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by

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L^2-FLOW OF ELASTIC CURVES WITH KNOT POINTS AND CLAMPED ENDS

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ABSTRACT. In this paper we investigate the L^2-flow of elastic non-closed curves in n-dimensional Euclidean spaces with knot points and two clamped ends. The L^2-flow corresponds to a fourth-order parabolic equation on each piece of curve between two successive knot points with certain dynamic (interior) boundary conditions at these interior knot points. For solutions of the L^2-flow, we prove that they are not only piecewise C^∞-smooth but also globally C^1-smooth at each fixed time t if the initial curves are both piecewise C^∞-smooth and globally C^1-smooth. Moreover, the asymptotic limit curves are piecewise C^∞-smooth but globally C^2-smooth. As an application, the L^2-flow of non-closed elastic curves in this article provides a new approach for the curve fitting problem

1. INTRODUCTION

Although geometric flows of elastic curves have been investigated by many researchers, most articles studying this subject in the literature only focus on the case of closed curves (e.g., [5], [11], [24], [26]). In fact, the case of non-closed (or open) elastic curves has been motivated by higher-order geometric variational problems (e.g., [27]); by mechanical modeling of biological polymers (e.g., [23]); by motion-planning problems in geometric control theory (see [9], [10], [22]); and nonlinear (poly-)spline interpolations (see [6]), [7], [8], [12], [17], [25]). However, there are relatively fewer articles investigating geometric flows of non-closed curves. Various (interior) boundary conditions for the case of non-closed curves have been proposed in the corresponding higher-order variational problems. These boundary conditions naturally have their parabolic versions in the corresponding evolution equations. One of the simplest boundary conditions for fourth-order parabolic equations is the so-called clamped boundary condition. For this case, we proposed a parabolic PDE approach in [18] to obtain the long-time existence of smooth solutions for the L^2-flow of non-closed elastic curves as long as the initial curves are smooth. Note that the L^2-flow of elastic curves in [18] is different from the so-called curve-straightening flow in [14], [15] by Langer and Singer and [19], [20] by Linnér, where a variational

approach using minimax method was applied. In this article, we continue the parabolic PDE approach in [18] to investigate the L^2-flow of non-closed elastic curves with prescribed "knot points" on the curves and with clamped boundary conditions at two end points. One may associate certain (interior) boundary conditions at the prescribed knot points for the L^2-flow of elastic curves. We would like to show in this paper that the result similar to the main theorem in [18] could be obtained if we impose proper dynamic "boundary conditions" at these interior knot points. As an application, the L^2-flow of non-closed elastic curves in this article provides a new approach for the curve fitting problem, which vaguely speaking is to find equilibrium configurations of elastic energy among the class of curves with given knot points and clamped ends.

Let $-1 = x_0 < x_1 < \cdots < x_N = +1$ be a partition of the closed interval $I = [-1, +1] \subset \mathbb{R}$, and $I_i = [x_{i-1}, x_i]$ for all $i \in \{1, ..., N\}$. Let $f \in \bigcup_{i=1}^{N} C^\infty((x_{i-1}, x_i), \mathbb{R}^n) \bigcap C^1([-1,1], \mathbb{R}^n)$ represent a piecewise smooth curve in $\mathbb{R}^n$. Denote by $ds = |\partial_x f|\, dx$ the arclength element, and $\partial_s = |\partial_x f|^{-1} \partial_x$ the arclength differentiation. Denote by $T = \partial_s f$ the unit tangent vector of f and $\kappa = \partial_s^2 f$ the curvature vector of f. For convenience, as we reparametrize a curve f by its arclength parameter, i.e., $\widetilde{f}(s) = (f \circ x)(s)$, we still denote the curve by $f = f(s)$. So is the same for the geometric functions induced from f, e.g., unit tangent T and curvature vector κ of f. For each piece of curve, we define the bending energy by

$$(1.1) \qquad \mathcal{E}[f_{|I_i}] := \int_{I_i} \frac{1}{2} |\kappa|^2 \, ds,$$

the (relaxed) stretching energy by

$$(1.2) \qquad \mathcal{L}[f_{|I_i}] := \int_{I_i} \frac{1}{2} |\kappa|^2 \, ds,$$

and the elastic energy by

$$(1.3) \qquad \mathcal{E}_\lambda[f_{|I_i}] := \mathcal{E}[f_{|I_i}] + \lambda \cdot \mathcal{L}[f_{|I_i}],$$

where the constant λ is also called *tension modulus* and is always chosen to be a positive constant. The bending energy corresponds to the so-called *Euler-Bernoulli* model of elastic rods in the literature. We define the total elastic energy of f by

$$(1.4) \qquad \mathcal{E}_\lambda[f] := \sum_{i=1}^{N} \mathcal{E}_\lambda[f_{|I_i}].$$

The stretching energy plays the role of penalty-term in (1.4) when the total length of curves tends to infinity (e.g. see [5] or [24]). Note that since the definition of bending energy in (1.1) contains the constant $\frac{1}{2}$, the stretching modulus λ used in this article might be different from some other articles in the literature (e.g., [14], [15]).

Denote by $\nabla_s \eta := (\partial_s \eta)^\perp$ the normal component of $\partial_s \eta$ when η is a vector field along f. By applying the first variational formula of $\mathcal{E}$ (the bending energy functional of curves) and $\mathcal{L}$ (the length integral of curves) in Lemma 2 below, we could set up a gradient flow equation of the elastic energy functional $\mathcal{E}_\lambda = \mathcal{E} + \lambda \cdot \mathcal{L}$ for $f : [0, t_1) \times \bigcup_{i=1}^{N} (x_{i-1}, x_i) \to \mathbb{R}^n$ by letting

$$(1.5) \qquad \partial_t f = -\nabla_{L^2} \mathcal{E}_\lambda[f] = -\nabla_s^2 \kappa - \frac{|\kappa|^2}{2} \kappa + \lambda \cdot \kappa$$

with the (interior) boundary conditions

$$(1.6) \qquad f(t, x_i) = f^{(i)}, \, \forall \, i \in \{0, ..., N\},$$

$$(1.7) \qquad T(t, x_i) = T^{(i)}, \, \forall \, i \in \{0, N\},$$

$$(1.8) \qquad \partial_t T(t, x_i) = [\triangle_{x_i} \kappa](t), \, \forall \, i \in \{1, ..., N-1\},$$

where $\bigcup_{i=0}^{N} \{f^{(i)}\} \subset \mathbb{R}^n$ is the set of prescribed fixed positions, $\{T^{(0)}, T^{(N)}\}$ is the set of prescribed unit constant vectors, and

$$[\triangle_{x_i} \kappa](t) := \left[\kappa(t, x_i^+) - \kappa(t, x_i^-)\right],$$
$$\kappa(t, x_0^-) := 0 =: \kappa(t, x_N^+).$$

The prescribed fixed points, $f^{(0)}, ..., f^{(N)}$, in condition (1.6) are called *knot points* (this terminology has been used in the spline theory) for the flow (1.5) of curve f. Moreover, from conditions (1.7) and (1.8), $f^{(0)}$ and $f^{(N)}$ are called the *clamped ends* or *clamped boundary condition* for the flow (1.5) of curve f. Note that one could write $\partial_t T(t, x_i) = \nabla_t T(t, x_i)$ in (1.8) because $T(t, x_i^-) = T(t, x_i^+) = T(t, x_i)$ implies that both $\kappa(t, x_i^+)$ and $\kappa(t, x_i^-)$ are perpendicular to the unit tangent vector $T(t, x_i)$ for all $i \in \{0, ..., N\}$. The above (interior) boundary conditions have also appeared in [8], where a variational approach was applied to interpolating prescribed knot points by nonlinear splines on $\mathbb{R}^2$.

Following the philosophy in [5], [18], [24], [26], we derive an argument for the long-time existence of solutions with certain smoothness in the L^2-flow (1.5) with boundary conditions (1.6), (1.7), (1.8) in this paper. Namely, differentiating both sides of (1.5) provides an "algebraic" structure in writing up differential inequalities for higher-order Sobolev semi-norms of curvature. These differential inequalities are the type of Gronwall's differential inequalities, which would imply the global bounds of higher-order Sobolev semi-norms of curvature. To derive these differential inequalities, one needs to apply integration by parts to derive the right form of integrals and to apply Gagliardo-Nirenberg type interpolation inequalities to take care of the terms of lesser-order. However, as one works on the case of non-closed elastic curves, the boundary terms generated from integration by parts create the difficulty in deriving Gronwall's differential inequalities. In [18], we found that the

difficulty in estimating the boundary terms could be avoided by working with the L^2-norm of covariant derivatives of curvature with respective to time-variable, e.g., $\|\nabla_t^m f\|_{L^2}$, instead of derivatives respective to arclength-variables, e.g., $\|\nabla_s^m \kappa\|_{L^2}$. In other words, the boundary terms generated from integration by parts vanish in the estimates of $\|\nabla_t^m f\|_{L^2}$ in [18]. Thus, we derive the type of Gronwall's differential inequalities for $\|\nabla_t^m f\|_{L^2}$, $\forall\, m \in \mathbb{Z}_+$. On the other hand, by Lemma 7 below and the Gagliardo-Nirenberg type interpolation inequalities, e.g. see Lemma 5 below, one could show that $\|\nabla_s^{4m-2} \kappa\|_{L^2}^2$ is bounded from above by the quantity of the form, $(1 + \epsilon^2) \cdot \|\nabla_t^m f\|_{L^2}^2 + C_\epsilon \left(\|\kappa\|_{L^2}^2 \right)$. Therefore, in [18], the main task is to estimate the term $\|\nabla_t^m f\|_{L^2}^2$. However, in this article, the interior (dynamic) boundary condition (1.8) generates non-zero terms, which however could not be treated as terms of lesser-order. Thus, one could not simply apply the estimates in [18] to carry out the proof of long-time existence. To overcome this difficulty, we observe that one could utilize the "algebraic" structure in the differential identity of

$$(1.9) \qquad \mathcal{Y}_m(t) := \sum_{i=1}^N \int_{I_i} |\nabla_t^m f|^2 \ ds + \sum_{i=0}^N |\nabla_t^m T(t, x_i)|^2$$

to derive the type of Gronwall's differential inequalities for $\mathcal{Y}_m(t)$, cf. the higher-order energy identity in Lemma 9. The Gronwall's differential inequality gives the uniform bounds of $\mathcal{Y}_m(t)$ for each $m \in \mathbb{Z}_+$. In fact, the "algebraic" structure also depends on how interior (dynamic) boundary conditions are given. Note that the (interior) boundary conditions (1.6), (1.7) and (1.8) often appear in the so-called path-planning problem or motion-planning problem. Based on the uniform bounds of $\mathcal{Y}_m(t)$, $\forall\, m \in \mathbb{Z}_+$, Theorem 1 gives the long-time existence and asymptotic behavior for the piecewise smooth solutions of L^2-flow (1.5). Notice that the knot points $\{f^{(0)}, ..., f^{(N)}\}$ are not necessarily distinct in Theorem 1, i.e., the condition (1.6) allows $f^{(i_1)} = f^{(i_2)}$ for some $i_1 \neq i_2$.

Theorem 1. *Let $\lambda \in (0, \infty)$ be a positive constant and $-1 = x_0 < x_1 < \cdots < x_N = +1$. Suppose $f_0 \in \bigcup_{i=1}^N C^\infty\left((x_{i-1}, x_i), \mathbb{R}^n\right) \bigcap C^1\left([-1, 1], \mathbb{R}^n\right)$ is an initial curve with non-zero finite length, i.e., $0 < \int_{I_i} |\partial_x f_0| dx < \infty$, $\forall\, i \in \{1, ..., N\}$, to the evolution equation (1.5) with the (interior) boundary conditions (1.6), (1.7) and (1.8).*

Then, there exists a global solution of the L^2-flow of $\mathcal{E}_\lambda[f]$ in (1.5) with the regularity, $f(t, \cdot) \in \bigcup_{i=1}^N C^\infty\left((x_{i-1}, x_i), \mathbb{R}^n\right) \bigcap C^1\left([-1, 1], \mathbb{R}^n\right)$, $\forall\, t \in (0, \infty)$. As $t \to \infty$, the family of curves $\{f_t\}$ subconverges to f_∞, an equilibrium of the energy functional $\mathcal{E}_\lambda$ (up to reparametrization by arclength). Moreover, $f_\infty \in \bigcup_{i=1}^N C^\infty\left((x_{i-1}, x_i), \mathbb{R}^n\right) \bigcap C^2\left((-1, 1), \mathbb{R}^n\right)$.

It is worth mentioning here that the so-called minimal-energy splines in [8], which correspond to our asymptotic curves in Theorem 1, are also piecewise C^∞-smooth and globally C^2-smooth. However, their formulation in [8] is a variational setting, which induces a second-order elliptic equation for the angle of tangent indicatrix of planar curves (thus the result is restricted to the case of planar curves).

For numerical implement of L^2-gradient flow of open (i.e., non-closed) elastic curves with various boundary conditions, the reader is referred to the recent articles by Barrett, Garcke and Nürnberg in [3], [4].

The rest of this article is arranged as the following. In Section 2, we collect some notation, terminologies, identities, estimates and previous results from [1], [5], [16] and [18] to keep this paper short and self-contained. The proof of Theorem 1 is contained in Section 3.

2. Preliminaries and Notation

Lemma 1 ([5, Lemma 2.1]). *Suppose ϕ is any normal field along f and $f : [0, \epsilon) \times I \to \mathbb{R}^n$ is a time dependent curve satisfying $\partial_t f = V + \varphi T$, where V is the normal velocity and $\varphi = \langle T, \partial_t f \rangle$. Then the following formulae hold.*

$$(2.1) \qquad \nabla_s \phi = \partial_s \phi + \langle \phi, \kappa \rangle T,$$

$$(2.2) \qquad \partial_t (ds) = (\partial_s \varphi - \langle \kappa, V \rangle) ds,$$

$$(2.3) \qquad \partial_t \partial_s - \partial_s \partial_t = (\langle \kappa, V \rangle - \partial_s \varphi) \partial_s,$$

$$(2.4) \qquad \partial_t T = \nabla_s V + \varphi \kappa,$$

$$(2.5) \qquad \partial_t \phi = \nabla_t \phi - \langle \nabla_s V + \varphi\, \kappa, \phi \rangle T,$$

$$(2.6) \qquad \nabla_t \kappa = \nabla_s^2 V + \langle \kappa, V \rangle \kappa + \varphi \nabla_s \kappa,$$

$$(2.7) \qquad (\nabla_t \nabla_s - \nabla_s \nabla_t) \phi = (\langle \kappa, V \rangle - \partial_s \varphi) \nabla_s \phi + \langle \kappa, \phi \rangle \nabla_s V - \langle \nabla_s V, \phi \rangle \kappa.$$

Notice that the formula of integration by parts for the covariant differentiation ∇_s is still applicable. This is because that, as ψ_1, ψ_2 are normal vector fields along a smooth curve, one has

$$(2.8) \qquad \partial_s \langle \psi_1, \psi_2 \rangle = \langle \nabla_s \psi_1, \psi_2 \rangle + \langle \psi_1, \nabla_s \psi_2 \rangle.$$

Lemma 2. *Suppose $f : I = [a, b] \to \mathbb{R}^n$ is a smooth curve in $\mathbb{R}^n$. Then for any perturbation of f, $f_\varepsilon (x) = f (x) + \varepsilon W (x)$, where $W \in C^\infty (I, \mathbb{R}^n)$, one has the following formulae:*

$$\frac{d}{d\varepsilon} \big\lfloor_{\varepsilon=0} \mathcal{L} [f_\varepsilon] = -\int_I \langle \kappa, W \rangle \ ds + [\langle T, W \rangle]_a^b,$$

$$\frac{d}{d\varepsilon} \big\lfloor_{\varepsilon=0} \mathcal{E} [f_\varepsilon] = \int_I \langle \nabla_s^2 \kappa + \frac{|\kappa|^2}{2} \kappa, W \rangle \ ds$$

$$+ \left[\langle T, W \rangle \cdot \frac{|\kappa|^2}{2} + \langle \kappa, \nabla_s (W - \langle W, T \rangle T) \rangle - \langle \nabla_s \kappa, W \rangle \right]_a^b.$$

Proof of Lemma 2. The proof is based on a direct computation by applying (2.2), (2.6), (2.8) and integration by parts. The reader can also find the details of this computation in the literature (e.g., [16]). $\qquad\square$

For normal vector fields $\phi_1, ..., \phi_k$ along f, we denote by $\phi_1 * \cdots * \phi_k$ a term of the type

$$\phi_1 * \cdots * \phi_k = \begin{cases} \langle \phi_{i_1}, \phi_{i_2} \rangle \cdots \langle \phi_{i_{k-1}}, \phi_{i_k} \rangle & \text{, for } k \text{ even,} \\ \langle \phi_{i_1}, \phi_{i_2} \rangle \cdots \langle \phi_{i_{k-2}}, \phi_{i_{k-1}} \rangle \cdot \phi_{i_k}, & \text{for } k \text{ odd,} \end{cases}$$

where $i_1, ..., i_k$ is any permutation of $1, ..., k$. Slightly more generally, we allow some of the ϕ_i to be functions, in which case the $*$-product reduces to multiplication. For a normal vector field ϕ along f, we denote by $P_\nu^\mu(\phi)$ any linear combination of terms of the type $\nabla_s^{i_1}\phi * \cdots * \nabla_s^{i_\nu}\phi$ with coefficients bounded by a universal constant, where $\mu = i_1 + \cdots + i_\nu$ is the total number of derivatives. Notice that the following formulae hold:

$$\begin{cases} \nabla_s \left(P_b^a(\phi) * P_d^c(\phi) \right) = \nabla_s P_b^a(\phi) * P_d^c(\phi) + P_b^a(\phi) * \nabla_s P_d^c(\phi), \\ \\ P_b^a(\phi) * P_d^c(\phi) = P_{b+d}^{a+c}(\phi), \ \nabla_s P_d^c(\phi) = P_d^{c+1}(\phi). \end{cases}$$

The following lemma from [1] is a one-dimensional version of standard interpolations on order of smoothness.

Lemma 3 ([1, Theorem 5.2]). *Let Ω be an interval in $\mathbb{R}$ and $u \in W^{m,p}(\Omega)$ for some $p \in [1,\infty)$, $m \in \mathbb{Z}_+$. Then for each $\epsilon_0 > 0$ there exists finite constants K and K', each depending on m, p, ϵ_0, such that*

$$(2.9) \qquad \|u\|_{W^{j,p}} \le K \left(\epsilon \|D^m u\|_{L^p} + \epsilon^{-j/(m-j)} \|u\|_{L^p} \right),$$

$$(2.10) \qquad \|u\|_{W^{j,p}} \le K' \left(\epsilon \|u\|_{W^{m,p}} + \epsilon^{-j/(m-j)} \|u\|_{L^p} \right),$$

$$(2.11) \qquad \|u\|_{W^{j,p}} \le 2K' \|u\|_{W^{m,p}}^{j/m} \|u\|_{L^p}^{(m-j)/m},$$

for any $j \in \{0, 1, ..., m-1\}$ and $\epsilon \in (0, \epsilon_0)$. Here, $\|u\|_{L^p} := \left(\int_\Omega |u|^p \right)^{1/p}$ is the L^p-norm, and $\|u\|_{W^{m,p}} := \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha u\|_{L^p}^p \right)^{1/p}$ is the standard Sobolev norm.

Below are interpolation inequalities for non-closed curves, which are modified from [5]. Note that in this article we still follow the notation in [5] to use the scale invariant Sobolev norms:

$$\|\kappa\|_{k,p} := \sum_{i=0}^{k} \left\| \nabla_s^i \kappa \right\|_p, \ \left\| \nabla_s^i \kappa \right\|_p := \mathcal{L}[f]^{i+1-1/p} \left(\int_I \left| \nabla_s^i \kappa \right|^p \, ds \right)^{1/p}.$$

Note that using scale invariant Sobolev norms is convenient as working with inequalities in geometric flows since domain of functions also depends on time.

Lemma 4 (modified from [5, Lemma 2.4]). *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}^n$ be a smooth curve. Then for any $k \in \mathbb{Z}_+ \cup \{0\}$, $p \geq 2$ and $0 \leq i < k$, we have*

$$(2.12) \qquad \left\| \nabla_s^i \kappa \right\|_p \leq c \left\| \kappa \right\|_2^{1-\alpha} \left\| \kappa \right\|_{k,2}^{\alpha} ,$$

where $\alpha = (i + \frac{1}{2} - \frac{1}{p})/k$ and $c = c(n, k, p)$.

Proof of Lemma 4. The proof is standard. Assume $\mathcal{L}[f] = 1$ and use the inequality $|\partial_s |\phi|| \leq |\nabla_s \phi|$ as ϕ is a normal vector field along f. The inequality can be easily derived by applying (2.8). Then the standard proof for the case of scalar functions as in Chapter 5 of [1] applies. $\qquad \square$

Lemma 5 (modified from [5, Proposition 2.5]). *For any term $P_\nu^\mu (\kappa)$ with $\nu \geq 2$ which contains only derivatives of κ of order at most $k - 1$, we have*

$$(2.13) \qquad \int_I |P_\nu^\mu(\kappa)| \ ds \leq c \cdot \mathcal{L}[f]^{1-\mu-\nu} \left\| \kappa \right\|_2^{\nu-\gamma} \left\| \kappa \right\|_{k,2}^{\gamma} ,$$

where $\gamma = \left(\mu + \frac{1}{2}\nu - 1 \right)/k$, $c = c(n, k, \nu)$. Moreover, if $\mu + \frac{1}{2}\nu < 2k + 1$, then $\gamma < 2$ and we have for any $\varepsilon > 0$

$$(2.14) \quad \int_I |P_\nu^\mu(\kappa)| \, ds \leq \varepsilon \cdot \int_I |\nabla_s^k \kappa|^2 ds + c_1 \cdot \left(\int_I |\kappa|^2 \, ds \right)^{\frac{\nu-\gamma}{2-\gamma}} + c_2 \cdot \left(\int_I |\kappa|^2 \, ds \right)^{\nu/2} ,$$

where $c_1 = c_1(n, k, \mu, \nu) \cdot \varepsilon^{\frac{-\gamma}{2-\gamma}}$ and $c_2 = c_2(n, k, \mu, \nu) \cdot \mathcal{L}[f]^{1-\mu-\nu/2}$.

Proof of Lemma 5. By Hölder's inequality and Lemma 4 with $p = \nu$, we obtain

$$\int_I \left| \nabla_s^{i_1} \kappa * \cdots * \nabla_s^{i_\nu} \kappa \right| ds \leq \mathcal{L}[f]^{1-\mu-\nu} \prod_{j=1}^{\nu} \left\| \nabla_s^{i_j} \kappa \right\|_\nu \leq c \cdot \mathcal{L}[f]^{1-\mu-\nu} \prod_{j=1}^{\nu} \left\| \kappa \right\|_2^{1-\alpha_j} \left\| \kappa \right\|_{k,2}^{\alpha_j} ,$$

where $i_1 + \cdots + i_\nu = \mu$, $\alpha_j = (i_j + \frac{1}{2} - \frac{1}{\nu})/k$, $c = c(n, k, \nu)$. Thus, $\alpha_1 + \cdots + \alpha_\nu = \gamma$, and (2.13) is proved. Now a standard interpolation inequality in Lemma 3 implies the interpolation inequality of scale invariant version

$$\left\| \kappa \right\|_{k,2}^2 \leq c(k) \left(\left\| \nabla_s^k \kappa \right\|_2^2 + \left\| \kappa \right\|_2^2 \right) .$$

Therefore, as $\gamma < 2$, we obtain

$$\text{R.H.S. of } (2.13) \leq c(n, k, \mu, \nu) \, \mathcal{L}[f]^{1-\mu-\nu} \left(\left\| \nabla_s^k \kappa \right\|_2^\gamma \left\| \kappa \right\|_2^{\nu-\gamma} + \left\| \kappa \right\|_2^\nu \right)$$
$$\leq c(n, k, \mu, \nu) \left(\left\| \nabla_s^k \kappa \right\|_{L^2}^\gamma \left\| \kappa \right\|_{L^2}^{\nu-\gamma} + \mathcal{L}[f]^{1-\mu-\nu/2} \left\| \kappa \right\|_{L^2}^\nu \right)$$
$$\leq \varepsilon \left\| \nabla_s^k \kappa \right\|_{L^2}^2 + c(n, k, \mu, \nu) \, \varepsilon^{\frac{-\gamma}{2-\gamma}} \left\| \kappa \right\|_{L^2}^{2\frac{\nu-\gamma}{2-\gamma}} + c(n, k, \mu, \nu) \, \mathcal{L}[f]^{1-\mu-\nu/2} \left\| \kappa \right\|_{L^2}^\nu .$$

$\square$

Lemma 6 ([5, Lemma 2.6]). *We have the identities*

$$\nabla_s \kappa - \partial_s \kappa = |\kappa|^2 T,$$

$$\nabla_s^m \kappa - \partial_s^m \kappa = \sum_{i=1}^{[\frac{m}{2}]} Q_{2i+1}^{m-2i}(\kappa) + \sum_{i=1}^{[\frac{m+1}{2}]} Q_{2i}^{m+1-2i}(\kappa) T.$$

This is similar to the previous notation using $P_\nu^\mu(\kappa)$ as $Q_\nu^\mu(\kappa)$ denotes the linear combination of terms $\partial_s^{i_1} \kappa * \cdots * \partial_s^{i_\nu} \kappa$ with $i_1 + \cdots + i_\nu = \mu$.

In order to simplify the terminology of summation in the lemma below, we introduce the notation,

$$(2.15) \qquad \sum_{[\![a,b]\!] \leq [\![A,B]\!]} P_b^a(\kappa) := \sum_{a=0}^{A} \sum_{b=1}^{2A+B-2a} P_b^a(\kappa),$$

where $[\![\mu,\nu]\!] := 2\mu + \nu$. For our convenience, let's call $[\![\mu,\nu]\!]$ the order of $P_\nu^\mu(\kappa)$. Thus, (2.15) stands for the sum of $P_b^a(\kappa)$ with order no greater than that of $P_B^A(\kappa)$.

Lemma 7 ([18, Lemma 8]). *Suppose $f : [0, t_1) \times I \to \mathbb{R}^n$ is a smooth solution of (1.5), and denote by $\phi_\ell := \nabla_s^\ell \kappa$. Then, for any $\ell \in \mathbb{Z}_+ \cup \{0\}$ and $k, m \in \mathbb{Z}_+$, we have the following formulae.*

(2.16)
$$\nabla_t^m f - (-1)^m \nabla_s^{4m-2} \kappa = P_3^{4m-4}(\kappa) + \cdots + P_1^0(\kappa) = \sum_{[\![a,b]\!] \leq [\![4m-4,3]\!]} P_b^a(\kappa)$$

(2.17)
$$\nabla_t^m T - (-1)^m \nabla_s^{4m-1} \kappa = P_3^{4m-3}(\kappa) + \cdots + P_1^0(\kappa) = \sum_{[\![a,b]\!] \leq [\![4m-3,3]\!]} P_b^a(\kappa)$$

$$(2.18) \quad \nabla_t^m \kappa - (-1)^m \nabla_s^{4m} \kappa = P_3^{4m-2}(\kappa) + \cdots + P_1^0(\kappa) = \sum_{[\![a,b]\!] \leq [\![4m-2,3]\!]} P_b^a(\kappa)$$

$$(2.19) \qquad \nabla_t^m P_\nu^\mu(\kappa) = P_\nu^{4m+\mu}(\kappa) + \cdots + P_1^0(\kappa) = \sum_{[\![a,b]\!] \leq [\![4m+\mu,\nu]\!]} P_b^a(\kappa)$$

$$(2.20) \quad \nabla_t^m \partial_s f - \nabla_s \nabla_t^m f = P_3^{4m-3}(\kappa) + \cdots + P_1^0(\kappa) = \sum_{[\![a,b]\!] \leq [\![4m-3,3]\!]} P_b^a(\kappa)$$

(2.21)
$$\nabla_t^m \nabla_s^k \phi_\ell - \nabla_s^k \nabla_t^m \phi_\ell = P_3^{4m+k+\ell-2}(\kappa) + \cdots + P_1^0(\kappa) = \sum_{[\![a,b]\!] \leq [\![4m+k+\ell-2,3]\!]} P_b^a(\kappa)$$

$$(2.22) \quad \partial_t^m (ds) = \left(P_2^{4m-2}(\kappa) + \cdots + P_1^0(\kappa) \right) ds = \left(\sum_{[\![a,b]\!] \leq [\![4m-2,2]\!]} P_b^a(\kappa) \right) ds$$

Proof. The proof from (2.16) to (2.21) has been shown in [18]. Thus, we only prove (2.22) here, which is an induction argument.

As $m = 1$, one proves (2.22) by applying (2.2) and (1.5). Suppose (2.22) holds for $m = k$, where k is any positive integer. Then,

$$\partial_t^{k+1} (ds) = \partial_t \left(\partial_t^k (ds) \right) = \partial_t \left(\left(P_2^{4k-2}(\kappa) + \cdots + P_1^0(\kappa) \right) ds \right)$$

$$= \partial_t \left(P_2^{4k-2}(\kappa) + \cdots + P_1^0(\kappa) \right) ds + \left(P_2^{4k-2}(\kappa) + \cdots + P_1^0(\kappa) \right) \partial_t (ds)$$

$$= \left(P_2^{4k+2}(\kappa) + \cdots + P_1^0(\kappa) \right) ds,$$

where the last equality comes from applying (2.19) and (2.2).

$\square$

3. Proof of the Main Result

3.1. The short-time existence.
In this section we treat the short-time existence of piecewise smooth solutions of the L^2-flow (1.5), i.e.,

$$(3.1) \qquad f(t, \cdot) \in \bigcup_{i=1}^{N} C^\infty \left((x_{i-1}, x_i), \mathbb{R}^n \right) \bigcap C^1 \left((-1, 1), \mathbb{R}^n \right),$$

with the interior boundary conditions (1.6), (1.7), (1.8). Short time solutions exist due to standard theory for evolution equations once we can reformulate the flow as a fourth-order parabolic equation.

There are two basic ingredients in the short-time existence argument:

(1) one uses tangential diffeomorphisms to replace the covariant derivative $\nabla_s^2 \kappa$, which is orthogonal to the tangential of the curve, by the standard 4th-order spatial derivative $\gamma^{-4} \partial_x^4 f$ of the position vector. Note, that as $\gamma = |\partial_x f|$ the highest order term is quasilinear. If we assume that the flow is started with an arclength parametrized initial curve it is possible to conclude the prefactor γ^{-1} will not get singular for a short time.

(2) the linear version of the 4th order parabolic equation has a unique, global solution with estimates.

(1): We observe that it is enough to solve the flow equation up to tangential directions as a solution $f(t, x)$ of $\partial_t f = V + \varphi \cdot \partial_x f$ on each subinterval $(x_-, x_+) \subset I$, where V is the normal velocity, i.e. $\langle V, \partial_x f \rangle = 0$. As we let φ solve the x-parametrized o.d.e.'s $\partial_t \xi = \varphi(t, \xi)$, $\xi(0, x) = x$, we find that $\tilde{f}(t, x) = f(t, \xi(t, x))$ solves the normal flow problem $\partial_t \tilde{f} = V$ as long as we can ensure that boundary points $x_\pm$ are stationary for the o.d.e., i.e. $\varphi(t, x_\pm) = 0$.

Recalling $\partial_s = \frac{1}{\gamma} \partial_x$, Lemma 1 implies

$$\nabla_s^2 \kappa = \gamma^{-4} \partial_x^4 f - \left\langle \gamma^{-4} \partial_x^4 f, T \right\rangle T - R,$$

where R is lower order compared with the 4th-order term. Therefore, it is sufficient to solve problem

$$(3.2) \quad \partial_t f = W = -\gamma^{-4} \partial_x^4 f - \frac{|\kappa|^2}{2} \kappa + \lambda \cdot \kappa + R = -\gamma^{-4} \partial_x^4 f + \text{terms of lesser-order}$$

with boundary conditions (1.6), (1.7), (1.8) at the knot points.

Observe that if a smooth solution f is found, then the definition of W in (3.2) and the Dirichlet boundary condition of f implies that W vanishes at the knot points, i.e., $W(t, x_\pm) = 0$. Thus the afore-mentioned tangential diffeomorphisms ξ leave $x_\pm$ invariant as needed.

(2): The quasilinear boundary value problem for each subinterval can be solved $(x_-, x_+) \subset I$ by applying standard higher-order parabolic theory, namely, one solves the linear problem and obtains a short-time solution to the nonlinear problem via successive approximations as the quasilinear structure allows to render the non-linearity as a compact perturbation for small times. The reader is referred to the excellent exposition [2], which treats the higher order quasi-linear case in Section 7. Special care has to be taken to set up the approximation procedure as the problem to be solved has a prescribed time derivative with boundary data from neighboring intervals.

What we have to further generalize in this approach is to allow the boundary conditions to interact between the neighboring intervals. This can be done by a further application of successive approximations, which treats the curvature data from the curve outside each individual interval as they are temporarily frozen. Namely, we fix a time $\varepsilon > 0$, define $f_0 := f(0, \cdot)$ and solve for $n > 0$ on each subinterval $(x_-, x_+) \subset I$ the quasilinear problem

$$\partial_t f_n = -\gamma_n^{-4} \partial_x^4 f_n + \text{terms involving } f_n \text{ of lesser-order}$$

on the time interval $(0, \varepsilon)$. The boundary conditions formed from the previous frozen f_{n-1} read

$$f_n(0, x) = f_0(x),$$
$$f_n(t, x_\pm) = f_\pm(x_\pm),$$
$$\partial_t T_n(t, x_\pm) = (Id - T_n T_n^t) \cdot \left[\kappa_{n-1}(t, x_\pm^+) - \kappa_{n-1}(t, x_\pm^-) \right],$$

where $T_n(t, x_\pm) = \frac{\partial_x f_n(t, x_\pm)}{\gamma_n(t, x_\pm)}$, $t \in (0, \varepsilon)$ and $f_\pm$ is a given vector for any $\pm \in \{+, -\}$. Here, the index n refers to geometric terms coming from the curve f_n, and accordingly the terms indexed by $n - 1$. In particular, we let $\kappa_{n-1}(t, y^\pm)$ denote the left/right-sided limits of the curvature vectors at the knot point y according to the curve f_{n-1}. Observe that only the curvature terms of the boundary data is entirely prescribed by the previous step. We used the projection $(Id - T_n T_n^t)$ in the equation of $\partial_t T_n$ because the previous step curvature does not need to be orthogonal to T_n. However, as n increases, convergence of T_n and κ_n would render these modification obsolete in the limit $n \to +\infty$.

As the two ordinary differential equations for $T_n(t, x_\pm)$, resp., can be solved separately from the PDE we can use their solution on some positive time interval

$t \in (0, \varepsilon)$ to set up a boundary condition for the PDE via prescribing

$$\frac{\partial_x f_n}{|\partial_x f_n|}(t, x_\pm) = T_n(t, x_\pm) \qquad \text{for all } t \in (0, \varepsilon).$$

Thus, the boundary value problem to be solved for f_n can be written in such a form that boundary values for $\partial_x f_n$ but not their time derivatives are prescribed. This reformulation makes the initial value problem accessible to the methods in Amann [2]. Note that while the PDE is fourth-order elliptic, the nonlinear operator to define the boundary data for T_n is acting only on spatial derivatives of f_{n-1} up to order $k < 3$.

Letting $n \to +\infty$ and allowing ε to be chosen arbitrarily small, we establish convergence to the unique short-time solution satisfying the full set of boundary conditions (1.6), (1.7), (1.8) using the quantitative estimates provided in [2].

3.2. The long-time existence.

To prove the long-time existence, we need to estimate the higher-order Sobolev semi-norms of curvature. We use an argument similar to the one used in [18]. Namely, we consider the evolution equation for $\nabla_t^m f$ and derive the equation

$$\nabla_t \nabla_t^m f = -\nabla_s^4 \nabla_t^m f + \text{ tensors of lesser-order}$$

for all $m \in \mathbb{Z}_+$. The difference here is that we need to manage a way to split the boundary terms, coming from applying integration by parts in the L^2 estimates of $\nabla_t^m f$ (these boundary terms vanish in the case of clamped boundary conditions), so that we derive the following differential equality,

$$(3.3) \qquad \frac{d}{dt}\left\{ \sum_{i=1}^{N} \int_{I_i} |\nabla_t^m f|^2 \, ds + \sum_{i=0}^{N} |\nabla_t^m T(\cdot, x_i)|^2 \right\} + 2 \cdot \sum_{i=1}^{N} \int_{I_i} |\nabla_s^2 \nabla_t^m \kappa|^2 \, ds$$

$$= \text{ terms of lesser-order,}$$

where $T(\cdot, x_i) = \partial_s f(\cdot, x_i)$ is the unit tangent vector of f at x_i. It is sufficient to keep track only of the scaling of the terms of lesser-order, instead of computing these terms explicitly, in (3.3). In other words, we only have to know the order of the derivatives involved such that the Gagliardo-Nirenberg type interpolation inequalities still apply to (3.3) to derive a differential inequality.

The energy identity in the following lemma is derived by a straightforward computation.

Lemma 8 (Energy Identity). *Suppose $f : (0, t_1) \times [-1, 1] \to \mathbb{R}^n$ is the solution of the L^2-flow (1.5) with the (interior) boundary conditions (1.6), (1.7), (1.8) and*

with the regularity $f(t, \cdot) \in \bigcup\limits_{i=1}^{N} C^\infty\left((x_{i-1}, x_i), \mathbb{R}^n\right) \bigcap C^1\left((-1, 1), \mathbb{R}^n\right)$. Then,

(3.4)
$$\frac{d}{dt}_{|t=0} \mathcal{E}_\lambda[f] = -\sum_{i=1}^{N} \int\limits_{I_i} |\partial_t f|^2 \ ds - \sum_{i=0}^{N} |\partial_t T|^2(t, x_i)$$

$$= -\sum_{i=1}^{N} \int\limits_{I_i} |\nabla_t f|^2 \ ds - \sum_{i=0}^{N} |\nabla_t T|^2(t, x_i).$$

Proof. From Lemma 2 and the evolution equation of f in (1.5), one derives on each interval I_i the equality,

(3.5)
$$\frac{d}{dt}_{|t=0} \int\limits_{I_i} \left(\tfrac{1}{2}|\kappa|^2 + \lambda\right) \ ds$$

$$= \int\limits_{I_i} \left\langle \nabla_s^2 \kappa + \tfrac{|\kappa|^2}{2}\kappa - \lambda \cdot \kappa, \partial_t f \right\rangle \ ds + \left[\langle \kappa, \nabla_s \nabla_t f \rangle + \langle (\lambda + \tfrac{1}{2}|\kappa|^2)T - \nabla_s \kappa, \partial_t f \rangle\right]_{|\partial I_i}$$

$$= -\int\limits_{I_i} |\partial_t f|^2 \ ds + \langle \kappa, \nabla_t T \rangle_{|\partial I_i} = -\int\limits_{I_i} |\partial_t f|^2 \ ds + \langle \kappa, \partial_t T \rangle_{|\partial I_i},$$

where the second equality comes from applying the boundary condition (1.6), the property $\partial_t f = \nabla_t f$, (2.3) and the last equality comes from using the property $\nabla_t T = \partial_t T$ (since $\langle \partial_t T, T \rangle = \tfrac{1}{2}\partial |T|^2 = 0$). Therefore, from (3.5) and the (interior) boundary conditions in (1.6), (1.7), (1.8), one derives the energy identity,

(3.6)
$$\frac{d}{dt}_{|t=0} \mathcal{E}_\lambda[f] = \frac{d}{dt}_{|t=0} \sum_{i=1}^{N} \left\{ \int\limits_{I_i} \left(\tfrac{1}{2}|\kappa|^2 + \lambda\right) \ ds \right\}$$

$$= -\sum_{i=1}^{N} \int\limits_{I_i} |\partial_t f|^2 \ ds - \sum_{i=0}^{N} |\partial_t T|^2(t, x_i).$$

$\square$

A classical theorem by John Milnor states that the total curvature of a closed curve f in $\mathbb{R}^n$ can be approximated by the limit of the total curvatures of inscribed polygons inscribed of f. Thus, the total curvature of a smooth closed curve in $\mathbb{R}^n$ is at least 2π (cf. [21] or the proof in [13, Theorem 2.34]). We adapt part of the proof of Milnor's theorem into the situation in Proposition 1 below.

Proposition 1. *Let* $f : I = [a, b] \to \mathbb{R}^n$ *be a continuous curve with the regularity* $f \in C^2((a, b), \mathbb{R}^n)$. *Denote by* $ds = \partial_x f \ dx$ *the arclength element of* f. *Assume* $f(a) = f(b)$, *then the total curvature of* f *is at least* π, *i.e.,*

(3.7)
$$\int_{x=a}^{b} |\kappa| \ ds > \pi.$$

Proof. Since we assume $f(a) = f(b)$, we have $\int_I T(s) \ ds = f(b) - f(a) = 0$. This implies that the tangent indicatrix T can't be contained in any hemisphere, $\mathbb{S}_+^{n-1}$.

Therefore, the spherical diameter of the spherical curve T is greater than one-half of the length of a great circle on the unit sphere $\mathbb{S}^{n-1}(1)$, i.e.,

$$(3.8) \qquad \operatorname{dist}_{\mathbb{S}^{n-1}(1)}(T(x_1), T(x_2)) > \pi,$$

for some $x_1, x_2 \in I$. From (3.8), we obtain (3.7).

$\square$

The formula in the following lemma could be thought as a "higher-order energy identity".

Lemma 9. *Suppose $f : (0, t_1) \times [-1, 1] \to \mathbb{R}^n$ is the solution of the L^2-flow (1.5) with the regularity $f(t, \cdot) \in \bigcup\limits_{i=1}^{N} C^\infty((x_{i-1}, x_i), \mathbb{R}^n) \bigcap C^1((-1, 1), \mathbb{R}^n)$ and with the (interior) boundary conditions in (1.6), (1.7), (1.8). Then, the quantity $\mathcal{Y}_m(t)$ defined in (1.9) satisfies the identity,*

$$(3.9)$$
$$\tfrac{d}{dt}\mathcal{Y}_m(t) + \sum_{i=1}^{N} \int_{I_i} |\nabla_t^m \kappa|^2 \, ds + \sum_{i=1}^{N} \int_{I_i} |\nabla_s^{4m} \kappa|^2 \, ds = \sum_{i=1}^{N} \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \, ds.$$

Proof. From (2.8), (2.2), (1.5), we have

$$(3.10) \qquad \tfrac{d}{dt}\tfrac{1}{2}\int_{I_i} |\nabla_t^m f|^2 \, ds = \int_{I_i} \langle \nabla_t^m f, \nabla_t^{m+1} f \rangle \, ds - \int_{I_i} \tfrac{1}{2}|\nabla_t^m f|^2 \cdot \langle \kappa, \partial_t f \rangle \, ds$$

$$= \int_{I_i} \langle \nabla_t^m f, \nabla_t^m(-\nabla_s^2 \kappa - \tfrac{|\kappa|^2}{2}\kappa + \lambda \cdot \kappa) \rangle \, ds - \int_{I_i} \tfrac{1}{2}|\nabla_t^m f|^2 \cdot \langle \kappa, \partial_t f \rangle \, ds$$

$$= -\int_{I_i} \langle \nabla_t^m f, \nabla_t^m \nabla_s^2 \kappa \rangle \, ds$$

$$- \int_{I_i} \left(\langle \nabla_t^m f, \nabla_t^m(\tfrac{|\kappa|^2}{2}\kappa - \lambda \cdot \kappa) \rangle + \tfrac{1}{2}|\nabla_t^m f|^2 \cdot \langle \kappa, \partial_t f \rangle \right) \, ds.$$

By applying (2.16) and (2.18), we have

$$(3.11) \qquad \int_{I_i} \left(\langle \nabla_t^m f, \nabla_t^m(\tfrac{|\kappa|^2}{2}\kappa - \lambda \cdot \kappa) \rangle + \tfrac{1}{2}|\nabla_t^m f|^2 \cdot \langle \kappa, \partial_t f \rangle \right) \, ds$$

$$= \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \, ds.$$

By applying (2.21) and integration by parts, we have

$$(3.12)$$
$$\int_{I_i} \langle \nabla_t^m f, \nabla_t^m \nabla_s^2 \kappa \rangle \, ds$$

$$= -\int_{I_i} \langle \nabla_s \nabla_t^m f, \nabla_t^m \nabla_s \kappa \rangle \, ds + \langle \nabla_t^m f, \nabla_t^m \nabla_s \kappa \rangle_{|\partial I_i} + \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \, ds$$

$$= -\int_{I_i} \langle \nabla_t^m T, \nabla_t^m \nabla_s \kappa \rangle \, ds + \langle \nabla_t^m f, \nabla_t^m \nabla_s \kappa \rangle_{|\partial I_i} + \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \, ds,$$

where the last equality comes from applying (2.20) and (2.21). Again, by applying (2.21) and integration by parts, we have

(3.13)
$$\int_{I_i} \langle \nabla_t^m T, \nabla_t^m \nabla_s \kappa \rangle \, ds$$

$$= -\int_{I_i} \langle \nabla_s \nabla_t^m T, \nabla_t^m \kappa \rangle \, ds + \langle \nabla_t^m T, \nabla_t^m \kappa \rangle_{|\partial I_i} + \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \, ds$$

$$= -\int_{I_i} \langle \nabla_t^m \kappa, \nabla_t^m \kappa \rangle \, ds + \langle \nabla_t^m T, \nabla_t^m \kappa \rangle_{|\partial I_i} + \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \, ds,$$

where the last equality comes from applying (2.17) and (2.18), i.e.,

(3.14)
$$\nabla_t^m \kappa = \nabla_s \nabla_t^m \partial_s f + \sum_{[\![a,b]\!] \leq [\![4m-2,3]\!]} P_b^a(\kappa).$$

Thus, from (3.10), (3.11), (3.12) and (3.13), we have

(3.15) $\frac{1}{2} \frac{d}{dt} \int_{I_i} |\nabla_t^m f|^2 \, ds + \int_{I_i} |\nabla_t^m \kappa|^2 \, ds$

$$= -\langle \nabla_t^m f, \nabla_t^m \nabla_s \kappa \rangle_{|\partial I_i} + \langle \nabla_t^m T, \nabla_t^m \kappa \rangle_{|\partial I_i} + \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \, ds.$$

From applying (2.18) to (3.15), we have

(3.16)
$$\frac{d}{dt} \int_{I_i} |\nabla_t^m f|^2 \, ds + 2 \cdot \int_{I_i} |\nabla_s^{4m} \kappa|^2 \, ds$$

$$= -2 \cdot \langle \nabla_t^m f, \nabla_t^m \nabla_s \kappa \rangle_{|\partial I_i} + 2 \cdot \langle \nabla_t^m T, \nabla_t^m \kappa \rangle_{|\partial I_i} + \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \, ds.$$

By taking the sum $\sum_{i=1}^{N}$ in (3.16) and applying the (interior) boundary conditions in (1.6), (1.7), (1.8), we obtain

(3.17) $\frac{d}{dt} \sum_{i=1}^{N} \int_{I_i} |\nabla_t^m f|^2 \, ds + 2 \cdot \sum_{i=1}^{N} \int_{I_i} |\nabla_s^{4m} \kappa|^2 \, ds$

$$= -2 \cdot \sum_{i=0}^{N} \langle \nabla_t^m T(\cdot, x_i), \nabla_t^m [\triangle_{x_i} \kappa] \rangle + \sum_{i=1}^{N} \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \, ds.$$

Note that, from (1.8), we have

$$\nabla_t^{m+1} T(\cdot, x_i) = \nabla_t^m [\triangle_{x_i} \kappa].$$

Therefore, from (3.17), we obtain

$$\frac{d}{dt} \left\{ \sum_{i=1}^{N} \int_{I_i} |\nabla_t^m f|^2 \, ds + \sum_{i=0}^{N} |\nabla_t^m T(\cdot, x_i)|^2 \right\} + 2 \cdot \sum_{i=1}^{N} \int_{I_i} |\nabla_s^{4m} \kappa|^2 \, ds$$

$$= \sum_{i=1}^{N} \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \, ds.$$

□

Proof of the long-time existence and asymptotics in Theorem 1. The short-time existence of piecewise smooth solutions of (1.5) with the (interior) boundary conditions (1.6), (1.7), (1.8) allows us to assume that f with the regularity in (3.1) is a solution of (1.5) up to $t_1 \in (0, +\infty)$. Below, we show that the long-time existence could be derived by an argument of contradiction.

Let $\delta \in (0, 1/2)$ and re-write (3.9) as

(3.18)

$$\tfrac{d}{dt} \mathcal{Y}_m(t) + \delta \cdot \mathcal{Y}_m(t) + 2 \cdot \sum_{i=1}^{N} \int_{I_i} |\nabla_s^{4m} \kappa|^2 \ ds$$

$$= \delta \cdot \sum_{i=1}^{N} \int_{I_i} |\nabla_t^m f|^2 \ ds + \delta \cdot \sum_{i=0}^{N} |\nabla_t^m T(\cdot, x_i)|^2 + \sum_{i=1}^{N} \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \ ds.$$

From this equation, we would like to derive a differential inequality for $\mathcal{Y}_m$ by estimating the terms of lesser-order.

Step 1° From the energy identity in (3.4), $\mathcal{E}_\lambda [f]$ is non-increasing as t increases and

$$\mathcal{E}[f] + \lambda \cdot \mathcal{L}[f] =: \mathcal{E}_\lambda [f] \leq \mathcal{E}_\lambda [f_0].$$

Thus, as $t \in [0, t_1)$,

(3.19)
$$\sum_{i=1}^{N} \int_{I_i} |\kappa|^2 \ ds \leq 2 \cdot \mathcal{E}_\lambda [f_0]$$

and

(3.20)
$$\mathcal{L}[f] \leq \frac{\mathcal{E}_\lambda [f]}{\lambda} \leq \frac{\mathcal{E}_\lambda [f_0]}{\lambda} := \mathcal{L}_+.$$

Note that $\mathcal{E}_\lambda [f_0] < +\infty$ from the assumption of the initial curve f_0.

Below, we claim that there is a positive constant $\mathcal{L}_- = \mathcal{L}_- \left(\mathcal{E}_\lambda[f_0], f^{(0)}, ..., f^{(N)} \right)$ such that the total length satisfies

(3.21)
$$\mathcal{L}[f](t) \geq \mathcal{L}_- \gneqq 0,$$

for all $t \in [0, t_1)$. Note that, from the assumption on the regularity of f, the tangent indicatrix satisfies $T \in \bigcup_{i=1}^{N} C^\infty((x_{i-1}, x_i), \mathbb{S}^{n-1}) \cap C^0((-1, 1), \mathbb{S}^{n-1})$.

Let $i \in \{1, ..., N\}$. Assume $f^{(i)} \neq f^{(i-1)}$, then it is obvious that

(3.22)
$$\mathcal{L}[f_{|_{I_i}}](t) \geq |f^{(i)} - f^{(i-1)}| \gneqq 0.$$

On the other hand, if we assume $f^{(i)} = f^{(i-1)}$, then we apply Proposition 1 to obtain

(3.23)
$$\int_{I_i} |\kappa| \ ds > \pi.$$

From (3.19), (3.23) and by applying Hölder's inequality, we obtain

(3.24)
$$\mathcal{L}[f_{|I_i}] = \int_{I_i} \, ds \geq \frac{\left(\int_{I_i} |\kappa| \, ds \right)^2}{\int_{I_i} |\kappa|^2 \, ds} > \frac{\pi^2}{2 \cdot \mathcal{E}_\lambda[f_0]} \gneqq 0.$$

From (3.22) and (3.24), there is a constant $\mathcal{L}_{-}^{(i)} = \mathcal{L}_{-}^{(i)}(\mathcal{E}_\lambda[f_0], f^{(i-1)}, f^{(i)})$ such that

(3.25)
$$\mathcal{L}[f_{|I_i}] \geq \mathcal{L}_{-}^{(i)} \gneqq 0.$$

Thus, we conclude from (3.25) that

(3.26)
$$\mathcal{L}[f] = \sum_{i=1}^{N} \mathcal{L}[f_{|I_i}] \geq \sum_{i=1}^{N} \mathcal{L}_{-}^{(i)} =: \mathcal{L}_{-} \gneqq 0,$$

where $\mathcal{L}_{-} = \mathcal{L}_{-}\left(\mathcal{E}_\lambda[f_0], f^{(0)}, ..., f^{(N)} \right)$.

Step 2° Since $T(t, x_i^-) = T(t, x_i^+)$ for all $i \in \{1, ..., N-1\}$, we may write

$$T(t, x_i) - T^{(0)} = \sum_{j=1}^{i} \left(T(t, x_j) - T(t, x_{j-1}) \right) = \sum_{j=1}^{i} \int_{I_j} \kappa \, ds$$

for any $x \in (-1, 1)$. Then, by taking the differentiation ∇_t^m on both side, we have

$$\nabla_t^m T(t, x_i) = \sum_{j=1}^{i-1} \int_{I_j} \nabla_t^m (\kappa \, ds) = \sum_{j=1}^{i-1} \int_{I_j} \sum_{m_1 + m_2 = m} C_{m_1}^m \cdot \nabla_t^{m_1} \kappa \cdot \partial_t^{m_2} (ds),$$

where $C_{m_1}^m = \frac{m!}{m_1! \cdot m_2!}$. From (2.18) and (2.22), we have

$$\nabla_t^{m_1} \kappa \cdot \partial_t^{m-m_1}(ds) = \begin{cases} \left((-1)^m \nabla_s^{4m} \kappa + \displaystyle\sum_{[\![a,b]\!] \leq [\![4m-2,3]\!]} P_b^a(\kappa) \right) ds, \text{ as } m_1 = m, \\[4mm] \left(\displaystyle\sum_{[\![a,b]\!] \leq [\![4m-2,3]\!]} P_b^a(\kappa) \right) ds, \text{ as } m_1 \in \{m-1, ..., 0\}. \end{cases}$$

Thus,

$$\sum_{m_1 + m_2 = m} C_{m_1}^m \cdot \nabla_t^{m_1} \kappa \cdot \partial_t^{m_2} (ds) = (-1)^m \nabla_s^{4m} \kappa \, ds + \sum_{[\![a,b]\!] \leq [\![4m-2,3]\!]} P_b^a(\kappa) \, ds,$$

where each constant $C_{m_1}^m$ has been absorbed by the notation $P_b^a(\kappa)$ as $m_1 < m$. Therefore,

$$|\nabla_t^m T(t, x_i)|^2$$

$$\leq C(N, m) \cdot \sum_{j=1}^{N} \left(\left(\int_{I_j} |\nabla_s^{4m} \kappa| \, ds \right)^2 + \sum_{[\![a,b]\!] \leq [\![4m-2,3]\!]} \left(\int_{I_j} |P_b^a(\kappa)| \, ds \right)^2 \right)$$

$$\leq C(N, m) \cdot \mathcal{L}[f] \cdot \sum_{j=1}^{N} \int_{I_j} \left| \nabla_s^{4m} \kappa \right|^2 \, ds$$

$$+ C(N, m) \cdot \mathcal{L}[f] \cdot \sum_{j=1}^{N} \sum_{[\![a,b]\!] \leq [\![8m-4,6]\!]} \int_{I_j} |P_b^a(\kappa)| \, ds,$$

where $C(N, m)$ is a constant depending only on N and m. Now, the term $\sum\limits_{i=0}^{N} |\nabla_t^m T(t, x_i)|^2$ on the R.H.S. of (3.18) can be estimated by

(3.27)
$$\sum_{i=0}^{N} |\nabla_t^m T(t, x_i)|^2 \leq C_0(N, m) \cdot \mathcal{L}[f] \cdot \sum_{i=1}^{N} \int_{I_i} |\nabla_s^{4m} \kappa|^2 \ ds$$

$$+ C_0(N, m) \cdot \mathcal{L}[f] \cdot \sum_{i=1}^{N} \sum_{[\![a,b]\!] \leq [\![8m-4,6]\!]} \int_{I_i} |P_b^a(\kappa)| \ ds.$$

Note, from (2.16), we have

(3.28)
$$\sum_{i=1}^{N} \int_{I_i} |\nabla_t^m f|^2 \ ds = \sum_{i=1}^{N} \sum_{[\![a,b]\!] \leq [\![8m-4,2]\!]} \int_{I_i} P_b^a(\kappa) \ ds.$$

Therefore, from (3.18), (3.27), (3.28), we obtain

(3.29)
$$\frac{d}{dt} \mathcal{Y}_m(t) + \delta \cdot \mathcal{Y}_m(t) + (2 - \delta \cdot C_0(N, m) \cdot \mathcal{L}[f]) \cdot \sum_{i=1}^{N} \int_{I_i} |\nabla_s^{4m} \kappa|^2 \ ds$$

$$\leq \delta \cdot C_0(N, m) \cdot \mathcal{L}[f] \cdot \sum_{i=1}^{N} \sum_{[\![a,b]\!] \leq [\![8m-4,6]\!]} \int_{I_i} |P_b^a(\kappa)| \ ds$$

$$+ \delta \cdot \sum_{i=1}^{N} \sum_{[\![a,b]\!] \leq [\![8m-4,2]\!]} \int_{I_i} P_b^a(\kappa) \ ds + \sum_{i=1}^{N} \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \ ds.$$

Step 3° From the upper bound of the total length $\mathcal{L}_+$ in (3.20), we may choose a sufficiently small $\delta > 0$ so that

(3.30)
$$\delta \cdot C_0(N, m) \cdot \mathcal{L}_+ \leq 1$$

and then (3.29) gives

(3.31)
$$\frac{d}{dt} \mathcal{Y}_m(t) + \delta \cdot \mathcal{Y}_m(t) + \sum_{i=1}^{N} \int_{I_i} |\nabla_s^{4m} \kappa|^2 \ ds$$

$$\leq (C_0(N, m) \cdot \mathcal{L}_+)^{-1} \sum_{i=1}^{N} \sum_{[\![a,b]\!] \leq [\![8m-4,2]\!]} \int_{I_i} P_b^a(\kappa) \ ds + \sum_{i=1}^{N} \sum_{[\![a,b]\!] \leq [\![8m-2,4]\!]} \int_{I_i} P_b^a(\kappa) \ ds.$$

From the interpolation inequality (2.14), the lower bound of total length in (3.21) and the upper bound of bending energy in (3.19), we have

R.H.S. of (3.31)
$$\leq c_0 \cdot \varepsilon \cdot \sum_{i=1}^{N} \int_{I_i} |\nabla_s^{4m} \kappa|^2 \ ds + c_0 \cdot C \left(\mathcal{E}_\lambda[f_0], \lambda, f^{(0)}, ..., f^{(N)}, T^{(0)}, T^{(N)}, N, m, n, \varepsilon \right),$$

where $c_0 := \max\{1, (C_0(N, m) \cdot \mathcal{L}_+)^{-1}\}$ is a constant depending only on N, m, $\mathcal{E}_\lambda[f_0]$, λ. Thus, by choosing a sufficiently small $\varepsilon > 0$ so that $c_0 \cdot \varepsilon < 1$, we obtain from (3.31) that

$$\frac{d}{dt} \mathcal{Y}_m(t) + \delta \cdot \mathcal{Y}_m(t) \leq C \left(\mathcal{E}_\lambda[f_0], \lambda, f^{(0)}, ..., f^{(N)}, T^{(0)}, T^{(N)}, N, m, n \right),$$

where $\delta = \delta(N, m, \lambda, \mathcal{E}_\lambda[f_0]) > 0$ is due to the choice of δ in (3.30) and the upper bound of the total length $\mathcal{L}_+$ in (3.20). From this Gronwall's type differential inequality, we derive the uniform upper bound of $\mathcal{Y}_m(t)$,

$$(3.32) \qquad \mathcal{Y}_m(t) \le \mathcal{Y}_m(0) + C\left(\mathcal{E}_\lambda[f_0], \lambda, f^{(0)}, ..., f^{(N)}, T^{(0)}, T^{(N)}, N, m, n\right),$$

for all $t \in [0, t_1)$. Therefore,

$$(3.33)$$
$$\sum_{i=1}^{N} \|\nabla_t^m f\|_{L^2(I_i)}^2 (t) \le C\left(\mathcal{Y}_m(0), \mathcal{E}_\lambda[f_0], \lambda, f^{(0)}, ..., f^{(N)}, T^{(0)}, T^{(N)}, N, m, n\right),$$

for all $t \in [0, t_1)$.

Step 4° Observe that, by using the evolution equation (1.5) on the calculation of $\nabla_t^m T(\cdot, x_i)$, the assumption on C^∞-smoothness of initial curve f_0 in Theorem 1 implies that $\{\mathcal{Y}_m(0)\}_{m \in \mathbb{Z}_+}$ is a sequence of finite non-negative numbers. For each fixed $i \in \{1, ..., N\}$, we could estimate $\|\nabla_s^{4m-2} \kappa\|_{L^2(I_i)}^2$ by applying the formula of $\nabla_t^m f$ in (2.16), the interpolation inequality in (2.14), the upper bound of total bending energy $\sum_{i=1}^{N} \|\kappa\|_{L^2(I_i)}^2$ in (3.19) and the upper bound of $\|\nabla_t^m f\|_{L^2(I_i)}^2$ in (3.33) to obtain

$$(3.34)$$
$$\|\nabla_s^{4m-2} \kappa\|_{L^2(I_i)}^2 (t)$$
$$\le C\left(\mathcal{Y}_m(0), \mathcal{E}_\lambda[f_0], \lambda, f^{(0)}, ..., f^{(N)}, T^{(0)}, T^{(N)}, N, m, n\right),$$

for any $t \in [0, t_1)$ and $m \in \mathbb{Z}_+$. Note that, for any $\ell \in \mathbb{Z}_+$, we may choose $m = m_\ell := [\![\frac{\ell+2}{4}]\!] + 1$, where the notation $[\![A]\!]$ represents the greatest integer part of real number A, so that $\ell < 4m_\ell - 2$. Thus, by applying Lemma 6, the interpolation inequality (2.14), the upper bound of total bending energy $\sum_{i=1}^{N} \|\kappa\|_{L^2(I_i)}^2$ in (3.19) and (3.34), we obtain

$$(3.35)$$
$$\left\|\nabla_s^\ell \kappa\right\|_{L^2(I_i)}^2 (t) + \left\|\partial_s^\ell \kappa\right\|_{L^2(I_i)}^2 (t)$$
$$\le C\left(\mathcal{Y}_{m_\ell}(0), \mathcal{E}_\lambda[f_0], \lambda, f^{(0)}, ..., f^{(N)}, T^{(0)}, T^{(N)}, N, m_\ell, n\right),$$

for any $t \in [0, t_1)$, $i \in \{1, ..., N\}$ and $\ell \in \mathbb{Z}_+$.

For any differentiable function $g : I_i = (x_{i-1}, x_i) \subset \mathbb{R} \to \mathbb{R}^n$, it is easy to see that

$$(3.36) \qquad \widetilde{g}(s) := g(s) - \left(\int_{\sigma \in I_i} d\sigma\right)^{-1} \left(\int_{\sigma \in I_i} g(\sigma)\, d\sigma\right)$$

satisfies $\int_{I_i} \widetilde{g}(s)\, ds = 0$ and

$$(3.37) \qquad \|\widetilde{g}\|_{L^\infty(I_i)} \le c(n) \cdot \|\partial_s \widetilde{g}\|_{L^1(I_i)}.$$

By letting $g = \kappa_{\ell-1} := \partial_s^{\ell-1}\kappa$ in (3.37) and (3.36), where $\ell \in \mathbb{Z}_+$, we derive

$$(3.38) \qquad \|\kappa_{\ell-1}\|_{L^\infty(I_i)} \leq c(n) \cdot \|\partial_s \kappa_{\ell-1}\|_{L^1(I_i)} + \left(\int_{I_i} ds\right)^{-1} \cdot \|\kappa_{\ell-1}\|_{L^1(I_i)}.$$

By applying Hölder's inequality to the R.H.S. of (3.38), we obtain
(3.39)

$$\left\|\partial_s^{\ell-1}\kappa\right\|_{L^\infty(I_i)} \leq c(n) \cdot \left(\int_{I_i} ds\right)^{1/2} \left\|\partial_s^\ell \kappa\right\|_{L^2(I_i)} + \left(\int_{I_i} ds\right)^{-1/2} \left\|\partial_s^{\ell-1}\kappa\right\|_{L^2(I_i)}.$$

From applying the uniform upper bound of total length in (3.20), the uniform lower bound of the length of each component in (3.25) and the estimates in (3.35), we obtain from (3.39),

$$(3.40) \quad \left\|\partial_s^{\ell-1}\kappa\right\|_{L^\infty(I_i)} \leq C\left(\mathcal{Y}_{m_\ell}(0), \mathcal{E}_\lambda[f_0], \lambda, f^{(0)}, ..., f^{(N)}, T^{(0)}, T^{(N)}, N, m_\ell, n\right),$$

which gives a uniform upper bound of $\left\|\partial_s^{\ell-1}\kappa\right\|_{L^\infty(I_i)}$ for each $\ell \in \mathbb{Z}_+$. However, this is a contradiction if we assume that the solution of (1.5) remains smooth on each $f_{|I_i}$ only up to a finite time t_1. Therefore, $f \in \bigcup\limits_{i=1}^{N} C^\infty((x_{i-1}, x_i), \mathbb{R}^n)$.

Step 5° For the asymptotic behavior of the flow, we choose a subsequence of curves $f(t, \cdot)$, which converges smoothly to a curve f_∞ on each open interval after reparametrization of arclength. Let

$$u(t) := \sum_{i=1}^{N} \int_{I_i} |\partial_t f|^2 \, ds,$$

and

$$v(t) := \sum_{i=0}^{N} |\nabla_t T|^2(t, x_i).$$

By applying (3.33), we derive the inequality,

$$(3.41) \qquad \left|\tfrac{d}{dt}u(t)\right| \leq C\left(\mathcal{Y}_1(0), \mathcal{Y}_2(0), \mathcal{E}_\lambda[f_0], \lambda, f^{(0)}, ..., f^{(N)}, T^{(0)}, T^{(N)}, N, n\right),$$

for all $t \in [0, \infty)$. Moreover, by applying the interpolation inequality in (2.14) and the upper bound of the total length in (3.20) to (3.27), we derive the inequality,

$$(3.42) \qquad \left|\tfrac{d}{dt}v(t)\right| \leq C\left(\mathcal{Y}_2(0), \mathcal{Y}_3(0), \mathcal{E}_\lambda[f_0], \lambda, f^{(0)}, ..., f^{(N)}, T^{(0)}, T^{(N)}, N, n\right),$$

for all $t \in [0, \infty)$. Thus, $w(t) := u(t) + v(t)$ satisfies
(3.43)

$$\left|\tfrac{d}{dt}w(t)\right| \leq C\left(\mathcal{Y}_1(0), \mathcal{Y}_2(0), \mathcal{Y}_3(0), \mathcal{E}_\lambda[f_0], \lambda, f^{(0)}, ..., f^{(N)}, T^{(0)}, T^{(N)}, N, n\right),$$

for all $t \in [0, \infty)$.

On the other hand, the energy identity in (3.4) implies that $w(t) \in L^1([0, \infty))$. Therefore, $w(t) \to 0$ as $t \to \infty$. Since both $u(t)$ and $v(t)$ are non-negative, from the definition of $w(t)$, $u(t) \to 0$ as $t \to \infty$, which implies that f_∞ is independent of t. Thus, from the equation of L^2-flow (1.5), f_∞ is an equilibrium of $\mathcal{E}_\lambda$ on each I_i with the uniform bound of any higher-order derivatives in (3.40), i.e., $f_\infty \in$

$\bigcup_{i=1}^{N} C^{\infty}((x_{i-1}, x_i), \mathbb{R}^n)$. Besides, from the interior boundary condition in (1.8), $\partial_s^2 f_\infty(x_i^-) = \partial_s^2 f_\infty(x_i^+)$ for all $i \in \{1, ..., N-1\}$. Therefore, $f_\infty \in C^2((-1, 1), \mathbb{R}^n)$.

$\square$

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