}{4}(\kappa_1^2+\kappa_2^2)$. Together with (103) the desired estimate (101) follows.

It remains to prove (102). Recalling the definition of $\mathcal{R}$ we find that $\zeta(\bar x,y)$ does not depend on y. Since ϕ, as a function in $H^1(S\times\mathcal{Y},\mathbb{R}^3)$, is periodic in y, we deduce that the difference $f=\phi-\zeta$ is periodic in y as well. By (97) we have $\psi+\zeta_1=\phi-\zeta=f$. Thus,

$$\Big(\psi(\bar x,1)+\zeta_1(\bar x,1)\Big) - \Big(\psi(\bar x,0)+\zeta_1(\bar x,0)\Big) = f(\bar x,1)-f(\bar x,0)=0$$

almost everywhere in S. Obviously, this implies $\psi(\cdot,1)-\psi(\cdot,0) = -\big(\zeta_1(\cdot,1)-\zeta_1(\cdot,0)\big)$ almost everywhere in $\bar x$ and (102) follows.

□

APPENDIX A. PROOF OF LEMMAS 4.2 AND 4.3

Proof of Lemma 4.2. The proof will be divided into six steps. The construction in Step 2 - Step 4 is adapted from [FJM02, MM03].

Step 1. Extension of u^h.

Set $\omega_* := (-2h, 1+2h)$ and $\Omega_* := \omega_*\times S$. We assert that there exists an extension $u^h_*\in H^1(\Omega_*,\mathbb{R}^3)$ of u^h such that

$$(104)\qquad \int_{\Omega_*} \operatorname{dist}^2(\nabla_h u^h_*,\mathrm{SO}(3))\,\mathrm{d}x \lesssim \int_\Omega \operatorname{dist}^2(\nabla_h u^h,\mathrm{SO}(3))\,\mathrm{d}x.$$

We postpone the argument to the end of the proof.

Step 2. Construction of the piecewise affine rotation field.

For $\xi \in h\mathbb{Z}$ define $Q(\xi) := [\xi, \xi+h)\times S$ and $\widehat{Q}(\xi) := [\xi-h, \xi+h)\times S$. Let Z denote the smallest subset of $h\mathbb{Z}$ such that $\Omega \subset \bigcup_{\xi\in Z} Q(\xi)$. We note that by construction we have

$$Q(\xi) \subset \widehat{Q}(\xi) \subset \Omega_* \qquad \text{for all } \xi \in Z. \tag{105}$$

For $\xi \in Z$ let $\mathfrak{r}(\xi)$ and $\widehat{\mathfrak{r}}(\xi)$ denote rotations with

$$\mathfrak{r}(\xi) \in \operatorname*{argmin}_{R\in\mathrm{SO}(3)} \int_{Q(\xi)} |\nabla_h u^h_*(x) - R|^2 \,\mathrm{d}x, \qquad \widehat{\mathfrak{r}}(\xi) \in \operatorname*{argmin}_{R\in\mathrm{SO}(3)} \int_{\widehat{Q}(\xi)} |\nabla_h u^h_*(x) - R|^2 \,\mathrm{d}x. \tag{106}$$

We define $R^h_{pc} : \mathbb{R} \to \mathrm{SO}(3)$ as follows. For every $s \in [\xi, \xi+h)$ with $\xi \in Z$, set $R^h_{pc}(s) := \mathfrak{r}(\xi)$. Then extend R^h_{pc} to $\mathbb{R}$ by constancy. By construction R^h_{pc} is piecewise constant and its jumpset is a subset of $h\mathbb{Z}^d$.

Step 3. Estimate of the approximation error.

We assert that

$$\int_\Omega |R^h_{pc} - \nabla_h u^h|^2 \,\mathrm{d}x \lesssim \int_\Omega \mathrm{dist}^2(\nabla_h u^h, \mathrm{SO}(3)) \,\mathrm{d}x. \tag{107}$$

For the proof introduce the rescaled map $v^h : \omega_* \times (hS) \to \mathbb{R}^3$, $v^h(x_1, \bar{x}) := u^h_*(x_1, \frac{1}{h}\bar{x})$. For all $\xi \in Z$ we have

$$\begin{aligned}\int_{U(\xi)} |\mathfrak{r}(\xi) - \nabla_h u^h_*|^2 \,\mathrm{d}x &= \frac{1}{h}\int_{(\xi,\xi+h)\times(hS)} |\mathfrak{r}(\xi) - \nabla v^h|^2 \,\mathrm{d}x \\ &\overset{\text{Theorem 4.1 \& (106)}}{\lesssim} \frac{1}{h}\int_{(\xi,\xi+h)\times(hS)} \mathrm{dist}^2(\nabla v^h, \mathrm{SO}(3)) \,\mathrm{d}x \\ &= \int_{Q(\xi)} \mathrm{dist}^2(\nabla_h u^h_*, \mathrm{SO}(3)) \,\mathrm{d}x.\end{aligned} \tag{108}$$

Because $(\xi, \xi+h)\times(hS)$ can be written as a translation and uniform dilation of the set $(0,1)\times S$, the constant in the inequality above indeed only depends on S and is independent of h. Similarly,

$$\int_{\widehat{Q}(\xi)} |\widehat{\mathfrak{r}}(\xi) - \nabla_h u^h_*|^2 \,\mathrm{d}x \lesssim \int_{\widehat{Q}(\xi)} \mathrm{dist}^2(\nabla_h u^h_*, \mathrm{SO}(3)) \,\mathrm{d}x. \tag{109}$$

Now, estimate (107) follows by summing (108) over $\xi \in Z$, the definition of R^h_{pc} and (104).

Step 4. Estimate of the variation of R^h_{pc}.

We prove that

$$h \sum_{\xi\in h\mathbb{Z}} |[[R^h_{pc}(\xi)]]|^2 \lesssim \int_{\Omega_*} \mathrm{dist}^2(\nabla_h u^h_*, \mathrm{SO}(3)) \,\mathrm{d}x. \tag{110}$$

First, we note that $\sum_{\xi\in h\mathbb{Z}} |[[R^h_{pc}(\xi)]]|^2 = \sum_{\xi\in Z^-} |\mathfrak{r}(\xi) - \mathfrak{r}(\xi-h)|^2$ where $Z^- = Z \setminus \{\min Z\}$. Hence, by (104), it suffices to show that

$$h\,|\mathfrak{r}(\xi) - \mathfrak{r}(\xi-h)|^2 \lesssim \int_{\widehat{Q}(\xi)} \mathrm{dist}^2(\nabla_h u^h_*, \mathrm{SO}(3)) \,\mathrm{d}x \qquad \text{for all } \xi \in Z^-. \tag{111}$$

This can be seen as follows: by construction, the set $Q(\xi)\cup Q(\xi-h)$ is contained in $\widehat{Q}(\xi)$ for all $\xi\in Z^-$. Thus,

$$\begin{aligned}
h|\mathfrak{r}(\xi)-\mathfrak{r}(\xi{-}h)|^2 &\lesssim h|\mathfrak{r}(\xi)-\widehat{\mathfrak{r}}(\xi)|^2+h|\widehat{\mathfrak{r}}(\xi)-\mathfrak{r}(\xi-h)|^2\\
&\lesssim \int_{Q(\xi)}|\mathfrak{r}(\xi)-\widehat{\mathfrak{r}}(\xi)|^2\,\mathrm{d}x+\int_{Q(\xi-h)}|\widehat{\mathfrak{r}}(\xi)-\mathfrak{r}(\xi-h)|^2\,\mathrm{d}x\\
&\leq \int_{Q(\xi)}|\mathfrak{r}(\xi)-\nabla_h u^h_*(x)|^2\,\mathrm{d}x+\int_{\widehat{Q}(\xi)}|\widehat{\mathfrak{r}}(\xi)-\nabla_h u^h_*(x)|^2\,\mathrm{d}x\\
&\quad+\int_{\widehat{Q}(\xi)}|\widehat{\mathfrak{r}}(\xi)-\nabla_h u^h_*(x)|^2\,\mathrm{d}x+\int_{Q(\xi-h)}|\mathfrak{r}(\xi-h)-\nabla_h u^h_*(x)|^2\,\mathrm{d}x\\
&\lesssim \int_{\widehat{Q}(\xi)}\operatorname{dist}^2(\nabla_h u^h_*,\mathrm{SO}(3))\,\mathrm{d}x.
\end{aligned}$$

In the last line we appealed to estimates (108) & (109) and used again that $Q(\xi)\cup Q(\xi-h)\subset\widehat{Q}(\xi)$. This completes the argument.

Step 5. Boundary condition.

Suppose that $u^h\in\mathcal{A}^h_{(v^0,R^0)}(\Omega)$, that is

(112) $$u^h(0,\bar{x})=v^0+hR^0\mathbf{d}_S(\bar{x})\qquad\text{almost everywhere in } S.$$

We assert that in this case R^h_{pc} can be chosen such that $R^h_{pc}=R^0$ in $[0,h)$. A close look to the estimates in Step 3 and 4 shows that it suffices to argue that

$$|R^h_{pc}(0)-R^0|^2\lesssim\frac{1}{h}\int_{Q(0)}|\nabla_h u^h-R^h_{pc}(0)|^2\,\mathrm{d}x.$$

This can be seen as follows: set

$$V_h:(0,1)\times S\to\mathbb{R}^3,\qquad V_h(x):=\frac{1}{h}u^h(hx_1,\bar{x})-R^h_{pc}(0)\big(\mathbf{d}_S(\bar{x})+x_1\mathbf{e}_1\big)+\bar{V}^h$$

where $\bar{V}^h\in\mathbb{R}^3$ is chosen such that $\int_{(0,1)\times S}V_h\,\mathrm{d}x=0$. By (112) we have

$$\begin{aligned}
\int_{\{0\}\times S}|V^h|^2\,\mathrm{d}\bar{x} &= \int_{\{0\}\times S}|(R^0-R^h_{pc}(0))\mathbf{d}_S(\bar{x})+h^{-1}u^0+\bar{V}^h|^2\,\mathrm{d}\bar{x}\\
&\geq \int_{\{0\}\times S}|(R^0-R^h_{pc}(0))\mathbf{d}_S(\bar{x})|^2\,\mathrm{d}\bar{x}.
\end{aligned}$$

The last estimate is valid by orthogonality (indeed, we have $\int_S\mathbf{d}_S\,\mathrm{d}\bar{x}=0$ and $h^{-1}u^0+\bar{V}^h$ is constant). Because $R^0-R^h_{pc}(0)$ is a difference of two rotations, we have $|R^0-R^h_{pc}(0)|^2\lesssim|(R^0-R^h_{pc}(0))(\mathbf{e}_2+\mathbf{e}_3)|^2$ and consequently

$$|R^0-R^h_{pc}(0)|^2\lesssim\int_{\{0\}\times S}|V^h|^2\,\mathrm{d}\bar{x}.$$

On the other hand, for $V^h\in H^1((0,1)\times S)$ the trace operator is continuous and we get

$$\begin{aligned}
\int_{\{0\}\times S}|V^h|^2\,\mathrm{d}\bar{x} &\lesssim ||V^h||^2_{H^1}\overset{\text{Poincaré}}{\lesssim}\int_{(0,1)\times S}|\nabla V^h|^2\,\mathrm{d}x\\
&= \frac{1}{h}\int_{(0,h)\times S}|\nabla_h u^h(x)-R^h_{pc}(0)|^2\,\mathrm{d}x.
\end{aligned}$$

Step 6. Proof of Step 1.

Assume that $h < 1$. We only discuss the extension of u^h to the domain $(-2h, 1) \times S$, since the extension to $(0, 1 + 2h) \times S$ can be constructed similarly. Let v^h denote the rescaled deformation

$$v^h : \omega \times (hS) \to \mathbb{R}^3, \qquad v^h(x_1, \bar{x}) := u^h(x_1, \tfrac{1}{h}\bar{x})$$

For $x \in (0,h)\times hS$, set $w(x) := v^h(x) - R^0 x$ where $R^0 \in \mathrm{SO}(3)$ minimizes the function $\mathrm{SO}(3) \ni R \mapsto \int_{(0,h)\times(hS)} |\nabla v^h - R|^2 \,\mathrm{d}x$. We extend w to the domain $\mathbb{R}\times hS$ by reflection and periodicity:

$$w(x_1, \bar{x}) := \begin{cases} w(x_1 + 2\xi, \bar{x}) & \text{for } x_1 \in (-2\xi, -2\xi + h) \text{ for some } \xi \in h\mathbb{Z}, \\ w(2\xi - x_1, \bar{x}) & \text{for } -x_1 \in (-2\xi, -2\xi + h) \text{ for some } \xi \in h\mathbb{Z}. \end{cases}$$

Note that the map w is $2h$-periodic in x_1 and satisfies

$$\begin{aligned} \int_{(-2h,0)\times hS} |\nabla w|^2 \,\mathrm{d}x &= 2\int_{(0,h)\times hS} |\nabla w|^2 \,\mathrm{d}x = 2\int_{(0,h)\times hS} |\nabla v - R^0|^2 \,\mathrm{d}x \\ &\lesssim \int\limits_{(0,h)\times(hS)} \mathrm{dist}^2(\nabla v^h(x), \mathrm{SO}(3))^2 \,\mathrm{d}x. \end{aligned}$$

The last estimate is a consequence of the geometric rigidity estimate and the constant only depends on the geometry of S, because the domain $(0,h)\times(hS)$ is obviously a uniform dilation of $(0,1)\times S$. Now, for $x \in (-2h, 1) \times S$ define

$$u^h_*(x) := \begin{cases} u^h(x) & \text{for } x \in \Omega \\ w(x_1, h\bar{x}) + R^0(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + hx_3\mathbf{e}_3) & \text{for } x \in (-2h, 0] \times S. \end{cases}$$

By construction we have

$$\int_{(-2h,1)\times S} \mathrm{dist}^2(\nabla_h u^h_*, \mathrm{SO}(3)) \,\mathrm{d}x \lesssim \int_\Omega \mathrm{dist}^2(\nabla_h u^h, \mathrm{SO}(3)) \,\mathrm{d}x.$$

□

Proof of Lemma 4.3. It suffices to prove the following statement. For all $\widetilde{R} \in \mathrm{SO}(3) \setminus \{I\}$ there exists a smooth function $R : [0,1] \to \mathrm{SO}(3)$ with $R(0) = I$, $R(1) = \widetilde{R}$, $R(s)^{\mathrm{t}} R_{,1}(s) = \widetilde{K}$ and

$$\sup_{s\in[0,1]} |R_{,1}(s)|^2 \lesssim |\widetilde{R} - I|^2. \tag{113}$$

Here comes the argument: For $\mathbf{e} \in \mathbb{R}^3$, $|\mathbf{e}| = 1$, and $\alpha \in \mathbb{R}$, let $\mathrm{Rot}(\mathbf{e}, \alpha)$ denote the rotation in $\mathrm{SO}(3)$ about axis $\mathbf{e} \in \mathbb{R}^3$ and angle α, that is

$$\mathrm{Rot}(\mathbf{e}, \alpha) := I + \sin\alpha N_{\mathbf{e}} + (1 - \cos\alpha) N_{\mathbf{e}}^2$$

where $N_{\mathbf{e}} \in \mathrm{Skew}(3)$ is defined by $N_{\mathbf{e}} a := \mathbf{e} \times a$ for all $a \in \mathbb{R}^3$. (The formula above is called Rodrigues' rotation formula). Let $\alpha \in [0, \pi]$ and $\mathbf{e} \in \mathbb{R}^3$, $|\mathbf{e}| = 1$, denote the unique angle and axis such that $\widetilde{R} = \mathrm{Rot}(\mathbf{e}, \alpha)$ and set

$$R(s) := \mathrm{Rot}(\mathbf{e}, s\alpha) \qquad \text{for } s \in [0,1].$$

Obviously, $R : [0,1] \to \mathrm{SO}(3)$ is smooth and $R(0) = I$, $R(1) = \widetilde{R}$. A direct computation shows that

$$R_{,1}(s) = \partial_s \mathrm{Rot}(\mathbf{e}, s\alpha) = \alpha(\cos(s\alpha) N_{\mathbf{e}} + \sin(s\alpha) N_{\mathbf{e}}^2).$$

Using that $N_{\mathbf{e}}^{\mathrm{t}} = N_{\mathbf{e}}^3 = -N_{\mathbf{e}}$, we easily get $(R^{\mathrm{t}} R_{,1})(s) = \alpha N_{\mathbf{e}} =: K$, which is a constant skew-symmetric matrix. It remains to prove (113). Since $R(s) \in \mathrm{SO}(3)$ we have

$$|R_{,1}(s)|^2 = |R^{\mathrm{t}}(s) R_{,1}(s)|^2 = |K|^2 = |\alpha N_{\mathbf{e}}|^2 = \alpha^2.$$

Since $\alpha^2 \sim 1-\cos\alpha$ for $\alpha \in [0,\pi]$, it suffices to prove that $|R-I|^2 \sim 2(1-\cos\alpha)$. Here comes the argument:

$$\begin{aligned}|R-I|^2 = \sup_{|b|=1} |(R-I)b|^2 &= \sup_{|b|=1} |\sin\alpha N_{\mathbf{e}} b + (1-\cos\alpha) N_{\mathbf{e}}^2 b|^2 \\ &\le \sin^2\alpha + (1-\cos\alpha)^2 \le 2(1-\cos\alpha)\end{aligned}$$

where the second line follows by orthogonality of $N_{\mathbf{e}} b$ and $N_{\mathbf{e}}^2 b$ for $b \in \mathbb{R}^3$. For the opposite estimate let $b \in \mathbb{R}^3$, $|b| = 1$, be orthogonal to $\mathbf{e}$. Then $|N_{\mathbf{e}} b| = |N_{\mathbf{e}}^2 b| = 1$ and

$$\begin{aligned}|R-I|^2 \ge |(R-I)b|^2 &= |\sin\alpha N_{\mathbf{e}} b + (1-\cos\alpha) N_{\mathbf{e}}^2 b|^2 \\ &= \sin^2\alpha + (1-\cos\alpha)^2 = 2(1-\cos\alpha).\end{aligned}$$

□

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