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Abstract In [Alexander, Bishop, Ghrist: Total curvature and simple pursuit on domains of curvature bounded above. Geom. Dedicata, 2010], the authors study a pursuit-evasion game in CAT(0) spaces and arrive at the following topological characterization of the underlying CAT(0) space: the space is compact if and only if the pursuer always wins. In the present note, we, however, give an example of an unbounded CAT(0) space where the pursuer always wins, and hence strongly disprove the above statement on compactness.

Keywords CAT(0) space · pursuit-evasion game · compactness

Introduction

Let us first recall that a geodesic ray in a CAT(0) space (X, d) is an isometry ρ from $[0, \infty)$ to X, and a geodesic from a point $x \in X$ to a point $y \in X$ is an isometry γ from the compact interval $[0, d(x, y)]$ into X such that $\gamma(0) = x$, and $\gamma(d(x, y)) = y$. If no confusion can occur, we do not distinguish between the geodesic γ and its geodesic segment

$$\gamma([0, d(x, y)]) \subset X,$$

and we denote both by $[x, y]$. Similarly for geodesic rays.

For more details on CAT(0) geometry, we refer the unfamiliar reader to [2]. For more information on pursuit-evasion games, see [1] and the references therein.

We consider the following pursuit-evasion game in a complete CAT(0) space (X, d), which is studied in [1]. The pursuer and the evader start at some points $p_0 \in X$ and $e_0 \in X$, respectively. At the n-th step, $n \in \mathbb{N}$, the evader moves from $e_{n-1} \in X$ to a point $e_n \in X$ with $d(e_{n-1}, e_n) \leq \delta$, where $\delta > 0$ is a given maximal step size. The pursuer moves from $p_{n-1} \in X$ to the point $p_n \in X$ which

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lies on the geodesic segment $[p_{n-1}, e_{n-1}]$, and $d(p_{n-1}, p_n) = \delta$. Here we assume $d(p_{n-1}, e_{n-1}) > \delta$, since otherwise the game would have been ended (the pursuer would have won) at the $(n-1)$-th step. Indeed, by the definition, the pursuer wins the game if there exists $k \in \mathbb{N}$ such that $d(p_k, e_k) \leq \delta$. Otherwise, the evader wins.

The counterexample

The following topological characterization of the underlying CAT(0) space comes from [1, Theorem 8].

Theorem 1 *Let X be a complete CAT(0) space where the pursuit-evasion game (described above) is being played. Then X is compact if and only if the pursuer always wins.*

Unfortunately, the proof given in [1] is not correct. Namely, a noncompact CAT(0) space in not necessarily unbounded (as, for instance, the closed unit ball in an infinite dimensional Hilbert space), and even if it is unbounded, it does not necessarily contain a geodesic ray (as, for instance, the space in Example 2).

It is worth mentioning that Theorem 1 is not used in the remainder of the paper [1], and hence does not affect the other results of [1].

We now give an example of a complete CAT(0) space where the pursuer always wins even though the space is unbounded (and hence noncompact). This shows that the conclusion of Theorem 1 cannot be true.

Example 2 *We define a metric space consisting of countebly many compact intervals $[0, j] \subset \mathbb{R}$, where $j \in \mathbb{N}$, glued together at the origin, in the following way. Denote*

$$U = \bigcup_{j \in \mathbb{N}} \{j\} \times [0, j],$$

and for two points $(k, s), (l, t) \in U$ define $(k, s) \sim (l, t)$ if $s = t = 0$. Take the quotient $W = U/\sim$, and define $d : W \times W \to \mathbb{R}$, for any $x = (k, s) \in W$, and $y = (l, t) \in W$, by

$$d(x, y) = \begin{cases} |s - t|, & \text{if } k = l, \\ s + t, & \text{if } k \neq l. \end{cases}$$

Then (W, d) is a complete CAT(0) space. It is also immediate that W is unbounded. On the other hand there is no geodesic ray in the space W.

We claim that the pursuer always wins if the pursuit-evasion game is played in W according to the rules described above. Indeed, the only nontrivial case is when, at the n-th step, the pursuer's position $p_n = (k_n, s_n)$, and the evader's position $e_n = (l_n, t_n)$ are such that

$$k_n \neq l_n, \quad s_n + t_n > \delta, \quad \text{and} \quad s_n < \delta.$$

Then, at the $(n+1)$-th step, the pursuer's position $p_{n+1} = (k_{n+1}, s_{n+1})$ satisfies

$$k_{n+1} = l_n, \quad s_{n+1} < t_n, \quad \text{and} \quad s_{n+1} < \delta.$$

If, at the $(n+1)$-th step, the evader's position $e_{n+1} = (l_{n+1}, t_{n+1})$ is such that $l_{n+1} \neq l_n = k_{n+1}$, then $s_{n+1} + t_{n+1} < \delta$. It implies $d(p_{n+1}, e_{n+1}) < \delta$, and hence the game is over with the pursuer winning. If, otherwise, the evader moves at the $(n+1)$-th step such that $l_{n+1} = l_n = k_{n+1}$, then $s_{n+1} < t_n < t_{n+1}$, and the evader gets trapped in the compact geodesic segment $[0, l_n]$. The pursuer then wins in at most $\lceil l_n/\delta \rceil$ steps. □

References

1. Alexander, S., Bishop, R., Ghrist, R.: Total curvature and simple pursuit on domains of curvature bounded above. Geom. Dedicata 149, 275–290 (2010)
2. Bridson, M.R., Haefliger, A.: Metric spaces of non-positive curvature. Springer-Verlag, Berlin (1999)