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Decomposition of Monomial Algebras:
Applications and Algorithms

by

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DECOMPOSITION OF MONOMIAL ALGEBRAS:
APPLICATIONS AND ALGORITHMS

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ABSTRACT. Considering finite extensions $K[A] \subseteq K[B]$ of positive affine semigroup rings over a field K we have developed in [1] an algorithm to decompose $K[B]$ as a direct sum of monomial ideals in $K[A]$. By computing the regularity of homogeneous semigroup rings from the decomposition we have confirmed the Eisenbud-Goto conjecture in a range of new cases not tractable by standard methods. Here we first illustrate this technique and its implementation in our MACAULAY2 package MONOMIALALGEBRAS by computing the decomposition and the regularity step by step for an explicit example. We then focus on ring-theoretic properties of simplicial semigroup rings. From the characterizations given in [1] we develop and prove explicit algorithms testing properties like Buchsbaum, Cohen-Macaulay, Gorenstein, normal, and seminormal, all of which imply the Eisenbud-Goto conjecture. All algorithms are implemented in our MACAULAY2 package.

1. INTRODUCTION

Let B be a positive affine semigroup, that is, B is a finitely generated subsemigroup of $\mathbb{N}^m$ for some m. Let K be a field and $K[B]$ the affine semigroup ring associated to B, which can be identified with the subring of $K[t_1, \ldots, t_m]$ generated by monomials $t^u := t_1^{u_1} \cdot \ldots \cdot t_m^{u_m}$, where $u = (u_1, \ldots, u_m) \in B$. Denote by $C(B)$ and by $G(B)$ the cone and the group generated by B. From now on let $A \subseteq B$ be positive affine semigroups with $C(A) = C(B)$. We will now discuss the decomposition of $K[B]$ into a direct sum of monomial ideals in $K[A]$. Observe that

$$K[B] = \bigoplus_{g \in G} K \cdot \left\{ t^b \mid b \in B \cap g \right\},$$

where $G := G(B)/G(A)$. Note that $C(A) = C(B)$ if and only if $K[B]$ is a finitely generated $K[A]$-module. From this it follows that G is finite, and we can compute the above decomposition since all summands are finitely generated. Moreover, there are shifts $h_g \in G(B)$ such that

$$I_g := K \cdot \left\{ t^{b-h_g} \mid b \in B \cap g \right\}$$

is a monomial ideal in $K[A]$. Thus,

$$K[B] \cong \bigoplus_{g \in G} I_g(-h_g)$$

as $\mathbb{Z}^m$-graded $K[A]$-modules (with $\deg t^b = b$). A detailed formulation of the algorithm computing the ideals I_g and shifts h_g and a more general version of the decomposition in the setup of cancellative abelian semigroup rings over an integral domain can be found in [1, Algorithm 1, Theorem 2.1].

Our original motivation for developing this decomposition was to provide a fast algorithm to compute the Castelnuovo-Mumford regularity $\operatorname{reg} K[B]$ of a homogeneous semigroup ring in order to test the Eisenbud-Goto conjecture [3]. Recall that the *Castelnuovo-Mumford regularity*

reg M of a finitely generated graded module M over a standard graded polynomial ring $R = K[x_1, \ldots, x_n]$ is defined as the smallest integer m such that every j-th syzygy module of M is generated by elements of degree $\leq m + j$. Moreover, B is called a *homogeneous semigroup* if there exists a group homomorphism deg : $G(B) \to \mathbb{Z}$ with $\deg b_i = 1$ for $i = 1, \ldots, n$, where $\mathrm{Hilb}(B) = \{b_1, \ldots, b_n\}$ is the minimal generating set of B; by reg $K[B]$ we mean its regularity with respect to the R-module structure which is given by the K-algebra homomorphism $R \twoheadrightarrow K[B]$, $x_i \mapsto t^{b_i}$.

The *toric Eisenbud-Goto conjecture* can be formulated as follows: let K be a field and B a homogeneous semigroup, then reg $K[B] \leq \deg K[B] - \operatorname{codim} K[B]$, where $\deg K[B]$ denotes the degree and $\operatorname{codim} K[B] := \dim_K K[B]_1 - \dim K[B]$ the codimension. Even this special case of the Eisenbud-Goto conjecture is largely open, for references on known results see [1, Section 4]. The regularity of $K[B]$ is usually computed from a minimal graded free resolution. If n is large this computation is very expensive, and hence it is impossible to test the conjecture systematically in high codimension using this method. However, choosing A to be generated by minimal generators $e_1, \ldots, e_d$ of $C(B)$ of degree 1 the regularity can be computed as

$$\operatorname{reg} K[B] = \max\{\operatorname{reg} I_g + \deg h_g \mid g \in G\},$$

where reg I_g denotes the regularity of I_g with respect to the canonical $T = K[x_1, \ldots, x_d]$-module structure given by $T \twoheadrightarrow K[A]$, $x_i \mapsto t^{e_i}$. Since the free resolution of every ideal I_g appearing has length at most $d - 1$, this computation is typically much faster than the traditional approaches. This enabled us to test the conjecture for a large class of homogeneous semigroup rings by using our regularity algorithm. See [1, Section 4] for details.

In Section 2, we illustrate, step by step, decomposition and regularity computation for an explicit example using our MACAULAY2 [4] package MONOMIALALGEBRAS [2]. We say that $K[B]$ is a *simplicial semigroup ring* if the cone $C(B)$ is simplicial. In Section 3, we focus on simplicial semigroup rings $K[B]$. Based on the characterizations of ring-theoretic properties given in [1, Proposition 3.1] we develop explicit algorithms for testing whether $K[B]$ is Buchsbaum, Cohen-Macaulay, Gorenstein, seminormal, or normal. We also discuss that, by known results, all these ring-theoretic properties imply the Eisenbud-Goto conjecture. The algorithms mentioned are implemented in our MACAULAY2 package.

2. DECOMPOSITION AND REGULARITY

Our MACAULAY2 package can be loaded by

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "MonomialAlgebras";
```

We discuss the decomposition at the example of the homogeneous semigroup $B \subset \mathbb{N}^3$ specified by a list of generators

```
i2 : B = {{4,0,0},{2,2,0},{2,0,2},{0,2,2},{0,3,1},{3,1,0},{1,1,2}};
```

As an input for our algorithm we encode this data in a multigraded polynomial ring

```
i3 : K = ZZ/101;
i4 : S = K[x_1 ..  x_7, Degrees=>B];
```

The command

```
i5 : dc = decomposeMonomialAlgebra S
o5 = HashTable{  (-1,1,0)  =>{ ideal (x_1,x_3),  (-1,1,0)  }
                  0         =>{ ideal 1,           0        } }
```

decomposes $K[B]$ over $K[A]$ where $A \subseteq B$ is generated by minimal generators of $C(B)$ with minimal coordinate sum; so in the example $A = \langle (4, 0, 0), (2, 2, 0), (2, 0, 2), (0, 2, 2), (0, 3, 1) \rangle$. The

keys of the hash table represent the elements of G and the values are the tuples (I_g, h_g), hence

$$(2.1) \qquad K[B] \cong \langle \overline{x_1}, \overline{x_3} \rangle \, (-(-1,1,0)) \oplus K[A]$$

as $\mathbb{Z}^3$-graded $K[A]$-modules; here we write $K[A] \cong T/J$ with $T = K[x_1, x_2, x_3, x_4, x_5]$ and $\overline{x_i}$ for the class of x_i. Note that the on-screen output of MACAULAY2 does not distinguish between the class and the representative. To compute $\operatorname{reg} I_g$ we will consider the standard grading on T:

```
i6 :  KA = ring first first values dc;
i7 :  T = newRing(ring ideal KA,Degrees=>{5:1});
i8 :  J = sub(ideal KA,T);
o8 :  ideal of T
i9 :  betti res J
              0  1  2
      total:  1  3  2
o9 =  0       1  .  .
      1       .  1  .
      2       .  2  2
o9 :  BettiTally
```

The usual approach would be to obtain $\operatorname{reg} K[B]$ from a minimal graded free resolution of the toric ideal I_B with respect to the standard grading.

```
i10 :  IB = monomialAlgebraIdeal S;
o10 :  ideal of S
i11 :  R = newRing(ring IB,Degrees=>{7:1});
i12 :  betti res sub(IB,R)
               0  1   2   3   4  5
       total:  1  8  15  13   6  1
o12 =  0       1  .   .   .   .  .
       1       .  6   8   3   .  .
       2       .  2   3   .   .  .
       3       .  .   4  10   6  1
o12 :  BettiTally
```

We observe that $\operatorname{reg} K[B] = 3$. With $\deg u = (u_1 + u_2 + u_3)/4$ by Equation (2.1) it holds

$$\operatorname{reg} K[B] = \max \left\{ \operatorname{reg} \langle \overline{x_1}, \overline{x_3} \rangle + 0, \ \operatorname{reg} K[A] + 0 \right\}.$$

By o9 we have $\operatorname{reg} K[A] = 2$. To compute $\operatorname{reg} \langle \overline{x_1}, \overline{x_3} \rangle$ do:

```
i13 :  I1 = first (values dc)#0
o13 =  ideal (x_1,x_3)
o13 :  ideal of KA
i14 :  g = matrix entries sub(gens I1, T);
                1        2
o15 :  Matrix T <--- T
i15 :  betti res image map(coker gens J, source g, g)
               0  1  2  3
       total:  2  5  4  1
o15 =  1       2  2  .  .
       2       .  .  .  .
       3       .  3  4  1
```

Hence $\operatorname{reg} \langle \overline{x_1}, \overline{x_3} \rangle = 3$ and therefore $\operatorname{reg} K[B] = 3$. Observe, that the resolution of $K[B]$ has length 5, whereas the ideals I_g have resolutions of length at most 3. The command

```
i16 :  regularityMA S
o16 :  {3, {{ideal (x_1,x_3), (-1,1,0)}}}
```

provides an implementation of this approach, also returning the tuples (I_g, h_g) where the maximum is achieved. By [1, Proposition 4.1] we have $\deg K[B] = \#G \cdot \deg K[A] = 10$ since

```
i17 :  degree J
o17 =  5
```

Moreover, $\operatorname{codim} K[B] = 4$ since $\dim K[B] = \dim C(B) = 3$. Hence the ring $K[B]$ satisfies the Eisenbud-Goto bound.

3. Algorithms for ring theoretic properties

In this section we focus on simplicial semigroup rings $K[B]$. Based on the characterizations given in [1, Proposition 3.1] we develop and prove explicit algorithms for testing whether $K[B]$ is Buchsbaum, Cohen-Macaulay, Gorenstein, seminormal, or normal. Note that, in the simplicial case, all these properties are independent of K, and they imply the Eisenbud-Goto conjecture by results of [7, 8, 6]. As an example, consider the following homogeneous simplicial semigroup $B \subset \mathbb{N}^3$ specified by the generators

```
i18 :  B = {{4,0,0},{0,4,0},{0,0,4},{1,0,3},{0,2,2},{3,0,1},{1,2,1}};
```

We compute the decomposition of $K[B]$ over $K[A]$, where $A = \langle (4,0,0), (0,4,0), (0,0,4) \rangle \subset B$ is generated again by minimal generators of $C(B)$ with minimal coordinate sum.

```
i19 :  S = K[x_1 .. x_7, Degrees=>B];
i20 :  decomposeMonomialAlgebra S
o20 :  HashTable{  (−1, 0, 1)   =>{ ideal 1,            (3, 0, 1)   }
                   (−1, 2, −1)  =>{ ideal 1,            (3, 2, 3)   }
                   (0, 2, 2)    =>{ ideal 1,            (0, 2, 2)   }
                   (1, 0, −1)   =>{ ideal 1,            (1, 0, 3)   }
                   (1, 2, 1)    =>{ ideal 1,            (1, 2, 1)   }
                   (2, 0, 2)    =>{ ideal (x_3,x_1,x_2),  (2, 0, 2)   }
                   (2, 2, 0)    =>{ ideal 1,            (2, 2, 4)   }
                   0            =>{ ideal 1,            0           } }
```

Hence

$$K[B] \cong K[A] \oplus K[A](-1)^4 \oplus K[A](-2)^2 \oplus \langle x_1, x_2, x_3 \rangle (-1)$$

with respect to the standard grading induced by $\deg u = (u_1 + u_2 + u_3)/4$. It follows that $\operatorname{depth} K[B] = 1$, thus, $K[B]$ is not Cohen-Macaulay. Hence $K[B]$ is also not normal by [5]. We can test seminormality via the following algorithm:

Algorithm 1 Seminormality test

Input: A simplicial semigroup $B \subseteq \mathbb{N}^m$.
Output: true if $K[B]$ is seminormal, false otherwise.
 1: Let $e_1, \ldots, e_d \in B$ be minimal generators of $C(B)$ with minimal coordinate sum, and set $A := \langle e_1, \ldots, e_d \rangle$.
 2: Compute $B_A := \{x \in B \mid x \notin B + (A \setminus \{0\})\}$ as described in [1, Algorithm 1, Step 1].
 3: **for all** $x \in B_A$ **do**
 4:    Solve the linear system of equations $\sum_{i=1}^d \lambda_i e_i = x$ for $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Q}^d$.
 5:    **if** $\|\lambda\|_\infty > 1$ **then return** false
 6: **return** true

Here, by $\|-\|_\infty$ we denote the maximum norm. Note that all λ_i are non-negative since $C(B)$ is a simplicial cone. Verifying in Step 5 the condition $\|\lambda\|_\infty \geq 1$ instead results in an algorithm which tests normality. Using our package we observe that $K[B]$ is not seminormal:

```
i21 :  isSeminormalMA B
o21 :  false
```

The Buchsbaum property can be tested by the following algorithm. We denote by $K[A]_+$ the homogeneous maximal ideal of $K[A]$.

Algorithm 2 Buchsbaum test

Input: A simplicial semigroup $B = \langle b_1, \ldots, b_n \rangle \subseteq \mathbb{N}^m$.
Output: `true` if $K[B]$ is Buchsbaum, `false` otherwise.
1: Let $e_1, \ldots, e_d \in B$ be minimal generators of $C(B)$ with minimal coordinate sum, and set $A := \langle e_1, \ldots, e_d \rangle$.
2: Using the (minimal) generators $e_1, \ldots, e_d$ of A decompose
$$K[B] \cong \bigoplus_{g \in G} I_g(-h_g),$$
where $I_g \subseteq K[A]$, $h_g \in G(B)$ and $G = G(B)/G(A)$ by [1, Algorithm 1].
3: **if** $\exists g \in G$ with $I_g \neq K[A]$ and $I_g \neq K[A]_+$ **then return** `false`
4: $H := \{h_g \mid g \in G \text{ with } I_g = K[A]_+\}$
5: $C := \{b_1, \ldots, b_n\} \setminus \{0, e_1, \ldots, e_d\}$
6: $H + C := \{h_g + b_i \mid h_g \in H, b_i \in C\}$
7: **return** `true` if $(H + C) \cap H = \emptyset$ and `false` otherwise.

Proof. By [1, Proposition 3.1] the ring $K[B]$ is Buchsbaum iff each ideal I_g is either equal to $K[A]$, or to $K[A]_+$ and $h_g + b \in B$ for all $b \in \text{Hilb}(B)$. So, Step 3 is correct and we may now assume that $I_g = K[A]$ or $I_g = K[A]_+$ for all $g \in G$. Recall that $I_g = \{t^{v-h_g} \mid v \in \Gamma_g\} K[A]$, where $\Gamma_g = \{x \in B_A \mid x \in g\}$. Moreover, note that $\{t^{v-h_g} \mid v \in \Gamma_g\}$ is always a minimal generating set of I_g and $h_g = \sum_{k=1}^d \min\{\lambda_k^v \mid v \in \Gamma_g\} e_k$, where $v = \sum_{k=1}^d \lambda_k^v e_k$ with $\lambda_k^v \in \mathbb{Q}$.

Since $h + e_k \in B_A$ for all $h \in H$ and all $k = 1, \ldots, d$, we have $H \cap B = \emptyset$. In case that $(H + C) \cap H \neq \emptyset$ we obtain $h + b \notin B$ for some $h \in H$ and some $b \in B \setminus \{0\}$, that is, $h + \text{Hilb}(B) \not\subseteq B$. Hence $K[B]$ is not Buchsbaum.

In case that $K[B]$ is not Buchsbaum, there is an $h \in H$ and some $b \in \text{Hilb}(B)$ such that $h + b \notin B$. It is now sufficient to show that $b \in C$ and $h + b \in H$. By the above argument, $b \in C$. Let $m_k = h + b + e_k$ for $k = 1, \ldots, d$. Suppose that $m_i \notin B_A$ for some $i \in \{1, \ldots, d\}$. Since $m_k - e_k \notin B$ for all $k = 1, \ldots, d$, necessarily $m_i - e_j \in B$ for some $j \neq i$. Consider $y = m_j - \sum_{k=1}^d n_k e_k \in B$ with $n_k \in \mathbb{N}$ such that $\sum_{k=1}^d n_k$ is maximal. By construction $y \in B_A$, moreover, $n_j = 0$ since $m_j - e_j \notin B$. In the same way if $x = m_i - e_j - \sum_{k=1}^d n_k e_k \in B$ with $\sum_{k=1}^d n_k$ maximal, then $x \in B_A$. Since $m_i, m_j \in g$ for some $g \in G$, we also have $x, y \in g$. Since $e_1, \ldots, e_d$ are linearly independent, we have $\lambda_j^y - \lambda_j^x \geq 2$. Moreover, since $t^{y-h_g}, t^{x-h_g} \in K[A]$ we get that t^{y-h_g} is not a linear form. Hence $I_g \neq K[A]$ and $I_g \neq K[A]_+$, thus, $m_k \in B_A$ for all $k = 1, \ldots, d$. We have $\#\Gamma_g \in \{1, d\}$ by minimality, hence $\Gamma_g = \{m_1, \ldots, m_d\}$. By construction, $h_g = h + b$ and $I_g = K[A]_+$, therefore $h + b \in H$. $\qquad\square$

Note that in Step 2 the shifts h_g and hence the ideals I_g are uniquely determined since $e_1, \ldots, e_d$ are linearly independent. This is not true for arbitrary generating sets. By

    `i22 : isBuchsbaumMA B`
    `o22 : true`

we conclude that $K[B]$ satisfies the Eisenbud-Goto conjecture by [7]. Note that we can read off from the decomposition the regularity and the Eisenbud-Goto bound: we have $\text{reg } K[A] = 0$ and $\text{reg}\langle x_1, x_2, x_3\rangle = 1$, therefore $\text{reg } K[B] = \max\{0, 1, 2, 1+1\} = 2$. Moreover, $\deg K[B]$ is the number of ideals which occur in the decomposition, hence $\deg K[B] - \text{codim } K[B] = 8 - 4 = 4$.

Note that, in case B is Buchsbaum, the regularity of $K[B]$ is independent of the field K since all ideals in the decomposition are equal to the homogeneous maximal ideal or to $K[A]$.

We finish this section by providing an algorithm for testing the Gorenstein property.

Algorithm 3 Gorenstein test

Input: A simplicial semigroup $B \subseteq \mathbb{N}^m$.
Output: `true` if $K[B]$ is Gorenstein, `false` otherwise.
 1: Let $e_1, \ldots, e_d \in B$ be minimal generators of $C(B)$ with minimal coordinate sum, and set
    $A := \langle e_1, \ldots, e_d \rangle$.
 2: Using the (minimal) generators $e_1, \ldots, e_d$ of A decompose
$$K[B] \cong \bigoplus\nolimits_{g \in G} I_g(-h_g),$$
    where $I_g \subseteq K[A]$, $h_g \in G(B)$ and $G = G(B)/G(A)$ by [1, Algorithm 1].
 3: **if** $\exists g \in G$ with $I_g \neq K[A]$ **then return** `false`
 4: $H := \{h_g \mid g \in G\}$
 5: **if** $h \in H$ with maximal coordinate sum is not unique **then return** `false`
 6: Let $h \in H$ with maximal coordinate sum.
 7: **while** $H \neq \emptyset$ **do**
 8:     Let $h_g \in H$
 9:     **if** $h - h_g \notin H$ **then return** `false`
10:     $H := H \setminus \{h_g, \ h - h_g\}$
11: **return** `true`

Proof. By [1, Proposition 3.1] the ring $K[B]$ is Gorenstein iff $I_g = K[A]$ for all $g \in G$ and H has a unique maximal element with respect to $\leq$ given by $x \leq y$ if there is a $z \in B$ such that $x + z = y$. Note that $H = B_A$ since $I_g = K[A]$ for all $g \in G$. If there is a maximal element $h \in H$, then this element has maximal coordinate sum. If H has more than one element with maximal coordinate sum, then H does not have a unique maximal element. To complete the proof we need to show that an element $h_g \in H$ satisfies $h_g \leq h$ iff $h - h_g \in H$. But this follows from the fact that if $x \notin B_A$ then $x + y \notin B_A$ for all $x, y \in B$. $\square$

Note that performing Steps 1–3 of Algorithm 3 (and returning `true` afterwards) gives a test for the Cohen-Macaulay property.

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