Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

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by

Philip Dowerk and Yurii Savchuk

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INDUCED $*$-REPRESENTATIONS AND C^*-ENVELOPES OF SOME QUANTUM $*$-ALGEBRAS.

PHILIP A. DOWERK AND YURII SAVCHUK

Abstract. We consider three quantum algebras: the q-oscillator algebra, the Podleś sphere and the q-deformed enveloping algebra of $su(2)$. To each of these $*$-algebras we associate certain partial dynamical system and perform the "Mackey analysis" of $*$-representations developed in [SS]. As a result we get the description of "standard" irreducible $*$-representations. Further, for each of these examples we show the existence of a "C^*-envelope" which is canonically isomorphic to the covariance C^*-algebra of the partial dynamical system. Finally, for the q-oscillator algebra and the q-deformed $\mathcal{U}(su(2))$ we show the existence of "bad" representations.

Introduction and preliminaries

The aim of this paper is to demonstrate a unified approach to the $*$-representation theory of various quantum algebras based on the techniques developed in [SS]. Most of the quantum $*$-algebras (e.g. non-compact quantum groups) possess unbounded $*$-representations. The main problem in the theory of unbounded $*$-representations is to define and classify the "well-behaved" $*$-representations of a given $*$-algebra. We recall two classical examples.

Example. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra, G be the corresponding simply connected Lie group and $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ be the complex enveloping $*$-algebra of $\mathfrak{g}$. A $*$-representation π of $\mathfrak{g}$ is called *integrable* if $\pi = dU$ for some unitary representation U of G. If $G \neq \mathbb{R}$ there exists a $*$-representation of $\mathfrak{g}$ which is not integrable and, moreover, cannot be extended to an integrable representation even in a larger Hilbert space, see [S1]. Already in the case $G = \mathbb{R}^2,\ \mathfrak{g} = \mathbb{C}[x_1, x_2]$ the category of all $*$-representations of $\mathfrak{g}$ is in a certain sense "very large" as shown in [S3, Section 9].

Example. Let W_n be the n-dimensional Weyl algebra. That is, W_n is a complex $*$-algebra generated by self-adjoint elements $p_i, q_i,\ i = 1, \dots, n$, satisfying $[p_i, q_j] = -\delta_{ij}\mathbf{i},\ [p_i, p_j] = [q_i, q_j] = 0$. A $*$-representation π of W_n is called integrable if $P_i = \overline{\pi(p_i)},\ Q_j = \overline{\pi(q_j)},\ i, j = 1, \dots, n$, are self-adjoint and the one-parameter unitary groups $e^{\mathbf{i}t_iP_i},\ e^{\mathbf{i}s_jQ_j}$ satisfy the Weyl commutation relations. Already for W_1 one

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can show the existence of "bad representations" and show that the category of all $*$-representations is again "very large", whereas the only integrable $*$-representations are sums of copies of the Schrödinger representation.

We investigate the following three $*$-algebras in details: the q-oscillator algebra $\mathcal{A}_q$ for $q > 0$, the q-deformed enveloping algebra $\mathcal{U}_q(su(2))$, $q > 0$ and the Podleś spheres $\mathcal{O}(S^2_{qr})$, $q \in (0,1)$, $r \in (0,\infty)$. The algebras $\mathcal{A}_q$ and $\mathcal{U}_q(su(2))$ are deformations of W_1 and $\mathcal{U}_{\mathbb{C}}(su(2))$ respectively, however, for both these algebras the notion of "integrability" cannot be generalized in a direct way. Instead of this we use the approach from [SS], which applies to all three algebras $\mathcal{A}_q$, $\mathcal{O}(S^2_{qr})$, $\mathcal{U}(su(2))$ as well as to their classical analogues. Let $\mathcal{A}$ denote one of these algebras. The basic idea is to find a natural $\mathbb{Z}$-grading $\mathcal{A}_k$, $k \in \mathbb{Z}$, for $\mathcal{A}$ such that $\mathcal{A}_0 =: \mathcal{B}$ is commutative. Further, we define the "positive" spectrum $\widehat{\mathcal{B}}^+$ of $\mathcal{B}$ as the set of those characters $\chi \in \widehat{\mathcal{B}}$ which satisfy $\chi(a^*a) \geq 0$ for all $a \in \mathcal{A}$, such that $a^*a \in \mathcal{B}$. The group grading of $\mathcal{A}$ defines a structure of a $*$-algebraic bundle in the sense of [FD], and there is a canonical partial action α of $\mathbb{Z}$ on $\widehat{\mathcal{B}}^+$. By means of the partial dynamical system $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$ we

- define well-behaved $*$-representations,
- show that the irreducible ones naturally correspond to the orbits of $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$,
- construct the dual partial action β on $C_0(\widehat{\mathcal{B}}^+)$ and the partial crossed product C^*-algebra $C_0(\widehat{\mathcal{B}}^+) \times_\beta \mathbb{Z}$ in the sense of [Ex]; using the Woronowicz's theory of affiliated operators, we establish a Morita equivalence between $C_0(\widehat{\mathcal{B}}^+) \times_\beta \mathbb{Z}$ and $\mathcal{A}$.

It turns out that every irreducible well-behaved representation of $\mathcal{A}$ is induced from a one-dimensional representation. Thereby, the induction procedure is the generalized Rieffel induction introduced and studied in [SS]. This result can be viewed as an analogue of the following theorem by Kirillov (see [Kir]): Every irreducible unitary representation of a nilpotent Lie group is induced from a one-dimensional representation of a certain subgroup.

In each of three cases the constructed crossed product C^*-algebra is of a special kind. Namely, the partial action of $\mathbb{Z}$ on $C_0(\widehat{\mathcal{B}}^+)$ is generated by a single partial automorphism Θ, see [Ex, McC] and Section 0.2. In this case the partial crossed product C^*-algebra coincides with the covariance C^*-algebra of the partial automorphism in the sense of [Ex, Definition 3.7]. In the case $\mathcal{A} = \mathcal{O}(S^2_{qr})$ all $*$-representations are bounded, hence well-behaved, and the crossed product C^*-algebra $C_0(\widehat{\mathcal{B}}^+) \times_\alpha \mathbb{Z}$ is isomorphic to the enveloping C^*-algebra of $C^*_{env}(\mathcal{A})$.

Finally, for $\mathcal{A}_q$ and for $\mathcal{U}_q(su(2))$ we show the existence of "bad" representations. More precisely, we prove the existence of a $*$-representation which is not well-behaved and cannot be extended to a well-behaved $*$-representation even in a larger Hilbert space. It generalizes the well-known results for W_1 and $\mathcal{U}(su(2))$.

Among the examples which can be analyzed in the same spirit include various bounded and unbounded $*$-algebras: quantum group algebras $SU_q(2)$, $SU_q(1,1)$, q-deformed $\mathcal{U}(su(1,1))$, different deformations of CAR and CCR, AF pre-C^*-algebras (see [Ex1]) etc.

0.1. **$*$-Algebras and $*$-representations.** By a $*$-*algebra* we mean a complex associative algebra $\mathcal{A}$ equipped with a mapping $a \mapsto a^*$ of $\mathcal{A}$ into itself, called the *involution* of $\mathcal{A}$, such that $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$, $(ab)^* = b^* a^*$ and $(a^*)^* = a$ for

$a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$. The unit of $\mathcal{A}$ (if it exists) will be denoted by $\mathbf{1}_{\mathcal{A}}$ or simply by $\mathbf{1}$. For every $*$-algebra $\mathcal{A}$ denote by $\sum \mathcal{A}^2$ the set of finite sums $\sum a_i^* a_i,\ a_i \in \mathcal{A}$.

Throughout this paper we use some terminology and results from unbounded representation theory in Hilbert space (see e.g. [S3]). We repeat some basic notions and facts. If T is a Hilbert space operator, $\mathcal{D}(T)$, $\overline{T}$ and T^* denote its domain, its closure and its adjoint, respectively. Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$. A *$*$-representation* of a $*$-algebra $\mathcal{A}$ on $\mathcal{D}$ is an algebra homomorphism π of $\mathcal{A}$ into the algebra $L(\mathcal{D})$ of linear operators on $\mathcal{D}$ such that $\langle \pi(a)\varphi, \psi \rangle = \langle \varphi, \pi(a^*)\psi \rangle$ for all $\varphi, \psi \in \mathcal{D}$ and $a \in \mathcal{A}$. We call $\mathcal{D}(\pi) := \mathcal{D}$ the *domain* of π and write $\mathcal{H}_\pi := \mathcal{H}$. Two $*$-representations π_1 and π_2 of $\mathcal{A}$ are *(unitarily) equivalent* if there exists an isometric linear mapping U of $\mathcal{D}(\pi_1)$ onto $\mathcal{D}(\pi_2)$ such that $\pi_2(a) = U\pi_1(a)U^{-1}$ for $a \in \mathcal{A}$. The *direct sum representation* $\pi_1 \oplus \pi_2$ acts on the domain $\mathcal{D}(\pi_1) \oplus \mathcal{D}(\pi_2)$ by $(\pi_1 \oplus \pi_2)(a) = \pi_1(a) \oplus \pi_2(a)$, $a \in \mathcal{A}$. A $*$-representation π is called *irreducible* if a direct sum decomposition $\pi = \pi_1 \oplus \pi_2$ is only possible when $\mathcal{D}(\pi_1) = \{0\}$ or $\mathcal{D}(\pi_2) = \{0\}$. For a $*$-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ we denote by $\mathrm{Res}_{\mathcal{B}}\pi$ its restriction to $\mathcal{B}$. The *graph topology* of π is the locally convex topology on the vector space $\mathcal{D}(\pi)$ defined by the norms $\varphi \mapsto \|\varphi\| + \|\pi(a)\varphi\|$, where $a \in \mathcal{A}$. If $\mathcal{D}(\overline{\pi})$ denotes the completion the $\mathcal{D}(\pi)$ in the graph topology of π, then $\overline{\pi}(a) := \overline{\pi(a)} \upharpoonright \mathcal{D}(\overline{\pi})$, $a \in \mathcal{A}$, defines a $*$-representation of $\mathcal{A}$ with domain $\mathcal{D}(\overline{\pi})$, called the *closure* of π. In particular, π is *closed* if and only if $\mathcal{D}(\pi)$ is complete in the graph topology of π. A $*$-representation π is called *non-degenerate* if $\pi(\mathcal{A})\mathcal{D}(\pi) := \mathrm{Lin}\,\{\pi(a)\varphi; a \in \mathcal{A}, \varphi \in \mathcal{D}(\pi)\}$ is dense in $\mathcal{D}(\pi)$ in the graph topology of π. If $\mathcal{A}$ is unital and π is non-degenerate, then we have $\pi(\mathbf{1}_{\mathcal{A}})\varphi = \varphi$ for all $\varphi \in \mathcal{D}(\pi)$. We say that π is *cyclic* if there exists a vector $\varphi \in \mathcal{D}(\pi)$ such that $\pi(\mathcal{A})\varphi$ is dense in $\mathcal{D}(\pi)$ in the graph topology of π. For a C^*-algebra $\mathfrak{A}$ and Hilbert space $\mathcal{H}$, denote by $\mathrm{Rep}(\mathfrak{A}, \mathcal{H})$ the category of non-degenerate $*$-representations of $\mathfrak{A}$ on $\mathcal{H}$. By $\mathrm{Rep}\mathfrak{A}$ denote the category of all non-degenerate $*$-representations of $\mathfrak{A}$.

We recall the induction procedure for $*$-representations of general $*$-algebras developed in [SS, Section 2] in a slightly more general context. However, we will not perform this procedure but use Proposition 1.2 to get the explicit formulas. Let $\mathcal{B} \subseteq \mathcal{A}$ be $*$-algebras. A linear map $p : \mathcal{A} \to \mathcal{B}$ is called a *bimodule projection* if $p(a^*) = p(a)^*,\ p(b_1 a b_2) = b_1 p(a) b_2,\ p(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{B}}$, for all $a \in \mathcal{A},\ b_1, b_2 \in \mathcal{B}$. Let ρ be a $*$-representation of $\mathcal{B}$. Denote by $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$ the quotient of $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{B}$ by the linear span of vectors $ab \otimes \varphi - a \otimes \rho(b)\varphi, a \in \mathcal{A}, b \in \mathcal{B}, \varphi \in \mathcal{D}(\rho)$. We say that ρ is *inducible* from $\mathcal{B}$ to $\mathcal{A}$ via p if the sesquilinear form

$$\langle \sum_k x_k \otimes \varphi_k, \sum_l y_l \otimes \psi_l \rangle_0 := \sum_{k,l} \langle \rho(p(y_l^* x_k))\varphi_k, \psi_l \rangle, \tag{0.1}$$

is positive semi-definite on $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$. Denote by $\mathcal{K}_\rho$ the kernel of $\langle \cdot, \cdot \rangle_0$. Then $\mathcal{D}_0 = \mathcal{A} \otimes_{\mathcal{B}} \mathcal{D}(\rho)/\mathcal{K}_\rho$ is an inner-product space. Define a $*$-representation π on $\mathcal{D}_0$ via

$$\pi(a)(\sum_i [a_i \otimes \varphi_i]) := \sum_i [a a_i \otimes \varphi_i],$$

where $\sum_i [a_i \otimes \varphi_i] \in \mathcal{D}_0$ denotes the image of $\sum_i a_i \otimes \varphi_i$ under the quotient mapping. Finally define $\mathrm{Ind}\rho$ to be the closure of π.

Our major application of the induction procedure will be in the following context. Let G be a discrete group and $\mathcal{A}$ be a *G-graded $*$-algebra*. That is, $\mathcal{A}$ is a direct sum of vector spaces $\mathcal{A}_g,\ g \in G$, such that

$$\mathcal{A}_g \cdot \mathcal{A}_h \subseteq \mathcal{A}_{g\cdot h} \text{ and } (\mathcal{A}_g)^* \subseteq \mathcal{A}_{g^{-1}} \text{ for } g,h \in G. \tag{0.2}$$

The elements of $\cup_{g\in G}\mathcal{A}_g$ are called *homogeneous.* For every subgroup $H \subseteq G$ the sum $\oplus_{g\in H}\mathcal{A}_g =: \mathcal{A}_H$ is a $*$-subalgebra of $\mathcal{A}$ and the canonical projection $p : \mathcal{A} \to \mathcal{A}_H$ is a bimodule projection. If $\mathcal{A}_e$ is commutative then a character $\chi : \mathcal{A}_e \to \mathbb{C}$ is inducible (via $p_e : \mathcal{A} \to \mathcal{A}_e$) if and only if $\chi(a^*a) \geq 0$ for all homogeneous $a \in \mathcal{A}$.

0.2. **Partial actions and partial crossed products.** The constructions and results of this subsection are taken from [Ex, McC]. A *partial action* of a discrete group G on a set X is a pair

$$\alpha = (\{\mathcal{D}_g\}_{g\in G}, \{\alpha_g\}_{g\in G}),$$

where $\mathcal{D}_g \subseteq X,\ g \in G$ are subsets and $\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$ are bijections such that

(i) $\alpha_g(\mathcal{D}_{g^{-1}} \cap \mathcal{D}_h) = \mathcal{D}_{gh} \cap \mathcal{D}_g,\ g,h \in G$,
(ii) $\alpha_{hg}(x) = \alpha_h(\alpha_g(x)),\ x \in \mathcal{D}_{g^{-1}} \cap \mathcal{D}_{g^{-1}h^{-1}}$,
(iii) $\mathcal{D}_e = X,\ \alpha_e = \mathrm{Id}_X$.

For a partial action $\alpha = (\{\mathcal{D}_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ on a topological space X we require in addition that $\mathcal{D}_g$ are open sets and $\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g,\ g \in G$ are homeomorphisms. We call (X, G, α) a *partial dynamical system (p.d.s.).*

For a partial action $\beta = (\{I_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ of G on a C^*-algebra $\mathfrak{B}$ we require in addition that $I_g,\ g \in G$ are closed two-sided ideals and $\beta_g : I_{g^{-1}} \to I_g$ are $*$-isomorphisms. We call $(\mathfrak{B}, G, \beta)$ a *partial C^*-dynamical system (C^*-p.d.s.).* For a p.d.s. (X, G, α) where X is a locally compact Hausdorff space we define the *dual* C^*-p.d.s. as follows. Put $\mathfrak{B} = C_0(X),\ I_g = C_0(\mathcal{D}_g)$ and define $\beta_g : I_{g^{-1}} \to I_g$ by

$$(\beta_g(f))(x) = f(\alpha_{g^{-1}}(x)),\ x \in \mathcal{D}_g, f \in I_{g^{-1}},\ g \in G.$$

Direct computations show that $\beta = (\{I_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is a partial action on $\mathfrak{B}$ and that $(\mathfrak{B}, G, \beta)$ is a C^*-p.d.s.

Let $(\mathfrak{B}, G, \beta),\ \beta = (\{I_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ be a C^*-p.d.s. The *partial crossed product C^*-algebra* $\mathfrak{A} = \mathfrak{B} \times_\beta G$ is the enveloping C^*-algebra of the $*$-algebra $\mathfrak{B}G$ defined as follows. $\mathfrak{B}G \subseteq \mathfrak{B} \otimes \mathbb{C}[G]$ is the linear span of the set $\{a \otimes g \mid a \in \mathcal{I}_g\}$, with multiplication and involution defined by

$$(a \otimes g)(b \otimes h) := \alpha_g(\alpha_{g^{-1}}(a)b) \otimes gh,\ (a \otimes g)^* := \alpha_{g^{-1}}(a^*) \otimes g^{-1}.$$

The examples of C^*-p.d.s. which appear below are of a special kind. Recall [Ex], that a *partial automorphism* of a C^*-algebra $\mathfrak{B}$ is a triple $\Theta = (\theta, I, J)$, where $I, J \subseteq \mathfrak{B}$ are closed two-sided ideals and $\theta : I \to J$, is a $*$-isomorphism. Set $I_0 = \mathfrak{B}$ and define $I_n,\ n \in \mathbb{Z}$, by induction

$$I_{n+1} = \left\{a \in J \mid \theta^{-1}(a) \in I_n\right\},\ \text{for } n \geq 0,$$
$$I_{n-1} = \{a \in I \mid \theta(a) \in I_n\},\ \text{for } n \leq 0.$$

In particular, $I = I_{-1}$ and $J = I_1$. It can be checked, see [Ex, Section 3], that the triple $(\mathfrak{B}, \mathbb{Z}, \beta)$, where $\beta = (\{I_n\}_{n\in\mathbb{Z}}, \{\theta^n\}_{n\in\mathbb{Z}})$ is a C^*-p.d.s. The partial crossed product algebra $\mathfrak{B} \times_\beta \mathbb{Z}$ is called the *covariance algebra of* $(\mathfrak{B}, \Theta)$ and is denoted by $C^*(\mathfrak{B}, \Theta)$. As in the case of a crossed-product by a $*$-automorphism, $*$-representations of $C^*(\mathfrak{B}, \Theta)$ are in one-to-one correspondence with covariant representations of the pair $(\mathfrak{B}, \Theta)$, see [Ex, Section 5]. In case of the C^*-p.d.s. defined by $(\mathfrak{B}, \Theta)$ a *covariant representation* $\pi \times u$ consists of a $*$-representation

$\pi : \mathfrak{B} \to B(\mathcal{H})$ and a partial isometry u, whose initial and final spaces are $\overline{\pi(I)\mathcal{H}}$ and $\overline{\pi(J)\mathcal{H}}$ respectively, so that

$$\pi(\theta(b)) = u\pi(b)u^*, \text{ holds for every } b \in I.$$

If the latter is satisfied, then $\pi \times u$ becomes a $*$-representation of $\mathfrak{B}\mathbb{Z}$, hence of $C^*(\mathfrak{B}, \mathbb{Z})$, via

$$(\pi \times u)(f \otimes k) = \pi(f)u^k, \text{ for } f \otimes k \in \mathfrak{B}\mathbb{Z},$$

where $u^{-k} = u^{*k}$ for $k \in \mathbb{N}$.

0.3. **Unbounded elements affiliated with C^*-algebras and C^*-envelopes.** The theory of unbounded elements affiliated with a C^*-algebra was developed in [Wor1], see also [Lan]. Let $\mathfrak{A}$ be a C^*-algebra and let T be a densely defined closed linear operator on $\mathfrak{A}$. Denote by $D(T) \subseteq \mathfrak{A}$ its domain[1]. The adjoint operator T^* is defined as follows. For $y, z \in \mathfrak{A}$ write $y \in D(T^*)$, $T^*y = z$ if $\langle Tx, y\rangle = \langle x, z\rangle$ holds for all $x \in D(T)$. Following [Lan] we say that T is *affiliated*[2] *with* $\mathfrak{A}$ and write $T\,\eta\,\mathfrak{A}$, if $D(T^*)$ and the range of $1 + T^*T$ are dense in $\mathfrak{A}$, see [Lan, Chapter 9].

Every non-degenerate $*$-representation of a C^*-algebra $\mathfrak{A}$ can be continued to the set $\mathfrak{A}^\eta$ of all operators affiliated with $\mathfrak{A}$. Namely, for every $\pi \in \mathrm{Rep}(\mathfrak{A}, \mathcal{H})$ and $T\,\eta\,\mathfrak{A}$, there exists a closed operator $\pi(T)\,\eta\,\pi(\mathfrak{A})$ with a core $\pi(D(T))\mathcal{H}$ such that

$$\pi(T)(\pi(a)\varphi) = \pi(Ta)\varphi, \text{ for all } \varphi \in \mathcal{H}, \ a \in D(T).$$

Moreover, if $D_0 \subseteq D(T)$ is a core of T, then $\pi(D_0)\mathcal{H}$ is a core of $\pi(T)$.

Definition 0.1. Let $\mathcal{A}$ be a $*$-algebra with a given category of $*$-representations $\mathrm{Rep}\mathcal{A}$ and fixed generators $a_1, \dots, a_n$. We will say that a C^*-algebra $\mathfrak{A}$ is a C^*-*envelope* of $\mathcal{A}$ if there exist affiliated elements $A_1, \dots, A_n\,\eta\,\mathfrak{A}$ such that

$$\pi(A_i) = \overline{\rho(a_i)}, \ i = 1, \dots, n. \tag{0.3}$$

defines an equivalence functor $\rho \mapsto \pi$ between $\mathrm{Rep}\mathcal{A}$ and $\mathrm{Rep}\mathfrak{A}$.

Remarks 1. If every $*$-representation of $\mathcal{A}$ is bounded, then there exists the enveloping C^*-algebra $C^*_{env}(\mathcal{A})$, which is obviously a C^*-envelope of $\mathcal{A}$.
2. In the last definition, the isomorphism class of $\mathfrak{A}$ depends a priori on the choice of the generators a_i and of the category $\mathrm{Rep}\mathcal{A}$. However, we cannot provide any example, where $\mathfrak{A}$ would depend on the generators a_i.

1. The orbit method

In this section we recall the orbit method developed in [SS]. Throughout the section G is a countable discrete group and $\mathcal{A}$ is a G-*graded* $*$-*algebra*. We assume that the $*$-subalgebra $\mathcal{B} := \mathcal{A}_e$ is commutative and denote by $\widehat{\mathcal{B}}$ the set of all characters of $\mathcal{B}$ (i.e. nontrivial $*$-homomorphisms $\chi : \mathcal{B} \to \mathbb{C}$). Further, we define the "positive" spectrum $\widehat{\mathcal{B}}^+ \subseteq \widehat{\mathcal{B}}$ to be the set of all characters $\chi \in \widehat{\mathcal{B}}$ which satisfy[3]

$$\chi(a^*a) \geq 0 \text{ for all homogeneous elements } a \in \mathcal{A}. \tag{1.1}$$

[1] Recall that $\mathcal{D}(\cdot)$ is domain of a Hilbert space operator.
[2] In [Lan] the term *regular operator* on $\mathfrak{A}$ is used.
[3] The theory developed in [SS] requires the additional condition $\chi(c^*d)\chi(d^*c) = \chi(c^*c)\chi(d^*d)$ for all $\chi \in \widehat{\mathcal{B}}^+, g \in G, c, d \in \mathcal{A}_g$, which holds automatically. It can be checked using the equation $(c^*cd^*d)^2 = (c^*cd^*d)(c^*dd^*c)$ which follows by commutativity of $\mathcal{B}$.

Lemma 1.1. *Assume that for every $g \in G$ there exists an element $a_g \in \mathcal{A}_g$ such that $\mathcal{A}_g = a_g \mathcal{B}$. Then $\chi \in \widehat{\mathcal{B}}$ belongs to $\widehat{\mathcal{B}}^+$ if and only if $\chi(a_g^* a_g) \geq 0$ for all $g \in G$.*

Proof. The "only if" part is clear. Assume that $\chi(a_g^* a_g) \geq 0$ for all $g \in G$. By assumption, if $c_g \in \mathcal{A}_g$, then $c_g = a_g b$ for some $b \in \mathcal{B}$. Hence

$$\chi(c_g^* c_g) = \chi(b^* a_g^* a_g b) = \chi(a_g^* a_g)\chi(b^* b) \geq 0.$$

□

The set $\widehat{\mathcal{B}}^+$ consists of those characters which satisfy (0.1), i.e. are inducible from $\mathcal{B}$ to $\mathcal{A}$ via p_e.

Definition 1.1. For $g \in G$ define[4]

$$\mathcal{D}_{g^{-1}} = \left\{ \chi \in \widehat{\mathcal{B}}^+ \mid \chi(a_g^* a_g) \neq 0 \text{ for some } a_g \in \mathcal{A}_g \right\}. \tag{1.2}$$

If $\chi \in \mathcal{D}_{g^{-1}}$ and $\chi(a_g^* a_g) \neq 0$ we set

$$(\alpha_g(\chi))(b) := \frac{\chi(a_g^* b a_g)}{\chi(a_g^* a_g)} \text{ for } b \in \mathcal{B}. \tag{1.3}$$

Direct computations (see [SS, Proposition 13]) show that $\alpha = (\{\alpha_g\}_{g \in G}, \{\mathcal{D}_g\}_{g \in G})$ is a well-defined partial action of G on $\widehat{\mathcal{B}}^+$. We will often write χ^g instead of $\alpha_g(\chi)$. For a character $\chi \in \widehat{\mathcal{B}}^+$ we denote by $\mathrm{Orb}\chi \subseteq \widehat{\mathcal{B}}^+$ its orbit under the partial action of G.

Proposition 1.2 (see Proposition 16 in [SS])**.** *Let $\chi \in \widehat{\mathcal{B}}^+$ and $\pi = \mathrm{Ind}\chi$ be the induced $*$-representation. For every $g \in G$ such that $\chi \in \mathcal{D}_{g^{-1}}$ fix an element $a_g \in \mathcal{A}_g$ such that $\chi(a_g^* a_g) \neq 0$. Then there exists an orthonormal base $\{e_g \mid \chi \in \mathcal{D}_{g^{-1}}\}$ in $\mathcal{D}(\pi)$ such that for $h \in G$ and $b_h \in \mathcal{A}_h$ we have*

$$\pi(b_h)e_g = \frac{\chi(a_{hg}^* b_h a_g)}{\chi(a_{hg}^* a_{hg})^{1/2} \chi(a_g^* a_g)^{1/2}} e_{hg}, \;\; \textit{if } \chi \in \mathcal{D}_{g^{-1}h^{-1}}$$

and $\pi(b_h)e_g = 0$ otherwise. In particular, if $b \in \mathcal{B}$, we have $\pi(b)e_g = \chi^g(b)e_g$.

For an element $b \in \mathcal{B}$ introduce its *"Gel'fand transform"*

$$\widehat{b} : \widehat{\mathcal{B}} \to \mathbb{C}, \; \widehat{b}(\chi) = \chi(b), \; \chi \in \widehat{\mathcal{B}}.$$

We equip $\widehat{\mathcal{B}}$ with the weak topology defined by $\left\{ \widehat{b} \mid b \in \mathcal{B} \right\}$ and the Borel structure generated by the open sets. By definition of $\widehat{\mathcal{B}}^+$ it is a closed subset of $\widehat{\mathcal{B}}$. It can be checked, that the partial action of G is topological. That is, $\mathcal{D}_g, \; g \in G$ are open sets, and $\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$ are homeomorphisms. Since G is countable and the one-point sets are closed, the G-orbits are Borel subsets of $\widehat{\mathcal{B}}^+$.

Definition 1.2. A closed $*$-representation π of $\mathcal{A}$ is called *well-behaved* if:

(i) the restriction $\mathrm{Res}_{\mathcal{B}}\pi$ of π to $\mathcal{B}$ is integrable and there exists a spectral measure E_π on $\widehat{\mathcal{B}}^+$ such that

$$\overline{\pi(b)} = \int_{\widehat{\mathcal{B}}^+} \widehat{b}(\chi) dE_\pi(\chi) \text{ for } b \in \mathcal{B}.$$

[4]In [SS] the notation $\alpha_g : \mathcal{D}_g \to \mathcal{D}_{g^{-1}}$ was used.

(ii) For all $a_g \in \mathcal{A}_g, g \in G$, and all Borel subsets $\Delta \subseteq \widehat{\mathcal{B}}^+$, we have

$$\pi(a_g)E_\pi(\Delta) \supseteq E_\pi(\alpha_g(\Delta \cap \mathcal{D}_{g^{-1}}))\pi(a_g). \tag{1.4}$$

A well-behaved $*$-representation π is *associated with an orbit* Orbχ if E_π is supported on the set Orbχ. Denote by Rep$\mathcal{A}$ the category of well-behaved $*$-representations.

By [SS, Proposition 17], relation (1.4) can be replaced with

$$u_g \int f(t) dE_\pi(t) \subseteq \int_{\mathcal{D}_g} f(\alpha_{g^{-1}}(t)) dE_\pi(t) \cdot u_g. \tag{1.5}$$

where u_g is the partial isometry in the polar decomposition $\overline{\pi(a_g)} = u_g c_g$, and f is any measurable function on $\widehat{\mathcal{B}}^+$. If f is bounded, then "$\subseteq$" becomes an equality.

In the next proposition we collect basic properties of well-behaved $*$-representations. For the proof see Propositions 18, 29 and Theorem 7 in [SS].

Proposition 1.3. *(i) Every bounded $*$-representation is well-behaved.*

(ii) If the partial action of G on $\widehat{\mathcal{B}}^+$ posesses a measurable countably separated section, then every irreducible well-behaved $*$-representation is associated with an orbit.

(iii) Condition (i) in Definition 1.2 holds automatically if $\mathcal{B}$ is countably generated, and the restriction of π on $\mathcal{B}$ is integrable, that is $\overline{\pi(a)}$ is normal for all $b \in \mathcal{B}$.

A measurable set Γ is *countably separated* if and only if there exist Borel sets B_k, $k \in \mathbb{N}$, $\Gamma \subseteq \bigcup_{k\in\mathbb{N}} B_k$ such that for arbitrary $x, y \in \Gamma$, $x \neq y$, we have $x \in B_{k_0}$, $y \notin B_{k_0}$ for some $k_0 \in \mathbb{N}$. A subset Γ containing exactly one point from each orbit is called a *section* of a partial dynamical system.

Recall, that for a subgroup $H \subseteq G$, $\mathcal{A}_H = \oplus_{g\in H}\mathcal{A}_g$ is a $*$-subalgebra of $\mathcal{A}$.

Theorem 1.4 (See Theorem 5 in [SS]). *Let $\chi \in \widehat{\mathcal{B}}^+$ be a character and let $H = \mathrm{St}\chi$ be its stabilizer group. Then the map*

$$\rho \mapsto \mathrm{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}}(\rho) = \pi$$

is a bijection from the set of unitary equivalence classes of inducible $*$-representations ρ of $\mathcal{A}_H$ for which

$$\mathrm{Res}_{\mathcal{B}}\rho \text{ corresponds to a multiple of the character } \chi \tag{1.6}$$

onto the set of unitary equivalence classes of well-behaved $*$-representations π of $\mathcal{A}$ associated with Orbχ. *A $*$-representation ρ satisfying* (1.6) *is bounded and inducible. Moreover, π is irreducible if and only if ρ is irreducible.*

The last theorem suggests the following algorithm for description of all irreducible well-behaved $*$-representations of $\mathcal{A}$:

- determine $\widehat{\mathcal{B}}$, $\widehat{\mathcal{B}}^+$, the partial action of G on $\widehat{\mathcal{B}}^+$ and a section $\Gamma \subseteq \widehat{\mathcal{B}}^+$,
- for each $\chi \in \Gamma$
  - if the stabilizer Stχ is trivial, compute Indχ,
  - otherwise find all irreducible representations ρ of $\mathcal{A}_{\mathrm{St}\chi}$ satisfying (1.6) and compute Indρ.

If the Proposition 1.3, (ii) applies, then we obtain all irreducible well-behaved representations of $\mathcal{A}$.

2. The q-Oscillator Algebra

By the *quantum harmonic oscillator* (*q-oscillator*) we mean the following relation

$$aa^* = \mathbf{1} + qa^*a, \ q > 0. \tag{2.1}$$

In this section, we use the notation of the q-calculus $[[k]]_q = 1 + q + \cdots + q^{k-1}$. Further, we put

$$F(t) := 1 + qt.$$

Clearly $F([[k]]_q) = [[k+1]]_q, \ k \in \mathbb{N}_0$.

In [CGP] the authors have obtained the following representations of (2.1) by Hilbert space operators:

- For every $q > 0$ the Fock representation π_F acting on the orthonormal base $\{\mathbf{e}_k\}_{k\in\mathbb{N}_0}$ as

$$\pi_F(a)\mathbf{e}_k = [[k]]_q^{1/2}\mathbf{e}_{k-1}, \ \pi_F(a^*)\mathbf{e}_k = [[k+1]]_q^{1/2}\mathbf{e}_{k+1}, \text{ where } \mathbf{e}_{-1} := 0. \tag{2.2}$$

- For $q \in (0, 1)$ the series of unbounded $*$-representations $\pi_\gamma, \gamma \in (0, 1]$ acting on the orthonormal base $\{\mathbf{e}_k\}_{k\in\mathbb{Z}}$ as

$$\pi_\gamma(a)\mathbf{e}_k = \left(\frac{1 + q^{\gamma+k}}{1 - q}\right)^{1/2}\mathbf{e}_{k+1}, \ \pi_\gamma(a^*)\mathbf{e}_k = \left(\frac{1 + q^{\gamma+k+1}}{1 - q}\right)^{1/2}\mathbf{e}_{k-1}. \tag{2.3}$$

- For $q \in (0, 1)$ the series of one-dimensional $*$-representations

$$\pi_\varphi(a) = e^{\mathbf{i}\varphi}(1 - q)^{-1/2}, \ \pi_\varphi(a^*) = e^{-\mathbf{i}\varphi}(1 - q)^{-1/2}, \ \varphi \in [0, 2\pi). \tag{2.4}$$

Using the orbit method described in the previous section, we classify all irreducible well-behaved $*$-representations of the *q-oscillator algebra*

$$\mathcal{A} = \mathbb{C}\langle a, a^* \mid aa^* = qa^*a + \mathbf{1}\rangle, \qquad q > 0.$$

We will see that the formulas for the irreducible well-behaved $*$-representations of $\mathcal{A}$ coincide with (2.2)–(2.4).

We now introduce the ingredients needed for the orbit method. Define the $\mathbb{Z}$-grading on $\mathcal{A}$ by setting $a \in \mathcal{A}_1, \ a^* \in \mathcal{A}_{-1}$ and put $\mathcal{B} := \mathcal{A}_0$. It is easily checked that $\mathcal{B} = \mathbb{C}[N]$, where $N = a^*a$, and $\mathcal{A}_n = a^n\mathcal{B}, \ \mathcal{A}_{-n} = a^{*n}\mathcal{B}$ for every $n \in \mathbb{N}$. Using induction on $k \in \mathbb{N}$ we obtain the relations

$$\begin{aligned} a^k a^{*k} &= \prod_{j=1}^{k} \left(q^j N + [[j]]_q\mathbf{1}\right), \ k \in \mathbb{N}. \\ a^{*k} a^k &= \prod_{j=0}^{k-1} \left(q^{-j} N + [[-j]]_q\mathbf{1}\right), \ k \in \mathbb{N}. \end{aligned} \tag{2.5}$$

Since $\mathcal{B} = \mathbb{C}[N]$, every character on $\mathcal{B}$ is of the form $\chi_t(N) = t \in \mathbb{R}$. In what follows we identify the space of all characters $\widehat{\mathcal{B}}$ with $\mathbb{R}$.

Proposition 2.1. *(i)* $\widehat{\mathcal{B}}^+ = \{[[k]]_q \mid k \in \mathbb{N}_0\}$ for $q \geq 1$,

$\widehat{\mathcal{B}}^+ = \{[[k]]_q \mid k \in \mathbb{N}_0\} \cup [1/(1-q), +\infty)$ for $q \in (0, 1)$.

(ii) The partial action $\alpha = (\{\mathcal{D}_n\}_{n\in\mathbb{Z}}, \{\alpha_n\}_{n\in\mathbb{Z}})$ is given as follows.

$$\mathcal{D}_{-n} = \{[[k]]_q \mid k \geq n\} \ \text{ if } q \geq 1,$$
$$\mathcal{D}_{-n} = \{[[k]]_q \mid k \geq n\} \cup [1/(1-q), \infty) \ \text{ if } q \in (0, 1).$$

If $\chi_t \in \mathcal{D}_{-n}$, then $\chi_t^n = \chi_{F^{-n}(t)}$. In particular, $\chi^n_{[[k]]_q} = \chi_{[[k-n]]_q}$ for $n \le k$.

Proof. (i) By Lemma 1.1 a character $\chi \in \widehat{\mathcal{B}}$ belongs to $\widehat{\mathcal{B}}^+$ if and only if $\chi(a^k a^{*k}) \geq 0,\ \chi(a^{*k}a^k) \geq 0$ for all $k \in \mathbb{N}$. Further, (2.5) implies that $a^n a^{*n} = \sum_{j=0}^n \alpha_j N^j$ for some $\alpha_j \geq 0$. Hence $\chi_t \in \widehat{\mathcal{B}}^+$ if and only if $\chi(a^{*k}a^k) = \chi\left(\prod_{j=0}^{k-1}\left(q^{-j}N + [[-j]]_q \mathbf{1}\right)\right) \geq 0$ for all $k \in \mathbb{N}$. The last system of inequalities is equivalent to

$$\prod_{j=0}^{k} (t - [[j]]_q) \geq 0 \text{ for all } k \in \mathbb{N}_0. \tag{2.6}$$

Consider first $q \geq 1$. Then $[[k]]_q \to \infty,\ k \to \infty$, and (2.6) is satisfied if and only if $t = [[k]]_q$ for some $k \in \mathbb{N}_0$. If $q \in (0,1)$, then $[[k]]_q \to \frac{1}{1-q},\ k \to \infty$, and every $t \geq \frac{1}{1-q}$ satisfies (2.6). For $t \in \left[0, \frac{1}{1-q}\right)$, (2.6) holds if and only if $t = [[k]]_q$ for some $k \in \mathbb{N}_0$.
(ii) One can verify by induction on $n \in \mathbb{N}$ that $F^n(t) = q^n t + [[n]]_q$ for all $n \in \mathbb{Z}$. Using (2.5), we obtain

$$\chi_t^n(N) = \frac{\chi_t(a^{*n}Na^n)}{\chi_t(a^{*n}a^n)} = \chi_t(q^{-n}N + [[-n]]_q) = \chi_{F^{-n}(t)}(N),$$

for $\chi_t \in \mathcal{D}_{-n},\ n \in \mathbb{N}$, and

$$\chi_t^{-n}(N) = \frac{\chi_t(a^n N a^{*n})}{\chi_t(a^n a^{*n})} = q^{-1}\chi_t(q^{n+1}N + [[n+1]]_q) - q^{-1} = \chi_{F^n(t)}(N).$$

for $\chi_t \in \mathcal{D}_n,\ n \in \mathbb{N}$. Inequalities (2.6) imply that for $q > 0$ and $t = [[k]]_q$, we have $\chi_t \in \mathcal{D}_{-n}$ if and only if $n \leq k$. In case $q \in (0,1)$ and $t \geq \frac{1}{1-q}$, we have $\chi_t \in \mathcal{D}_{-n}$ for all $n \in \mathbb{Z}$. □

Using Proposition 2.1, we conclude that the stabilizer $\mathrm{St}\chi_t$ of $\chi_t \in \widehat{\mathcal{B}}^+$ is trivial except for the case $t = 1/(1-q)$, where the stabilizer is $\mathbb{Z}$. Define the subset $\Gamma \subseteq \widehat{\mathcal{B}}^+$ as

$$\Gamma = \{0\} \cup \left\{\frac{1}{1-q}\right\} \cup \left\{\frac{1+q^\gamma}{1-q} \mid \gamma \in (0,1]\right\}, \quad \text{if } q \in (0,1),$$
$$\Gamma = \{0\}\,, \quad \text{if } q \geq 1.$$

Direct computations using Proposition 2.1 show that each orbit under the partial action of $\mathbb{Z}$ on $\widehat{\mathcal{B}}^+$ intersects Γ in exactly one point, i.e. Γ is a section of the partial action. The topology on $\widehat{\mathcal{B}}^+$ is induced from the standard topology on $\mathbb{R}$. Hence Γ is countably separated and measurable. By Proposition 1.3(ii) every irreducible well-behaved $*$-representation of $\mathcal{A}$ is associated to some $\mathrm{Orb}\chi,\ \chi \in \Gamma$. For $\chi \in \Gamma$ we consider three cases.

(i) Case $\chi = \chi_0$. Since the stabilizer of χ is trivial, the only irreducible well-behaved $*$-representation associated to $\mathrm{Orb}\chi$ is (up to unitary equivalence) $\pi_F := \mathrm{Ind}\chi$. Using Proposition 1.2 and relations (2.5), we calculate the action of π_F on the orthonormal basis $\{e_{-n}\}_{n\in\mathbb{N}_0}$ of the representation space $\mathcal{H}_{\pi_F}$.

$$\pi_F(a)e_{-n} = \frac{\chi(a^{n-1}aa^{*n})}{\chi(a^{n-1}a^{*(n-1)})^{1/2}\chi(a^na^{*n})^{1/2}}e_{-n+1} = \frac{\chi(a^na^{*n})^{1/2}}{\chi(a^{n-1}a^{*(n-1)})^{1/2}}e_{-n+1}$$
$$= q^{n/2}\left(\chi(N) - [[-n]]_q\right)^{1/2} e_{-n+1} = [[n]]_q^{1/2}e_{-n+1},$$
$$\pi_F(a^*)e_{-n} = \frac{\chi_0(a^{n+1}a^*a^{*n})}{\chi_0(a^{n+1}a^{*(n+1)})^{1/2}\chi_0(a^na^{*n})^{1/2}}e_{-n-1} = \frac{\chi_0(a^{n+1}a^{*(n+1)})^{1/2}}{\chi_0(a^na^{*n})^{1/2}}e_{-n-1}$$
$$= q^{(n+1)/2}(\chi(N) - [[-n-1]]_q)^{1/2}e_{-n-1} = [[n+1]]_q^{1/2}e_{-n-1},$$

where $e_1 := 0$ and $n \in \mathbb{N}_0$. It exists for any $q > 0$ and is bounded if and only if $q \in (0, 1)$.

(*ii*) Case $\chi = \chi_{\frac{1+q^\gamma}{1-q}}$, $\gamma \in (0, 1]$. The stabilizer of χ is again trivial, thus $\pi_\gamma := \mathrm{Ind}\chi_{\frac{1+q^\gamma}{1-q}}$ is the only irreducible well-behaved $*$-representation associated to $\mathrm{Orb}\chi$. We calculate the action of $\pi_\gamma(a)$ respectively $\pi_\gamma(a^*)$ using Proposition 1.2 and relations (2.5). For $n \in \mathbb{Z}$ we have

$$\pi_\gamma(a)e_n = \frac{\chi(a^{*(n+1)}aa^n)}{\chi(a^{*(n+1)}a^{n+1})^{1/2}\chi(a^{*n}a^n)^{1/2}}e_{n+1} = \frac{\chi(a^{*(n+1)}a^{n+1})^{1/2}}{\chi(a^{*n}a^n)^{1/2}}e_{n+1}$$
$$= \left(q^{-n}\frac{1+q^\gamma}{1-q} + \frac{1-q^{-n}}{1-q}\right)^{1/2} e_{n+1} = \left(\frac{1+q^{\gamma-n}}{1-q}\right)^{1/2} e_{n+1}.$$

In the same way we obtain

$$\pi_\gamma(a^*)e_n = \left(\frac{1+q^{\gamma-n+1}}{1-q}\right)^{1/2} e_{n-1},\ \text{for } n \in \mathbb{Z}.$$

Note that π_γ is not bounded for every $\gamma \in (0, 1]$.

(*iii*) Case $\chi = \chi_{\frac{1}{1-q}}$. The stabilizer group H of $\chi_{\frac{1}{1-q}}$ is $\mathbb{Z}$. Let ρ be an irreducible $*$-representation of $\mathcal{A}$ satisfying (1.6). Since $\chi(aa^* - a^*a) = 0$, we have $\rho(a)\rho(a^*) = \rho(a^*)\rho(a)$. By Schur's Lemma ρ is one-dimensional. For $\lambda \in \mathbb{C}$ such that $\lambda = \rho(a)$ we get $|\lambda|^2 = \rho(aa^*) = \rho(\mathbf{1} + qa^*a) = 1 + q\,|\lambda|^2$. Hence $\rho = \rho_\varphi$, for some $\varphi \in [0, 2\pi)$, where $\rho_\varphi(a) = e^{\mathrm{i}\varphi}(1-q)^{-1/2}$. Since $\mathcal{A}_H = \mathcal{A}$, $\pi_\varphi := \mathrm{Ind}_{\mathcal{A}_H\uparrow\mathcal{A}}\rho_\varphi$ is equivalent to ρ_φ and we have

$$\pi_\varphi(a) = e^{\mathbf{i}\varphi}(1-q)^{-1/2},\ \pi_\varphi(a^*) = e^{-\mathbf{i}\varphi}(1-q)^{-1/2},\ \varphi \in [0, 2\pi).$$

By Theorem 1.4 these are all (up to unitary equivalence) irreducible well-behaved $*$-representations of $\mathcal{A}$. Moreover, putting $\mathbf{e}_k := e_{-k}$, $k \in \mathbb{Z}$, we see that the above formulas coincide with (2.2), (2.3) and (2.4) respectively. We have proved the following

Theorem 2.2. *Every irreducible well-behaved $*$-representation of the q-oscillator algebra, $q > 0$, is induced from a one-dimensional $*$-representation.*

2.1. **Existence of bad $*$-representations.** In this subsection we prove the existence of a $*$-representation π of $\mathcal{A}$ which is not well-behaved and which cannot be continued to a well-behaved representation in a possibly larger Hilbert space. The idea is similar to the proof of [S1, Theorem 4.1].

Lemma 2.3. *The polynomial*

$$p := (N-1)(N-(1+q)) \in \mathbb{C}[N]$$

is positive in every well-behaved $*$-representation of $\mathcal{A}$ and $p \notin \sum \mathcal{A}^2$.

Proof. We first show that every element of $\sum \mathcal{A}^2 \cap \mathcal{B}$ is of the form

$$\sum_{k=0}^{n} a^{*k} a^k \cdot p_k^* p_k, \text{ where } p_k \in \mathbb{C}[N],\ n \in \mathbb{N}. \tag{2.7}$$

Indeed, an element $b \in \mathcal{B}$ belongs to $\sum \mathcal{A}^2$ if and only if $b = \sum b_j^* b_j$, where $b_j \in \mathcal{A}_{k_j}$, $k_j \in \mathbb{Z}$. Since $\mathcal{A}_k = a^k \cdot \mathcal{B}$ for every $k \in \mathbb{Z}$ (here $a^{-k} = a^{*k}$ for $k > 0$), we obtain $b = \sum a^{*k} a^k \cdot s_k$, where $s_k \in \sum \mathcal{B}^2$. It is a well-known fact, that every positive polynomial in $\mathbb{C}[N]$ is a single square $r^* r$. Hence $s_k = p_k^* p_k,\ p_k \in \mathbb{C}[N]$, for $k \in \mathbb{Z}$. Furthermore, relations (2.5) imply $a^n a^{*n} \in \sum \mathcal{B}^2 + a^* a \sum \mathcal{B}^2$, which proves (2.7).

Let π be a well-behaved $*$-representation of $\mathcal{A}$ with associated spectral measure E_π. Since $\operatorname{supp} E_\pi \subseteq \widehat{\mathcal{B}}^+$ and $p \geq 0$ on $\widehat{\mathcal{B}}^+$, we have $\pi(p) = \int_{\widehat{\mathcal{B}}^+} p(\lambda) dE_\pi(\lambda) \geq 0$.

Assume to the contrary that $p \in \sum \mathcal{A}^2$. Since the degree of $p(N)$ in $\mathbb{C}[N]$ is 2, we get by (2.7)

$$p = f^* f + a^* a \cdot g^* g + a^{*2} a^2 \cdot h^* h = f^* f + N g^* g + N(N-1) h^* h$$

for some polynomials $f, g, h \in \mathbb{C}[N]$, where $\deg f \leq 1,\ \deg g = 0$ and $\deg h = 0$, that is, g and h are constant. Setting $N := 1$, we obtain $|f(1)|^2 + |g|^2 = 0$, i.e. $g = 0,\ f(1) = 0$. Setting $N := 1 + q$, we get $|f(1+q)|^2 + q(1+q)|h|^2 = 0$ which implies $h = f(1+q) = 0$. Since $\deg f \leq 1,\ f \equiv 0$, i.e. $p \equiv 0$, a contradiction. □

For the prove of the next theorem we will need the following technical result, see [S2, Lemma 2].

Lemma 2.4. *Let $\mathcal{A}$ be a unital $*$-algebra which has a faithful $*$-representation π (that is, $\pi(a) = 0$ implies that $a = 0$) and is a union of a sequence of finite dimensional subspaces E_n, $n \in \mathbb{N}$. Assume that for each $n \in \mathbb{N}$ there exists a number $k_n \in \mathbb{N}$ such that the following is satisfied: If $a \in \sum \mathcal{A}^2$ is in E_n, then we can write a as a finite sum $\sum a_j^* a_j$ such that all a_j are in E_{k_n}.*
Then the cone $\sum \mathcal{A}^2$ is closed in $\mathcal{A}$ with respect to the finest locally convex topology on $\mathcal{A}$.

Theorem 2.5. *There exists a $*$-representation π of the q-oscillator algebra $\mathcal{A}$ which cannot be extended to a well-behaved representation in a possibly larger Hilbert space.*

Proof. Since $p \notin \sum \mathcal{A}^2$ and $\sum \mathcal{A}^2$ is closed by Lemma 2.4, there exists a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ such that $\varphi(\sum \mathcal{A}^2) \geq 0$ and $\varphi(p) < 0$ by the Hahn-Banach Theorem. Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be its GNS-construction (see [S3, Section 8.6.]). Assume to the contrary, that that π_φ has a well-behaved extension, say π. Then $\langle \pi(p)\xi_\varphi, \xi_\varphi \rangle = \langle \pi_\varphi(p)\xi_\varphi, \xi_\varphi \rangle = \varphi(p) < 0$. On the other hand, $\pi(p) \geq 0$ by Lemma 2.3, a contradiction. □

2.2. **C^*-envelope of the q-oscillator algebra.** In this subsection we show that $\mathcal{A}$, considered with the category Rep$\mathcal{A}$ and generators a, a^* has a C^*-envelope $\mathfrak{A}$ in the sense of Definition 0.1. For let $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ be the C^*-p.d.s. dual to $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$. More precisely, define the partial action $\beta = (\{I_k\}_{k\in\mathbb{Z}}, \{\beta_k\}_{k\in\mathbb{Z}})$ on $C_0(\widehat{\mathcal{B}}^+)$ by setting $I_k := C_0(\mathcal{D}_k)$ and

$$(\beta_k(f))(t) := f(\alpha_{-k}(t)) = f(F^k(t)), \text{ for } f \in I_{-k},\ t \in \mathcal{D}_k.$$

Proposition 2.1 implies that the C^*-p.d.s. $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ is defined by the partial automorphism $\Theta = (\theta, I, J)$, where

$$I = I_{-1},\ J = I_1 = C_0(\widehat{\mathcal{B}}^+),\ (\theta(f))(t) = (\beta_1(f))(t) = f(1+qt),\ f \in I_{-1}.$$

We define

$$\mathfrak{A} := C^*(C_0(\widehat{\mathcal{B}}^+), \Theta) = C_0(\widehat{\mathcal{B}}^+) \times_\beta \mathbb{Z}.$$

Theorem 2.6. *Consider the q-oscillator algebra $\mathcal{A}$ with generators a, a^* and the category of well-behaved representations* Rep$\mathcal{A}$. *Then $\mathfrak{A}$ is a C^*-envelope of $\mathcal{A}$.*

Proof. Let $\mathfrak{A}_0$ be the linear hull of

$$\{f \otimes k \in \mathfrak{A} \mid k \in \mathbb{Z},\ \mathrm{supp} f \subseteq \mathcal{D}_k \text{ is compact}\}.$$

$\mathfrak{A}_0$ is obviously dense in $\mathfrak{A}$. For $f \otimes k \in \mathfrak{A}_0$ define

$$A(f(t) \otimes k) = \sqrt{1+qt} f(1+qt) \otimes (k+1),$$

and

$$A^*(f(t) \otimes k) = \sqrt{t} f(q^{-1}t - q^{-1}) \otimes (k-1)$$

Then A and A^* are densely defined linear operators on $\mathfrak{A}$ and their closures, denoted again by A and A^*, are adjoint to each other. For $f \otimes k \in \mathfrak{A}_0$ we have

$$A^*A(f(t) \otimes k) = A^*(\sqrt{\alpha_{-1}(t)} f(\alpha_{-1}(t)) \otimes (k+1)) = tf(t) \otimes k.$$

The last equation shows that the range of $I + A^*A$ is dense in $\mathfrak{A}_0 \subseteq \mathfrak{A}$, so that A is affiliated with $\mathfrak{A}$. By [Wor1, Theorem 1.4.] the adjoint A^* is also affiliated with $\mathfrak{A}$. We show the correspondence (0.3) between the generators $a, a^* \in \mathcal{A}$ and affiliated elements $A, A^* \,\eta\, \mathfrak{A}$.

By [Ex, Theorem 5.6] every $*$-representation of $\mathfrak{A}$ is given by a covariant representation $\pi \times u$ of $(C_0(\widehat{\mathcal{B}}^+), \Theta)$. Here $\pi : C_0(\widehat{\mathcal{B}}^+) \to B(\mathcal{H}_\pi)$ is a $*$-representation of $C_0(\widehat{\mathcal{B}}^+)$ and u is a partial isometry on $\mathcal{H}_\pi$ satisfying $\pi(\theta(b)) = u\pi(b)u^*$ for every $b \in I$. By the spectral theory of commutative C^*-algebras, there exists a unique spectral measure E_π on $\widehat{\mathcal{B}}^+$ such that

$$\pi(f) = \int_{\widehat{\mathcal{B}}^+} f(t) dE_\pi(t),\ f \in C_0(\widehat{\mathcal{B}}^+).$$

By definition of θ for $f \in I_{-1} = C_0(\mathcal{D}_{-1})$ we have

$$u\left(\int f dE_\pi\right) u^* = u\pi(f)u^* = \pi(\theta(f)) = \int f(1+qt) dE_\pi(t).$$

Multiplying the latter by u from the right and remembering that the initial space of u is $\overline{\pi(I_{-1})\mathcal{H}_\pi} = E_\pi(\mathcal{D}_{-1})\mathcal{H}_\pi$ we get

$$u \int f dE_\pi = \int f(1+qt) dE_\pi(t) \cdot u, \text{ for } f \in I_{-1}. \tag{2.8}$$

The extension of $(\pi \times u)$ to A and A^* is given by

$$(\pi \times u)(A) = u \int \sqrt{t} dE_\pi,\ (\pi \times u)(A^*) = \int \sqrt{t} dE_\pi(t) \cdot u^* \tag{2.9}$$

Indeed, for every $f\otimes k\in\mathfrak{A}_0$ we have

$$u\int\sqrt{t}dE_\pi\cdot((\pi\times u)(f\otimes k))=u\int\sqrt{t}dE_\pi\cdot\int fdE_\pi\cdot u^k=$$
$$=u\int\sqrt{t}f(t)dE_\pi\cdot u^k=\int\sqrt{1+qt}f(1+qt)dE_\pi\cdot u^{k+1}=(\pi\times u)(A(f\otimes k)).$$

Since $\mathfrak{A}_0\subseteq\mathfrak{A}$ is a core of A, $\pi(\mathfrak{A}_0)\mathcal{H}_\pi$ is a core of A, and we get the first part of (2.9). The second part follows from $(\pi\times u)(A^*)=((\pi\times u)(A))^*$.

Let ρ be a well-behaved $*$-representation of $\mathcal{A}$, and E_ρ be the corresponding spectral measure on $\widehat{\mathcal{B}}^+\subseteq\mathbb{R}_+$. Further, let $\overline{\rho(a)}=u_1c_1$ be the polar decomposition of $\overline{\rho(a)}$. Since a^*a is the generator of $\mathcal{B}$, E_ρ coincides with the spectral measure of $\overline{\rho(a^*a)}=\rho(a)^*\overline{\rho(a)}$. Hence

$$\overline{\rho(a)}=u_1\int\sqrt{t}dE_\rho,\ \text{and}\ \overline{\rho(a^*)}=\int\sqrt{t}dE_\rho\cdot u_1^*. \tag{2.10}$$

Since $\ker u_1=\ker c_1$, the initial space of u_1 is the range of $E_\rho(\widehat{\mathcal{B}}^+\setminus\{0\})=E_\rho(\mathcal{D}_{-1})$. Further, we have $u_1c_1^2u_1^*=1+qc_1^2$, which implies that $\ker u_1^*$ is trivial, so that the final space of u_1 is $E_\rho(\widehat{\mathcal{B}}^+)=E_\rho(\mathcal{D}_1)$. Applying (1.5) to $f\in C_0(\mathcal{D}_{-1})$ we obtain

$$u_1\rho(f)u_1^*=u_1\int f(t)dE_\rho(t)\cdot u_1^*=\int f(\alpha_{-1}(t))dE_\rho(t)\cdot u_1^*u_1=$$
$$=\int f(1+qt)dE_\rho(t)=\rho(\theta(f)),$$

i.e. $(\rho_{\mathcal{B}}\times u_1)$, where $\rho_{\mathcal{B}}$ is the restriction of ρ to $\mathcal{B}$, defines a covariant representation of $(C_0(\widehat{\mathcal{B}}^+),\Theta)$.

The correspondence (0.3) between π and ρ follows now by comparing (2.9) with (2.10). □

Remark. In [Wor2, Section 3] the author shows that the operators p,q of the Weyl algebra W_1 generate a C^*-algebra $\mathfrak{A}$ in the sense of the Definition 3.1 therein, and that $\mathfrak{A}$ is the algebra of compact operators. It corresponds to the fact that the C^*-envelope of q-CCR with $q=1$ is isomorphic to the partial crossed product $C_0(\mathbb{N}_0)\times_\alpha\mathbb{Z}\simeq K(l^2(N_0))$.

3. The Podleś Sphere

In this section we investigate $*$-representations of the Podles' sphere $\mathcal{O}(S^2_{qr})$. We consider only the case $q\in(0,1)$, $r\in(0,\infty)$. The cases $r=0$, $r=\infty$ can be treated similarly. Recall [Pd] that $\mathcal{A}:=\mathcal{O}(S^2_{qr})$ is the unital $*$-algebra generated by $a=a^*,b,b^*$ and defining relations

$$ab=q^{-2}ba,\ ab^*=q^2b^*a,\ b^*b=a-a^2+r\mathbf{1},\ bb^*=q^2a-q^4a^2+r\mathbf{1}. \tag{3.1}$$

The defining relations imply that every $*$-representation of $\mathcal{A}$ is bounded and hence well-behaved by Proposition 1.3 (i). In [Pd] the following irreducible $*$-representations of $\mathcal{A}$ were obtained.

- Two infinite-dimensional $*$-representations $\pi_\pm$ which act on an orthonormal base $\{\mathbf{e}_k\}_{k\in\mathbb{N}_0}$ of the representation space $\mathcal{H}_\pm$ by

$$\pi_\pm(a)\mathbf{e}_k = q^{2k}\lambda_\pm \mathbf{e}_k,\ \pi_\pm(b)\mathbf{e}_k = \left(q^{2k}\lambda_\pm - (q^{2k}\lambda_\pm)^2 + r\right)^{1/2}\mathbf{e}_{k-1},$$
$$\pi_\pm(b^*)\mathbf{e}_k = \left(q^{2(k+1)}\lambda_\pm - (q^{2(k+1)}\lambda_\pm)^2 + r\right)^{1/2}\mathbf{e}_{k+1},\ \mathbf{e}_{-1} := 0,$$

where $\lambda_\pm := \frac{1}{2} \pm (r + \frac{1}{4})^{1/2}$.
- The series of one-dimensional $*$-representations $\pi_\varphi,\ \varphi \in [0, 2\pi)$,

$$\pi_\varphi(a) = 0,\ \pi_\varphi(b) = e^{\mathrm{i}\varphi} r^{1/2},\ \pi_\varphi(b^*) = e^{-\mathrm{i}\varphi} r^{1/2}.$$

Using (3.1) and induction on $n \in \mathbb{N}$ we obtain the following relations

$$\begin{aligned} &ab^n = q^{-2n} b^n a,\ ab^{*n} = q^{2n} b^{*n} a. \\ &b^{*n}b^n = \prod_{j=1}^{n}\left(q^{-2(j-1)}a - q^{-4(j-1)}a^2 + r\right),\ b^n b^{*n} = \prod_{j=1}^{n}\left(q^{2j}a - q^{4j}a^2 + r\right). \end{aligned} \tag{3.2}$$

Define a $\mathbb{Z}$-grading on $\mathcal{A}$ by setting $a \in \mathcal{A}_0,\ b \in \mathcal{A}_1$ and $b^* \in \mathcal{A}_{-1}$. Using the defining relations one easily derives

$$\mathcal{B} := \mathcal{A}_0 = \mathrm{Lin}\{a^l b^{*m} b^m \mid l, m \in \mathbb{N}_0\},\quad \mathcal{A}_n := b^n\mathcal{B},\quad \mathcal{A}_{-n} := b^{*n}\mathcal{B},$$

where $n \in \mathbb{N}_0$. Further, relations (3.2) imply that $\mathcal{B} = \mathbb{C}[a]$, hence $\widehat{\mathcal{B}} = \{\chi_t \mid t \in \mathbb{R}\}$ where $\chi_t(a) = t$. As in the previous section we identify $\widehat{\mathcal{B}}$ with $\mathbb{R}$.

Proposition 3.1. *(i)* $\widehat{\mathcal{B}}^+ = \{\chi_{m,+}\}_{m\in\mathbb{N}_0} \cup \{\chi_{m,-}\}_{m\in\mathbb{N}_0} \cup \{\chi_\infty\}$, *where* $\chi_{m,\pm}$ *denotes* $\chi_t,\ t = q^{2m}\lambda_\pm$ *and* χ_∞ *denotes* $\chi_t,\ t = 0$.

(ii) The partial action $\alpha = (\{\mathcal{D}_n\}_{n\in\mathbb{Z}}, \{\alpha_n\}_{n\in\mathbb{Z}})$ *is given as follows:*

$$\mathcal{D}_{-n} = \{\chi_{m,\pm} \mid m \geq n\} \cup \{\chi_\infty\},\ \text{and}\ \chi^n_{m,\pm} = \chi_{m-n,\pm},\ \chi^n_\infty = \chi_\infty.$$

Proof. (i) Lemma 1.1 and relations (3.2) imply that $\chi_t,\ t \in \mathbb{R}$ belongs to $\widehat{\mathcal{B}}^+$ if and only if the following inequalities are satisfied for all $n \in \mathbb{N}$:

$$\begin{aligned} \chi_t(b^{*n}b^n) &= \prod_{k=0}^{n-1}(q^{-2k}t - q^{-4k}t^2 + r) \geq 0, \\ \chi_t(b^n b^{*n}) &= \prod_{k=1}^{n}(q^{2k}t - q^{4k}t^2 + r) \geq 0. \end{aligned} \tag{3.3}$$

Assume $q^{-2k}t - q^{-4k}t^2 + r > 0$ for all $k \in \mathbb{N}_0$. Since $q^{-2k} \to +\infty,\ k \to +\infty$, it is possible only if $t = 0$, i.e. $\chi_t = \chi_\infty$. If $t \neq 0$ we get $q^{-2k}t - q^{-4k}t^2 + r = 0$ for some $k \in \mathbb{N}_0$, whence

$$t = \frac{-q^{-2k} \pm \sqrt{q^{-4k} + 4q^{-4k}r}}{-2q^{-4k}} = q^{2k}\lambda_\mp.$$

One can easily check that every $t = q^{2k}\lambda_\pm,\ k \in \mathbb{N}_0$, satisfies (3.3).

(ii) Relations (3.3) imply that $\mathcal{D}_{-n} = \{\chi_{m,\pm} \mid m \geq n\} \cup \{\chi_\infty\}$. Assume that $\chi_{m,\pm} \in \mathcal{D}_{-n}$, where $n \in \mathbb{N}_0$. Using relations (3.2) we obtain

$$\chi^n_{m,\pm}(a) = \frac{\chi_{m,\pm}(b^{*n}ab^n)}{\chi_{m,\pm}(b^{*n}b^n)} = \frac{\chi_{m,\pm}(b^{*n}b^n)\chi_{m,\pm}(q^{-2n}a)}{\chi_{m,\pm}(b^{*n}b^n)} = \chi_{m-n,\pm}(a).$$

For χ_∞ we have $\chi_\infty(b^{*n}b^n) = \chi_\infty(b^n b^{*n}) = r^n \neq 0$ for all $n \in \mathbb{Z}$ by equations (3.3). Hence $\chi_\infty \in \mathcal{D}_n$ for all $n \in \mathbb{Z}$ and $\chi_\infty^n(a) = 0$. □

Let Γ be the subset $\{\chi_{0,+}, \chi_{0,-}, \chi_\infty\} \subseteq \widehat{\mathcal{B}}^+$. Obviously Γ is a measurable countably separated section of the p. d. s. $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$. We calculate all irreducible $*$-representations associated with $\mathrm{Orb}\chi,\ \chi \in \Gamma$.

(*i*) Case $\chi_{0,\pm}$. The stabilizer of $\chi_{0,\pm}$ is trivial by Proposition 3.1, (ii). Put $\pi_\pm := \mathrm{Ind}\chi_{0,\pm}$. We use Proposition 1.2 to compute the action of $\pi_\pm$ on the orthonormal base $\{e_{-k}\}_{k\in\mathbb{N}_0}$.

$$\begin{aligned}\pi_\pm(b)e_{-k} &= \left(q^{2k}\chi_{0,\pm}(a) - q^{4k}\chi_{0,\pm}(a^2) + r\right)^{1/2} e_{-k+1} \\ &= \left(q^{2k}\lambda_\pm - \left(q^{2k}\lambda_\pm\right)^2 + r\right)^{1/2} e_{-k+1}, \\ \pi_\pm(b^*)e_{-k} &= \left(q^{2(k+1)}\lambda_\pm - \left(q^{2(k+1)}\lambda_\pm\right)^2 + r\right)^{1/2} e_{-k-1}, \\ \pi_\pm(a)e_{-k} &= \chi_{0,\pm}^{-k}(a) = q^{2k}\lambda_\pm e_{-k}.\end{aligned}$$

(*ii*) Case χ_∞. The stabilizer group H of χ is $\mathbb{Z}$. Let ρ be an irreducible $*$-representation of $\mathcal{A}_H$ satisfying (1.6). Since $\chi(bb^* - b^*b) = 0$, we have $\rho(b)\rho(b^*) = \rho(b^*)\rho(b)$. By Schur's Lemma ρ is one-dimensional. For $\lambda \in \mathbb{C}$ such that $\lambda = \rho(b)$ we get $|\lambda|^2 = \rho(bb^*) = r$. Hence $\rho = \rho_\varphi$, for some $\varphi \in [0, 2\pi)$, where $\rho_\varphi(b) = e^{\mathrm{i}\varphi} r^{1/2}$. Since $\mathcal{A}_H = \mathcal{A}$, $\pi_\varphi := \mathrm{Ind}_{\mathcal{A}_H\uparrow\mathcal{A}}\rho_\varphi$ is equivalent to ρ_φ and we get

$$\pi_\varphi(a) = 0,\ \pi_\varphi(b) = e^{\mathbf{i}\varphi} r^{1/2},\ \pi_\varphi(b^*) = e^{-\mathbf{i}\varphi} r^{1/2},\ \varphi \in [0, 2\pi).$$

By Theorem 1.4 these are all, up to unitary equivalence, irreducible $*$-representations of $\mathcal{A}$. Setting $\mathbf{e}_k := e_{-k},\ k \in \mathbb{N}_0$, we see that these coincide with the ones found in [Pd]. In particular, we have the following

Theorem 3.2. *Every irreducible $*$-representation of the Podleś sphere $\mathcal{O}(S^2_{qr})$, $q \in (0,1)$, $r \in (0, \infty)$ is induced from a one-dimensional $*$-representation.*

In the remaining part of this section we describe the enveloping C^*-algebra of $\mathcal{O}(S^2_{qr})$. For let $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ be the C^*-p.d.s. dual to $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$ as defined in the Subsection 0.2. Note, that the sets $\mathcal{D}_k,\ k \in \mathbb{Z}$, are compact, hence $I_k := C(D_k)$. By definition of β we have

$$(\beta_k(f))(t) = f(\alpha_{-k}(t)) = f(q^{2k}t),\ f \in I_{-k}, k \in \mathbb{Z}. \tag{3.4}$$

It is easily seen from the description of α in the Proposition 3.1 that the partial action $\beta = (\{I_k\}_{k\in\mathbb{Z}}, \{\beta_k\}_{k\in\mathbb{Z}})$ is defined by the partial automorphism $\Theta = (\theta, I, J)$, where $\theta = \beta_1,\ I = I_{-1},\ J = I_1 = \mathcal{A}$.

Theorem 3.3. *The enveloping C^*-algebra $\mathfrak{A}$ of $\mathcal{A}$ is isomorphic to the covariance algebra $C^*(C(\widehat{\mathcal{B}}^+), \Theta) \simeq C(\widehat{\mathcal{B}}^+) \times_\beta \mathbb{Z}$*

Proof. The proof goes similarly to the proof of the Theorem 2.6 by replacing the η-relation with $\in$-relation.

We first define $*$-homomorphism $\epsilon : \mathcal{A} \to C(\widehat{\mathcal{B}}^+) \times_\beta \mathbb{Z}$ by setting

$$\epsilon(a) = t \otimes 0,\ \epsilon(b) = (q^2t - q^4t^2 + r)^{1/2} \otimes 1,\ \epsilon(b^*) = (t - t^2 + r)^{1/2} \otimes (-1).$$

Direct computations using (3.4) show that $\epsilon(a), \epsilon(b), \epsilon(b^*)$ satisfy the defining relations of $\mathcal{O}(S^2_{qr})$, that is, ϵ is well-defined. Every representation π of $\mathcal{A}$ is bounded, hence well-behaved by Proposition 1.3. That is, π gives rise to a covariant representation $\pi|_{\mathcal{B}} \times u$, where u is the partial isometry in the polar decomposition $\pi(b) = uc$. On the other hand, every $*$-representation of $\mathfrak{A}$ is given by a covariant representation of the partial automorphism Θ. This proves the correspondence (0.3) for the representations of $\mathcal{A}$ and $\mathfrak{A}$. □

4. The Quantum Algebra $\mathcal{U}_q(su(2))$

In this section $\mathcal{A}$ is the q-deformed enveloping $*$-algebra $\mathcal{U}_q(su(2)),\ q > 0,\ q \neq 1$, which is generated by E, F, K, K^{-1} satisfying the following defining relations

$$KK^{-1} = K^{-1}K = \mathbf{1}, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$
$$[E, F] = EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$
$$E^* = FK, \quad F^* = K^{-1}E, \quad K^* = K.$$

In this section, we use the standard notation $[n] \equiv [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, where $n \in \mathbb{Z}$ and $q \neq 0$. Further X^0 denotes $\mathbf{1}$ if X is one of the four generators E, F, K, K^{-1}.

In [VS] the authors considered the family of irreducible $*$-representations $\left\{\pi_{\omega,l} \mid \omega = \pm 1,\ l \in \frac{1}{2}\mathbb{N}_0\right\}$ of $\mathcal{U}_q(su(2))$. The representation $\pi_{\omega,l}$ acts on an orthonormal base $\{\mathbf{e}_m\}_{m=-l,\dots,l}$ of the representation space as follows:

$$\begin{aligned} \pi_{\omega,l}(K)\mathbf{e}_m &= \omega q^{2m}\mathbf{e}_m, \\ \pi_{\omega,l}(E)\mathbf{e}_m &= q^{m+1}\sqrt{[l-m][l+m+1]}\,\mathbf{e}_{m+1}, \\ \pi_{\omega,l}(F)\mathbf{e}_m &= \omega q^{-m}\sqrt{[l+m][l-m+1]}\,\mathbf{e}_{m-1}. \end{aligned} \tag{4.1}$$

We will show that every irreducible well-behaved representation of $\mathcal{A}$ is unitarily equivalent to $\pi_{\omega,l}$ for some $l \in \frac{1}{2}\mathbb{N}_0,\ \omega = \pm 1$.

Define a $\mathbb{Z}$-grading of $\mathcal{A}$ by setting $E \in \mathcal{A}_1,\ F \in \mathcal{A}_{-1}$ and $K, K^{-1} \in \mathcal{A}_0$. Then

$$\mathcal{B} := \mathcal{A}_0 = \mathrm{Lin}\{F^lK^mE^l \mid l \in \mathbb{N}_0,\ m \in \mathbb{Z}\} = \mathrm{Lin}\{E^lK^mF^l \mid l \in \mathbb{N}_0,\ m \in \mathbb{Z}\}.$$

The $*$-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is commutative and is equal to $\mathbb{C}[EF, K, K^{-1}] = \mathbb{C}[C_q, K, K^{-1}]$. For $n \in \mathbb{N}_0$, we have

$$\mathcal{A}_n = E^n\mathcal{B} = \mathrm{Lin}\{E^{n+l}K^mF^l \mid l \in \mathbb{N}_0,\ m \in \mathbb{Z}\},$$
$$\mathcal{A}_{-n} = F^n\mathcal{B} = \mathrm{Lin}\{F^{n+l}K^mE^l \mid l \in \mathbb{N}_0,\ m \in \mathbb{Z}\}.$$

One can verify by a direct computation that the *quantum Casimir element* C_q is a central element in $\mathcal{A}$, where

$$C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2}.$$

The following lemma can be easily proved by induction.

Lemma 4.1. *For every $n \in \mathbb{N}$ we have*

$$(i)\ [E, F^n] \equiv EF^n - F^nE = [n]F^{n-1}[K; 1-n],$$
$$(ii)\ [E^n, F] \equiv E^nF - FE^n = [n]E^{n-1}[K; n-1],$$

where we set $[K; l] := (q^lK - q^{-l}K^{-1})/(q - q^{-1})$ *for* $l \in \mathbb{Z}$.

This lemma implies the following relations:

$$E^nF^n = \prod_{j=1}^{n} (EF + [j-1][K;-j]),$$
$$F^nE^n = \prod_{j=1}^{n} (EF - [j][K;j-1]),\ n \in \mathbb{N}. \tag{4.2}$$

Since $\mathcal{B} = \mathbb{C}[C_q, K, K^{-1}]$, every character $\chi \in \widehat{\mathcal{B}}$ is equal to some $\chi_{st} \in \widehat{\mathcal{B}},\ (s,t) \in \mathbb{R} \times \mathbb{R}\backslash\{0\}$ where

$$\chi_{st}(C_q) = s, \quad \chi_{st}(K) = t.$$

Proposition 4.2. *(i) A character $\chi_{st} \in \widehat{\mathcal{B}}$ belongs to $\widehat{\mathcal{B}}^+$ if and only if*

$$t = \pm q^{m-n} \text{ and } s = \frac{\pm q^{m+n+1} \pm q^{-m-n-1}}{(q-q^{-1})^2}, \text{ where } m, n \in \mathbb{N}_0.$$

In particular,

$$\widehat{\mathcal{B}}^+ = \{\chi_{m,n,+} \mid m, n \in \mathbb{N}_0\} \cup \{\chi_{m,n,-} \mid m, n \in \mathbb{N}_0\},$$

where $\chi_{m,n,\pm} = \chi_{st}$ with s, t from above.
(ii) The partial action $\alpha = (\{\mathcal{D}_n\}_{n\in\mathbb{Z}}, \{\alpha_n\}_{n\in\mathbb{Z}})$ is given as follows:

$$\mathcal{D}_{-k} = \{\chi_{m,n,\pm} \mid -m \le k \le n\},\ \text{and } \chi^k_{m,n,\pm} = \chi_{m+k,n-k,\pm}.$$

Proof. (i) Lemma 1.1 and Equations (4.2) imply that $\chi \in \widehat{\mathcal{B}}^+$ if and only if the following inequalities are satisfied for arbitrary $k \in \mathbb{N}$:

$$\chi(E^kF^kK^{-k}) = \prod_{j=1}^{k} \chi(EF + [j-1][K;-j])\chi(K)^{-k} \ge 0. \tag{4.3}$$

$$\chi(F^kE^kK^k) = \prod_{j=1}^{k} \chi(EF - [j][K;j-1])\chi(K)^k \ge 0, \tag{4.4}$$

We show that there exist $m, n \in \mathbb{N}_0$ such that

$$\chi(EF + [m][K;-m-1]) = 0, \tag{4.5}$$
$$\chi(EF - [n+1][K;n]) = 0. \tag{4.6}$$

Assume the contrary, i.e. $\chi(EF + [k][K;-k-1]) \ne 0$ for all $k \in \mathbb{N}_0$. Suppose $t > 0$. Then, by (4.3),

$$\chi(EF) \ge -[k]\chi([K;-k-1]) = \frac{(q^{-2k-1} - q^{-1})t + (q^{2k+1} - q)t^{-1}}{(q-q^{-1})^2}.$$

Such an value $\chi(EF) \in \mathbb{R}$ cannot exist, since $q^{-2k-1} \to \infty$ for $k \to \infty$ if $q \in (0,1)$, respectively $q^{2k+1} \to \infty$ for $k \to \infty$ if $q > 1$. Analogously one obtains a contradiction for $t < 0$, using inequalities (4.3). Thus, $\chi(EF + [m][K;-m-1]) = 0$ for some $m \in \mathbb{N}_0$. Similarly one can prove that $\chi(EF - [n+1][K;n]) = 0$ for some

$n \in \mathbb{N}_0$, using inequalities (4.4). Subtracting (4.6) from (4.5) yields

$$\begin{aligned} & [m]\chi([K;-m-1]) = -[n+1]\chi([K;n]) \\ \iff & (q^{-1}-q^{-2m-1})t - (q^{2m+1}-q)t^{-1} = \\ & = (q^{-1}-q^{2n+1})t - (q^{-2n-1}-q)t^{-1} \\ \iff & t^2 = \frac{q^{2m+1}-q^{-2n-1}}{q^{2n+1}-q^{-2m-1}} = \frac{q^{2m}(q-q^{-2m-2n-1})}{q^{2n}(q-q^{-2m-2n-1})} \\ \iff & t = \pm q^{m-n}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \chi(C_q) &= [n+1]\chi([K;n]) + \frac{q^{-1}\chi(K)+q\chi(K^{-1})}{(q-q^{-1})^2} \\ &= \frac{q^{2n+1}t + q^{-2n-1}t^{-1}}{(q-q^{-1})^2} = \frac{\pm q^{m+n+1} \pm q^{-m-n-1}}{(q-q^{-1})^2}. \end{aligned}$$

(ii) Observe that $\chi_{m,n,\pm}(E^kF^kK^{-k}) \neq 0$ if and only if $k \le m$ by (4.3) and (4.5). Analogously, $\chi_{m,n,\pm}(F^kE^kK^k) \neq 0$ if and only if $k \le n$ by (4.4) and (4.6). This implies that $\chi_{m,n,\pm} \in \mathcal{D}_{-k}$ if and only if $-m \le k \le n$. Now suppose $k \in \{0,1,\ldots,n\}$. Since C_q commutes with E, F, we have

$$\begin{aligned} \chi^k_{m,n,\pm}(K) &= \frac{\chi_{m,n,\pm}(E^{*k}KE^k)}{\chi_{m,n,\pm}(E^{*k}E^k)} = \frac{\chi_{m,n,\pm}(E^{*k}E^kq^{2k}K)}{\chi_{m,n,\pm}(E^{*k}E^k)} = q^{2k}\chi_{m,n,\pm}(K), \\ \chi^k_{m,n,\pm}(C_q) &= \frac{\chi_{m,n,\pm}(E^{*k}C_qE^k)}{\chi_{m,n,\pm}(E^{*k}E^k)} = \chi_{m,n,\pm}(C_q). \end{aligned}$$

Analogously, if $k \in \{-m, -m+1, \ldots, 0\}$ we have

$$\begin{aligned} \chi^k_{m,n,\pm}(K) &= \frac{\chi_{m,n,\pm}(F^{*k}KF^k)}{\chi_{m,n,\pm}(F^{*k}F^k)} = q^{-2k}\chi_{m,n,\pm}(K), \\ \chi^k_{m,n,\pm}(C_q) &= \chi_{m,n,\pm}(C_q). \end{aligned}$$

Hence, if $\chi^k_{m,n,\pm}$ is defined, then $\chi^k_{m,n,\pm}(K) = \pm q^{(m+k)-(n-k)} = \chi_{m+k,n-k,\pm}(K)$ and $\chi^k_{m,n,\pm}(C_q) = \chi_{m,n,\pm}(C_q)$. □

In particular, the previous proposition shows that for each $\chi \in \widehat{\mathcal{B}}^+$ the stabilizer $\mathrm{St}\chi$ is trivial. We set

$$\Gamma := \{\chi_{0,n,+} \mid n \in \mathbb{N}_0\} \cup \{\chi_{n,-} \mid n \in \mathbb{N}_0\}.$$

As in Section 2, we conclude that Γ is a measurable countably separated section of the partial action. Using Proposition 4.2 we conclude that $\mathrm{Orb}\chi_{0,n,\pm}$ consists of $n+1$ elements and hence $\mathrm{Ind}\chi_{0,n,\pm}$ has dimension $n+1$ by Proposition 1.2, where $\chi_{0,n,\pm} \in \Gamma$. Now put $l := \frac{n}{2}$ and $\pi_{\omega,l} := \mathrm{Ind}\chi_{0,n,\pm}$, where $\omega = \pm 1$. Let $\{e_{l+m}\}_{m=-l,-l+1,\ldots,l}$ be an orthonormal base of the representation space $\mathcal{H}_{\pi_{l,\pm}}$ of $\pi_{l,\pm}$. For notational convenience, we put $e_{l+1} := 0$, and $e_{-l-1} := 0$.

Using Proposition 1.2, relations (4.2), Proposition 4.2 and the facts that $\chi_{0,n,\pm}(EF) = 0$, $\chi_{0,n,\pm}([K;l+m]) = -\omega[l-m]$, we obtain the action of $\pi_{\omega,l}$

on the base vectors e_{l+m}.

$$\begin{aligned}
\pi_{\omega,l}(K)e_{l+m} &= \chi^{l+m}_{0,n,\pm}(K)e_{l+m} = \chi_{l+m,n-l-m,\pm}(K)e_{l+m} = \omega q^{2m} e_{l+m}, \\
\pi_{\omega,l}(E)e_{l+m} &= \frac{\chi_{0,n,\pm}(E^{*(l+m+1)}E^{l+m+1})}{\left(\chi_{0,n,\pm}(E^{*(l+m+1)}E^{l+m+1})\chi_{0,n,\pm}(E^{*(l+m)}E^{l+m})\right)^{1/2}} e_{l+m+1} \\
&= \left(\frac{\chi_{0,n,\pm}(E^{*(l+m+1)}E^{l+m+1})}{\chi_{0,n,\pm}(E^{*(l+m)}E^{l+m})}\right)^{1/2} e_{l+m+1} \\
&= \left(q^{(l+m+1)(l+m+2)-(l+m)(l+m+1)}\right)^{1/2} \\
&\quad \times \left(\chi_{0,n,\pm}(EF-[l+m+1][K;l+m])\chi_{0,n,\pm}(K)\right)^{1/2} e_{l+m+1} \\
&= q^{m+1}\sqrt{[l-m][l+m+1]}e_{l+m+1}, \\
\pi_{\omega,l}(F)e_{l+m} &= \pi_{\omega,l}(E^*K^{-1})e_{l+m} = \omega q^{-2m}\pi_{\omega,l}(E)^* e_{l+m} \\
&= \omega q^{-m}\sqrt{[l+m][l-m+1]}e_{l+m-1}.
\end{aligned}$$

Putting $\mathbf{e}_m := e_{l+m},\ m = -l, \dots, l$, we see that all irreducible well-behaved $*$-representations of the quantum algebra $\mathcal{U}_q(su(2))$ are unitarily equivalent to the irreducible well-behaved $*$-representation $\pi_{\omega,l}$, given by the formulas (4.1), for some $\omega \in \{-1,+1\}$ and $l \in \frac{1}{2}\mathbb{N}_0$. In particular, all irreducible $*$-representations of $\mathcal{U}_q(su(2))$ are bounded. Summarizing the above discussion, we obtain the following

Theorem 4.3. *Every irreducible well-behaved $*$-representation of $\mathcal{U}_q(su(2))$, $q \in \mathbb{R}^+\backslash\{1\}$, is induced from a one-dimensional $*$-representation.*

Similarly to Lemma 2.3 and Theorem 2.5 one can prove the following Lemma and Theorem.

Lemma 4.4. *The polynomial*

$$(EF - [2][K;1])(EF - [3][K;2]) \in \mathbb{C}[EF, K, K^{-1}]$$

is positive in every well-behaved $*$-representation of $\mathcal{A}$ and is not of the form $\sum_{k=1}^n a_k^* a_k$ for $a_k \in \mathcal{A}$.

Theorem 4.5. *There exists a $*$-representation of $\mathcal{U}_q(su(2))$, $q \in \mathbb{R}^+\backslash\{1\}$, which has no well-behaved extension in a possibly larger Hilbert space.*

Let Rep$\mathcal{A}$ denote the category of well-behaved non-degenerate representations of $\mathcal{A}$. Then $\mathcal{A}$, considered with Rep$\mathcal{A}$ and generators E, F, K, K^{-1}, has a C^*-envelope $\mathfrak{A}$ in the sense of Definition 0.1. As in Section 2, let $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ be the C^*-p.d.s. dual to $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$. The description of the p.d.s. $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$ in Proposition 4.2 implies that the C^*-p.d.s. $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ is defined by the partial automorphism $\Theta = (\theta, I, J)$, where $I = I_{-1},\ J = I_1,\ (\theta(f))(t) = (\beta_1(f))(t),\ f \in I_{-1}$.

The proof of the following theorem is completely analogous to the proof of Theorem 2.6.

Theorem 4.6. *Consider the q-deformed enveloping algebra $\mathcal{A} = \mathcal{U}_q(su(2))$ with generators E, F, K, K^{-1} and the category of well-behaved representations* Rep$\mathcal{A}$. *Then the covariance algebra $\mathfrak{A} := C^*(C_0(\widehat{\mathcal{B}}^+), \Theta)$ is a C^*-envelope of $\mathcal{A}$.*

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MPI Leipzig, Germany

E-mail address: dowerk@mis.mpg.de, p.dowerk@gmx.de

Universität Leipzig, Mathematisches Institut Johannisgasse 26, 04103 Leipzig, Germany

E-mail address: savchuk@math.uni-leipzig.de