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by

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Improved bounds on negativity of superpositions

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Abstract

We consider an alternative formula for the negativity based on a simple generalization of the concurrence. We use the formula to bound the amount of entanglement in a superposition of two bipartite pure states of arbitrary dimension. Various examples indicate that our bounds are tighter than the previously known results.

Keywords:superpostions;pure states;negativity

1 Introduction

The study of entanglement in a superposition as a function of its terms was initiated by Linden, Popescu, and Smolin [1]. With the use of entanglement of formation, it was observed that the superposition of two separable states can give an entangled state, while the superposition of two entangled states can give a separable one. The literature comprises now a number of papers devoted to bound the amount of entanglement in a superposition [2, 3, 4, 5, 6, 7, 8, 9]. The present note is a contribution in this direction. Our main tool is a generalization of the concurrence [10, 11, 12]. When restricted to a special case, this gives an alternative formula for the negativity. We apply the formula to study entanglement in superpositions of two bipartite pure states of arbitrary dimension. We derive simple but compact relationships between the negativity of a superposition and that of its terms for biorthogonal and generic pure states. In analytical and numerical examples based on randomly generated states, our bounds turn out to perform better than previous results obtained in [3, 4, 9]. The expression for the negativity is presented in Section 2. The bounds are in Section 3. Examples are in Section 4.

2 Negativity

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces of dimension d_1 and d_2, respectively. The *concurrence* (see [12]) of a pure bipartite state $\rho_{AB} = |\psi\rangle\langle\psi|$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ is defined as $C(|\psi\rangle) := \sqrt{2\,(1 - Tr\rho_A^2)}$. We denote by ρ_S, with $S = A, B$, the reduced density operator. It is well-known that a pure state is separable if and only if its concurrence is zero. The square of the concurrence can be written as

$$C^2(|\psi\rangle) = 2\left(1 - Tr\rho_A^2\right) = \sum_{m,n=1}^{D_1,D_2} |C_{mn}|^2, \tag{1}$$

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where $D_1 = d_1(d_1-1)/2$, $D_2 = d_2(d_2-1)/2$, and $C_{mn} = \langle\psi|\widetilde{\psi}_{mn}\rangle$. Here, $|\widetilde{\psi}_{mn}\rangle := (L_m \otimes L_n)|\psi^*\rangle$, with $L_m, m = 1, ..., D_1$, $L_n, n = 1, ..., D_2$ being the generators of the groups $SO(d_1)$ and $SO(d_2)$, respectively [13]. Since $|\psi\rangle = \sum_{i,j=1}^{d_1,d_2} \psi_{ij}|ij\rangle$ ($\psi_{ij} \in \mathbb{C}$), with respect to the computational bases of $\mathcal{H}_A$ and $\mathcal{H}_B$, we have the following equivalent form of the concurrence:

$$C(|\psi\rangle) = \left(4\sum_{i<j,k<l}^{d_1,d_2} |\psi_{ik}\psi_{jl} - \psi_{il}\psi_{jk}|^2\right)^{1/2}.$$

By noticing that this is an ℓ_2-norm, a straightforward generalization would be

$$C_p(|\psi\rangle) := \left(2^p\sum_{i<j,k<l}^{d_1,d_2} |\psi_{ik}\psi_{jl} - \psi_{il}\psi_{jk}|^p\right)^{\frac{1}{p}}. \tag{2}$$

For our purposes, we shall only consider the case $p = 1$, which is arguably the simplest to analyze. Thus, let the ℓ_1-norm concurrence

$$C_1(|\psi\rangle) := \left(2\sum_{i<j,k<l}^{d_1,d_2} |\psi_{ik}\psi_{jl} - \psi_{il}\psi_{jk}|\right) = \sum_{m,n=1}^{D_1,D_2} |C_{mn}|, \tag{3}$$

where C_{mn} is as in Eq. (1). For a pure state ρ, if the eigenvalues of ρ_A are $\lambda_1, ..., \lambda_n$ ($\lambda_1 \geq ... \geq \lambda_n$) then $C^2(|\psi\rangle) = \sum_{i,j=1}^{n} \lambda_i\lambda_j$. It follows that

$$C_1(|\psi\rangle) = \sum_{i,j=1}^{n} \sqrt{\lambda_i\lambda_j}. \tag{4}$$

This expression is nothing but the negativity of the pure state [14]. Recall that the *negativity* of ρ_{AB} is defined as $N(|\psi\rangle) = (\|\rho_{T_A}\|_1 - 1) = (\mathrm{Tr}(\rho_{T_A}\rho_{T_A}^{\dagger})^{1/2} - 1)$. Here ρ_{T_A} is the partial transpose of ρ_{AB} with respect to the subsystem A. It is well-known that he negativity is an entanglement monotone. In what follows, we shall use the standard notation $N(|\psi\rangle)$.

3 Bounds

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces of dimension N. We consider two states $|\psi\rangle, |\phi\rangle \in \mathcal{H} \cong \mathcal{H}_A \otimes \mathcal{H}_B$. Let $\{|1\rangle, ..., |N\rangle\}$ be the computational basis of $\mathcal{H}_A$ (resp. $\mathcal{H}_B$). We write $|\psi\rangle = \sum_{1\leq i,j\leq N} \psi_{ij}|ij\rangle$ and $|\phi\rangle = \sum_{1\leq i,j\leq N} \phi_{ij}|ij\rangle$. A superposition of $|\psi\rangle$ and $|\phi\rangle$ is simply

$$|\gamma\rangle := \alpha|\psi\rangle + \beta|\phi\rangle = \sum_{1\leq i,j\leq N} (\alpha\psi_{ij} + \beta\phi_{ij})|ij\rangle, \tag{5}$$

where $|\alpha|^2 + |\beta|^2 = 1$. We say that $|\psi\rangle$ and $|\phi\rangle$ are *biorthogonal* (see [1]), when

$$Tr_A[Tr_B(|\psi\rangle\langle\psi|)Tr_B(|\phi\rangle\langle\phi|)] = Tr_B[Tr_A(|\psi\rangle\langle\psi|)Tr_A(|\phi\rangle\langle\phi|)] = 0.$$

By applying local unitary transformations, $|\psi\rangle = \sum_{i=1}^{N_1} \psi_{ii}|ii\rangle$ and $|\phi\rangle = \sum_{i=N_1+1}^{N} \phi_{ii}|ii\rangle$.

Proposition 3.1 *Let $|\psi\rangle$ and $|\phi\rangle$ be biorthogonal states. Let $|\chi\rangle = \alpha|\psi\rangle + \beta|\phi\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. Then*

$$N(|\chi\rangle) = |\alpha|^2 N(\psi) + |\beta|^2 N(\phi) + 2|\alpha||\beta|\sqrt{N(\psi)N(\phi)}.$$

Proof. Since $|\psi\rangle = \sum_{i=1}^{N_1} \psi_{ii}|ii\rangle$ and $|\phi\rangle = \sum_{i=N_1+1}^{N} \phi_{ii}|ii\rangle$, then $|\chi\rangle = \sum_{i=1}^{N_1} \alpha\psi_{ii}|ii\rangle + \sum_{i=N_1+1}^{N} \beta\phi_{ii}|ii\rangle := \sum_{i=1}^{N} \gamma_{ii}|ii\rangle$. By Eq. (3),

$$N(\chi) = \sum_{l,m,r,s=1}^{N} |\gamma_{rm}\gamma_{sl} - \gamma_{rl}\gamma_{sm}|.$$

By disregarding the zero terms and rearranging the sum of the product terms, we have

$$\begin{aligned} N(\chi) &= \sum_{l,m=1}^{N} |\gamma_{mm}\gamma_{ll} - \gamma_{ml}\gamma_{lm}| = \sum_{l,m=1:l\neq m}^{N} |\gamma_{mm}\gamma_{ll} - \gamma_{ml}\gamma_{lm}| \\ &= \sum_{l,m=1:l\neq m}^{N} |\gamma_{mm}c_{ll}| = \sum_{m=1}^{N} |\gamma_{mm}| \sum_{l\neq m} |\gamma_{ll}| \end{aligned}$$

The following equation is straightforward algebra:

$$\begin{aligned} N(\chi) &= \left(\sum_{m=1}^{N} |\gamma_{mm}|\right)^2 - \sum_{m=1}^{N} |\gamma_{mm}|^2 \\ &= \left(\sum_{i=1}^{N_1} |\alpha\psi_{ii}|ii\rangle| + \sum_{i=N_1+1}^{N} |\beta\phi_{ii}|ii\rangle|\right)^2 - \sum_{i=1}^{N_1} |\alpha\psi_{ii}|ii\rangle|^2 + \sum_{i=N_1+1}^{N} |\beta\phi_{ii}|ii\rangle|^2 \\ &= |\alpha|^2 \left[\left(\sum_{i=1}^{N_1} |\psi_{ii}|ii\rangle|\right)^2 - \sum_{i=1}^{N_1} |\psi_{ii}|ii\rangle|^2\right] + |\beta|^2 \left[\left(\sum_{i=N_1}^{N} |\phi_{ii}|ii\rangle|\right)^2 - \sum_{i=N_1}^{N} |\phi_{ii}|ii\rangle|^2\right] \\ &+ 2|\alpha||\beta| \sum_{i=1}^{N_1} |\psi_{ii}|ii\rangle| \sum_{i=N_1+1}^{N} |\phi_{ii}|ii\rangle| \\ &= |\alpha|^2 N(\psi) + |\beta|^2 N(\phi) + 2|\alpha||\beta|\sqrt{N(\psi)N(\phi)}. \end{aligned}$$

■

Proposition 3.1 is about biorthogonal states. Let us now focus on generic pure states in $\mathcal{H}$. Again, by Eq. (3),for a generic pure state $|\chi\rangle = \sum_{1\leq i,j\leq N} \gamma_{ij}|ij\rangle$, we have

$$N(\chi) := \sum_{l,m,r,s=1}^{N} |\gamma_{rm}\gamma_{sl} - \gamma_{rl}\gamma_{sm}| = \sum_{m,r,l,s=1}^{N} |\langle\chi|L_{rs} \otimes L_{ml}|\chi\rangle|, \tag{6}$$

where $\{L_{ij}\}_{1\leq i,j\leq N}$ are the generators of the group $SO(N)$.

Proposition 3.2 *Let $|\psi\rangle$ and $|\phi\rangle$ be generic pure states. Let $|\chi\rangle = \alpha|\psi\rangle + \beta|\phi\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. Then,*

$$||\chi||^2 N(\chi') \leq |\alpha|^2 N(\psi) + |\beta|^2 N(\phi) + 2|\alpha\beta| \sum_{m,r,l,s=1}^{N} |\langle\psi|L_{rs} \bigotimes L_{ml}|\phi\rangle| \tag{7}$$

$$||\chi||^2 N(\chi') \geq \max \left\{ \begin{array}{l} |\alpha|^2 N(\psi) - |\beta|^2 N(\phi) - 2|\alpha\beta| \sum |\langle\psi|L_{rs} \bigotimes L_{ml}|\phi\rangle|, \\ -|\alpha|^2 N(\psi) + |\beta|^2 N(\phi) - 2|\alpha\beta| \sum |\langle\psi|L_{rs} \bigotimes L_{ml}|\phi\rangle|, \\ -|\alpha|^2 N(\psi) - |\beta|^2 N(\phi) + 2|\alpha\beta| \sum |\langle\psi|L_{rs} \bigotimes L_{ml}|\phi\rangle| \end{array} \right. \tag{8}$$

where $||\chi||^2 := \langle\chi|\chi\rangle$, and $\chi' := \frac{1}{||\chi||} * \chi$ is the normalized state.

Proof. From triangular inequality, we have

$$\|\chi\|^2 N(\chi') = \sum_{m,r,l,s=1}^{N} |\alpha^2(\psi_{rm}\psi_{sl} - \psi_{rl}\psi_{sm}) + \beta^2(\phi_{rm}\phi_{sl} - \phi_{rl}\phi_{sm}) + \alpha\beta(\psi_{rm}\phi_{sl} + \phi_{rm}\psi_{sl} - \psi_{rl}\phi_{sm} - \phi_{rl}\psi_{sm})|$$

$$\leq |\alpha|^2 N(\psi) + |\beta|^2 N(\phi) + |\alpha\beta| \sum_{m,r,l,s=1}^{N} |\psi_{rm}\phi_{sl} + \phi_{rm}\psi_{sl} - \psi_{rl}\phi_{sm} - \phi_{rl}\psi_{sm}|$$

$$= |\alpha|^2 N(\psi) + |\beta|^2 N(\phi) + 2|\alpha\beta| \sum_{m,r,l,s=1}^{N} |\langle\psi|L_{rs} \bigotimes L_{ml}|\phi\rangle|.$$

The lower bound can be proved similarly:

$$\|\chi\|^2 N(\chi') \geq |\alpha|^2 N(\psi) - |\beta|^2 N(\phi) - |\alpha\beta| \sum_{m,r,l,s=1}^{N} |\psi_{rm}\phi_{sl} + \phi_{rm}\psi_{sl} - \psi_{rl}\phi_{sm} - \phi_{rl}\psi_{sm}|$$

$$= |\alpha|^2 N(\psi) - |\beta|^2 N(\phi) - 2|\alpha\beta| \sum_{m,r,l,s=1}^{N} |\langle\psi|L_{rs} \bigotimes L_{ml}|\phi\rangle|.$$

The other two lower bounds can be easy proved in the same way. ■

4 Examples

We give here some examples to show that the bounds obtained in Eqs. (7) and (8) are close to the true value of the negativity. A comparison of our bound with results of [3],[4] and [9] is done by making use of random states, which have been generated with a computer algebra system [15]. Additionally, some analytical examples are also useful to put in evidence the advantages of our method. In the following, we consider the state $|\chi\rangle = a|\psi\rangle + b|\phi\rangle$ for various choices of $|\psi\rangle$ and $|\phi\rangle$.

1. Let $|\psi\rangle = (0.2266, 0.2941, 0.8821, 0.2897)^T$ and $|\phi\rangle = (0.2758, 0.4802, 0.5380, 0.6354)^T$.

2. Let $|\psi\rangle = (0.3915, 0.1285, 0.0627, 0.4252, 0.2975, 0.3002, 0.1044, 0.2509, 0.2296, 0.1863, 0.3012, 0.0934, 0.2545, 0.2205, 0.1836, 0.2430)^T$ and $|\phi\rangle = (0.3041, 0.2115, 0.1197, 0.2700, 0.3389, 0.3721, 0.0577, 0.3292, 0.3316, 0.0205, 0.2922, 0.0644, 0.0290, 0.1473, 0.2138, 0.3777)^T$.

3. Let $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|11\rangle + \frac{1}{2}|22\rangle$ and $|\phi\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{2}|11\rangle + \frac{1}{2}|22\rangle$.

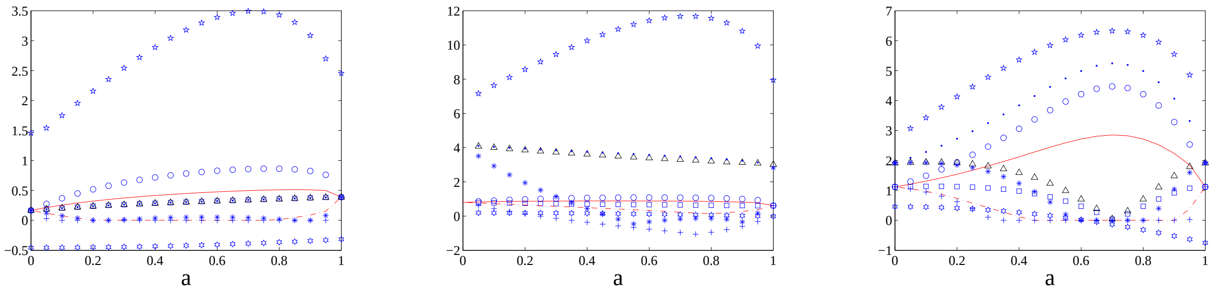
4. Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\phi\rangle = \frac{1}{\sqrt{2}}(|22\rangle - |33\rangle)$, with $|0\rangle = (1,0,0,0)^T$,...,$|3\rangle = (0,0,0,1)^T$, and $a = b = 2^{-1/2}$. The lower and upper bound for the negativity of $|\chi\rangle$ given in [4] are 0 and 8, respectively. The bounds obtained from our result are 1 and 3. The bounds for the usual concurrence given in [9] and [3] are $-2, 2$ and $-4, 4$, respectively. In the case of generic a and b, [4] gives 0 and $4 + 8a\sqrt{1-a^2}$; our bounds are instead

$$\max\{-1 + 2a^2 - 4a\sqrt{1-a^2}, 1 - 3a^2 - 2a\sqrt{1-a^2}, -1 + 4a\sqrt{1-a^2}\}$$

   and $(1 + 4a\sqrt{1-a^2})$. Thus, our upper bound is always smaller than the one in [4] for all $0 \leq a \leq 1$; our lower bound is larger than the one in [4] for $0 \leq a \leq 0.966$ and $0.973 \leq a \leq 1$.

5. Let $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$ and $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle + |1,0\rangle)$ be the usual Bell states. Our upper and lower bounds are 1 and $\max\{-1, 1-2a^2, 2a^2-1\}$; the upper and lower bounds in [4] are $4a\sqrt{1-a^2}+1$ and 0. Hence, our bounds are tighter.

The bounds for the first three points are illustrated in the figure (left/center/right, respectively). The coefficients a and b are free (here we choose $b = \sqrt{1-a^2}$ for example 1,2,4,5; and $b = -\sqrt{1-a^2}$ for point 3). Our bounds are represented by the dotted line (upper) and the '*' line (lower); the bound in [9], by the red solid line (upper) and the red '-.' line (lower); the bound in [3], by the 'O' line (upper) and the '+' line (lower); the bound in [4], by the lines with pentagrams (upper) and hexagrams (lower). The true value of the negativity is represented by $\triangle$ and the concurrence by $\square$.



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