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by

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Abstract We consider the reduced model for thin-film devices in stationary micromagnetics proposed in [4, DeSimone, Kohn, Müller, Otto, Schäfer 2001]. In the case of *soft* material, one of the energy contributions is negligible, and the problem becomes degenerate. The analysis and the numerical scheme recently developed in [7, Ferraz-Leite, Melenk, Praetorius 2011] are not satisfactory in this case. In the present work, we overcome the degeneracy and extend the numerical scheme by introducing a stabilizing energy term. Convergence of the method is established and a numerical experiment concludes the paper.

1 Model problem and introduction

We consider the model that was proposed in [3, 4] for the simulation of thin ferromagnetic films: Let $\Omega = \omega \times (0,t)$ be a thin ferromagnetic sample. The surface $\omega \subseteq \mathbb{R}^2$ is a Lipschitz domain with $\mathrm{diam}(\omega) \sim 1$. We model the sample as this screen and neglect the thickness $t \ll 1$.

We are interested in the effective behavior of the ferromagnetic material when exposed to an in-plane exterior field $\mathbf{h}_{\mathrm{ext}}$. For simplicity we assume $\mathbf{h}_{\mathrm{ext}} \in \mathbb{R}^2$ to be constant. The magnetization $\mathbf{m} : \omega \to \mathbb{R}^2$ is an in-plane vector field. In the full 3-dimensional model due to Landau and Lifschits [8], the magnetization is of constant length $|\mathbf{m}(x)| = 1$. In the reduced thin-film model, however, this constraint relaxes to the convex admissibility condition $|\mathbf{m}(x)| \leq 1$ almost everywhere in ω.

Let V denote the simple-layer operator on ω corresponding to the Laplacian in 3D, i.e.

$$(V\varphi)(x) := \frac{1}{4\pi} \int_\omega \frac{\varphi(y)}{|x-y|}\, dy, \quad x \in \omega. \tag{1}$$

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We recall that $V \in L(\widetilde{H}^{-1/2}(\omega); H^{1/2}(\omega))$ is elliptic and the induced norm $\|\varphi\|_V^2 := \langle \varphi, V\varphi \rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)}$ is an equivalent norm on the negative fractional order Sobolev space $\widetilde{H}^{-1/2}(\omega) := (H^{1/2}(\omega))^*$, see e.g. [10, 11].

The steady states of the magnetization $\mathbf{m}$ in presence of the applied field $\mathbf{h}_{\text{ext}}$ are minimizers of the quadratic energy functional

$$e(\mathbf{m}) = \frac{1}{2}\|\nabla\cdot\mathbf{m}\|_V^2 + \frac{q}{2}\|\mathbf{m}_2\|_{L^2(\omega)}^2 - (\mathbf{h}_{\text{ext}}, \mathbf{m})_{L^2(\omega)^2}. \tag{2}$$

The first term is the so-called stray-field energy. In the thin-film model, the magnetostatic potential $u : \mathbb{R}^3 \to \mathbb{R}$ is given as simple-layer potential of the negative divergence of $\mathbf{m}$, i.e.

$$u(x) = -\frac{1}{4\pi}\int_\omega \frac{\nabla\cdot\mathbf{m}(y)}{|x-y|}\,dy, \quad x \in \mathbb{R}^3 \setminus \overline{\omega}. \tag{3}$$

This representation leads to the energy contribution $\|\nabla\cdot\mathbf{m}\|_V^2$. The second term models a crystalline anisotropy of the material. Here, we consider the uniaxial case where the magnetization is favored to align with the first in-plane axis, and the energy contribution is scaled with a material-parameter $q \geq 0$. Finally, the linear term favors alignment of $\mathbf{m}$ along the applied field $\mathbf{h}_{\text{ext}}$.

In [6, 7] the authors establish an appropriate Hilbert space setting and analyze well-posedness of the model problem. The energy space for the magnetization is $\mathscr{H} := \{\mathbf{m} \in L^2(\omega)^2 \,|\, \nabla\cdot\mathbf{m} \in \widetilde{H}^{-1/2}(\omega), \mathbf{m}\cdot\mathbf{n} = 0 \text{ on } \partial\omega\}$, where $\mathbf{n}$ denotes the outer normal vector of $\omega \subseteq \mathbb{R}^2$. The space is naturally equipped with the norm $\|\mathbf{m}\|_{\mathscr{H}} := (\|\mathbf{m}\|_{L^2}^2 + \|\nabla\cdot\mathbf{m}\|_{\widetilde{H}^{-1/2}}^2)^{1/2}$. The admissible set is $\mathscr{A} := \{\mathbf{m} \in \mathscr{H} \,|\, |\mathbf{m}| \leq 1\}$.

Theorem 1 ([7, Theorem 11]). *The energy $e(\mathbf{m})$ of (2) admits a minimizer $\mathbf{m}^*$ in $\mathscr{A}$. The minimizer depends continuously on the data $\mathbf{h}_{\text{ext}}$ with respect to the energy semi-norm $|||\mathbf{m}||| := (\|\nabla\cdot\mathbf{m}\|_V^2 + q\|\mathbf{m}_2\|_{L^2(\omega)}^2)^{1/2}$. For $q > 0$, the quantity $|||\mathbf{m}|||$ is a (not equivalent) norm on $\mathscr{H}$, and the minimizer $\mathbf{m}^*$ is uniquely determined.*

In [5, 7], a penalty method in the spirit of [1] was proposed to solve the model problem. Although the analysis formally covers the entire parameter regime $q \geq 0$, it is tailored for the case $q > 0$. From a mathematical point of view the model is better justified for *soft* ferromagnetic films [3], i.e. $q = 0$. In this case, however, $|||\mathbf{m}|||$ is not positive definite; minimizers $\mathbf{m}^*$ are not uniquely determined; the problem becomes degenerate. In the present work, we closely analyze this specific case $q = 0$. First, we establish uniqueness of a minimum-norm solution $\mathbf{m}^{**}$. Then, for some $\delta > 0$, we introduce the stabilized energy

$$e^\delta(\mathbf{m}) := \frac{1}{2}\|\nabla\cdot\mathbf{m}\|_V^2 + \frac{\delta}{2}\|\mathbf{m}\|_{L^2(\omega)}^2 - (\mathbf{h}_{\text{ext}}, \mathbf{m})_{L^2(\omega)}. \tag{4}$$

This stabilized energy admits a unique minimizer $\mathbf{m}^\delta$ in $\mathscr{A}$, and we prove convergence

$$\mathbf{m}^\delta \rightharpoonup \mathbf{m}^{**} \quad \text{as} \quad \delta \to 0. \tag{5}$$

2 Degeneracy and the penalty method

In this Section we recall in short the penalty scheme from [7] for the approximation of solutions $\mathbf{m}^*$. We point out the difficulties and restrictions in the analysis that arise for $q = 0$. In this case the energy functional (2) reads

$$e(\mathbf{m}) = \frac{1}{2}\|\nabla\cdot\mathbf{m}\|_V^2 - (\mathbf{h}_{\text{ext}}, \mathbf{m})_{L^2(\omega)^2}. \tag{6}$$

As was already observed in [4], the constraint $\mathbf{m}\cdot\mathbf{n} = 0$ on $\partial\omega$ and integration by parts yield

$$\int_\omega \mathbf{h}_{\text{ext}}\cdot\mathbf{m}\,dx = -\int_\omega (\mathbf{h}_{\text{ext}}\cdot x)\nabla\cdot\mathbf{m}\,dx, \tag{7}$$

and hence

$$e(\mathbf{m}) = e(-\nabla\cdot\mathbf{m}) = \frac{1}{2}\|\nabla\cdot\mathbf{m}\|_V^2 + (\mathbf{h}_{\text{ext}}\cdot x, -\nabla\cdot\mathbf{m})_{L^2(\omega)}. \tag{8}$$

We use the notation $\nabla^\perp\psi = (-\partial_2\psi, \partial_1\psi)^T$ and observe $\nabla\cdot(\nabla^\perp\psi) = 0$. Let $\mathbf{m}^*$ be a minimizer of the energy (6). Then, any stream function $\psi \in H^1(\omega)$ such that $(\mathbf{m}^* + \nabla^\perp\psi) \in \mathscr{A}$ yields another admissible minimizer $\mathbf{m}^* + \nabla^\perp\psi$. In general we cannot expect the solution $\mathbf{m}^*$ to be unique. But since $\|\varphi\|_V$ is an equivalent norm on $\widetilde{H}^{-1/2}(\omega)$, at least the divergence of a minimizer $\nabla\cdot\mathbf{m}^*$ is unique.

We stress that the rough geometry of $\mathscr{A}$ does not allow a straight forward use of projection based schemes. In order to compute a numerical approximation to some minimizer $\mathbf{m}^*$, we first propose a penalty scheme on the continuous level and then discretize the resulting problem conformingly. We define the positive part function

$$(u)_+(x) := \begin{cases} u(x), & \text{if } u(x) \geq 0 \\ 0, & \text{else} \end{cases} \tag{9}$$

and use the fact that the energy $e(\mathbf{m})$ is well-defined and smooth on the full space $\mathscr{H}$. Given $\varepsilon > 0$, we seek a minimizer $\mathbf{m}^\varepsilon \in \mathscr{H}$ of the penalized energy

$$e_\varepsilon(\mathbf{m}) := e(\mathbf{m}) + \frac{1}{2\varepsilon}\|(|\mathbf{m}| - 1)_+\|_{L^2(\Omega)}^2. \tag{10}$$

The penalized energy functional (10) is convex and coercive in $\mathscr{H}$ and, hence, admits a (non-unique) minimizer $\mathbf{m}^\varepsilon \in \mathscr{H}$.

To solve the unconstrained non-linear penalized problem, we discretize the energy space $\mathscr{H}$ conformingly: Let $\mathscr{T}_h$ denote a regular triangulation of ω in the sense of Ciarlet [2] with mesh-size $h > 0$. We denote by $RT^0(\mathscr{T}_h)$ the space of lowest-order Raviart-Thomas finite element functions [9]. This is a natural conforming discretization of $\mathscr{H}$ [6], and since $RT^0(\mathscr{T}_h) \subseteq \mathscr{H}$ is a closed subspace, we immediately conclude existence of a (non-unique) minimizer

$$\mathbf{m}_h^\varepsilon = \operatorname{argmin}_{RT^0(\mathscr{T}_h)} e_\varepsilon(\mathbf{m}_h). \tag{11}$$

In [7] the authors prove that this approach—penalization and conforming discretization—yields an unconditionally convergent scheme, but the notion of convergence for $q = 0$ is quite unsatisfying.

Theorem 2 ([7, Theorem 11]). *Let* $(h_n)_{n\in\mathbb{N}}$ *and* $(\varepsilon_n)_{n\in\mathbb{N}}$ *be arbitrary positive zero sequences. A sequence of minimizers* $\mathbf{m}_n \in RT^0(\mathscr{T}_{h_n})$ *of the penalized energy* $e_{\varepsilon_n}(\cdot)$ *from (10) satisfies convergence in the following sense: Any subsequence* $(\mathbf{m}_k)_{k\in\mathbb{N}} \subseteq (\mathbf{m}_n)_{n\in\mathbb{N}}$ *contains a convergent subsequence* $(\mathbf{m}_\ell)_{\ell\in\mathbb{N}} \subseteq (\mathbf{m}_k)_{k\in\mathbb{N}}$ *whose limit is a minimizer* $\mathbf{m}^* \in \mathscr{A}$ *of the energy* $e(\cdot)$ *from (2). Convergence holds with respect to the weak topology of* $\mathscr{H}$ *and the topology induced by the semi-norm* $|||\mathbf{m}||| := (\|\nabla\cdot\mathbf{m}\|_V^2 + q\|\mathbf{m}_2\|_{L^2(\omega)}^2)^{1/2}$ *i.e.*

$$\mathbf{m}_\ell \rightharpoonup \mathbf{m}^* \qquad \text{and} \qquad |||\mathbf{m}_\ell - \mathbf{m}^*||| \to 0. \tag{12}$$

Moreover, the entire sequence of energies converges:

$$e(\mathbf{m}_n) \to e(\mathbf{m}^*) \quad \text{and} \quad e_{\varepsilon_n}(\mathbf{m}_n) \to e(\mathbf{m}^*). \tag{13}$$

For $q > 0$, *the minimizer* $\mathbf{m}^*$ *is uniquely determined and we have convergence of the full sequence, i.e. (12) holds with* $\mathbf{m}_\ell$ *replaced by* $\mathbf{m}_n$.

Remark 1. The degeneracy in the case $q = 0$ has the following consequences:

- To compute a minimizer $\mathbf{m}_n \in RT^0(\mathscr{T}_{h_n})$ we use a damped Newton-method. The Hessian of the penalized energy reads

$$\mathrm{Hess}(e(\mathbf{m}_n))(\mathbf{v})(\mathbf{w}) = \langle\nabla\cdot\mathbf{v}, V(\nabla\cdot\mathbf{w})\rangle + (f_\varepsilon(\mathbf{m}_n)\mathbf{v}, \mathbf{w})_{L^2(\omega)} \tag{14}$$

  with some non-linearity $f_\varepsilon(\mathbf{m}_n)$ that corresponds to the penalty term. This is in general not a positive definite form, but merely positive *semi-definite*.
- Convergence is mathematically only ensured for some subsequence $\mathbf{m}_\ell$. The numerical approximations could oscillate between different minimizers $\mathbf{m}^*$ of the continuous and constrained problem. In experiments, we observe that different initial values for the Newton algorithm lead to different minimizers.

3 A stabilized approximation

In this Section we define the minimum norm minimizer and propose a stabilized scheme to compute it. The minimum norm solution is of special physical relevance. It allows reconstruction of certain microstructural phenomena from the effective magnetization computed by the reduced model, cf. [4].

3.1 Minimum norm solution and stabilized energy functional

As already mentioned, the quantity $\sigma^* = -\nabla \cdot \mathbf{m}^*$ is uniquely determined from the model problem (6). We minimize the L^2-norm of $\mathbf{m} \in \mathscr{H}$ under the side constraints $\mathbf{m} \in \mathscr{A}$ and $-\nabla \cdot \mathbf{m} = \sigma^*$ and define the solution $\mathbf{m}^{**}$ as minimum norm minimizer.

Proposition 1. *Let $\sigma \in \widetilde{H}^{-1/2}(\omega)$ be given. Then there is a uniquely determined $\mathbf{m}^{**} \in \mathscr{A}_\sigma := \{\mathbf{m} \in \mathscr{A} \mid -\nabla \cdot \mathbf{m} = \sigma\}$ such that $\|\mathbf{m}^{**}\|_{L^2(\omega)} \le \|\mathbf{m}\|_{L^2(\omega)}$ for $\mathbf{m} \in \mathscr{A}_\sigma$.*

Proof. We recall that $\mathscr{A}$ is a closed and convex subset of $\mathscr{H}$, cf. [7]. The set of functions $\mathbf{m} \in \mathscr{H}$ that satisfy $-\nabla \cdot \mathbf{m} = \sigma$ can be written as $\mathscr{H}_\sigma := \{\mathbf{m} \in \mathscr{H} \mid \mathbf{m} = \nabla\phi + \mathbf{w} \,\text{with}\, \nabla \cdot \mathbf{w} = 0\}$. The function $\phi \in H^1(\omega)$ is given as the (up to constants uniquely determined) solution to the Neumann problem

$$-\Delta\phi = \sigma \in \widetilde{H}^{-1}(\omega) \supseteq \widetilde{H}^{-1/2}(\omega), \tag{15}$$

$$\frac{\partial\phi}{\partial\mathbf{n}} = 0 \in \widetilde{H}^{-1/2}(\partial\omega). \tag{16}$$

The admissibility set $\mathscr{A}_\sigma := \mathscr{A} \cap \mathscr{H}_\sigma$ is closed and convex with respect to the norm topology of $\mathscr{H}$. The functional $\mathbf{m} \mapsto \|\mathbf{m}\|_{L^2}^2$ is obviously strictly convex, continuous on $\mathscr{H}$, and coercive on $\mathscr{A}_\sigma$. □

From a numerical point of view this approach is not suitable. First, it doesn't provide uniqueness of $\mathbf{m}^*$ in the first step for the computation of σ^*. And second, the side constraint $-\nabla \cdot \mathbf{m} = \sigma^*$ seems numerically inconvenient. Instead, we introduce the stabilized energy functional

$$e^\delta(\mathbf{m}) = \frac{1}{2}\|\nabla \cdot \mathbf{m}\|_V^2 + \frac{\delta}{2}\|\mathbf{m}\|_{L^2(\omega)}^2 - (\mathbf{h}_{\mathrm{ext}}, \mathbf{m})_{L^2(\omega)}. \tag{17}$$

It depends on a (small) parameter $\delta > 0$ and we seek to compute minimizers $\mathbf{m} \in \mathscr{A}$ of the energy (17).

Proposition 2. *There is a unique minimizer $\mathbf{m}^\delta \in \mathscr{A}$ of the energy (17).*

Proof. We define the δ-norm

$$|\!|\!|\mathbf{m}|\!|\!|_\delta^2 = \|\nabla \cdot \mathbf{m}\|_V^2 + \delta\|\mathbf{m}\|_{L^2(\omega)}^2 \tag{18}$$

and observe that it is an equivalent norm on $\mathscr{H}$. We seek to minimize

$$e^\delta(\mathbf{m}) = \frac{1}{2}|\!|\!|\mathbf{m}|\!|\!|_\delta^2 - F(\mathbf{m}) \tag{19}$$

where $F \in L(\mathscr{H}, \mathbb{R})$ is defined through $F(\mathbf{m}) = (\mathbf{h}_{\mathrm{ext}}, \mathbf{m})_{L^2(\omega)}$. Obviously this is a strictly convex, continuous, and coercive minimization problem. Since $\mathscr{A}$ is a closed and convex subset of $\mathscr{H}$, the problem has a unique solution $\mathbf{m}^\delta \in \mathscr{A}$. □

3.2 Convergence

We now establish convergence of $\mathbf{m}^\delta$ to the minimum norm solution $\mathbf{m}^{**}$ as $\delta \to 0$. The proof is organized in two steps: First, energy bounds provide weak convergence to some minimizer $\mathbf{m}^*$. From that, we obtain convergence of the quantity $\sigma^\delta = -\nabla\cdot\mathbf{m}^\delta$ to the uniquely determined $\sigma^* = -\nabla\cdot\mathbf{m}^*$ in the $\widetilde{H}^{-1/2}(\omega)$-norm. Second, the stabilized energy functional (17) provides the lower bound $\|\mathbf{m}^\delta\|_{L^2(\omega)} \le \|\mathbf{m}^{**}\|_{L^2(\omega)}$. Together, we get that each accumulation-point $\widetilde{\mathbf{m}}$ of the sequence $\mathbf{m}^\delta$ is a minimizer of the energy (6) such that $\|\widetilde{\mathbf{m}}\|_{L^2(\omega)} \le \|\mathbf{m}^{**}\|_{L^2(\omega)}$, which proves $\widetilde{\mathbf{m}} = \mathbf{m}^{**}$.

Before stating the convergence of the proposed scheme, we recall two simple Lemmas. The proofs are left to the reader.

Lemma 1. *Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence and $x \in X$ some element of the metric space (X,d). Assume that each subsequence $(x_k)_{k\in\mathbb{N}} \subseteq (x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_\ell)_{\ell\in\mathbb{N}} \subseteq (x_k)_{k\in\mathbb{N}}$ such that $\lim_{\ell\to\infty} = x$. Then we already have $\lim_{n\in\mathbb{N}} x_n = x$.* □

Lemma 2. *Let $(u_n)_{n\in\mathbb{N}} \subseteq H$ be a sequence in a Hilbert space $(H, \|\cdot\|_H)$. Let $(\cdot,\cdot)_H$ denote a continuous semi-scalar product with induced semi-norm $|\cdot|_H = (\cdot,\cdot)_H^{1/2}$. Assume that the sequence has a weak limit $u_n \rightharpoonup u$ and that the semi-norm converges $\lim_{n\in\mathbb{N}} |u_n|_H = |u|_H$. Then we already have $\lim_{n\to\infty} |u_n - u|_H = 0$.* □

Theorem 3. *Let $(\delta_n)_{n\in\mathbb{N}}$ be some positive zero sequence, i.e. $\delta_n > 0$ and $\lim_{n\to\infty}\delta_n = 0$. Then, the sequence $(\mathbf{m}^{\delta_n})_{n\in\mathbb{N}} \subseteq \mathscr{A}$ of minimizers of the stabilized energy (17) satisfies*

$$\|\nabla\cdot(\mathbf{m}^\delta - \mathbf{m}^{**})\|_{\widetilde{H}^{-1/2}(\omega)} \to 0, \qquad \lim_{n\to\infty}\mathbf{m}^{\delta_n} = \mathbf{m}^{**}, \tag{20}$$

where the second statement holds with respect to the weak topology of $\mathscr{H}$.

Proof. We may assume without loss of generality $\delta_n \le 1$. From $e^\delta(\mathbf{m}) = e(\mathbf{m}) + \frac{\delta}{2}\|\mathbf{m}\|^2_{L^2(\omega)}$ and the fact that $\mathbf{m}^{\delta_n} \in \mathscr{A}$ is the minimizer of e^{δ_n}, we conclude

$$e(\mathbf{m}^{\delta_n}) \le e^{\delta_n}(\mathbf{m}^{\delta_n}) \le e^{\delta_n}(\mathbf{m}^{**}) \le e^1(\mathbf{m}^{**}) = K. \tag{21}$$

From coercivity of $e(\cdot)$ we obtain boundedness of $(\mathbf{m}^{\delta_n})_{n\in\mathbb{N}}$ in $\mathscr{H}$. Therefore, each subsequence $(\mathbf{m}^{\delta_k})_{k\in\mathbb{N}} \subseteq (\mathbf{m}^{\delta_n})_{n\in\mathbb{N}}$ has a weakly convergent subsequence $(\mathbf{m}^{\delta_\ell})_{\ell\in\mathbb{N}} \subseteq (\mathbf{m}^{\delta_k})_{k\in\mathbb{N}}$. We choose one such subsequence $(\mathbf{m}^{\delta_\ell})_{\ell\in\mathbb{N}}$ and denote its weak limit by $\widetilde{\mathbf{m}} \leftharpoonup \mathbf{m}^{\delta_\ell}$. Trivially, $e(\mathbf{m}^{**}) \le e(\widetilde{\mathbf{m}})$. Weak lower semicontinuity of $e(\cdot)$ yields the converse inequality

$$e(\widetilde{\mathbf{m}}) \le \liminf_{\ell\in\mathbb{N}} e(\mathbf{m}^{\delta_\ell}) \le \liminf_{\ell\in\mathbb{N}} e^{\delta_\ell}(\mathbf{m}^{\delta_\ell}) \le \liminf_{\ell\in\mathbb{N}} e^{\delta_\ell}(\mathbf{m}^{**}) = e(\mathbf{m}^{**}), \tag{22}$$

i.e. the weak limit $\widetilde{\mathbf{m}}$ is a minimizer of the energy $e(\cdot)$.

From the uniqueness of $\sigma^* = -\nabla\cdot\mathbf{m}^{**}$, we conclude $-\nabla\cdot\widetilde{\mathbf{m}} = \sigma^*$. Recall that

$$(\mathbf{m},\mathbf{w})_\sigma := \langle\nabla\cdot\mathbf{m}, V(\nabla\cdot\mathbf{w})\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)}, \tag{23}$$

defines a continuous semi-scalar product on the energy space $\mathscr{H}$. We denote the induced continuous semi-norm by $|\cdot|_\sigma$. The energy $e(\cdot)$ can be written as

$$e(\mathbf{m}) = \frac{1}{2}|\mathbf{m}|_\sigma - (\mathbf{h}_{\text{ext}}, \mathbf{m})_{L^2(\omega)}. \tag{24}$$

The mapping $\mathbf{m} \mapsto (\mathbf{h}_{\text{ext}}, \mathbf{m})_{L^2(\omega)}$ is a linear and continuous functional on $\mathscr{H}$. From $e(\widetilde{\mathbf{m}}) \leq \liminf_{\ell\in\mathbb{N}} e(\mathbf{m}^{\delta_\ell})$ and $e^{\delta_\ell}(\mathbf{m}^{\delta_\ell}) \leq e^{\delta_\ell}(\mathbf{m}^{**}) \to e(\mathbf{m}^{**})$, we conclude convergence of the energy $\lim_{\ell\to\infty} e(\mathbf{m}^{\delta_\ell}) = e(\mathbf{m}^{**})$. From the weak convergence we have $(\mathbf{h}_{\text{ext}}, \mathbf{m}^{\delta_\ell})_{L^2(\omega)} \to (\mathbf{h}_{\text{ext}}, \widetilde{\mathbf{m}})_{L^2(\omega)}$, and we finally obtain convergence of the semi-norm $\lim_{\ell\to\infty} |\mathbf{m}^{\delta_\ell}|_\sigma = |\mathbf{m}^{**}|_\sigma$. With Lemma 2, we conclude $\nabla\cdot\mathbf{m}^{\delta_n} \to \nabla\cdot\mathbf{m}^{**} \in \widetilde{H}^{-1/2}(\omega)$. Consider the sequence $(\sigma_n := -\nabla\cdot\mathbf{m}^{\delta_n})_{n\in\mathbb{N}}$. According to Lemma 1, we have proven $\lim_{n\to\infty}\sigma_n = \sigma^* \in \widetilde{H}^{-1/2}(\omega)$.

Next, we show that the stabilized energy functional in fact approximates the minimum norm solution from the set of available choices. We use the fact that $\mathbf{m}^{**}$ and $\mathbf{m}^{\delta_n}$ are minimizers of the corresponding energy functionals to see

$$e(\mathbf{m}^{**}) \leq e(\mathbf{m}^{\delta_n}) \quad \text{and} \quad e(\mathbf{m}^{\delta_n}) + \frac{\delta}{2}\|\mathbf{m}^{\delta_n}\|^2_{L^2(\omega)} \leq e(\mathbf{m}^{**}) + \tfrac{\delta}{2}\|\mathbf{m}^{**}\|^2_{L^2(\omega)}. \tag{25}$$

This means that

$$\frac{\delta}{2}\left(\|\mathbf{m}^{\delta_n}\|^2_{L^2(\omega)} - \|\mathbf{m}^{**}\|^2_{L^2(\omega)}\right) \leq e(\mathbf{m}^{**}) - e(\mathbf{m}^{\delta_n}) \leq 0, \tag{26}$$

and hence $\|\mathbf{m}^{\delta_n}\|_{L^2(\omega)} \leq \|\mathbf{m}^{**}\|_{L^2(\omega)}$. Now consider again the weakly convergent subsequence $(\mathbf{m}^{\delta_\ell})_{\ell\in\mathbb{N}}$ whose weak limit $\widetilde{\mathbf{m}}$, as we have seen before, is a minimizer. The L^2-norm is convex and continuous on $\mathscr{H}$. It is, thus, weakly lower semicontinuous, and we easily conclude $\|\widetilde{\mathbf{m}}\|_{L^2} \leq \liminf_\ell \|\mathbf{m}^{\delta_\ell}\|_{L^2} \leq \|\mathbf{m}^{**}\|_{L^2}$. Since we know $\nabla\cdot\widetilde{\mathbf{m}} = \nabla\cdot\mathbf{m}^{**}$ and according to Proposition 1, this means $\widetilde{\mathbf{m}} = \mathbf{m}^{**}$. Finally, Lemma 1 yields $\lim\limits_{n\to\infty}\mathbf{m}^{\delta_n} = \mathbf{m}^{**} \in \mathscr{H}$ in the weak topology. □

Remark 2. We only established convergence of the sequence of continuous minimizers $\mathbf{m}^\delta$. Similar ideas as in [7] can be used to prove that a sequence $\mathbf{m}_n = \mathbf{m}_{h_n}^{\varepsilon_n,\delta_n}$ of discrete minimizers of the stabilized and penalized problem convergences

$$\mathbf{m}_n \to \mathbf{m}^{**} \quad \text{as} \quad (h_n, \varepsilon_n, \delta_n) \to (0,0,0) \tag{27}$$

without additional assumptions on the zero sequences $h_n, \varepsilon_n, \delta_n$. A close analysis of the Euler-Lagrange equations as in [5]—a topic that would exceed the scope of this paper—leads to the heuristic choice of $h \sim \varepsilon \sim \delta$, at least for uniform meshes.

We perform a numerical experiment with $\omega = (-0.5, 0.5) \times (-0.1, 0.1)$ and $\mathbf{h}_{\text{ext}} = (1, -0.3)^T$. This choice ensures that data is smooth, no symmetries arise, and the constraint $|\mathbf{m}| \leq 1$ is active on a large subdomain. Following the heuristics, we choose $\varepsilon = \delta = h$. To estimate various error quantities, we computed a reference solution on a mesh with $h = 7.8125 \cdot 10^{-4}$ and $1,964,544$ degrees of freedom.

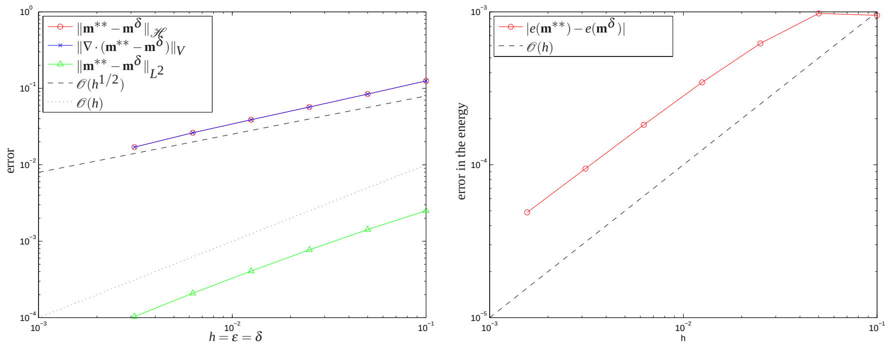

Fig. 1 Left: error $(\mathbf{m}^{**} - \mathbf{m}^{\delta})$ measured in the $\mathscr{H}$-, V-, and L^2-norm. Right: error in the energy $|e(\mathbf{m}^{**}) - e(\mathbf{m}^{\delta})|$. All quantities are plotted over the mesh-size h. The parameters are $\varepsilon = \delta = h$.

Figure 1 shows the results: The dominant error contribution is the error of the divergence in the V-norm $\|\nabla \cdot (\mathbf{m}^{**} - \mathbf{m}^{\delta})\|_V$ that decays at a rate of $\mathscr{O}(h^{1/2})$. The error measured in the L^2-norm is of higher order. The energy is approximated with $\mathscr{O}(h)$.

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