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$C^{1,1}$ REGULARITY FOR DEGENERATE ELLIPTIC OBSTACLE PROBLEMS IN MATHEMATICAL FINANCE

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Abstract. The Heston stochastic volatility process is a degenerate diffusion process where the degeneracy in the diffusion coefficient is proportional to the square root of the distance to the boundary of the half-plane. The generator of this process with killing, called the elliptic Heston operator, is a second-order, degenerate-elliptic partial differential operator, where the degeneracy in the operator symbol is proportional to the distance to the boundary of the half-plane. In mathematical finance, solutions to obstacle problem for the elliptic Heston operator correspond to value functions for perpetual American-style options on the underlying asset. With the aid of weighted Sobolev spaces and weighted Hölder spaces, we establish the optimal $C^{1,1}$ regularity (up to the boundary of the half-plane) for solutions to obstacle problems for the elliptic Heston operator when the obstacle functions are sufficiently smooth.

Contents

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1. Introduction

In [4], the authors established the existence and uniqueness of a solution in a weighted Sobolev space[1], $u \in H^2(\mathscr{O}, \mathfrak{w})$, to the obstacle problem,

$$\begin{aligned} \min\{Au - f, u - \psi\} &= 0 \quad \text{a.e. on } \mathscr{O}, \\ u &= g \quad \text{on } \Gamma_1, \end{aligned} \tag{1.1}$$

for the *Heston operator* [13],

$$Au := -\frac{y}{2}\left(u_{xx} + 2\varrho\sigma u_{xy} + \sigma^2 u_{yy}\right) - \left(r - q - \frac{y}{2}\right) u_x - \kappa(\theta - y)u_y + ru, \tag{1.2}$$

on a subdomain $\mathscr{O}$ (possibly unbounded) of the upper half-plane $\mathbb{H} := \mathbb{R}\times(0,\infty)$, where $f : \mathscr{O} \to \mathbb{R}$ is a source function, $g : \Gamma_1 \to \mathbb{R}$ prescribes a Dirichlet boundary condition along $\Gamma_1 := \mathbb{H} \cap \partial\mathscr{O}$, and $\psi : \mathscr{O} \cup \Gamma_1 \to \mathbb{R}$ is an obstacle function which is compatible with g in the sense that $\psi \leq g$ on Γ_1. The differential operator A given in (1.2) is elliptic on $\mathscr{O}$ but becomes degenerate along $\bar\Gamma_0$, where Γ_0 denotes the interior of $\{y = 0\} \cap \partial\mathscr{O}$. Because $\kappa\theta > 0$ (see assumption (1.3) below), *no boundary condition* is prescribed along the portion $\bar\Gamma_0$ of the boundary $\partial\mathscr{O} = \bar\Gamma_0 \cup \Gamma_1$ of $\mathscr{O}$.

The operator A is the generator of the two-dimensional Heston stochastic volatility process with killing, a degenerate diffusion process well known in mathematical finance and a paradigm for a broad class of degenerate diffusion processes. The coefficients defining A in (1.2) are constants assumed throughout this article to obey

$$\sigma \neq 0, \quad -1 < \varrho < 1, \quad r \geq 0, \quad q \geq 0, \quad \kappa > 0, \quad \theta > 0, \tag{1.3}$$

while their financial meaning is described in [13]. For a detailed introduction to the Heston operator and the obstacle problem (1.1), we refer the reader to our article [4].

In this article, we will establish $C^{1,1}_s$ regularity on $\mathscr{O} \cup \Gamma_0$ and a priori $C^{1,1}_s$ estimates for the solution u to (1.1) on subdomains $U \Subset \mathscr{O} \cup \Gamma_0$. We use $C^{1,1}_s$ to indicate a weighted Hölder norm and corresponding Hölder space which are distinct from the usual $C^{1,1}$ Hölder norm and Hölder space and which take into account the degeneracy of the operator, A, along $y = 0$ — see section 2 for their definition. In the case of a uniformly elliptic operator on a bounded domain, interior $C^{1,1}$ regularity was established by Brezis and Kinderlehrer [2] (see also [11, Theorem 1.4.1] for a statement of their result and an exposition of their proof), while global $C^{1,1}$ regularity, given a Dirichlet boundary condition, was established by Jensen [14] (see also [23, Theorem 4.38] for a statement of his result and an exposition of his proof), recalling that [11, p. 23], for a bounded domain $U \subset \mathbb{R}^n$, one has $W^{2,\infty}(U) = C^{1,1}(\bar U)$. To the best of the authors' knowledge, however, our article is the first to establish $C^{1,1}$ regularity of a solution to an obstacle problem defined by a *degenerate* elliptic operator, despite the importance of this question in applications to American-style option pricing problems for asset prices modeled by stochastic volatility processes [13].

For interior $C^{1,1}$ regularity, the case of a uniformly elliptic operator on a bounded domain reduces, by standard methods (see, for example, [1, 23]), to the case of the Laplace operator

[1]See section 2 for its definition.

and ingenious techniques introduced by Caffarelli [3] greatly simplify the proof of interior $C^{1,1}$ regularity for solutions to an obstacle problem in this case. We shall adapt Caffarelli's approach in our article but, because of the degeneracy of our operator, A, along $y=0$, careful consideration must be given to the different scaling of the equation near $y=0$. This scaling is reflected in the use of the *cycloidal* distance function, $s(\cdot,\cdot)$, defined in section 2 and of weighted Sobolev, Hölder and $C^{1,1}$ spaces. Weighted Sobolev and Hölder spaces have been introduced previously (see, for example, [4, 5, 7, 16]) in order to obtain sharp estimates for solutions to equations involving degenerate elliptic operators of the form (1.2) and their parabolic analogues.

Let $B_\rho(Q_0) := \{P \in \mathbb{R}^2 : \operatorname{dist}(P,Q_0) < \rho\}$ denote the open ball with center $Q_0=(p_0,q_0)\in\mathbb{R}^2$ and radius $\rho>0$, and set

$$B^+_\rho(Q_0) := B_\rho(Q_0)\cap\mathbb{H}. \tag{1.4}$$

For a given radius $R_0>0$ and for any R obeying $0<R<R_0$, we denote

$$V := B^+_{R_0}(Q_0) \quad\text{and}\quad U := B^+_R(Q_0). \tag{1.5}$$

Throughout our article, we shall assume that $Q_0=(p_0,q_0)\in\bar{\mathbb{H}}$ with $0\le q_0\le\Lambda$, for a positive constant[2] Λ. We shall abuse notation slightly and let Γ_0 denote the interiors of $\partial\mathbb{H}\cap\partial\mathscr{O}$, $\partial\mathbb{H}\cap\partial V$, or $\partial\mathbb{H}\cap\partial U$ when we write $\mathscr{O}\cup\Gamma_0$, $V\cup\Gamma_0$, or $U\cup\Gamma_0$, respectively. The definitions of the weighted Hölder spaces, $C^\alpha_s(\bar V)$, $C^{2+\alpha}_s(\bar V)$ and $C^{1,1}_s(\bar V)$, which we require for the statement of the main result of this article below are collected in section 2.

Theorem 1.1 (Optimal regularity)**.** *Let $R_0>0$ and $\Lambda>0$ and suppose $Q_0=(p_0,q_0)\in\bar{\mathbb{H}}$ with $0\le q_0\le\Lambda$. Let V be as in* (1.5)*. Assume that $u\in H^2(V,\mathfrak{w})\cap C(\bar V)$ is a solution to the obstacle problem,*

$$\min\{Au-f, u-\psi\}=0 \quad a.e.\ on\ V, \tag{1.6}$$

with $\psi\in C^{2+\alpha}_s(\bar V)$ and $f\in C^\alpha_s(\bar V)$, for some $\alpha\in(0,1)$. Then, $u\in C^{1,1}_s(V\cup\Gamma_0)$ and there is a constant C, depending on α, R_0, Λ, and the coefficients of the operator A, such that if U is as in (1.5) *with $R=R_0/2$, then*

$$\|u\|_{C^{1,1}_s(\bar U)} \le C\left(\|u\|_{C(\bar V)} + \|f\|_{C^\alpha_s(\bar V)} + \|\psi\|_{C^{1,1}(\bar V)}\right). \tag{1.7}$$

Theorem 1.1 immediately yields

Corollary 1.2 (Optimal regularity)**.** *Let $\mathscr{O}\subset\mathbb{H}$ be a bounded domain. Assume that $u\in H^2(\mathscr{O},\mathfrak{w})$ is a solution to the obstacle problem* (1.6) *on $\mathscr{O}$ with $\psi\in C^{2+\alpha}_s(\bar{\mathscr{O}})$ and $f\in C^\alpha_s(\bar{\mathscr{O}})$, for some $\alpha\in(0,1)$. Then, $u\in C^{1,1}_s(\mathscr{O}\cup\Gamma_0)$ and, for each precompact subdomain $\mathscr{O}'\Subset\mathscr{O}\cup\Gamma_0$, there is a constant C, depending on $\alpha,\mathscr{O}',\mathscr{O}$, and the coefficients of the operator A, such that*

$$\|u\|_{C^{1,1}_s(\bar{\mathscr{O}}')} \le C\left(\|u\|_{C(\bar{\mathscr{O}})} + \|f\|_{C^\alpha_s(\bar{\mathscr{O}})} + \|\psi\|_{C^{1,1}(\bar{\mathscr{O}})}\right). \tag{1.8}$$

[2] Note that A in (1.2) is uniformly elliptic on $B_{R_0}(Q_0)$ when $q_0>R_0$, and results concerning regularity of solutions to (1.6) are then standard [11, 23] and so, for the purpose of this article, we could choose $\Lambda=R_0$ without loss of generality.

Our proof of Theorem 1.1 proceeds by adapting ideas of Caffarelli in [3]; see also an exposition by Petrosyan in [19]. However, because our operator is degenerate, careful consideration must be given to the difference of the scaling of the equation in regions close (y small) and away ($y \geq \rho > 0$) from the portion of the boundary, $\{y = 0\} \cap \partial V$, where A becomes degenerate.

1.1. **Generalizations.** When the main result of our article (Theorem 1.1) is combined with Jensen's global $C^{1,1}$ regularity theorem [14], we see that $H^2(\mathscr{O}, \mathfrak{w})$ solutions, u, to (1.1) actually belong to $C^{1,1}_s(\mathscr{O}) \cap C^{1,1}(\mathscr{O} \cup \Gamma_1)$ under hypotheses on f and ψ analogous to those stated in Theorem 1.1. By making further use of methods in [10], it should follow that $u \in C^{1,1}_{s,\mathrm{loc}}(\bar{\mathscr{O}})$. Moreover, there is good reason to believe that results on the regularity of the free boundary for the obstacle problem defined by a non-degenerate elliptic or parabolic operator extend to degenerate operators of the kind considered in this article; see [20] and references therein for the non-degenerate elliptic case and [17, 18] and references therein for the non-degenerate parabolic case. We shall leave consideration of these extensions to our future articles.

The solution, u, to (1.1) can be interpreted as the value function for a perpetual American-style option with payoff function, ψ [15]. The $C^{2+\alpha}_s(\bar{V})$ regularity property assumed for the obstacle function, ψ, in the statement of Theorem 1.1 does not reflect the more typical Lipschitz regularity for ψ encountered in applications to mathematical finance, such as $\psi(x, y) = \max\{E - e^x, 0\}$, where E is a positive constant, in the case of a put option. Nevertheless, simple examples in this context [22, §8.3] and results of [17, 18] suggest that the solution, u, should nevertheless have the optimal $C^{1,1}_s$ regularity even when $\psi = \max\{E - e^x, 0\}$. Again, we shall leave consideration of this question to our future articles.

We have chosen, in this article, to work with our model, the Heston operator A, because of its relevance to mathematical finance and reliance on results in our previous work [4] and that of Feehan and Pop [9, 7, 10, 21]. However, we expect that the $C^{1,1}_s$ regularity result and a priori estimate in Theorem 1.1 may be easily generalized to higher dimensions and degenerate elliptic operators on $\mathbb{R}^{n-1} \times (0, \infty)$ with variable coefficients,

$$Au = -x_n a_{ij} u_{x_i x_j} - b_i u_{x_i} + cu,$$

under the assumptions that (a_{ij}) is strictly elliptic, $b_n \geq \nu > 0$, for some constant $\nu > 0$, and $c \geq 0$ and all coefficients are Hölder continuous of class $C^\alpha_s(\bar{V})$, for some $\alpha \in (0, 1)$. See [8] for an analysis with applications to probability theory based on parabolic operators of this type.

1.2. **Outline of the article.** For the convenience of the reader, we provide a brief outline of the article. We begin in §2 by reviewing our definitions of weighted Hölder spaces [5] and weighted Sobolev spaces [4] which we shall need for this article. In §3, we review results from [6, 7, 9, 10] concerning existence, uniqueness, and regularity of solutions to the elliptic Heston equation on bounded subdomains of the upper half-plane; see also [5]. In §4, we develop the key pointwise growth estimates (see Propositions 4.1 and 4.4) for solutions to the obstacle problem for the elliptic Heston operator. We conclude in §5 with the proof of our main result, Theorem 1.1.

1.3. **Notation.** Throughout the rest of the article we will set $Lu := -Au$, where A is given by (1.2) and we work with L instead to facilitate comparisons with the methods of Caffarelli [3] and the sign conventions therein. The operator L is then given by

$$Lu = \frac{y}{2}\left(u_{xx} + 2\varrho\sigma u_{xy} + \sigma^2 u_{yy}\right) + \left(r - q - \frac{y}{2}\right)u_x + \kappa\left(\theta - y\right)u_y - ru, \tag{1.9}$$

with coefficients which satisfy the assumption (1.3).

We let $C = C(*,\ldots,*)$ denote a constant which depends at most on the quantities appearing on the parentheses. In a given context, constants denoted by $C, C', \cdots$ and so on may have different values depending on the same set of arguments and may increase from one inequality to the next. Constants with values denoted by $K, K', \cdots$ and so on are reserved for quantities which remain fixed. We let $C(L)$ denote a constant which may depend on one or more of the constant coefficients of the operator L (that is, $q, r, \kappa, \theta, \varrho, \sigma$).

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2. Weighted Sobolev and Hölder spaces

In [4] the authors defined the following weighted Sobolev spaces of functions on a possibly unbounded domain $\mathscr{O} \subset \mathbb{H}$.

Definition 2.1 (Weighted Sobolev spaces)**.** Let $L^2(\mathscr{O}, \mathfrak{w})$ denote the Hilbert space of Borel measurable functions, $u : \mathscr{O} \to \mathbb{R}$, such that

$$\|u\|_{L^2(\mathscr{O},\mathfrak{w})} := \left(\int_{\mathscr{O}} u^2\, \mathfrak{w}\, dx\, dy\right)^{1/2} < \infty,$$

with weight function $\mathfrak{w}(x,y) := y^{\beta-1}e^{-\gamma|x|-\mu y}$, for $(x,y) \in \mathbb{H}$, where $\beta := 2\kappa\theta/\sigma^2$ and $\mu := 2\kappa/\sigma^2$ and the constant $\gamma > 0$ depends only on the coefficients of A. We define the vector space,

$$H^2(\mathscr{O}, \mathfrak{w}) := \left\{u \in L^2(\mathscr{O}, \mathfrak{w}) :\ y|D^2u|,\ (1+y)|Du|,\ (1+y)^{1/2}u \in L^2(\mathscr{O}, \mathfrak{w})\right\},$$

where $Du = (u_x, u_y)$ and $D^2u = (u_{xx}, u_{xy}, u_{yx}, u_{yy})$ are defined in the sense of distributions.

When equipped with the norm,

$$\|u\|_{H^2(\mathscr{O},\mathfrak{w})} := \left(\int_{\mathscr{O}} \left(y^2|D^2u|^2 + (1+y)^2|Du|^2 + (1+y)u^2\right)\, \mathfrak{w}\, dx\, dy\right)^{1/2},$$

one finds that $H^2(\mathscr{O}, \mathfrak{w})$ is a Hilbert space and, noting that $\mathscr{O}$ has dimension two, $H^2(\mathscr{O}, \mathfrak{w}) \subset C(\mathscr{O} \cup \Gamma_1)$ via the embedding theorem for standard, unweighted Sobolev spaces [4], but elementary examples show that functions in $H^2(\mathscr{O}, \mathfrak{w})$ need not be continuous up to $\bar{\Gamma}_0$.

We next define weighted $C^{1,1}$ and Hölder norms on a bounded domain $\mathscr{O} \subset \mathbb{H}$.

Definition 2.2 ($C^{1,1}_s$ norm and Banach space)**.** We say that $u \in C^{1,1}_s(\bar{\mathscr{O}})$ if u belongs to $C^{1,1}(\mathscr{O}) \cap C^1(\bar{\mathscr{O}})$ and

$$\|u\|_{C^{1,1}_s(\bar{\mathscr{O}})} := \|yD^2u\|_{L^\infty(\mathscr{O})} + \|Du\|_{C(\bar{\mathscr{O}})} + \|u\|_{C(\bar{\mathscr{O}})} < \infty.$$

Also, we say that $u \in C^{1,1}_s(\mathscr{O} \cup \Gamma_0)$, if $u \in C^{1,1}_s(\bar{U})$ for any subdomain $U \Subset \mathscr{O} \cup \Gamma_0$.

We recall the definition of the distance function, $s(\cdot,\cdot)$ on $\mathbb{H}$, equivalent to the distance function defined by the *cycloidal metric*, $y^{-1}(dx^2 + dy^2)$ on $\mathbb{H}$, and introduced by Daskalopoulos and Hamilton in [5] and by H. Koch in [16],

$$s(z, z_0) := \frac{|x - x_0| + |y - y_0|}{\sqrt{y} + \sqrt{y_0} + \sqrt{|x - x_0| + |y - y_0|}}, \quad \forall z = (x, y), z_0 = (x_0, y_0) \in \mathbb{H}. \tag{2.1}$$

This is the natural metric for our degenerate equation; see [5] for a discussion. The following weighted Hölder spaces were introduced by Daskalopoulos and Hamilton in [5].

Definition 2.3 (C^α_s and $C^{2+\alpha}_s$ norms and Banach spaces)**.** Given $\alpha \in (0, 1)$, we say that $u \in C^\alpha_s(\bar{\mathscr{O}})$ if $u \in C(\bar{\mathscr{O}})$ and

$$\|u\|_{C^\alpha_s(\bar{\mathscr{O}})} := \|u\|_{C(\bar{\mathscr{O}})} + \sup_{\substack{z_1, z_2 \in \mathscr{O} \\ z_1 \neq z_2}} \frac{|u(z_1) - u(z_2)|}{s(z_1, z_2)^\alpha} < \infty.$$

We say that $u \in C^{2+\alpha}_s(\bar{\mathscr{O}})$ if u has continuous first and second derivatives, Du, D^2u, in $\mathscr{O}$, and Du, yD^2u extend continuously up to the boundary, $\partial\mathscr{O}$, and the extensions belong to $C^\alpha_s(\bar{\mathscr{O}})$. We denote

$$\|u\|_{C^{2+\alpha}_s(\bar{\mathscr{O}})} := \|u\|_{C^\alpha_s(\bar{\mathscr{O}})} + \|Du\|_{C^\alpha_s(\bar{\mathscr{O}})} + \|yD^2u\|_{C^\alpha_s(\bar{\mathscr{O}})}.$$

We say that[3] $u \in C^\alpha_s(\mathscr{O} \cup \Gamma_0)$ if $u \in C^\alpha_s(\bar{U})$ for every subdomain $U \Subset \mathscr{O} \cup \Gamma_0$ and similarly that $u \in C^{2+\alpha}_s(\mathscr{O} \cup \Gamma_0)$ if $u \in C^{2+\alpha}_s(\bar{U})$ for every subdomain $U \Subset \mathscr{O} \cup \Gamma_0$.

One can show that $C^{2+\alpha}_s(\bar{\mathscr{O}})$ is a Banach space when equipped with the norm in Definition 2.3.

Remark 2.4. On any bounded subdomain $U \subset \mathbb{H}$ we have,

$$c\,|z - z_0| \leq s(z, z_0) \leq \sqrt{|z - z_0|}, \tag{2.2}$$

for some positive constant $c := c(\mathrm{diam}(U))$ depending only on the diameter of U. Hence, $C^\alpha(\bar{U}) \subset C^\alpha_s(\bar{U}) \subset C^{\alpha/2}(\bar{U})$.

[3] In [5, p. 901], when defining the spaces $C^\alpha_s(\mathscr{A})$ and $C^{2+\alpha}_s(\mathscr{A})$, it is assumed that $\mathscr{A}$ is a compact subset of the closed half-plane, $\{y \geq 0\}$.

3. Schauder existence, uniqueness, and regularity results

We collect some known results for solutions to the degenerate elliptic equation,

$$Lv = f \quad \text{on } V, \tag{3.1}$$

where V is as in (1.5). These results will be used in the proof of Theorem 1.1. Theorems 3.1, 3.2, 3.3, and 3.4 are proved in [10] and may be viewed as analogues of [12, Theorems 6.2, 6.6, 6.13 and 6.14] and a generalization of [5, Theorem I.1.1].

Theorem 3.1 (A priori Schauder interior estimate)**.** [10] *Given $f \in C^{\alpha}_s(\bar{V})$, where V is as in* (1.5)*, and a solution $u \in C^{2+\alpha}_s(V \cup \Gamma_0) \cap C(\bar{V})$ to*

$$Lv = f \quad \text{on } V,$$

there is a constant, C, depending at most on α, R, R_0, Λ, and the coefficients of L, such that if U is as in (1.5)*, then*

$$\|v\|_{C^{2+\alpha}_s(\bar{U})} \leq C\left(\|u\|_{C(\bar{V})} + \|f\|_{C^{\alpha}_s(\bar{V})}\right). \tag{3.2}$$

Theorem 3.2 (Existence of a solution to a Dirichlet problem with continuous boundary data)**.** [10] *Given $f \in C^{\alpha}_s(V \cup \Gamma_0)$, where V is as in* (1.5)*, and $g \in C(\overline{\mathbb{H} \cap \partial V})$, there exists a unique solution $u \in C^{2+\alpha}_s(V \cup \Gamma_0) \cap C(\bar{V})$ to the Dirichlet problem*

$$Lv = f \quad \text{on } V \quad \text{and} \quad v = g \quad \text{on } \mathbb{H} \cap \partial V. \tag{3.3}$$

Theorem 3.3 (A priori global Schauder estimate)**.** [10] *Given $f \in C^{\alpha}_s(\bar{V})$ and $g \in C^{2+\alpha}_s(\bar{V})$ and a solution $u \in C^{2+\alpha}_s(\bar{V})$ to the Dirichlet problem* (3.3)*, there is a constant, C, depending at most on α, R_0, Λ, and the coefficients of L, such that*

$$\|v\|_{C^{2+\alpha}_s(\bar{V})} \leq C\left(\|u\|_{C(\bar{V})} + \|f\|_{C^{\alpha}_s(\bar{V})} + \|g\|_{C^{2+\alpha}_s(\bar{V})}\right). \tag{3.4}$$

Theorem 3.4 (Existence of a solution to a Dirichlet problem)**.** [10] *Given $f \in C^{\alpha}_s(\bar{V})$ and $g \in C^{2+\alpha}_s(\bar{V})$, there exists a unique solution $u \in C^{2+\alpha}_s(\bar{V})$ to the Dirichlet problem* (3.3)*.*

The preceding results easily imply the following consequence when combined with a regularity theorem from [9] and a maximum principle estimate from [6].

Proposition 3.5 (Regularity and interior Schauder estimate)**.** *Let $f \in C^{\alpha}_s(V \cup \Gamma_0)$ and let $v \in H^2(V, \mathfrak{w})$ be a solution to*

$$Lv = f \quad \text{a.e. on } V.$$

Then, $v \in C^{2+\alpha}_s(V \cup \Gamma_0)$. Moreover, if $f \in C^{\alpha}_s(\bar{V})$, there is a constant, C, depending at most on α, R, R_0, Λ, and the coefficients of L, such that if U is as in (1.5)*, then*

$$\|v\|_{C^{2+\alpha}_s(\bar{U})} \leq C\left(\|v\|_{C(\bar{V})} + \|f\|_{C^{\alpha}_s(\bar{V})}\right). \tag{3.5}$$

Proof. Choose R_1 obeying $R \leq R_1 < R_0$ and let $V_1 := B^+_{R_1}(Q_0)$, so that $U \subsetneqq V_1 \Subset V \cup \Gamma_0$. Then $f \in C^\alpha_s(\bar V_1)$ and we may choose $w \in C^{2+\alpha}_s(\bar V_1)$ to be the unique solution to $Lw = f$ on V_1 and $w = 0$ on $\mathbb{H} \cap \partial V_1$ provided by Theorem 3.4. Moreover, $v_0 := v - w \in H^2(V_1, \mathfrak{w})$ is a solution to $Lv_0 = 0$ a.e. on V_1 and so, by [9], we have $v_0 \in C^\infty(V_1 \cup \Gamma_0)$ and thus $v = v_0 + w \in C^{2+\alpha}_s(V_1 \cup \Gamma_0)$. Since R_1 is arbitrary, we obtain $v \in C^{2+\alpha}_s(V \cup \Gamma_0)$, as desired. Now fix $R_1 = (R + R_0)/2$ and observe that (3.2) gives

$$\|v_0\|_{C^{2+\alpha}_s(\bar U)} \leq C\|v_0\|_{C(\bar V_1)}.$$

Setting $\mathfrak{B}_1 := \{u \in C^{2+\alpha}_s(\bar V_1) : u = 0 \text{ on } \mathbb{H} \cap \partial V_1\}$ and $\mathfrak{B}_1 := C^\alpha_s(\bar V_1)$, then Theorem 3.4 implies that $L : \mathfrak{B}_1 \to \mathfrak{B}_2$ is bijective and bounded and thus $L^{-1} : \mathfrak{B}_2 \to \mathfrak{B}_1$ is bounded by the open mapping theorem. Hence, there is a constant, C, depending at most on α, R, R_0, Λ, and the coefficients of L, such that

$$\|w\|_{C^{2+\alpha}_s(\bar V_1)} \leq C\|L^{-1}w\|_{C^\alpha_s(\bar V_1)} = C\|f\|_{C^\alpha_s(\bar V_1)}.$$

Therefore, by combining the preceding estimates,

$$\begin{aligned}
\|v\|_{C^{2+\alpha}_s(\bar U)} &\leq \|v_0\|_{C^{2+\alpha}_s(\bar U)} + \|w\|_{C^{2+\alpha}_s(\bar U)} \\
&\leq C\left(\|v_0\|_{C(\bar V_1)} + \|f\|_{C^\alpha_s(\bar V_1)}\right) \\
&\leq C\left(\|v\|_{C(\bar V_1)} + \|w\|_{C(\bar V_1)} + \|f\|_{C^\alpha_s(\bar V_1)}\right) \\
&\leq C\left(\|v\|_{C(\bar V_1)} + \|f\|_{C^\alpha_s(\bar V_1)}\right),
\end{aligned}$$

and thus (3.5) follows. □

Remark 3.6 (Alternative proofs of regularity in Proposition 3.5)*.* The regularity assertion, $v \in C^{2+\alpha}_s(V \cup \Gamma_0)$, also follows by using Theorem 3.2 instead of Theorem 3.4 in the proof of Proposition 3.5.

We can avoid relying on the regularity results in [9] if we are given $v \in H^2(V, \mathfrak{w}) \cap C(\bar V)$. Indeed, Theorem 3.2 provides a unique solution $\tilde v \in C^{2+\alpha}_s(V \cup \Gamma_0) \cap C(\bar V)$ to $L\tilde v = f$ on V and $\tilde v = v$ on $\mathbb{H} \cap \partial V$. But then $\tilde v \in H^2(V, \mathfrak{w})$ and by the weak maximum principle for L acting on functions in $H^2(V, \mathfrak{w})$ [6, Lemma 6.13 & Theorem 8.8], we must have $v = \tilde v$ a.e. on V and thus $v \in C^{2+\alpha}_s(V \cup \Gamma_0) \cap C(\bar V)$.

Remark 3.7 (Alternative proof of the interior Schauder a priori estimate in Proposition 3.5)*.* We can instead choose $w \in C^{2+\alpha}_s(V \cup \Gamma_0) \cap C(\bar V)$ to be the unique solution to $Lw = f$ on V and $w = 0$ on $\mathbb{H} \cap \partial V$ provided by Theorem 3.2. If we also assume that the coefficient r in (1.9) is positive, then the weak maximum principle estimate [6, Proposition 2.19 & Theorem 5.1] gives

$$\|w\|_{C(\bar V)} \leq \frac{1}{r}\|f\|_{C(\bar V)},$$

while (3.4) yields

$$\|w\|_{C^{2+\alpha}_s(\bar U)} \leq C\left(\|w\|_{C(\bar V)} + \|f\|_{C^\alpha_s(\bar V)}\right).$$

We now obtain $v \in C_s^{2+\alpha}(V \cup \Gamma_0) \cap C(\bar{V})$ and the interior Schauder a priori estimate (3.5) by combining the preceding observations for w with those for v_0 in the proof of Proposition 3.5 but without a need to appeal to Theorem 3.4.

The following weak and strong maximum principles are shown in [6]. Recall that if $v \in C_s^{2+\alpha}(\mathscr{O} \cup \Gamma_0)$, then $Dv \in C(\mathscr{O} \cup \Gamma_0)$ and $yD^2v \in C(\mathscr{O} \cup \Gamma_0)$ (by definition) while $yD^2v = 0$ on Γ_0 (see [5, Proposition I.12.1] or [8, Lemma 3.1]).

Theorem 3.8 (Weak maximum principle for the Heston operator)**.** [6, Theorem 5.1] *Let $v \in C_s^{2+\alpha}(\mathscr{O} \cup \Gamma_0) \cap C(\bar{\mathscr{O}})$ be a subsolution, $Lv \geq 0$ on $\mathscr{O}$ and $v \leq 0$ on $\mathbb{H} \cap \partial\mathscr{O}$, for a bounded domain $\mathscr{O} \subset \mathbb{H}$. Then $v \leq 0$ on $\mathscr{O}$.*

Theorem 3.9 (Strong maximum principle for the Heston operator)**.** [6, Theorem 4.3] *Let $v \in C_s^{2+\alpha}(\mathscr{O} \cup \Gamma_0) \cap C(\bar{\mathscr{O}})$ be a subsolution, $Lv \geq 0$ on $\mathscr{O}$, for a bounded, connected domain $\mathscr{O} \subset \mathbb{H}$. If v achieves its maximum value at a point $P \in \mathscr{O} \cup \Gamma_0$ and, in addition, $v(P) \geq 0$ if $r > 0$ (where r is the coefficient of L in* (1.9)*), then v must be a constant on $\bar{\mathscr{O}}$.*

We finish this section by showing how to reduce to the case $f = 0$ in Theorem 1.1.

Proposition 3.10 (Reduction to a homogeneous obstacle problem)**.** *We may assume, without loss of generality, that $f = 0$ on V in Theorem 1.1.*

Proof. Let $v \in C_s^{2+\alpha}(\bar{V})$ be the solution to the Dirichlet problem $Lv = f$ on V and $v = 0$ on $\mathbb{H} \cap \partial V$ (its existence follows from Theorem 3.4). It follows from (3.4) that

$$\|v\|_{C_s^{2+\alpha}(\bar{V})} \leq C\left(\|v\|_{C(\bar{V})} + \|f\|_{C^\alpha(\bar{V})}\right),$$

and hence (see the proof of (3.5))

$$\|v\|_{C_s^{2+\alpha}(\bar{V})} \leq C\|f\|_{C^\alpha(\bar{V})}, \tag{3.6}$$

where in (3.6) we use C to denote a constant which depends at most on α, R_0, Λ and the coefficients of L.

Now if u is a solution to the obstacle problem (1.1) on V as in Theorem 1.1, then $\bar{u} := u - v$ is a solution to the obstacle problem (1.1) on V with source function $\bar{f} = 0$ on V and obstacle $\bar{\psi} := \psi - v$ on V. If Theorem 1.1 is proved for $\bar{f} = 0$ in place of f on V, then $\bar{u} \in C_s^{1,1}(\bar{U})$ and the estimate (1.7) for $\bar{u}$ yields

$$\|\bar{u}\|_{C_s^{1,1}(\bar{U})} \leq C\left(\|\bar{u}\|_{C(\bar{V})} + \|\bar{\psi}\|_{C^{1,1}(\bar{V})}\right).$$

But $u = \bar{u} + v \in C_s^{1,1}(\bar{U})$ and we obtain the estimate (1.7) for u from the preceding inequality and the estimate (3.6) for v. $\square$

Because of the reduction in Proposition 3.10, we may assume without loss of generality that $u \in H^2(V, \mathfrak{w}) \cap C(\bar{V})$ is a solution to the obstacle problem (1.1) with obstacle function $\psi \in C_s^{2+\alpha}(\bar{V})$ and $f = 0$ on V, that is,

$$\min\{Lu, u - \psi\} = 0 \quad \text{a.e. on } V. \tag{3.7}$$

We make this assumption for the remainder of this article.

4. Supremum bounds

We will assume, throughout this section, that u is a solution to the obstacle problem (3.7) on V, where V is as in (1.5), and that all the assumptions of Theorem 1.1 hold. Adopting the terminology of mathematical finance, we call

$$\mathscr{C}(u) = \{P \in V \cup \Gamma_0 : u(P) > \psi(P)\} \tag{4.1}$$

the *continuation region* (or non-coincidence set),

$$\mathscr{E}(u) = \{P \in V \cup \Gamma_0 : u(P) = \psi(P)\} \tag{4.2}$$

the *exercise region* (or coincidence set), and

$$\mathscr{F}(u) = (V \cup \Gamma_0) \cap \partial\mathscr{C}(u) \tag{4.3}$$

the *free boundary* (or optimal exercise boundary, as it is known in mathematical finance). From (3.7) and (4.1), we see that

$$Lu \leq 0 \quad \text{a.e. on } V \quad \text{and} \quad Lu = 0 \quad \text{on } \mathscr{C}(u). \tag{4.4}$$

Since $Lu = 0$ on $\mathscr{C}(u)$, it follows from Proposition 3.5 that u is of class $C^{2+\alpha}_s$ on $\mathscr{C}(u)$. (Actually one may also easily see that u is of class C^∞ on $\mathscr{C}(u)$.)

We will establish sharp growth estimates from above on $u - \psi$ near free boundary points $P_0 \in \mathscr{F}(u)$. Because of the degeneracy of our operator L, we will need to scale our estimates in different ways, depending on the distance of P_0 from the boundary portion, $\bar{\mathscr{O}} \cap \partial\mathbb{H} = \{y = 0\}$. Similar estimates in the non-degenerate case, where L is the Laplace operator, Δ, were established by Caffarelli in [3].

The first such estimate, in Proposition 4.1, concerns with free boundary points $P_0 = (x_0, y_0) \in \mathscr{F}(u)$ with $y_0 > 0$. To simplify the notation we will assume that $0 < y_0 < 1$. The estimate near any free boundary point $P_0 = (x_0, y_0) \in \mathscr{F}(u)$ with $y_0 > 1$ can be shown similarly. We have the following analogue of [3, Lemma 2]; see also [19, Lemma 1.6] (where $L = \Delta$ and $\psi = 0$).

Proposition 4.1 (Quadratic growth of solution near free boundary and away from degenerate boundary)**.** *Let u be as in Theorem 1.1 and let $P_0 = (x_0, y_0) \in \mathscr{F}(u) \cap V$ with $0 < y_0 < 1$. Then there are constants $0 < \rho_0 < 1$ and $0 < C < \infty$, depending at most on the coefficients of L, such that if $B_{\rho_0 y_0}(P_0) \Subset V$, then*

$$\sup_{B_{\rho y_0/2}(P_0)} (u - \psi) \leq C y_0 \rho^2 \|\psi\|_{C^{1,1}(\bar{B}_{\rho_0 y_0}(P_0))}, \quad \forall \rho < \rho_0. \tag{4.5}$$

Remark 4.2. We shall establish (4.5) with the aid of certain auxiliary functions, ζ in (4.9) and w in (4.19), defined on balls $\bar{B}_{\rho y_0}(P_0)$.

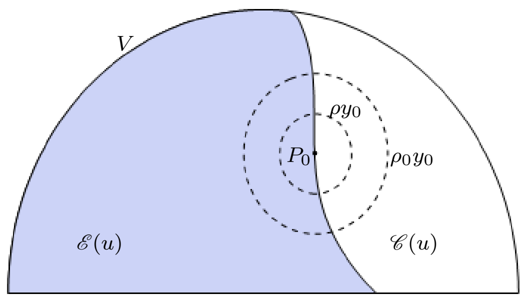

FIGURE 4.1. Regions in the proof of Proposition 4.1 for estimating the growth of a solution near the free boundary and away from the degenerate boundary.

We begin by observing that since $B_{\rho_0 y_0}(P_0) \Subset V$ by assumption, the operator L is uniformly elliptic on $B_{\rho_0 y_0}(P_0) \Subset \mathbb{H}$. Consider the linear approximation,

$$l_{P_0}(x,y) := \psi(P_0) + D\psi(P_0)\cdot(x-x_0, y-y_0), \quad (x,y)\in\mathbb{R}^2, \tag{4.6}$$

to our obstacle function ψ at P_0. A direct calculation shows that

$$|L(l_{P_0})| \leq M \quad \text{on } B_{\rho_0 y_0}(P_0), \tag{4.7}$$

where, noting that $0<\rho_0<1$ and $0<y_0<1$ as in the hypotheses of Proposition 4.1 and that $\|\psi\|_{C^1(\bar{B}_{\rho_0 y_0}(P_0))} \leq \|\psi\|_{C^{1,1}(\bar{B}_{\rho_0 y_0}(P_0))}$,

$$M := K\|\psi\|_{C^{1,1}(\bar{B}_{\rho_0 y_0}(P_0))}, \tag{4.8}$$

and the constant $K>0$ depends at most on the coefficients of L. For $0<\rho<\rho_0$, let $\zeta \in C^{2+\alpha}(\bar{B}_{\rho y_0}(P_0))$ be the unique solution (assured by [12, Theorem 6.14]) to the elliptic boundary value problem,

$$\begin{cases} L\zeta = L(l_{P_0}) & \text{on } B_{\rho y_0}(P_0),\\ \zeta = 10My_0\rho^2 & \text{on } \partial B_{\rho y_0}(P_0).\end{cases} \tag{4.9}$$

The next lemma provides sharp bounds from above and below on ζ in terms of ρ and the constant M in (4.9).

Lemma 4.3 (Quadratic growth of an auxiliary function near free boundary and away from degenerate boundary)**.** *The function $\zeta \in C^{2+\alpha}(\bar{B}_{\rho y_0}(P_0))$ in (4.9) satisfies the bound,*

$$My_0\rho^2 \leq \zeta \leq 14My_0\rho^2 \quad \text{on } B_{\rho y_0}(P_0), \quad 0<\rho<\rho_0, \tag{4.10}$$

where M is as in (4.8) and $\rho_0<1$ is a constant depending at most on the coefficients of L.

Before proceeding to the proof of Lemma 4.3, we consider the effect of rescaling on the operator L. Observe that, for any $u \in C^2(\mathbb{H})$, if

$$v(x,y) := u(x_0 + y_0 x, y_0 + y_0 y), \qquad (\bar{x}, \bar{y}) := (x_0 + y_0 x, y_0 + y_0 y),$$

then

$$\begin{aligned}(Lu)(\bar{x},\bar{y}) = \frac{y_0 + y_0 y}{2} y_0^{-2} \left(v_{xx} + 2\varrho\sigma v_{xy} + \sigma^2 v_{yy}\right)(x,y) + \left(r - q - \frac{y_0(1+y)}{2}\right) y_0^{-1} v_x(x,y) \\ + \kappa\left(\theta - y_0(1+y)\right) v_y(x,y) - rv(x,y),\end{aligned}$$

and therefore,

$$y_0 (Lu)(\bar{x},\bar{y}) = (L_{y_0} v)(x,y), \quad \forall (x,y) \in \mathbb{H}, \tag{4.11}$$

where

$$\begin{aligned}(L_{y_0} v)(x,y) := \frac{1+y}{2} \left(v_{xx} + 2\varrho\sigma v_{xy} + \sigma^2 v_{yy}\right)(x,y) + \left(r - q - \frac{y_0(1+y)}{2}\right) v_x(x,y) \\ + \kappa\left(\theta - y_0(1+y)\right) v_y(x,y) - r y_0 v(x,y), \quad \forall (x,y) \in \mathbb{H}.\end{aligned} \tag{4.12}$$

We now proceed to the

Proof of Lemma 4.3. Since the ellipticity constant for L depends on y_0, we shall use the rescaling in (4.11). Note that the operator L_{y_0} is uniformly elliptic on $B_{1/2}$, since

$$\frac{1}{4} < \frac{1+y}{2} < \frac{3}{4} \quad \text{on} \quad B_{1/2}, \tag{4.13}$$

and the coefficients of L_{y_0} are bounded by a constant (recall that $y_0 < 1$) depending at most on the coefficients of L. Let

$$\bar{\zeta}(x,y) := \frac{1}{M y_0} \zeta(x_0 + y_0 x, y_0 + y_0 y), \quad \forall (x,y) \in B_\rho, \tag{4.14}$$

with ζ as in (4.9). It follows from (4.7) and (4.9) that $\bar{\zeta}$ satisfies

$$\begin{aligned}|L_{y_0}\bar{\zeta}| \le 1 \quad \text{on } B_\rho, \\ \bar{\zeta} = 10\rho^2 \quad \text{on } \partial B_\rho,\end{aligned} \tag{4.15}$$

since (4.11) yields

$$(L_{y_0}\bar{\zeta})(x,y) = \frac{1}{M y_0} y_0 (L\zeta)(\bar{x},\bar{y}) = \frac{1}{M}(L\zeta)(\bar{x},\bar{y}).$$

We will show that

$$\rho^2 \le \bar{\zeta} \le 14\rho^2 \quad \text{on } B_\rho, \tag{4.16}$$

provided $\rho < \rho_0$, with $\rho_0 < 1$ a constant depending at most on the coefficients of L, and this will conclude the proof of the lemma, since (4.10) follows from (4.14) and (4.16).

To this end, we consider the barrier function,

$$\vartheta(x,y) := a x^2, \quad \forall (x,y) \in \mathbb{R}^2, \tag{4.17}$$

for different choices of constants $a \in \mathbb{R}$ and compute that

$$(L_{y_0}\vartheta)(x,y) = a\left[(1+y) + 2\left(r - q - \frac{y_0(1+y)}{2}\right)x - ry_0x^2\right].$$

Since $1/2 < 1+y < 3/2$ on $B_{1/2}$, by choosing $\rho < \rho_0$, with $\rho_0 < 1$ a constant depending at most on the coefficients r, q of L, and using $(x,y) \in B_\rho$ and recalling that $0 < y_0 < 1$, we can ensure that

$$L_{y_0}\vartheta < \frac{1}{4}a \quad \text{on } B_\rho, \quad \text{if } a < 0 \quad \text{and} \quad L_{y_0}\vartheta > \frac{1}{4}a \quad \text{on } B_\rho, \quad \text{if } a > 0. \tag{4.18}$$

Choose $a = -8$ and set $w := \bar\zeta + \vartheta - \rho^2$. By combining (4.15) and (4.18) and using the definition (4.12) of L_{y_0}, we obtain

$$L_{y_0}w \leq 1 - 2 + r\rho^2 y_0 < 0 \quad \text{on } B_\rho,$$

if $\rho^2 < 1/r$ (remember that $y_0 < 1$). On the other hand, since $\bar\zeta = 10\rho^2$ on ∂B_ρ by (4.15) and using (4.17), we see that

$$\begin{aligned} w &= \bar\zeta + \vartheta - \rho^2 \\ &\geq 10\rho^2 - 8\rho^2 - \rho^2 > 0 \quad \text{on } \partial B_\rho. \end{aligned}$$

Therefore, the weak maximum principle for L_{y_0} on B_ρ implies that

$$\bar\zeta + \vartheta - \rho^2 \geq 0 \quad \text{on } B_\rho.$$

Since $\vartheta = -8x^2 \leq 0$ on $\mathbb{R}^2$, we conclude that $\bar\zeta \geq \rho^2 - \vartheta \geq \rho^2$ in B_ρ.

We will now estimate $\bar\zeta$ from above. This time we take $a = 4$ and setting $z := \bar\zeta + \vartheta$, we now find from (4.15) and (4.18) that

$$L_{y_0}z > -1 + \frac{1}{4}4 = 0 \quad \text{on } B_\rho.$$

But (4.15) and the definition (4.17) give $\vartheta = 4x^2 \leq 4\rho^2$ on B_ρ and

$$z = \bar\zeta + \vartheta \leq 10\rho^2 + 4\rho^2 = 14\rho^2 \quad \text{on } \partial B_\rho,$$

and so the weak maximum principle for L_{y_0} on B_ρ shows that $\bar\zeta \leq 14\rho^2$ on B_ρ. This finishes the proof of (4.16), and hence concludes the proof of our lemma. □

Proof of Proposition 4.1. We shall follow the proof of Lemma 2 in [3]. Our case is more difficult since linear functions are not solutions to the equation $Lu = 0$. In addition, our operator L has variable coefficients and our scaling depends on the ellipticity constant of the operator L on V, which is comparable to y_0.

With l_{P_0} given by (4.6) and $0 < \rho < \rho_0$ and $\zeta \in C^{2+\alpha}(\bar B_{\rho y_0}(P_0))$ the function defined by (4.9), we set

$$w := u - l_{P_0} + \zeta \in H^2(B_{\rho y_0}(P_0)) \cap C(\bar B_{\rho y_0}(P_0)) \tag{4.19}$$

and observe that

$$w = (u - \psi) + (\psi - l_{P_0} + \zeta) \geq \psi - l_{P_0} + \zeta \quad \text{on } \bar B_{\rho y_0}(P_0),$$

since $u \geq \psi$ on $\bar{B}_{\rho y_0}(P_0)$. By Taylor's theorem,

$$|\psi(x,y) - l_{P_0}(x,y)| \leq 2y_0^2\rho^2\|\psi\|_{C^{1,1}(\bar{B}_{\rho y_0}(P_0))}, \quad \forall (x,y) \in \bar{B}_{\rho y_0}(P_0). \tag{4.20}$$

Since $\zeta \geq My_0\rho^2$ by (4.10) and $y_0 < 1$ (by hypothesis in Proposition 4.1), we conclude that

$$\begin{aligned} w &\geq -2y_0^2\rho^2\|\psi\|_{C^{1,1}(\bar{B}_{\rho y_0}(P_0))} + My_0\rho^2 \\ &\geq -2y_0^2\rho^2\|\psi\|_{C^{1,1}(\bar{B}_{\rho_0 y_0}(P_0))} + My_0\rho^2 \quad (\text{since } \rho < \rho_0) \\ &> 0 \quad \text{on } \bar{B}_{\rho y_0}(P_0) \quad (\text{by definition of } M), \end{aligned}$$

provided that the constant K in the definition (4.8) of M is chosen large enough that $K > 2$. Also, since $L\zeta = L(l_{P_0})$ by (4.9), we have

$$Lw = Lu \leq 0 \quad \text{a.e. on } B_{\rho y_0}(P_0),$$

where the inequality follows from (4.4). Let us now split w as

$$w = w_1 + w_2, \tag{4.21}$$

where $w_1 \in C^{2+\alpha}(B_{\rho y_0}(P_0)) \cap C(\bar{B}_{\rho y_0}(P_0))$ is the unique solution (assured by [12, Theorem 6.13]) to

$$\begin{cases} Lw_1 = 0 & \text{on } B_{\rho y_0}(P_0), \\ w_1 = w & \text{on } \partial B_{\rho y_0}(P_0). \end{cases} \tag{4.22}$$

(Note that $w = u - l_{P_0} + \zeta$ belongs to $C(\partial B_{\rho y_0}(P_0))$.) By the weak maximum principle, we have

$$0 \leq w_1 \leq w \quad \text{on } \bar{B}_{\rho y_0}(P_0), \tag{4.23}$$

(since $w > 0$ on $\bar{B}_{\rho y_0}(P_0)$), and hence

$$0 \leq w_2 \leq w \quad \text{on } \bar{B}_{\rho y_0}(P_0). \tag{4.24}$$

The inequality (4.23) obeyed by w_1 and the definition (4.19) of w yield,

$$w_1(P_0) \leq w(P_0) = \zeta(P_0),$$

and thus, by (4.10),

$$w_1(P_0) \leq 14My_0\rho^2. \tag{4.25}$$

Consider the rescaled solution,

$$\bar{w}_1(x,y) := w_1(x_0 + y_0x, y_0 + y_0y), \quad \forall (x,y) \in \bar{B}_\rho, \tag{4.26}$$

and observe that the function $w_1 \in C^{2+\alpha}(B_\rho) \cap C(\bar{B}_\rho)$, by (4.11) and (4.22), satisfies the uniformly elliptic equation,

$$L_{y_0}\bar{w}_1 = 0 \quad \text{on } B_\rho.$$

The Harnack inequality [12, Corollary 9.25 & Equation (9.47)], the definition (4.26) of $\bar{w}_1$, and the inequality (4.25) imply the estimate,

$$\sup_{B_{\rho/2}} \bar{w}_1 \leq C' \inf_{B_{\rho/2}} \bar{w}_1 \leq C'\bar{w}_1(0) = C'w_1(P_0) \leq CMy_0\rho^2,$$

for constants C' and $C = 14C'$ which depend at most on the coefficients of L, but are independent of y_0, and the constant M is given by (4.8). Hence, by (4.26),

$$\sup_{B_{\rho y_0/2}(P_0)} w_1 \leq CMy_0\rho^2, \quad 0 < \rho < \rho_0. \tag{4.27}$$

We will next bound w_2 on $B_{\rho y_0}(P_0)$, taking care to note that (like the regularity of u in Theorem 1.1) w_2 only belongs to $H^2(B_{\rho y_0}(P_0)) \cap C(\bar{B}_{\rho y_0}(P_0))$. Recall that $0 \leq w_2 \leq w$ on $B_{\rho y_0}(P_0)$ by (4.24) and that $w_2 = 0$ on $\partial B_{\rho y_0}(P_0)$ by (4.21) and (4.22). Assume that $P_1 = (x_1, y_1)$ is a maximum point for the function w_2 on the closure of the ball $B_{\rho y_0}(P_0)$ and that $w_2(P_1) > 0$. Then, $P_1 \in B_{\rho y_0}(P_0)$ and we consider two cases.

Case 1 ($P_1 \in \mathscr{E}(u)$)**.** If $P_1 \in \mathscr{E}(u)$ (where $u = \psi$), then $u(P_1) = \psi(P_1)$ and hence, by the inequalities (4.10), (4.20), (4.24), and definition (4.19) of w, we have

$$w_2(P_1) \leq w(P_1) = \psi(P_1) - l_{P_0}(P_1) + \zeta(P_1) \leq 16My_0\rho^2,$$

provided the constant K in the definition (4.8) of M is chosen large enough that $K > 2$.

Case 2 ($P_1 \in \mathscr{C}(u)$)**.** If $P_1 \in \mathscr{C}(u)$ (where $u > \psi$) then, since $Lw_2 = 0$ on the open set $\mathscr{C}(u) \cap B_{\rho y_0}(P_0)$ and w_2 achieves an interior maximum there, the strong maximum principle [12, Theorem 3.5] implies that w_2 must be constant on the connected component of $\mathscr{C}(u) \cap B_{\rho y_0}(P_0)$ containing P_1. Since $w_2 = 0$ on $\partial B_{\rho y_0}(P_0)$ and $w_2(P_1) > 0$ by assumption, it follows that $w_2(P_1) = w_2(P_2)$ for some point $P_2 \in \mathscr{E}(u) \cap B_{\rho y_0}(P_0)$. (Recall that, by hypothesis, $P_0 \in \mathscr{F}(u)$ and so $\mathscr{E}(u) \cap B_{\rho y_0}(P_0)$ is non-empty.) Thus, by the inequalities (4.10), (4.20), (4.24), and definition (4.19) of w, we have

$$w_2(P_1) = w_2(P_2) \leq w(P_2) = \psi(P_2) - l_{P_0}(P_2) + \zeta(P_2) \leq 16My_0\rho^2,$$

provided the constant K in the definition (4.8) of M is chosen large enough that $K > 2$.

By combining the two cases and recalling that $w_2 \leq w_2(P_1)$ on $B_{\rho y_0}(P_0)$, we obtain

$$\sup_{B_{\rho y_0/2}(P_0)} w_2 \leq 16My_0\rho^2, \quad 0 < \rho < \rho_0. \tag{4.28}$$

By combining the supremum bounds (4.27) and (4.28) for w_1 and w_2, respectively, we obtain

$$w \leq CMy_0\rho^2 \quad \text{on } B_{\rho y_0/2}(P_0), \quad 0 < \rho < \rho_0, \tag{4.29}$$

where C depends at most on the coefficients of L, and M is given by (4.8). This shows, in particular, by (4.10) and (4.19), that

$$u - l_{P_0} \leq CMy_0\rho^2 \quad \text{on } B_{\rho y_0/2}(P_0),$$

where C depends at most on the coefficients of L. Now, again using (4.20), we have

$$u - \psi = u - l_{P_0} + l_{P_0} - \psi \leq CMy_0\rho^2 \quad \text{on } B_{\rho y_0/2}(P_0), \quad 0 < \rho < \rho_0,$$

for a possibly larger constant C that depends at most on the coefficients of L, and this gives the desired bound (4.5). □

We will next establish a supremum bound for the solution, u, which holds near $y = 0$ and is independent of the y_0 coordinate of the point P_0.

Proposition 4.4 (Linear growth of solution near free and degenerate boundaries)**.** *Let u be as in Theorem 1.1 and let $P_0 = (x_0, y_0) \in \mathscr{F}(u) \cap V$ with $0 \leq y_0 < \theta/4$, where $\theta > 0$ is a coefficient of L in* (1.9)*. Then, there are a constant $0 < \rho_0 < 1$ and a constant $0 < C < \infty$, depending at most on the coefficients of L, such that if $B^+_{\rho_0}(P_0) \Subset V \cup \Gamma_0$, then*

$$\sup_{B^+_{\rho/2}(P_0)} (u - \psi) \leq C\rho \|\psi\|_{C^{1,1}(\bar{B}^+_{\rho_0}(P_0))}, \quad 0 < \rho < \rho_0. \tag{4.30}$$

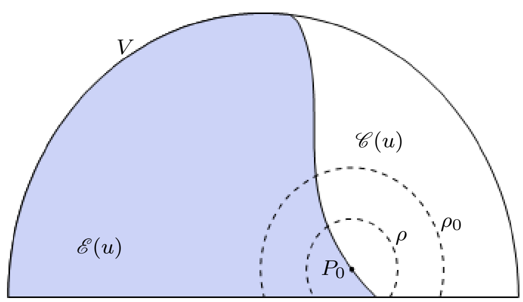

FIGURE 4.2. Regions in the proof of Proposition 4.4 for estimating the growth of a solution near the free boundary and near the degenerate boundary.

Our proof of Proposition 4.4 follows the pattern of the proof of Proposition 4.1. However, we shall use a different scaling. Observe that

$$|L\psi| \leq N\kappa\theta \quad \text{on } B^+_{\rho_0}(P_0), \tag{4.31}$$

where $\kappa > 0, \theta > 0$ are coefficients of L in (1.9) and

$$N := K' \|\psi\|_{C^{1,1}(\bar{B}^+_{\rho_0}(P_0))}, \tag{4.32}$$

where K' is a constant which depends at most on the coefficients of L in (1.9) (remember that $0 \leq y_0 < \theta/4$ and $0 < \rho_0 < 1$).

For $0 < \rho < \rho_0$, let $\xi \in C^{2+\alpha}_s(B^+_\rho(P_0)) \cap C(\bar{B}^+_\rho(P_0))$ be the solution to the boundary value problem,

$$\begin{cases} L\xi = L\psi & \text{on } B^+_\rho(P_0), \\ \xi = 10N\rho & \text{on } \mathbb{H} \cap \partial B_\rho(P_0), \end{cases} \tag{4.33}$$

provided by Theorem 3.2 or 3.4.

Lemma 4.5 (Linear growth of an auxiliary function near free and degenerate boundaries). *The function ξ given by* (4.33) *satisfies the bound,*

$$N\rho \leq \xi \leq 20N\rho \quad \text{on } B_\rho^+(P_0), \quad 0 < \rho < \rho_0, \tag{4.34}$$

where N is as in (4.31) *and $\rho_0 < 1$ is a constant depending at most on the coefficients of L.*

Proof. We first establish the bound from above. We set

$$z := \xi + 2N(y - y_0 + \rho) - 20N\rho \in C_s^{2+\alpha}(B_\rho^+(P_0)) \cap C(\bar{B}_\rho^+(P_0)),$$

and use (1.9) to compute that

$$\begin{aligned} Lz &= L\xi + 2N\kappa(\theta - y) - 2Nr(y - y_0 + \rho) + 20Nr\rho \\ &\geq -N\kappa\theta + 2N\kappa(\theta - y) - 2Nr(y - y_0 + \rho) + 20Nr\rho \quad \text{(by (4.31) and (4.33))} \\ &\geq -N\kappa\theta + N\kappa\theta + 16Nr\rho \\ &\geq 0 \quad \text{on } B_\rho^+(P_0), \end{aligned}$$

if $0 \leq y_0 \leq \theta/4$ and $\rho \leq \rho_0$ with $\rho_0 \leq \min\{\theta/4, 1\}$ and noting that $0 \leq y < y_0 + \rho \leq \theta/2$. On the other hand, since $\xi = 10N\rho$ on $\mathbb{H} \cap \partial B_\rho^+(P_0)$ by (4.33), we have

$$\begin{aligned} z &= \xi + 2N(y - y_0 + \rho) - 20N\rho \\ &\leq 10N\rho + 4N\rho - 20N\rho \\ &\leq 0 \quad \text{on } \mathbb{H} \cap \partial B_\rho^+(P_0). \end{aligned}$$

Hence, the weak maximum principle for L on $B_\rho^+(P_0)$ (Theorem 3.8), implies that $z \leq 0$ on $B_\rho^+(P_0)$, which implies the desired upper bound in (4.34),

$$\xi = z - 2N(y - y_0 + \rho) + 20N\rho \leq 20N\rho \quad \text{on } B_\rho^+(P_0),$$

since $y - y_0 + \rho \geq -\rho + \rho = 0$ on $B_\rho^+(P_0)$.

For the bound from below, we now set

$$z := \xi - 4N(y - y_0 + \rho) - N\rho,$$

and use (1.9) to compute that

$$\begin{aligned} Lz &= L\xi - 4N\kappa(\theta - y) + 4Nr(y - y_0 + \rho) + Nr\rho \\ &\leq N\kappa\theta - 4N\kappa(\theta - y) + 4Nr(y - y_0 + \rho) + Nr\rho \quad \text{(by (4.31) and (4.33))} \\ &\leq N\kappa\theta - 2N\kappa\theta + 9Nr\rho \\ &\leq N\kappa\theta - 2N\kappa\theta + N\kappa\theta \\ &\leq 0 \quad \text{on } B_\rho^+(P_0), \end{aligned}$$

if $0 \leq y_0 \leq \theta/4$ and $\rho \leq \rho_0$ with $\rho_0 \leq \min\{\theta/4, \kappa\theta/(9r), 1\}$ and noting that $0 \leq y < y_0 + \rho \leq \theta/2$. On the other hand, since $\xi = 10N\rho$ on $\mathbb{H} \cap \partial B^+_\rho(P_0)$ by (4.33), we have

$$\begin{aligned} z &= \xi - 4N(y - y_0 + \rho) - N\rho \\ &\geq 10N\rho - 8N\rho - N\rho \\ &\geq 0 \quad \text{on } \mathbb{H} \cap \partial B^+_\rho(P_0). \end{aligned}$$

The weak maximum principle for L on $B^+_\rho(P_0)$ (Theorem 3.8) once more shows that $z \geq 0$ on $B^+_\rho(P_0)$. We conclude that

$$\xi = z + 4N(y - y_0 + \rho) + N\rho \geq N\rho \quad \text{on } B^+_\rho(P_0),$$

provided that $\rho < \rho_0$, with $\rho_0 < 1$ depending at most on the coefficients of L. This yields the desired upper bound in (4.34) and finishes the proof of the lemma. □

We will now give the proof of Proposition 4.4.

Proof of Proposition 4.4. We give an argument which is similar to the one used in the proof of Proposition 4.1 but we scale our estimate differently and use Lemma 4.5 instead of Lemma 4.3. We set

$$w := u - \psi + \xi \in H^2(B^+_\rho(P_0), \mathfrak{w}) \cap C(\bar{B}^+_\rho(P_0)), \tag{4.35}$$

with $\xi \in C^{2+\alpha}_s(B^+_\rho(P_0)) \cap C(\bar{B}^+_\rho(P_0))$ given by (4.33). Then w satisfies

$$Lw = Lu \quad \text{a.e. on } B^+_\rho(P_0). \tag{4.36}$$

Let us now split w as $w = w_1 + w_2$, where $w_1 \in C^{2+\alpha}_s(B^+_\rho(P_0)) \cap C(\bar{B}^+_\rho(P_0))$ (whose existence is assured by Theorem 3.2) is defined by

$$\begin{cases} Lw_1 = 0 & \text{on } B^+_\rho(P_0), \\ w_1 = w & \text{on } \mathbb{H} \cap \partial B^+_\rho(P_0). \end{cases} \tag{4.37}$$

By the weak maximum principle for L on $B^+_\rho(P_0)$ (Theorem 3.8) and the fact that $w \geq \xi \geq 0$ on $B^+_\rho(P_0)$, we have

$$0 \leq w_1 \leq w \quad \text{on } B^+_\rho(P_0), \tag{4.38}$$

and thus

$$0 \leq w_2 \leq w \quad \text{on } B^+_\rho(P_0). \tag{4.39}$$

From (4.34), (4.35), (4.38), and the fact that $u(P_0) = \psi(P_0)$, we see that

$$w_1(P_0) \leq w(P_0) = \xi(P_0) \leq 20N\rho \quad \text{on } B^+_\rho(P_0). \tag{4.40}$$

Set $(x_\rho, y_\rho) := (x_0 + \rho y, y_0 + \rho y)$ and consider the rescaled solution,

$$\bar{w}_1(x, y) := w_1(x_0 + \rho x, y_0 + \rho y), \quad (x, y) \in B_1 \cap \{y_\rho \geq 0\},$$

which satisfies the equation,

$$L_\rho \bar{w}_1 = 0 \quad \text{on } B_1 \cap \{y_\rho > 0\}, \tag{4.41}$$

where (compare (4.12))

$$\begin{aligned} L_\rho v := \frac{y_\rho}{2\rho}\left(v_{xx} + 2\varrho\sigma v_{xy} + \sigma^2 v_{yy}\right) + (r - q - y_\rho)v_x \\ + \kappa(\theta - y_\rho)v_y - r\rho v, \quad \forall v \in C^\infty(\mathbb{H}), \end{aligned} \tag{4.42}$$

and using the fact that (compare (4.11))

$$\rho(Lw_1)(x_\rho, y_\rho) = (L_\rho \bar{w}_1)(x, y).$$

From (4.41), the Harnack inequality[4] [7, Theorem 1.16] yields the estimate

$$\sup_{B_{1/2}(0)\cap\{y_\rho>0\}} \bar{w}_1 \leq C \inf_{B_{1/2}(0)\cap\{y_\rho>0\}} \bar{w}_1 \leq C\bar{w}(0),$$

for a constant, C, depending at most on the coefficients of L. Combining the preceding inequality with (4.40) yields

$$\sup_{B^+_{\rho/2}(P_0)} w_1 \leq Cw(P_0) \leq 20CN\rho,$$

that is,

$$\sup_{B^+_{\rho/2}(P_0)} w_1 \leq 20CN\rho, \quad 0 < \rho < \rho_0, \tag{4.43}$$

for a constant, ρ_0, depending at most on the coefficients of L.

We will next bound w_2 on $B^+_\rho(P_0)$, following the same reasoning as in the proof of Proposition 4.1. Recall that $0 \leq w_2 \leq w$ on $B^+_\rho(P_0)$ by (4.39) and $w_2 = 0$ on $\mathbb{H}\cap\partial B^+_\rho(P_0)$ by (4.37). Assume that $P_1 = (x_1, y_1)$ is a maximum point for the function w_2 on $\bar{B}^+_\rho(P_0)$ and that $w_2(P_1) > 0$. Therefore, $P_1 \in B^+_\rho(P_0) \cup \Gamma_0$, where (by our convention) $\Gamma_0 = \{y = 0\} \cap \partial B^+_\rho(P_0)$.

Case 1 **($P_1 \in \mathscr{E}(u)$).** If $P_1 \in \mathscr{E}(u) \cap (B^+_\rho(P_0) \cup \Gamma_0)$, then $u(P_1) = \psi(P_1)$. Recalling that $w = u - \psi + \xi$ by (4.35), we conclude from (4.34) and (4.39) that at P_1 we have the bound

$$w_2(P_1) \leq w(P_1) = \xi(P_1) \leq 20N\rho,$$

provided $\rho < \rho_0$.

Case 2 **($P_1 \in \mathscr{C}(u)$).** If $P_1 \in \mathscr{C}(u)\cap(B^+_\rho(P_0)\cup\Gamma_0)$ (where $u > \psi$), then since $w_2 \in H^2(B^+_\rho(P_0), \mathfrak{w})\cap C(\bar{B}^+_\rho(P_0))$ obeys

$$Lw_2 = 0 \quad \text{a.e. on } \mathscr{C}(u) \cap B^+_\rho(P_0),$$

by (4.4), (4.36), and (4.37), the regularity result in Proposition 3.5 implies that w_2 also belongs to $C^{2+\alpha}_s(\mathscr{C}(u)\cap(B^+_\rho(P_0)\cup\Gamma_0))$. But w_2 achieves a positive maximum at $P_1 \in \mathscr{C}(u)\cap(B^+_\rho(P_0)\cup\Gamma_0)$ and so the strong maximum principle (Theorem 3.9) implies that w_2 must be constant on the connected component of $\mathscr{C}(u)\cap(B^+_\rho(P_0)\cup\Gamma_0)$ containing P_1. Since $w_2 = 0$ on $\mathbb{H}\cap\partial B^+_\rho(P_0)$ and $w_2(P_1) > 0$, it follows that $w_2(P_1) = w_2(P_2)$, for some point P_2 with $P_2 \in \mathscr{E}(u)\cap(B^+_\rho(P_0)\cup\Gamma_0)$.

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[4] See also [16, Theorem 4.5.3] for a version of the Harnack inequality for the linearization of the parabolic porous medium equation.

(Recall that, by hypothesis, $P_0 \in \mathscr{F}(u)$ and so $\mathscr{E}(u) \cap (B^+_\rho(P_0) \cup \Gamma_0)$ is non-empty.) We conclude that by (4.34), (4.35), (4.39), and the fact that $u(P_2) = \psi(P_2)$,

$$w_2(P_1) = w_2(P_2) \le w(P_2) = \xi(P_2) \le 20N\rho,$$

provided $\rho < \rho_0$.

Combining the two cases and recalling that $w_2 \le w_2(P_1)$ on $\bar{B}^+_\rho(P_0)$, by definition of P_1, yields

$$\sup_{B^+_\rho(P_0)} w_2 \le 20N\rho, \quad 0 < \rho < \rho_0. \tag{4.44}$$

Combining the estimates (4.43) and (4.44), respectively, for w_1 and w_2 yields

$$\sup_{B^+_{\rho/2}(P_0)} w \le C'N\rho, \quad 0 < \rho < \rho_0,$$

for a constant C' which depends at most on the coefficients of L, and N is as in (4.32). This yields the desired bound (4.30). □

Corollary 4.6 (Linear growth of solution near free and degenerate boundaries)**.** *Under the hypotheses of Proposition 4.4, there are a constant* $0 < \rho_0 < 1$ *and a constant* $0 < C < \infty$*, depending at most on the coefficients of* L*, such that*

$$\sup_{B^+_{\rho/2}(P_0)} (u - \psi(P_0)) \le C\rho\|\psi\|_{C^{1,1}(\bar{B}^+_{\rho_0}(P_0))}, \quad 0 < \rho < \rho_0. \tag{4.45}$$

5. Proof of main theorem

We will establish in this section the $C^{1,1}_s$ regularity of our solution, u, in Theorem 1.1. For a much simpler example — interior $C^{1,1}$ regularity for a solution, u, to $\min\{\Delta u - 1, u\} = 0$ on a bounded domain in $\mathbb{R}^n$ — but one which conveys some of the flavor of our proof of Theorem 1.1, see the proof of Theorem 1.1 in [19, p. 11], which is based in turn on ideas of Caffarelli [3].

Proof of Theorem 1.1. Because of Proposition 3.10 we may assume without loss of generality that $u \in H^2(V, \mathfrak{w}) \cap C(\bar{V})$ is a solution to the *homogeneous* obstacle problem (3.7) with obstacle function $\psi \in C^{2+\alpha}(\bar{V})$ and $f = 0$ on V. Recall that $V = B^+_{R_0}(Q_0)$ is as in (1.5), for some $R_0 > 0$ and $Q_0 = (p_0, q_0) \in \bar{\mathbb{H}}$. We may also assume without loss of generality that

$$0 < R_0 \le 1,$$

and also that

$$\Lambda = 1 \quad \text{and} \quad 0 \le q_0 \le 1,$$

since L is uniformly elliptic on $V = B_{R_0}(Q_0)$ when $q_0 > 1$ and standard results imply that $u \in C^{1,1}(\bar{V})$ [23, Theorem 4.38].

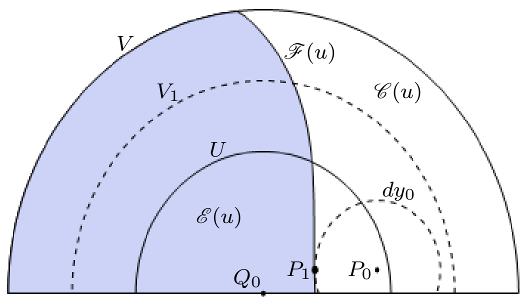

FIGURE 5.1. Regions in the proof of Theorem 1.1 for estimating the $C_s^{1,1}$ norm of a solution.

Let $P_0 = (x_0, y_0) \in \mathscr{C}(u) \cap \bar{U}$, where $U = B^+_{R_0/2}(Q_0)$ as in (1.5) with $R = R_0/2$; see Figure 5.1. Assuming without loss of generality that[5]

$$0 < y_0 \leq 1, \tag{5.1}$$

we will establish the bound

$$y_0|D^2u(P_0)| + |Du(P_0)| + |u(P_0)| \leq C\left(\|u\|_{C(\bar{V})} + \|\psi\|_{C^{1,1}(\bar{V})}\right), \tag{5.2}$$

where the constant $C = C(L, R_0)$ may depend R_0 and the coefficients of L. Since the constant C will not depend on y_0 (if y_0 obeys (5.1)) this will provide the desired $C_s^{1,1}$ bound on u up to $y = 0$. Set

$$V_1 := B^+_{3R_0/4}(Q_0).$$

Since u is continuous on $\bar{V}$, the exercise region, $\mathscr{E}(u)$, as defined in (4.2), is a relatively closed subset of $V \cup \Gamma_0$. We may suppose without loss of generality that

$$\mathscr{E}(u) \cap V_1 \neq \varnothing.$$

Otherwise, $V_1 \subset \mathscr{C}(u)$, where $\mathscr{C}(u)$ is the continuation region, as defined in (4.1), and because $Lu = 0$ on $\mathscr{C}(u)$, Theorem 1.1 would follow immediately from Proposition 3.5. Now let d be the maximum number such that

$$B_{dy_0}(P_0) \cap \mathscr{E}(u) \cap \bar{V}_1 = \varnothing. \tag{5.3}$$

Then there exists at least one point[6]

$$P_1 = (x_1, y_1) \in \partial B_{dy_0}(P_0) \cap \mathscr{F}(u) \cap \bar{V}_1. \tag{5.4}$$

[5]Our assumptions so far that $P_0 \in \bar{U}$ and $R_0 = 1$ ensure $0 \leq y_0 \leq 3/2$, but standard results apply when $y_0 \geq 1$.

[6]We alert the reader that in section 4 we use P_0 to denote a point in $\mathscr{F}(u)$ whereas in this section we use P_0 to denote a point in $\mathscr{C}(u)$ and P_1 to denote a point in $\mathscr{F}(u)$.

Since $P_0 \in \bar{U}$ and $P_1 \in \bar{V}_1$, we have $0 < dy_0 \leq 5R_0/4$.

Throughout this section, we let $0 < \rho_0 < 1$ denote the smaller of the two constants in Propositions 4.1 and 4.4 and, by replacing ρ_0 with a smaller constant if needed, we may assume that

$$0 < \rho_0 < \min\{1, R_0/5\}. \tag{5.5}$$

We shall distinguish between three situations. We begin with the first situation.

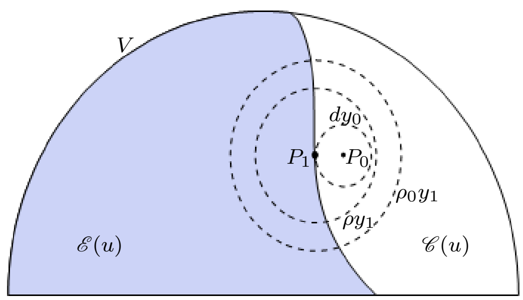

FIGURE 5.2. Regions for Step 1 (where $d \leq \rho_0/4$) of the proof of Theorem 1.1 for estimating the $C^{1,1}_s$ norm of a solution near the free boundary and away from the degenerate boundary.

Step 1 ($d \leq \rho_0/4$)**.** Since $\rho_0 < 1$, we have $B_{\rho_0 y_1}(P_1) \Subset \mathbb{H}$ (unless $y_1 = 0$), while $P_1 \in \bar{V}_1$ implies $\operatorname{dist}(P_1, \mathbb{H} \cap \partial V) \geq R_0/4$; since we also have $\rho_0 < R_0/5$ by (5.5), we may conclude that

$$B_{\rho_0 y_1}(P_1) \Subset V \quad \text{if } 0 < y_1 < \frac{5}{4}. \tag{5.6}$$

Since $P_1 = (x_1, y_1) \in \partial B_{dy_0}(P_0)$ and $P_0 = (x_0, y_0)$, we have $|y_1 - y_0| \leq y_0 d < y_0/4$ and thus

$$3y_0/4 < y_1 < 5y_0/4, \tag{5.7}$$

and hence y_0 and y_1 are comparable. In particular, we have $0 < y_1 < 5/4$ by (5.1), and hence by (5.6) (see Figure 5.2), we see that

$$B_{\rho_0 y_1}(P_1) \Subset V.$$

Set $\rho := 4d$ and let $\zeta \in C^{2+\alpha}(\bar{B}_{\rho y_1}(P_1))$ be the function defined by (4.9) (with P_0 and y_0 replaced by P_1 and y_1, respectively), that is,

$$\begin{cases} L\zeta = L(l_{P_1}) & \text{on } B_{\rho y_1}(P_1), \\ \zeta = 10 M y_1 \rho^2 & \text{on } \partial B_{\rho y_1}(P_1), \end{cases} \tag{5.8}$$

where (compare (4.8))

$$M := K\|\psi\|_{C^{1,1}(\bar{B}_{\rho_0 y_1}(P_1))}, \tag{5.9}$$

and K is a constant which depends at most on the coefficients of L, and the following inequality holds (compare (4.7))

$$|L\zeta| = |L(l_{P_1})| \leq M \quad \text{on } B_{\rho y_1}(P_1). \tag{5.10}$$

It follows from (4.10) that

$$My_1\rho^2 \leq \zeta \leq 14My_1\rho^2 \quad \text{on } B_{\rho y_1}(P_1). \tag{5.11}$$

Moreover, since $P_1 = (x_1, y_1) \in \partial B_{dy_0}(P_0)$ and $3y_0/4 < y_1$ and $d = \rho/4 < 1/4$, we have $\operatorname{dist}(P_1, P_0) = dy_0$ and $2dy_0 \leq 8dy_1/3 = 2\rho y_1/3 \leq \rho y_1$, and thus (see Figure 5.2)

$$B_{dy_0}(P_0) \subset B_{\rho y_1}(P_1). \tag{5.12}$$

Therefore, ζ is also defined on $\bar{B}_{dy_0}(P_0)$ and satisfies the bounds (5.11) on $B_{dy_0}(P_0)$ with ρ replaced by $d = \rho/4$:

$$\frac{1}{16}My_1d^2 \leq \zeta \leq \frac{14}{16}My_1d^2 \quad \text{on } B_{dy_0}(P_0),$$

and thus, applying (5.7),

$$\frac{3}{64}My_0d^2 \leq \zeta \leq \frac{70}{64}My_0d^2 \quad \text{on } B_{dy_0}(P_0). \tag{5.13}$$

As in the proof of Proposition 4.1, we set

$$w := u - l_{P_1} + \zeta \in H^2(B_{\rho y_1}(P_1)) \cap C(\bar{B}_{\rho y_1}(P_1)). \tag{5.14}$$

The inequality (4.29) (with the role of $B_{\rho y_0/2}(P_0)$ there replaced by $B_{\rho y_1/2}(P_1)$) yields

$$0 \leq w \leq CMy_1\rho^2 \quad \text{on } B_{\rho y_1/2}(P_1),$$

and thus, since $y_1 \leq 5y_0/4$ by (5.7) and $\rho = 4d$ and $B_{dy_0/2}(P_0) \subset B_{\rho y_1/2}(P_1)$ by (5.12),

$$0 \leq w \leq CMy_0d^2 \quad \text{on } B_{dy_0/2}(P_0), \tag{5.15}$$

for a larger constant C depending at most on the coefficients of L and where M is as in (5.9) (compare (4.8)).

Because $B_{dy_0}(P_0) \subset \mathscr{C}(u)$, we have $Lu = 0$ on $B_{dy_0}(P_0)$, while $L(l_{P_1}) = L\zeta$ on $B_{\rho y_1}(P_1)$ by (5.8). It follows that

$$Lw = 0 \quad \text{on } B_{dy_0}(P_0), \tag{5.16}$$

since $B_{dy_0}(P_0) \subset B_{\rho y_1}(P_1)$ by (5.12). Consider now the rescaled solution, $\bar{w} \in C^{2+\alpha}(\bar{B}_d)$, given by

$$\bar{w}(x, y) := w(x_0 + y_0x, y_0 + y_0y), \quad \forall (x, y) \in \bar{B}_d, \tag{5.17}$$

to the uniformly elliptic equation

$$L_{y_0}\bar{w} = 0 \quad \text{on } B_d, \tag{5.18}$$

where $B_d = B_d(0,0)$ and L_{y_0} is given by (4.12). The classical Schauder interior estimates for strictly elliptic equations [12, Corollary 6.3] yield

$$d\|D\bar{w}\|_{C(\bar{B}_{d/4})} + d^2\|D^2\bar{w}\|_{C(\bar{B}_{d/4})} \leq C\|\bar{w}\|_{C(\bar{B}_{d/2})},$$

for a constant C depending at most on the coefficients of L (noting that $d \leq 1/4$). Combining the preceding inequality with the inequalities (5.15) for w implies the bounds

$$\frac{1}{d}|D\bar{w}(0)| + |D^2\bar{w}(0)| \leq \frac{C}{d^2}\|\bar{w}\|_{C(\bar{B}_{d/2})} \leq CMy_0,$$

where C depends at most on the coefficients of L, and M is as in (5.9). Hence, since $D\bar{w}(0) = y_0 Dw(P_0)$ and $D^2\bar{w}(0) = y_0^2 D^2 w(P_0)$ by (5.17), we obtain

$$\frac{y_0}{d}|Dw(P_0)| + y_0^2|D^2w(P_0)| \leq CMy_0.$$

We conclude that

$$|Dw(P_0)| + y_0|D^2w(P_0)| \leq CM, \tag{5.19}$$

Similarly, the rescaled function $\bar{\zeta} \in C^{2+\alpha}(\bar{B}_d)$ given by

$$\bar{\zeta}(x,y) := \zeta(x_0 + y_0 x, y_0 + y_0 y), \quad \forall (x,y) \in \bar{B}_d \tag{5.20}$$

satisfies the uniformly elliptic equation (see (4.11))

$$L_{y_0}\bar{\zeta} = y_0 L\zeta = y_0 L(l_{P_1}) = y_0 f_1 \quad \text{on } B_d,$$

where $f_1 := L(l_{P_1})$ is a smooth, linear function with

$$\begin{aligned}\|f_1\|_{C^1(B_d)} &\leq C\left(|\psi(P_0)| + |D\psi(P_0)|\right)\\ &\leq C\|\psi\|_{C^{1,1}(\bar{B}_{dy_0}(P_0))} \leq C\|\psi\|_{C^{1,1}(\bar{B}_{\rho_0 y_1}(P_1))}\\ &= CM,\end{aligned}$$

while C depends at most on the coefficients of L, and M is as in (5.9). Define $\eta \in C^{2+\alpha}(\bar{B}_1)$ by

$$\bar{\zeta}(x,y) =: d^2\eta(x/d, y/d), \quad (\bar{x},\bar{y}) := (x/d, y/d) \in B_1,$$

The function η obeys

$$\begin{aligned}(L_{y_0}\bar{\zeta})(x,y) &= \frac{1+y}{2}\left(\bar{\zeta}_{xx} + 2\varrho\sigma\bar{\zeta}_{xy} + \sigma^2\bar{\zeta}_{yy}\right)(x,y) + \left(r - q - \frac{y_0(1+y)}{2}\right)\bar{\zeta}_x(x,y)\\ &\quad + \kappa\left(\theta - y_0(1+y)\right)\bar{\zeta}_y(x,y) - ry_0\bar{\zeta}(x,y)\\ &= \frac{1+d\bar{y}}{2}\left(\eta_{\bar{x}\bar{x}} + 2\varrho\sigma\eta_{\bar{x}\bar{y}} + \sigma^2\eta_{\bar{y}\bar{y}}\right)(\bar{x},\bar{y}) + d\left(r - q - \frac{y_0(1+d\bar{y})}{2}\right)\eta_{\bar{x}}(\bar{x},\bar{y})\\ &\quad + d\kappa\left(\theta - y_0(1+d\bar{y})\right)\eta_{\bar{y}}(\bar{x},\bar{y}) - ry_0 d^2\eta(\bar{x},\bar{y})\\ &=: L_{y_0,d}\eta(\bar{x},\bar{y}), \quad \forall (\bar{x},\bar{y}) \in B_1,\end{aligned}$$

and

$$L_{y_0,d}\eta(\bar{x},\bar{y}) = y_0 f_1(x,y) = y_0 f_1(d\bar{x}, d\bar{y}) =: y_0\bar{f}_1(\bar{x},\bar{y}), \quad \forall (\bar{x},\bar{y}) \in B_1.$$

We have $Df_1(x,y) = d^{-1}D\bar{f}_1(\bar{x},\bar{y})$ and so, noting that $0 < d \leq 1/4$,

$$\begin{aligned}\|\bar{f}_1\|_{C^1(B_1)} &= \|\bar{f}_1\|_{C(\bar{B}_1)} + \|D\bar{f}_1\|_{C(\bar{B}_1)} \\ &= \|f_1\|_{C(\bar{B}_d)} + d\|Df_1\|_{C(\bar{B}_d)} \leq CM.\end{aligned}$$

Applying the classical Schauder interior estimates [12, Corollary 6.3] to the solution η to $L_{y_0,d}\eta = \bar{f}_1$ on B_1 gives

$$\begin{aligned}\|D\eta\|_{C(\bar{B}_{1/2})} + \|D^2\eta\|_{C(\bar{B}_{1/2})} &\leq C\left(\|\eta\|_{C(\bar{B}_1)} + y_0\|\bar{f}_1\|_{C^1(B_1)}\right) \\ &\leq C\left(\|\eta\|_{C(\bar{B}_1)} + My_0\right),\end{aligned}$$

for a constant C depending at most on the coefficients of L (recall that $0 < y_0 \leq 1$). Therefore, on B_d,

$$d^{-1}\|D\bar{\zeta}\|_{C(\bar{B}_{d/2})} + \|D^2\bar{\zeta}\|_{C(\bar{B}_{d/2})} \leq C\left(d^{-2}\|\bar{\zeta}\|_{C(\bar{B}_d)} + My_0\right),$$

for a constant C depending at most on the coefficients of L. Combining the preceding inequality with the bound (5.13) for ζ yields

$$\begin{aligned}d^{-1}|D\bar{\zeta}(0)| + |D^2\bar{\zeta}(0)| &\leq C\left(d^{-2}\|\bar{\zeta}\|_{C(\bar{B}_d)} + My_0\right) \\ &= C\left(d^{-2}\|\zeta\|_{C(\bar{B}_{dy_0}(P_0))} + My_0\right) \\ &\leq CMy_0,\end{aligned}$$

for a larger constant C, but depending at most on the coefficients of L. Hence, since $D\bar{\zeta}(0) = y_0 D\zeta(P_0)$ and $D^2\bar{\zeta}(0) = y_0^2 D^2\zeta(P_0)$ by (5.20), we obtain

$$d^{-1}y_0|D\zeta(P_0)| + y_0^2|D^2\zeta(P_0)| \leq CMy_0,$$

and thus, noting that $4 \leq d^{-1}$,

$$|D\zeta(P_0)| + y_0|D^2\zeta(P_0)| \leq CM. \tag{5.21}$$

Recalling that $w = u - l_{P_1} + \zeta$ by (5.14), we conclude from (5.19) and (5.21) that

$$|Du(P_0)| + y_0|D^2u(P_0)| \leq CM,$$

where M is as in (5.9) and so (5.2) holds for this step.

We consider the second situation.

Step 2 ($d > 1$)**.** We shall consider two cases.

Case 1 ($d > 1$ and $dy_0 \geq \rho_0/4$)**.** Since $dy_0 \geq \rho_0/4$ for this case (see Figure 5.3)

$$B^+_{\rho_0/4}(P_0) \subset B^+_{dy_0}(P_0) \cap V \subset \mathscr{C}(u) \cap V,$$

and so $Lu = 0$ on $B^+_{\rho_0/4}(P_0)$. The Schauder estimate (3.5) therefore yields

$$\|u\|_{C^{1,1}_s(\bar{B}^+_{\rho_0/8}(P_0))} \leq C\|u\|_{C^{2+\alpha}_s(\bar{B}^+_{\rho_0/8}(P_0))} \leq C\|u\|_{C(\bar{B}^+_{\rho_0/4}(P_0))}, \tag{5.22}$$

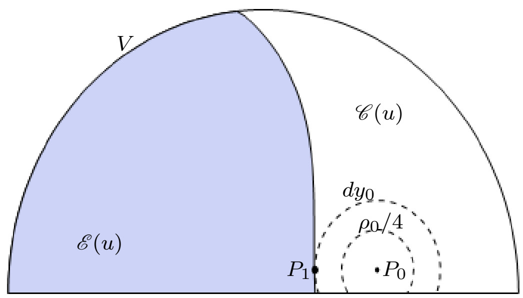

FIGURE 5.3. Regions for Case 1 of Step 2 (where $d > 1$ and $dy_0 \geq \rho_0/4$) of the proof of Theorem 1.1 for estimating the $C^{1,1}_s$ norm of a solution near the free boundary and near the degenerate boundary.

with a constant, C, depending at most on α, ρ_0, and the coefficients of L, recalling that we have chosen $0 < y_0 \leq 1$ by our assumption (5.1) for this section (and thus $\Lambda = 1$). This yields the desired bound (5.2) for this case.

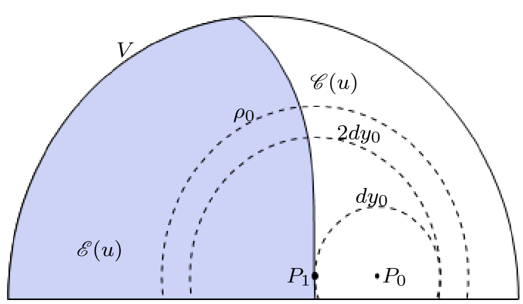

FIGURE 5.4. Regions for Case 2 of Step 2 (where $d > 1$ and $dy_0 < \rho_0/4$) of the proof of Theorem 1.1 for estimating the $C^{1,1}_s$ norm of a solution near the free boundary and near the degenerate boundary. (The radii $2dy_0$ and ρ_0 are not drawn to scale, since $2dy_0 < \rho_0/2$ for this case.)

Case 2 ($d > 1$ and $dy_0 < \rho_0/4$)**.** Since $dy_0 < \rho_0/4$, and also $P_1 \in \bar{V}_1$ and $\operatorname{dist}(P_1, P_0) = dy_0$ by (5.4) and $\rho_0 < R_0/2$ by (5.5), we have (see Figure 5.4)

$$B^+_{dy_0}(P_0) \subset B^+_{2dy_0}(P_1) \subset B^+_{\rho_0}(P_1) \Subset V \cup \Gamma_0,$$

Thus, it follows from (4.45) (with P_0 replaced by P_1 and $\rho = 4dy_0 < \rho_0$) and Taylor's theorem (since $\operatorname{dist}(P_1, P_0) = dy_0$) that

$$\begin{aligned}
\sup_{B^+_{dy_0}(P_0)} (u - \psi(P_0)) &\leq \sup_{B^+_{2dy_0}(P_1)} (u - \psi(P_0)) \\
&\leq \sup_{B^+_{2dy_0}(P_1)} (u - \psi(P_1)) + |\psi(P_1) - \psi(P_0)| \\
&\leq C dy_0 \|\psi\|_{C^{1,1}(\bar{B}^+_{\rho_0}(P_0))},
\end{aligned}$$

for a constant, C, depending at most on the coefficients of L, and hence

$$\sup_{B^+_{dy_0}(P_0)} (u - \psi(P_0)) \leq C dy_0 \|\psi\|_{C^{1,1}(\bar{V})}. \tag{5.23}$$

We now consider the function

$$w := u - \psi(P_0) \in C^{2+\alpha}(B^+_{dy_0}(P_0)) \cap C(\bar{B}^+_{dy_0}(P_0)), \tag{5.24}$$

which satisfies the equation

$$Lw = -L\psi(P_0) = r\psi(P_0) \quad \text{on } B^+_{dy_0}(P_0),$$

since $B_{dy_0}(P_0) \cap \mathscr{E}(u) = \varnothing$. By defining the rescaled function

$$\bar{w}(x,y) := \frac{1}{dy_0} w(x_0 + dy_0 x, y_0 + dy_0 y), \quad \forall (x,y) \in D,$$

on $D := B_1 \cap \{y > -1/d\} = B_1 \cap \{y_d > 0\}$, we see that

$$L_d \bar{w} = r\psi(P_0) \quad \text{on } D, \tag{5.25}$$

with (compare (4.11) and (4.12))

$$L_d \bar{w} := y_d \left(\bar{w}_{xx} + 2\varrho\sigma \bar{w}_{xy} + \sigma^2 \bar{w}_{yy}\right) + \left(r - q\frac{dy_0 y_d}{2}\right) \bar{w}_x + \kappa \left(\theta - dy_0 y_d\right) \bar{w}_y - r dy_0 \bar{w},$$

and $y_d := 1/d + y$. The operator L_d becomes degenerate at $y_d = 0$ or equivalently $y = -1/d$ which explains why the domain of consideration in the new variables is the intersection, D.

It follows from the bound (5.23) that

$$\|\bar{w}\|_{C(\bar{D})} \leq C \|\psi\|_{C^{1,1}(\bar{V})},$$

for a constant, C, depending at most on the coefficients of L. Denote $D_{1/2} := B_{1/2} \cap \{y_d > 0\}$. Hence, combining the preceding inequality with the Schauder estimate (3.5) for the solution $\bar{w}$

to the equation (5.25) and noting that $y_d = 1/d + y \geq 1/d$ on $\{y \geq 0\}$, yields the bound

$$\begin{aligned}\frac{1}{d}|D^2\bar{w}(0,0)| + |D\bar{w}(0,0)| &\leq \|y_d D^2\bar{w}\|_{C(\bar{D}_{1/2})} + \|D\bar{w}\|_{C(\bar{D}_{1/2})}\\ &\leq \|\bar{w}\|_{C^{2+\alpha}_s(\bar{D}_{1/2})}\\ &\leq C\left(\|\bar{w}\|_{C(\bar{D})} + rw(P_0)\right)\\ &\leq C\|\psi\|_{C^{1,1}(\bar{V})},\end{aligned}$$

recalling that $w(P_0) = dy_0\bar{w}(0,0)$; here, C is a constant depending at most on the coefficients of L. Since $D^2\bar{w}(0,0) = dy_0 D^2 w(P_0)$ and $D\bar{w}(0,0) = Dw(P_0)$, we obtain

$$y_0|D^2 w(P_0)| + |Dw(P_0)| \leq C\|\psi\|_{C^{1,1}(\bar{V})},$$

and thus, by (5.24),

$$y_0|D^2 u(P_0)| + |Du(P_0)| \leq C\|\psi\|_{C^{1,1}(\bar{V})},$$

for a possibly larger constant, C, but depending at most on the coefficients of L. This implies (5.2) for this case.

We consider the third situation.

Step 3 ($\rho_0/4 < d \leq 1$)**.** This is the simplest situation. As in Step 2, we consider two cases.

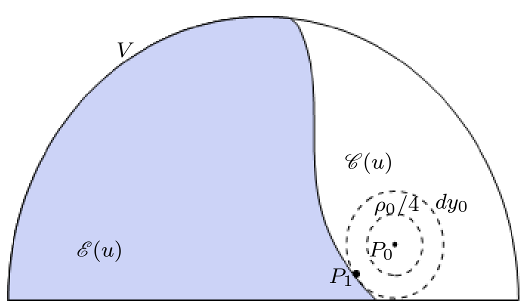

FIGURE 5.5. Regions for Case 1 of Step 3 (where $\rho_0/4 < d \leq 1$ and $dy_0 \geq \rho_0/4$) of the proof of Theorem 1.1 for estimating the $C^{1,1}_s$ norm of a solution near the free boundary and away from the degenerate boundary.

Case 1 ($\rho_0/4 < d \leq 1$ and $dy_0 \geq \rho_0/4$)**.** When $dy_0 \geq \rho_0/4$ (see Figure 5.5), we have $Lu = 0$ in $B^+_{\rho_0/4}(P_0)$, the estimate (5.22) for u holds, and (5.2) follows in this case.

Case 2 ($\rho_0/4 < d \leq 1$ and $dy_0 < \rho_0/4$)**.** We now assume that $dy_0 < \rho_0/4$ (see Figure 5.6). We consider

$$w := u - \psi(P_0) \in C^{2+\alpha}(B_{dy_0}(P_0)) \cap C(\bar{B}_{dy_0}(P_0)), \tag{5.26}$$

and the rescaled function,

$$\bar{w}(x,y) := \frac{1}{y_0} w(x_0 + y_0 x, y_0 + y_0 y), \quad (x,y) \in B_d,$$

which satisfies (compare (4.11))

$$\bar{L}\bar{w} = r\psi(P_0) \quad \text{on } B_d,$$

with (compare (4.12))

$$\begin{aligned}\bar{L}\bar{w} := \frac{1+y}{2}\left(\bar{w}_{xx} + 2\varrho\sigma\bar{w}_{xy} + \sigma^2\bar{w}_{yy}\right) + \left(r - q - \frac{y_0(1+y)}{2}\right)\bar{w}_x \\ + \kappa\left(\theta - y_0(1+y)\right)\bar{w}_y - ry_0\bar{w}.\end{aligned}$$

The operator $\bar{L}$ is strictly elliptic on B_d with ellipticity constant bounded below by a positive constant depending at most on the coefficients of L. In addition, since $dy_0 < \rho_0/4$, the bound (5.23) applies (irrespective of whether $d \leq 1$ or $d > 1$) to give

$$|w| \leq Cdy_0\|\psi\|_{C^{1,1}(\bar{V})} \quad \text{on } B_{dy_0}(P_0),$$

and thus

$$|\bar{w}| \leq Cd\|\psi\|_{C^{1,1}(\bar{V})} \quad \text{on } B_d,$$

for a constant, C, depending at most on the coefficients of L. Combining the preceding estimate

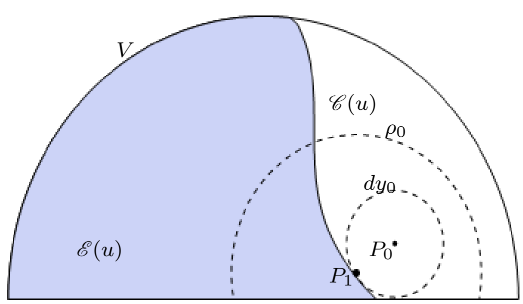

FIGURE 5.6. Regions for Case 2 of Step 3 (where $\rho_0/4 < d \leq 1$ and $dy_0 < \rho_0/4$) of the proof of Theorem 1.1 for estimating the $C^{1,1}_s$ norm of a solution near the free boundary and away from the degenerate boundary. (The radii dy_0 and ρ_0 are not drawn to scale, since $dy_0 < \rho/4$ for this case.)

with the classical Schauder interior estimate [12, Corollary 6.3] gives

$$\begin{aligned}|D^2\bar{w}(0)| + |D\bar{w}(0)| &\le \|\bar{w}\|_{C^{2+\alpha}(\bar{B}_{d/2})}\\ &\le C\left(\|\bar{w}\|_{C(\bar{B}_d)} + |\psi(P_0)|\right)\\ &\le C\|\psi\|_{C^{1,1}(\bar{V})},\end{aligned}$$

again for a constant, C, depending at most on the coefficients of L (recall that $\rho_0/4 \le d \le 1$ in this case and that ρ_0 depends at most on the coefficients of L). Hence,

$$y_0|D^2 w(P_0)| + |Dw(P_0)| \le C\|\psi\|_{C^{1,1}(\bar{V})},$$

since $D^2\bar{w}(0,0) = y_0 D^2 w(P_0)$ and $D\bar{w}(0,0) = Dw(P_0)$. Thus, by (5.26),

$$y_0|D^2 u(P_0)| + |Du(P_0)| \le C\|\psi\|_{C^{1,1}(\bar{V})},$$

for a possibly larger constant, C, and (5.2) follows in this case too.

This completes the proof of Theorem 1.1. □

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