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by

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LINEAR AND PROJECTIVE BOUNDARIES IN HNN-EXTENSIONS AND DISTORTION PHENOMENA

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Abstract. The linear boundary and the projective boundary have recently been introduced by Krön, Lehnert, Seifter and Teufl [6] as a quasi-isometry invariant boundary of Cayley graphs of finitely generated groups, but also as a more general concept in metric spaces.

An element of the linear boundary of a Cayley graph is an equivalence class of forward orbits $g^\infty = \{g^i : i \in \mathbb{N}\}$ of non-torsion elements g of the group G. Two orbits are equivalent when they stay sublinearly close to each other. For a formal definition see below. The elements of the projective boundary are obtained by taking cyclic subgroups $g^{\pm\infty} = \{g^i : i \in \mathbb{Z}\}$ instead of forward orbits. The boundaries are then obtained by equipping these points at infinity with an angle metric. A typical example is the $(n-1)$-dimensional sphere as linear boundary of $\mathbb{Z}^n$. Its projective boundary is the $(n-1)$-dimensional projective space.

The diameter of these boundaries is always at most 1. We show that for all finitely generated groups, the distance between the antipodal points g^∞ and $g^{-\infty}$ in the linear boundary is bounded from below by $\sqrt{1/2}$. But these distances can actually be smaller than 1: we give an example of a one-relator group—a derivation of the Baumslag-Gersten group—which has an infinitely iterated HNN-extension as an isometrically embedded subgroup. In this example, there is an element g for which the distance between g^∞ and $g^{-\infty}$ is less or equal $\sqrt{12/17}$.

We also give an example of a group with elements g and h such that $g^\infty = h^\infty$, but $g^{-\infty} \neq h^{-\infty}$. Furthermore, we introduce a notion of average-case-distortion – called growth of elements – and compute an explicit positive lower bound for the distances between points g^∞ and h^∞ which are limits of group elements g and h with different growth.

1. Introduction

One of the most important classes of groups studied in Geometric Group Theory is the class of word-hyperbolic groups (also referred to as Gromov-hyperbolic groups). Word-hyperbolic groups admit several geometric tools which can be used to derive algebraic properties. Since in Geometric Group Theory the focus lies on the large-scale geometry of the group, these tools are only defined up to quasi-isometries. An important large-scale invariant of a hyperbolic group is its Gromov-boundary. The present work is part of a program to understand up to which extent one can generalize this concept to arbitrary finitely generated groups.

A new concept of quasi-isometry invariant boundaries of metric spaces has recently been introduced by Krön, Lehnert, Seifter and Teufl [6]. It is related to a concept due to Bonnington, Richter and Watkins [1]. This concept is rather general and for instance, Tits' boundary of a CAT(0) space (see [2, Section 9]) fits into it, after a small modification. See [6] for a more detailed discussion of this relationship.

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We will not recall the full concept for metric spaces, because here, we are only interested in two applications to Cayley graphs of finitely generated groups, namely the linear and the projective boundary, which we shall introduce next.

Let G be a group generated by a set X. The Cayley graph $\Gamma = (V, E) = \mathrm{Cay}(G, X)$ is the graph with vertex set $V = G$ and edge set $E = \{\{g, h\} : g^{-1}h \in X\}$. Let d be the graph metric of Γ. That is, $d(g, h)$ is the length of the shortest path in Γ from g to h.

For $g \in G$ of infinite order let $g^\infty := \{g^n : n \in \mathbb{N}\}$ denote the cyclic subsemigroup generated by g. We also call g^∞ the *forward orbit* of g. Let $g^{\pm\infty} := \{g^k : k \in \mathbb{Z}\}$ denote the cyclic subgroup generated by g, and we call $g^{\pm\infty}$ the *orbit* of g. The *backward orbit* $g^{-\infty}$ is defined analogously.

Let $\mathcal{C}G$ and $\mathcal{C}^+G$ denote the family of infinite orbits or infinite forward orbits, respectively. That is, we set

$$\mathcal{C}G := \{g^{\pm\infty} : g \in G,\ |g| = \infty\}$$

and

$$\mathcal{C}^+G := \{g^\infty : g \in G,\ |g| = \infty\}.$$

We want to measure the distance between two orbits as if it were an angle. For this, fix $\alpha > 0$ and $c \in \mathbb{N}$, and call the set

$$\alpha \cdot g^\infty + c := \{v \in G : \exists n \in \mathbb{N} \text{ such that } d(v, g^n) \le \alpha \cdot d(1, g^n) + c\}$$

the (α, c)*–cone* around g^∞. In other words, the (α, c)*–cone* around g^∞ is the union of all balls with center g^n and radius $\alpha \cdot d(1, g^n) + c$. Analogously we define the (α, c)-cone around $g^{\pm\infty}$ as

$$\alpha \cdot g^{\pm\infty} + c := \{v \in G : \exists k \in \mathbb{Z} \text{ so that } d(v, g^k) \le \alpha \cdot d(1, g^k) + c\}.$$

We write $h^\infty \in \alpha \cdot g^\infty + c$ if $h^n \in \alpha \cdot g^\infty + c$ for all $n \in \mathbb{N}$ and define $h^{\pm\infty} \in \alpha \cdot g^{\pm\infty} + c$ analogously. For $x, y \in \mathcal{C}G$ or $x, y \in \mathcal{C}^+G$ set

$$s(x, y) := \inf\{\alpha \in \mathbb{R} : \exists c \in \mathbb{N} \text{ such that } x \in \alpha \cdot y + c \text{ and } y \in \alpha \cdot x + c\}.$$

If $s(x, y) = 0$ then we call x and y *linearly equivalent*, this is an equivalence relation. We call two elements g and h *forward equivalent* if $g^\infty \sim h^\infty$ and *backward equivalent* if $g^{-\infty} \sim h^{-\infty}$.

It is easy to check that the function s is well defined on the set of equivalence classes and that the square root $t = \sqrt{s}$ is a metric on the quotient $\mathcal{C}^+G/{\sim}$ and on $\mathcal{C}G/{\sim}$, respectively. The completion of the metric space $(\mathcal{C}^+G/{\sim}, t)$ is called the *linear boundary* $\mathcal{L}G$ of G, the completion of the metric space $(\mathcal{C}G/{\sim}, t)$ is called the *projective boundary* $\mathcal{P}G$ of G, or strictly speaking of G with respect to the generator X. Although the elements of the linear/projective boundary are equivalence classes of (forward) orbits $g^{(\pm)\infty}$, and not the (forward) orbits themselves, we shall slightly abuse notation and write $g^{(\pm)\infty}$ instead of $[g^{(\pm)\infty}]_\sim$ also for an element of the linear or projective boundary.

If G is finitely generated and we change the finite set of generators then the resulting quotient spaces are bi-Lipschitz equivalent and hence the boundaries are homeomorphic. But the values of s and t depend on the choice of generators. Moreover, by definition it is clear that the diameter of $\mathcal{L}G$ and of $\mathcal{P}G$ is at most 1. For more details we refer to [6].

The linear boundary of finitely generated nilpotent groups is (homeomorphic to) the disjoint union of spheres with dimensions d_i, which correspond to the free abelian quotients of rank $d_i + 1$ in the central series, and the projective boundary is (homeomorphic to) the disjoint union of projective spaces of the same dimension; see [6]. The latter fact relies on the observation that in the case of a nilpotent group the distance $t(g^\infty, h^\infty)$ equals the distance of the inverse elements $t(g^{-\infty}, h^{-\infty})$ for all $g^\infty, h^\infty \in \mathcal{L}G$. Thus the space $\mathcal{P}G$ can be obtained identifying each element with its inverse without changing distances (that is, $t_{\mathcal{P}G}(g^{\pm\infty}, h^{\pm\infty}) = t_{\mathcal{L}G}(g^\infty, h^\infty)$ holds for all $g, h \in G$).

One might guess that this yields a general method to construct the projective boundary but the results in Section 3 show that this is not the case. In general it is not even true that $g^\infty = h^\infty$ implies $g^{-\infty} = h^{-\infty}$ hence the projective boundary is not necessarily a quotient of the linear boundary.

Theorem 1.1. *There is a group H with elements g_1 and g_2 which are forward-equivalent but not backward-equivalent.*

The proof of Theorem 1.1 is given in Section 3.

Knowing of these counterintuitive phenomena regarding the distances of forward orbits compared to the distances of backward orbits, it is natural to ask whether the 'algebraic antipodal' $g^{-\infty}$ of $g^\infty \in \mathcal{C}^+G$ is also the metric antipodal. In other words, one would like to know whether $t(g^\infty, g^{-\infty})$ is always 1 or if at least this distance is universally bounded away from 0. We show that the answer to the first question is negative, but that there is a positive lower bound for $t(g^\infty, g^{-\infty})$.

Theorem 1.2.

(a) For any finitely generated group G and any $g \in G$ of infinite order we have $t(g^\infty, g^{-\infty}) \geq \sqrt{1/2}$.

(b) There exists a finitely generated group G which has an element g such that $t(g^\infty, g^{-\infty}) \leq \sqrt{12/17}$.

The proof of this result will span from Section 4 to Section 6. While the proof of the first part of Theorem 1.2 is not overly complicated, the proof of the second part is quite lengthy and takes up most of these three sections. There, we give an example of a family of groups as in Theorem 1.2(b). The groups in question are derivations of the so called Baumslag-Gersten group and in order to prove our theorem we have to understand some of the intrinsic geometry of these groups.

As we will see, the geometry of a cyclic subgroup can be very different from the usual geometry of the group of integers. This phenomenon is known as distortion and leads to one of the asymptotic invariants studied by Gromov in his seminal book [5]. For an element h of a group G generated by the finite set X let $|h|_X$ denote the length of the shortest word representing h in letters of $X^\pm$, where $X^\pm = \{x \in G : x \in X \text{ or } x \in X^{-1}\}$. Gromov defines the distortion function for a subgroup H generated by the finite set Y as:

$$\Delta_G^H(r) := \frac{1}{r} \max\{|h|_Y : h \in H,\ |h|_X \leq r\}.$$

This function measures something like a worst-case distortion and can easily be superexponential, for instance in the group G_p of Theorem 4.2. Such examples suggest that the factor $1/r$ is a bit artificial and in fact nowadays most authors follow the definition of Farb [3] who defined the distortion function just as $\Delta_G^H(r) := \max\{|h|_Y : h \in H,\ |h|_X \leq r\}$.

In the context of this work we are interested in the distortion of cyclic subgroups (or even cyclic subsemigroups). But as we would like to view these subgroups just as a set rather than as a sequence, worst-case considerations do not seem appropriate. A better fitting concept will be a kind of average-case distortion for cyclic subgroups—called growth of elements—which we define as follows:

Definition 1.3. Let G be a group generated by the finite set X and let $g \in G$. The *growth* of g is the function $w_g(n) : \mathbb{N} \to \mathbb{N}$ which counts the number of elements of the type g^i in the ball $B_1(n)$ of radius n around 1:

$$w_g(n) := |\{i \in \mathbb{Z} : |g^i|_X \leq n\}|.$$

Note that for the group $H = \langle g \rangle$ our growth function $w_g(n)$ measures the number of elements of H in the ball of radius r around 1, while Gromow's distortion $\Delta_G^H(r)$ determines the absolute value of the maximum of all i such that g^i still lies in this ball. There are some easy bounds on the growth. First of all, balls in Cayley graphs grow at most exponentially fast. Namely, it is easy to see that the upper bound $w_g(n) \leq |B_1(n)| \leq (2|X|-1)(2|X|)^{n-1}$ holds. Less obvious but still straightforward is the fact, that for all $k \in \mathbb{N}$ we have

$$w_g(kn) \geq k \cdot w_g(n).$$

For instance the groups which will be defined in Theorem 4.2 contain elements with exponential growth function, and in free nilpotent groups of class c the growth function of a central element is equivalent to n^c. The results of Olshanskii and Sapir [8] on length functions of subgroups, which are a very precise measure for distortion phenomena, suggest that there exist a broad variety of growth functions for elements. It seems natural to ask the following question:

Problem 1.4. *Can two elements g and h of a group, whose forward orbits are linearly equivalent, have growths of different order?*

In Section 2 we will give a partial solution to this problem. If g is an element of exponential growth, then there is even a minimal distance between $g^{\pm\infty}$, and any other orbit of $\mathcal{P}G$ of an element h of the group which has a different growth. This minimal distance depends on the number of generators of G and the growth functions of g and h. Our lower bound also holds for the minimal distance in $\mathcal{L}G$.

Theorem 1.5. *For every $d \in \mathbb{N}$, $\delta > 1$ and $\gamma > \delta$ there is a $t_{min} = t_{min}(d, \gamma, \delta)$ such that for each group G that is generated by d elements, and any $g, h \in G$ with $w_g(n) \in \omega(\gamma^n)$ and $w_h(n) \in o(\delta^n)$ we have that*

$$t(g^{\pm\infty}, h^{\pm\infty}) \geq t_{min} \quad \text{and} \quad t(g^{\infty}, h^{\infty}) \geq t_{min}.$$

In order to be able to speak of the growth of an element of a group without fixing a generating set, we consider equivalence classes of growth functions rather than explicit functions. Functions $f, g : \mathbb{N} \to \mathbb{N}$ are called *weakly equivalent* if there exist constants c_1, c_2 such that

$$\begin{aligned} g(n) &\leq c_1 f(c_1 n + c_2) + c_2 \text{ and} \\ f(n) &\leq c_1 g(c_1 n + c_2) + c_2 \end{aligned}$$

hold. If X and Y are finite generating sets for G, then $\mathrm{Cay}(G, X)$ and $\mathrm{Cay}(G, Y)$ are bi-Lipschitz equivalent and therefore the growth function of g with respect to X and the growth function of g with respect to Y are weakly equivalent. Note that this equivalence separates exponential functions from sub-exponential functions and hence having an exponential growth function is a property of the group element which is independent of the chosen generating set.

We say that an element of a finitely generated group has *exponential growth* if there is a generating set S of G such that the growth function of g with respect to S is exponential (by the preceding paragraph, this holdss then so for any generating set S). Now, Theorem 1.5 immediately gives the following corollary.

Corollary 1.6. *If g is an element of a finitely generated group that has exponential growth, then every element h with $g^\infty = h^\infty$ (or with with $g^{\pm\infty} = h^{\pm\infty}$) also has exponential growth.*

Before we start let us fix some further notation. Throughout the paper G will be a group generated by a (usually finite) set X. The free monoid over the alphabet $X^{\pm}$ will be denoted X^* and ℓ is the length function on X^*. The assumption that X is a generating set of G implies the existence of a surjective monoid homomorphism $\pi : X^* \to G$ and it is straightforward that for $g, h \in G$ we have

$$d(g,h) = \min\{\ell(w) : w \in X^*,\ \pi(x) = g^{-1}h\}.$$

Using this fact, we mostly work with representing words for group elements. We will use the shorthand notation $w_1 =_G w_2$ for $\pi(w_1) = \pi(w_2)$ whereas $w_1 = w_2$ means that the two words as elements of X^* are equal.

We assume that the reader is familiar with the concept of HNN-extensions and in particular with Britton's Lemma which most of our considerations concerning Part (b) of Theorem 1.2 rely on. Britton's Lemma can be used to derive a normal form for elements in HNN-extensions and gives a necessary condition for a word to represent the identity. The standard references for these results (and many other facts on HNN-extensions) are [7] and [10].

2. Distortion phenomena

The present section is dedicated to the aforementioned distortion phenomena. We prove Theorem 1.5.

Proof of Theorem 1.5. We will only show the result for the elements of the projective boundary, that is, we show the existence of a number t_{min} such that for each group G that is generated by d elements, and any $g, h \in G$ with $w_g(n) \in \omega(\gamma^n)$ and $w_h(n) \in o(\delta^n)$, the inequality $t(g^{\pm\infty}, h^{\pm\infty}) \geq t_{min}$ holds. The other part can be shown analogously.

We assume that $t(g^{\pm\infty}, h^{\pm\infty}) < 1$, since otherwise 1 is the desired bound.

Since $w_g(n) \in \omega(\gamma^n)$ and $w_h(n) \in o(\delta^n)$ there exist constants N_0, c_1, c_2, such that for all $n > N_0$ it holds:

$$w_g(n) \geq c_1 \cdot \gamma^n \text{ and } w_h(n) \leq c_2 \cdot \delta^n \tag{1}$$

Let $n > N_0$, let $\alpha \in \mathbb{R}$ s.t. $1 > \alpha > t(g^{\pm\infty}, h^{\pm\infty})^2 = s(g^\infty, h^\infty)$.

By definition there exists a constant c such that for all $i \geq 0$ there exists a $j = j(i)$ such that

$$g^i \in B_{\alpha d(1,h^j)+c}(h^j).$$

If $d(1, g^i) \leq n$ then by the triangle-inequality,

$$d(1,h^j) \leq d(1,g^i) + d(g^i h^j) \leq n + \alpha d(1,h^j) + c$$

and thus $d(1,h^j) \leq \frac{n+c}{1-\alpha}$.

Set $I := \{i \in \mathbb{Z} : d(1,g^i) < n\}$, and set $J := \{j \in \mathbb{Z} : d(1,h^j) \leq \frac{n+c}{1-\alpha}\}$. Then for each $i \in I$ we have $i(j) \in J$. By (1), $|I| \geq c_1\gamma^n$ and $|J| \leq c_2\delta^{\frac{n+c}{1-\alpha}}$, and the latter is smaller than $c_3\delta^{\frac{n}{1-\alpha}}$ for some constant c_3. Hence, by the pigeon-hole principle, there exists a j, such that

$$|B_{\alpha d(1,h^j)+c}(h^j)| \geq \frac{c_1 \cdot \gamma^n}{c_3 \cdot \delta^{\frac{n}{1-\alpha}}} = \frac{c_1}{c_3}\left(\frac{\gamma}{\delta^{\frac{1}{1-\alpha}}}\right)^n.$$

On the other hand $|B_{\alpha d(1,h^j)+c}(h^j)|$ is bounded above by a power of the number of generators d, namely by

$$|B_{\alpha d(1,h^j)+c}(h^j)| \leq 2d \cdot (2d-1)^{\alpha d(1,h^j)+c-1}.$$

We obtain the inequality

$$\begin{aligned}
\frac{c_1}{c_3}\left(\frac{\gamma}{\delta^{\frac{n}{1-\alpha}}}\right)^n &\leq 2d\cdot(2d-1)^{\alpha d(1,h^j)+c-1}\\
&\leq 2d\cdot(2d-1)^{\alpha\frac{n+c}{1-\alpha}+c-1}\\
&= c_4\cdot(2d-1)^{\frac{\alpha n}{1-\alpha}}\\
&= c_4\cdot\left((2d-1)^{\frac{\alpha}{1-\alpha}}\right)^n,
\end{aligned}$$

for $c_4 = 2d\cdot(2d-1)^{\frac{\alpha c}{1-\alpha}+c-1}$. This has to be true for arbitrary large values of n, which is possible only if

$$\begin{aligned}
& \frac{\gamma}{\delta^{\frac{1}{1-\alpha}}} && \leq (2d-1)^{\frac{\alpha}{1-\alpha}}\\
\Leftrightarrow\quad & \gamma^{1-\alpha} && \leq (2d-1)^{\alpha}\cdot\delta\\
\Leftrightarrow\quad & \ln\gamma-\alpha\cdot\ln\gamma && \leq \alpha\cdot\ln(2d-1)+\ln\delta\\
\Leftrightarrow\quad & \frac{\ln\gamma-\ln\delta}{\ln(2d-1)+\ln\gamma} && \leq \alpha\\
\Leftrightarrow\quad & \log_{(2d-1)\gamma}\frac{\gamma}{\delta} && \leq \alpha.
\end{aligned}$$

Note that $\frac{\gamma}{\delta} < (2d-1)\gamma$ and therefore this lower bound is less than 1. We obtain the lower bound $t(g^{\pm\infty}, h^{\pm\infty}) \geq \sqrt{\log_{(2d-1)\gamma}\frac{\gamma}{\delta}}$. □

The complete answer to Problem 1.4 remains open. In addition it might be an interesting project to completely understand the relationship between the usual distortion of cyclic subgroups and the growth of the generating element. It obviously happens that cyclic subgroups of different distortion yield elements of the same growth type but whether it can also be the other way around is an open question.

3. Forward- vs. backward-equivalence

In this section we will construct a group H that contains elements g_1 and g_2 for which $g_1^{\infty} \sim g_2^{\infty}$ but $g_1^{-\infty} \not\sim g_2^{-\infty}$. The group H is an iterated HNN-extension of a cyclic group (generated by the element a) with stable letters s, t, x given by the presentation

$$H = \left\langle a, s, t, x \,\middle|\, t^{-1}at = a^2, s^{-1}as = a^2, x^{-1}sx = s^2 \right\rangle. \tag{2}$$

Thus H is isomorphic to a free product with amalgamation $H = H_1 *_{\langle a\rangle} H_2$ where H_1 is the Baumslag-Solitar group $BS(1,2) = \langle a, t | t^{-1}at = a^2\rangle$ and $H_2 = \langle a, s, x | s^{-1}as = a^2, x^{-1}sx = s^2\rangle$ is an HNN-extension of $BS(1,2) = \langle a, s | s^{-1}as = a^2\rangle$ with associated subgroups $\langle s\rangle$ and $\langle s^2\rangle$.

We use the group H to prove Theorem 1.1.

Proof of Theorem 1.1. We have to show that H contains elements g_1 and g_2 which are forward-equivalent but not backward-equivalent. We do this for $g_1 := t$ and $g_2 := at$.

First of all, we estimate the distance $d_H(1, g_i^k)$ for $k \in \mathbb{Z}$. In all defining relations of presentation (2) the exponent sum of t is zero, hence any word representing t^k needs at least $|k|$ times the letter t (or t^{-1} if $k < 0$). So the word t^k is geodesic and

$$d(1, g_1^k) = |k|. \tag{3}$$

The same argument yields that

$$|k| \leq d(1, g_2^k) \leq 2|k|, \tag{4}$$

which will be a sufficient approximation for our purpose.

Let $k > 0$. We can use the relation $t^{-1}at = a^2$, which is the same as $at = ta^2$, to see that

$$g_2^k = t^k a^{2^{k+1}-2}. \tag{5}$$

By definition, the distance $d_H(g_1^k, g_2^k)$ is the same as

$$d_H(1, g_1^{-k} g_2^k) = d_H(1, t^{-k}t^k a^{2^{k+1}-2}) = d_H(1, a^{2^{k+1}-2}). \tag{6}$$

One easily checks that

$$a^{2^{k+1}-2} = s^{-(k+1)} a s^{k+1} a^{-2}. \tag{7}$$

Hence to obtain an upper bound for $d_H(1, g_1^{-k} g_2^k)$ we need to find a good upper bound for $d_H(1, s^k)$. Let $k_m k_{m-1} \dots k_0$ be the binary code for k (that is, $k_i \in \{0, 1\}$ and $k_m = 1$). Then, because of the relation $x^{-1}sx = s^2$, it holds that $(\prod_{i=0}^{m-1} s^{k_i} x^{-1}) s x^m = s^k$. The fact that $m = \lfloor \log_2 k \rfloor$ gives us the upper bound $d_H(1, s^k) \le 3 \cdot \lfloor \log_2 k \rfloor + 1$. Thus by (6) and (7),

$$d_H(1, g_1^{-k} g_2^k) \le 6 \cdot \lfloor \log_2(k+1) \rfloor + 5. \tag{8}$$

In order to show that $d^X_{\mathcal{LH}}(g_1^\infty, g_2^\infty) = 0$, we now fix an $\alpha > 0$ and show that $d_{\mathcal{L}H}(g_1^\infty, g_2^\infty) \le \alpha$. To do so, by (4), it suffices to show that there exists a constant $c = c(\alpha)$ such that for each k there exist k_1 and k_2 such that $d_H(g_1^k, g_2^{k_1}) < \alpha \cdot k_1 + c$ and $d_H(g_2^k, g_1^{k_2}) < \alpha \cdot k_2 + c$. Choosing $k_1 = k_2 = k$ and using (8), this breaks down to the statement that there exists a constant $c = c(\alpha)$ such that

$$6 \cdot \lfloor \log_2(k+1) \rfloor + 5 \le \alpha \cdot k + c,$$

which is obviously true. This shows that g_1 and g_2 are forward-equivalent.

We shall now show that g_1 and g_2 are not backward-equivalent. In fact, we claim that $d_{\mathcal{L}H}(g_1^{-\infty}, g_2^{-\infty}) = 1$. For this, by (3), it suffices to show that for each $c \in \mathbb{N}$ there exists an $l' \in \mathbb{N}$ such that for all $l \in \mathbb{N}$ the inequality

$$d_H(g_2^{-l'}, g_1^{-l}) > 1 \cdot d(1, g_1^l) + c = \ell + c$$

holds. Set $l' := c + 2$. By definition, and because of the relation $t^{-1}a^{-1}t = a^{-2}$, we have

$$d_H(g_2^{-l'}, g_1^{-l}) = d_H(1, g_1^l g_2^{-l'}) = d_H(1, t^l a^{-(2^{l'+1}-2)} t^{-l'}),$$

where for the last inequality we used (5).

Now, $h = t^l a^{-(2^{l'+1}-2)} t^{-l'}$ is an element of the subgroup H_1 and we can try to simplify it within the presentation of this group. Using $2^{l'} - 1$ times the relation $t^{-1}a^{-1}t = a^{-2}$ we obtain that $h = t^{l-1} a^{-2^{l'}+1} t^{-l'+1}$. In this word the number of a^{-1}'s is odd and therefore the t's and t^{-1}'s cannot cancel out (moving a t from left to right through a power of a's halves this power). For this reason any word in H_1 representing h has to contain at least $(l-1) + (l'-1)$ times the letters t or t^{-1}.

The group H is a free product with amalgamation of H_1 and H_2 and because no power of t is contained in the cyclic subgroup generated by a. Also, any word in H representing h has to contain at least $(l-1) + (l'-1)$ times the letters t or t^{-1}. So,

$$d_H(g_1^{-l}, g_2^{-l'}) \ge (l-1) + (l'-1) > 1 \cdot l + c,$$

as desired. □

4. The distance between g^∞ and $g^{-\infty}$

The remainder of this paper is devoted to the proof of Theorem 1.2. We split it into two parts. First we show in Theorem 4.1 the easier lower bound, the distance between two elements g^∞ and $g^{-\infty} = (g^{-1})^\infty$ of the linear boundary of a finitely generated group G. The more difficult part of Theorem 1.2 is obtained from Theorem 4.2, which shows that there are examples of groups with elements g where the distance between g^∞ and $g^{-\infty}$ is strictly smaller than 1. The proof of Theorem 4.2 will continue in Sections 5 and 6.

But let us first show the easier bound:

Theorem 4.1. *Let g be an element of a finitely generated group of infinite order. Then $t(g^\infty, g^{-\infty}) \geq 1/\sqrt{2}$.*

Proof. Any ball in a group with respect to a finite generating set is finite. Hence

$$\lim_{i\to\infty} d(1, g^i) = \infty. \tag{9}$$

Suppose $\alpha \in \mathbb{R}$ is such that $s(g^\infty, g^{-\infty}) < \alpha$. Then there is a $c \in \mathbb{N}$ such that for each i there exists an $m(i) \in \mathbb{N}$ with

$$\begin{aligned} d(g^{-i}, g^{m(i)}) &\leq \alpha \cdot d(1, g^{m(i)}) + c \\ &\leq \alpha \cdot \Big(d(1, g^{-i}) + d(g^{-i}, g^{m(i)})\Big) + c, \end{aligned}$$

using the triangle-inequality. By (9), there is an increasing sequence $(i_n)_{n\geq 1}$ such that

$$d(1, g^k) > d(1, g^{i_n}) \tag{10}$$

for all $k > i_n$. Thus

$$\begin{aligned} \alpha &\geq \frac{d(g^{-i_n}, g^{m(i_n)}) - c}{d(1, g^{-i_n}) + d(g^{-i_n}, g^{m(i_n)})} \\ &= \frac{d(1, g^{i_n + m(i_n)}) - c}{d(1, g^{i_n}) + d(1, g^{i_n + m(i_n)})} \\ &\overset{(10)}{\geq} \frac{d(1, g^{i_n + m(i_n)}) - c}{2 \cdot d(1, g^{i_n + m(i_n)})} \\ &= \frac{1}{2}\Big(1 - \frac{c}{d(1, g^{i + m(i_n)})}\Big). \end{aligned}$$

Since this inequality is valid for all i_n, $n \in \mathbb{N}$, and because of (9), we obtain that $\alpha \geq 1/2$. As α may be chosen arbitrarily close to $s(g^\infty, g^{-\infty})$, this implies that $s(g^\infty, g^{-\infty}) \geq 1/2$, and thus, $t(g^\infty, g^{-\infty}) \geq 1/\sqrt{2}$. □

We now turn to the proof of the second part of Theorem 1.2. This proof is rather tedious and will span over the remainder of this section and the following two sections.

Theorem 4.2. *Let $p \geq 20$. In the group $G_p = \langle a, t | t^{-1}a^{-1}tat^{-1}at = a^p \rangle$ it holds that $t(a^\infty, a^{-\infty}) \leq \sqrt{12/17}$.*

Remark 4.3. The group G_p from Theorem 4.2 has a perhaps more natural description: Consider the Baumslag-Solitar group $BS(1,p) = \langle a, x | x^{-1}ax = a^p \rangle$ and build the HNN-extension with associated subgroups $\langle a \rangle$ and $\langle x \rangle$. The resulting group is isomorphic to G_p. Furthermore, if we replace the p in the presentation by the number 2 we obtain what is called the Baumslag-Gersten group G_2. This group was constructed by Gersten [4] (see also [9]) as an example of a group with Dehn function $\sim {}^n2$.

Remark 4.4. From now on we consider $p \geq 20$ to be a fixed number. We chose a lower bound of 20 for the sake of brevity of the arguments. However, this is not the best possible bound for p. As a matter of fact we believe the theorem to hold for all $p \geq 2$.

We already remarked that the remainder of this section and the following two sections are devoted to the somewhat lengthy proof of Theorem 4.2. The main aim of rest of the present section is to introduce certain short geodesic words w_k of G_p, which represent large powers of a. The words w_k will later be used to show that $t(a^\infty, a^{-\infty})$ is bounded from above by $\sqrt{12/17}$.

For the sake of simplicity, let us shift our attention for a moment from G_p to the infinitely generated group G' which shall be defined next. First, set

$$\begin{aligned} G_i^k &:= \langle a_i, \ldots, a_k \mid a_j^{-1} a_{j-1} a_j = a_{j-1}^p, j = i+1, i+2, \ldots, k \rangle \\ G_i^\infty &:= \langle a_i, a_{i+1} \ldots \mid a_j^{-1} a_{j-1} a_j = a_{j-1}^p, j = i+1, i+2, \ldots \rangle, \end{aligned}$$

and set $G' := G_0^\infty$. Then $G_i^k = \langle A_i^k \rangle_{G'}$ and $G_i^\infty = \langle A_i \rangle_{G'}$ where

$$A_i^k := \{a_i, \ldots, a_k\} \quad \text{and} \quad A_i^\infty := \bigcup_{k>i} A_i^k.$$

Using the isomorphism $\varphi(a_j) = a_{j+i}$ we see that

$$G_i^\infty \cong G' \text{ and } G_j^k \cong G_{j+i}^{k+i} \text{ for all } i, j, k \in \mathbb{N}. \tag{11}$$

We shall now embed G' in G_p. By (11), the subgroup generated by the elements $\{a_i, a_{i+1}, a_{i+2} \ldots\}$ is isomorphic to G'. Therefore we can construct the ascending HNN-extension G associated to φ. Then

$$G = \langle t, a_i (i = 0, 1, 2 \ldots) \mid a_{i+1}^{-1} a_i a_{i+1} = a_i^p, t^{-1} a_i t = a_{i+1} \rangle.$$

Substituting a_0 by a and applying Tietze-transformations we obtain the presentation from Theorem 4.2:

$$G = G_p = \langle a, t \mid t^{-1} a^{-1} t a t^{-1} a t = a^p \rangle.$$

So G is in fact a one-relator group on two generators. Even if the elements a_i do no longer belong to our set of generators, we will still use the notation a_i for the element $t^{-i} a t^i$. In order to prove Theorem 4.2 we are only interested in distances between powers of a, hence elements of the subgroup G'. Such words have to contain the same number of letters t and t^{-1}. Moreover, they can be written entirely in letters a_i using the following rewriting process:

Let v be a word in $\{a^\pm, t^\pm\}^*$ as above. We replace every a by the letter a_i and every a^{-1} by a_i^{-1}, where i is the difference of the number of t^{-1}'s and the number of t's ahead of this a or a^{-1}, respectively. Afterwards we delete all letters $t^\pm$ to obtain the word $v' \in \{a_i^\pm\}_{i\in\mathbb{N}}^*$. For example $v = t^{-2} a t^4 a^2 t^{-3} a^{-5} t a$ becomes $v' = a_2 (a_{-2})^2 (a_1)^5 a_0$.

If the word v is (freely) reduced, we can recover it out of v' just by replacing a_i by $t^{-i} a t^i$ and a_i^{-1} by $t^{-i} a^{-1} t^i$, respectively, and freely reducing the result then. This defines a bijection ψ between the reduced words in $\{a_i^\pm\}_{i\in\mathbb{N}}$ and the reduced words in $\{a^\pm, t^\pm\}^*$ that have the same number of letters t and t^{-1}.

We proceed to defining the words w_k which shall be used as 'shortcuts' to go from large negative powers to large postive powers of a in the proof of Theorem 4.2. Our definition of the w_k will rely on the words w_k' in G' representing large powers of a_0 which we define first.

For this, first note that

$$
\begin{aligned}
a_{i+1}^{-k}\, a_i\, a_{i+1}^{k} \quad &=_{G'} \quad a_{i+1}^{-(k-1)}\, a_i^p\, a_{i+1}^{k-1} \\
&=_{G'} \quad (a_{i+1}^{-(k-1)}\, a_i\, a_{i+1}^{k-1})^p \\
&=_{G'} \quad ((a_{i+1}^{-(k-2)}\, a_i\, a_{i+1}^{k-2})^p)^p \\
&=_{G'} \quad (a_{i+1}^{-(k-2)}\ a_i\ a_{i+1}^{k-2})^{p^2} \\
&=_{G'} \quad \ldots \\
&=_{G'} \quad a_i^{p^k}.
\end{aligned}
$$

Now set $w_0' = a_0$ and obtain the word w_i' by conjugating all letters a_{i-1} in the word w_{i-1}' with a_i. Let ${}^n p$ denote the tower of length n of pth powers (often called tetration of p by n), e.g. ${}^3 p = p^{p^p}$. (Note that by convention $a^{b^c} = a^{(b^c)}$, not $(a^b)^c$.) Set

$$
\begin{aligned}
w_0' \quad &:= \quad a_0 \\
w_1' \quad &:= \quad a_1^{-1} a_0 a_1 \\
&=_{G'} \quad a_0^p \\
w_2' \quad &:= \quad (a_2^{-1} a_1^{-1} a_2)\ a_0\ (a_2^{-1} a_1 a_2) \\
&=_{G'} \quad a_1^{-p} a_0 a_1^p \\
&=_{G'} \quad a_0^{{}^2 p} \\
w_3' \quad &:= \quad (a_3^{-1} a_2^{-1} a_3)\ a_1^{-1}\ (a_3^{-1} a_2 a_3)\ a_0\ (a_3^{-1} a_2^{-1} a_3)\ a_1\ (a_3^{-1} a_2 a_3) \\
&=_{G'} \quad (a_2^{-p} a_1^{-1} a_2^p)\ a_0\ (a_2^{-p} a_1 a_2^p) \\
&=_{G'} \quad a_1^{-{}^2 p}\ a_0\ a_1^{{}^2 p} \\
&=_{G'} \quad a_0^{{}^3 p} \\
&\vdots \\
w_n' \quad &:= \quad \ldots =_{G'} a_0^{{}^n p}.
\end{aligned}
$$

Notice that the word w_k' only consists of $2^{k+1} - 1$ letters.

Finally, let $w_i := \psi(w_i')$. Then:

$$
\begin{aligned}
w_0 \quad &= \quad a \\
w_1 \quad &= \quad t^{-1} a^{-1} t a t^{-1} a t \\
w_2 \quad &= \quad \overbrace{t^{-2} a^{-1} t\, \cancel{t}}^{\psi(a_2^{-1})} \overbrace{\cancel{t^{-1}}\ a^{-1} \cancel{t}}^{\psi(a_1^{-1})} \overbrace{\cancel{t^{-1}} t^{-1} a t^2}^{\psi(a_2)} \overbrace{a t^{-2} a^{-1} t a t^{-1} a t^2}^{\psi(a_0\ a_2^{-1}\ a_1\ a_2)} \\
&= \quad t^{-1} w_1^{-1} t\ a\ t^{-1} w_1 t \\
\vdots \quad & \quad \quad \vdots
\end{aligned}
$$

There is a nice recursion formula for w_i:

$$
w_{i+1} \quad = \quad t^{-1} w_i^{-1} t\ a\ t^{-1} w_i t.
$$

This implies that the length of w_k is given by the recursion formula $l(w_{i+1}) = 2 \cdot l(w_i) + 5$ and therefore

$$l(w_k) \quad = \quad 3 \cdot 2^{k+1} - 5. \tag{12}$$

Our proof of Theorem 4.2 will follow from the next two lemmas.

Lemma 4.5. *The words w_k are geodesic.*

Lemma 4.5 will be proved in Section 5.

The second key ingredient in the proof of Theorem 4.2 is Lemma 4.6, to be stated next, and to be proved in Section 6. We employ the well-known Kronecker delta $\delta_{m,n}$, which, here for numbers $n, m \in \mathbb{Z}[\frac{1}{2}]$, takes the value 1 if $m = n$, and 0 otherwise.

Lemma 4.6. *Let $k \in \mathbb{N}$, $n \in \mathbb{Z}$ such that $d(1, a^n) =: d_n < 3 \cdot 2^{k+1} - 5$. Then*

$$d(1, a^{{}^k p - n}) \geq 3 \cdot 2^{k+1} - 5 + \min\{d_n, 3 \cdot 2^k - 5\} - (1 - \delta_{k,1}) \min\{d_n, 2^{k-1}\}.$$

Furthermore, if $k \geq 2$ and no geodesic word representing a^n contains a letter t (which is easily seen to be equivalent to $|n| < \frac{p+7}{2}$), then

$$d(1, a^{{}^k p - n}) \geq 3 \cdot 2^{k+1} - 5 + d_n - \delta_{|n|, \frac{p+6}{2}}.$$

Postponing the proofs of Lemma 4.5 and Lemma 4.6 to the next two sections we first show how they imply Theorem 4.2:

Proof of Theorem 4.2. Observe that it suffices to show that for all $\alpha > 12/17$ there is a c such that the elements a^{-n} are contained in the (α, c)-cones of a^∞. Then by symmetry, the reciprocal is true as well, showing that the distance between a^∞ and $a^{-\infty}$ is at most $\sqrt{12/17}$. Let $\alpha > 12/17$ and set $c := 5$.

Let $n > 0$. Now, let $k = k(n)$ be the unique positive integer such that

$$3 \cdot 2^{k+1} - 5 > d(1, a^n) \geq 3 \cdot 2^k - 5.$$

We define $h = h(n) := {}^k p - n$. Hence, by Lemma 4.5 and by (12),

$$d(a^{-n}, a^h) = d(1, a^{{}^k p}) = 3 \cdot 2^{k+1} - 5. \tag{13}$$

Using Lemma 4.6 we obtain

$$d(1, a^h) > 3 \cdot (2^{k+1} + 2^k) - 2^{k-1}.$$

By (13) this shows that

$$\begin{aligned} d(a^{-n}, a^h) &= 3 \cdot 2^{k+1} - 5 \\ &= 12/17 \cdot (3 \cdot (2^{k+1} + 2^k) - 2^{k-1}) - 60/17 \\ &< \alpha \cdot (3 \cdot (2^{k+1} + 2^k) - 2^{k-1}) + c \\ &\leq \alpha d(1, a^h) + c, \end{aligned}$$

and thus a^{-n} lies in the (α, c)–cone around a. □

5. The words w'_k and w_k are geodesic

The main aim of this section is to prove Lemma 4.5, namely that the words w_k are geodesic in G. This will be obtained by a series of results on the groups G_i^k and G_i^∞. A bit outside our way towards Lemma 4.5, we will also sketch a proof for the fact that the words w'_k are geodesic in G' (Lemma 5.5).

The other important results of this section will be Lemmas 5.6 and 5.7 which are used in the proof of our main theorem, Theorem 4.2.

We start by showing a number of rather easy lemmas.

Lemma 5.1. *Let $k > i$ and let w be a reduced word in G_i^k with $w =_{G_i^k} a_i^n$. Then there are words v_α, $\alpha = 1, \dots n$, in G_i^k such that*
(a) $w = a_i^{\ell_0} v_1 a_i^{\ell_1} v_2 a_i^{\ell_2} \dots v_n$,
(b) $v_\alpha =_{G_i^k} a_{i+1}^{\beta_\alpha}$ for some $\beta_\alpha \in \mathbb{Z}$, and
(c) $\prod_{\alpha=1}^n v_\alpha =_{G_i^k} 1$.

Proof. For fixed i, we use induction on k. For $k = i + 1$ the group G_i^k is the Baumslag-Solitar group $BS(1, p)$ for which the above statement is well-known.

So suppose $k > i + 1$, and assume the statement true for $k - 1$. The group G_i^k is an HNN-extension of G_i^{k-1} with associated subgroups $\langle a_{k-1} \rangle$ and $\langle a_{k-1}^p \rangle$ and stable letter a_k. As $w a_i^{-n} =_{G_i^k} 1$, Britton's lemma implies that w contains $a_k^{-1} a_{k-1}^{\ell} a_k$ or $a_k a_{k-1}^{p\ell} a_k^{-1}$ as a subword. Replacing such a subword by $a_{k-1}^{p\ell}$ or a_{k-1}^{ℓ}, respectively, we obtain a word with less occurrences of a_k which represents a_i^n. Repeating this procedure several times, if necessary, we arrive at a word $w' \in (A_i^{k-1})^*$ still representing a_i^n. By the induction hypothesis, w' has the form $a_i^{\ell_0} v_1 a_i^{\ell_1} v_2 a_i^{\ell_2} \dots v_n$ with $v_\alpha =_{G_i^{k-1}} a_i^{\beta_\alpha}$. Since all replacements have been made inside the words v_α, w also has the desired form. The statement follows. □

Lemma 5.2. *Let $i \in \mathbb{N}$. Any geodesic word in G' representing an element of G_i^i is an element of $(A_i^\infty)^*$.*

Proof. By (11), it is sufficient to show that for fixed $i \in \mathbb{N}$, the letter a_0 and its inverse a_0^{-1} are not contained in any geodesic word representing an element of $\langle a_i \rangle$. Let $v =_{G'} a_i^l$ be a geodesic word in G' containing a_0 or a_0^{-1} and set $\hat{G} := G' / \langle\langle a_0 \rangle\rangle$. The word v' which is obtained from v by deleting all occurrences of a_0 and of a_0^{-1} in v and the word a_i^l represent the same element in $\hat{G}$. Therefore $w := v' \circ a_i^{-l} \in \langle\langle a_0 \rangle\rangle$.

Now, w does not contain the letter $a_0^{\pm}$ (as v' and $a_i^{-\ell}$ do not) and due to the fact that the only relation involving the letter $a_0^{\pm}$ is $a_1^{-1} a_0 a_1 = a_0^p$, this implies that $w = 1$ and $v' =_{G'} a_i^l$. This contradicts our assumption that v is geodesic. □

Corollary 5.3. *The subgroups G_i^∞ are undistorted in G'. That is, for the generators considered above, the distances of elements of G_i^∞ is the same in G_i^∞ as in G'.*

Proof. Since subwords of geodesic are geodesic, any geodesic word in $(A_0)^*$ representing an element of G_i^∞ can be divided in geodesic subwords each representing elements of G_j^j for some $j \geq i$. Because of Lemma 5.2, none of these subwords can contain a letter of A_0^{i-1}. Therefore $w \in (A_i^\infty)^*$. □

Lemma 5.4. *Let $k \geq i \geq 0$. Any geodesic word in G' containing the letter $a_k^{\pm}$ and representing an element of $\langle a_{k-i} \rangle$ has at least length $2^{i+1} - 1$.*

Proof. Let v be a geodesic word representing an element of $\langle a_{k-i} \rangle$ and we may assume $k = \max\{j : a_j \pm \text{ is contained in } v\}$. We prove the statement by induction on i. Let $i = 0$. A word containing $a_k^{\pm}$ has at least length $1 = 2^{0+1} - 1$. Now assume the statement to be true for $i = n - 1$.

Let v be a geodesic word representing a_{k-n}^{ℓ} and containing the letter $a_k \pm$. According to Lemma 5.1, $v = v_1 a_{k-n}^{\ell_1} v_2 \dots a_{k-n}^{\ell_m} v_{m+1}$ where each $v_\alpha =_{G'} a_{k-n+1}^{\beta_\alpha}$ for some β_α and the product $v_1 v_2 \dots v_{m+1} = 1$. Since v contains a letter $a_k^{\pm}$, there exists an α, such that v_α contains $a_k^{\pm}$. Since v_α is geodesic, it has by induction hypothesis length at least $2^n - 1$. Since $v' = (\prod_{\gamma=\alpha+1}^m v_\gamma)(\prod_{\gamma=1}^{\alpha-1} v_\gamma)$ is a word representing v_α^{-1} this word cannot be shorter than the geodesic word v_α and also contains at least $2^n - 1$ letters. All in all, since v contains at least 1 letter a_{k-n} we obtain that the length of v is at least $2 \cdot (2^n - 1) + 1 = 2^{n+1} - 1$. □

In particular the last lemma shows that there exist no geodesic word containing $a_k^{\pm}$ and representing an element of $\langle a_0 \rangle$, which is shorter than w'_k. And in fact the following lemma, which will not be needed in the course of this paper, holds:

Lemma 5.5. *The word w'_k is a geodesic word in G'.*

Proof. The word w'_k represents the element $a_0^{{}^k p}$ and has length $2^{k+1} - 1$. So, by Lemma 5.4 for $i = k$ we only have to show that every geodesic word representing $a_0^{{}^k p}$ has to contain the letter a_k. This can again be done by induction on k. The statement is obviously true for $k = 0$. Because we won't need this statement later on, we leave the proof of the induction step, which can be done following the lines of the proof to Lemma 5.7, to the reader. □

In contrast to the situation in G' the product $w_i w_j$ for $i \neq j$ won't be freely reduced. Nevertheless in the group G the analogue of Lemma 5.4 also holds.

Lemma 5.6. *Let $k \geq 0$. Let w be a geodesic word representing a non-trivial element of $\langle a_0 \rangle$ such that $w' = \psi^{-1}(w)$ contains the letter $a_k^{\pm}$. The length of w is at least $3 \cdot 2^{k+1} - 5$.*

If in addition $\ell(w) = 3 \cdot 2^{k+1} - 5$, then $w =_G a^{\pm({}^k p)}$.

Proof. Without loss of generality we may assume that $k = \max\{j : a_j^{\pm} \text{ is contained in } w'\}$. We prove the statement by induction on k. For $k = 0$ the statement is trivial.

For $k > 0$, Lemma 5.1 yields that $w' = v'_0 a_1^{\ell_1} v'_1 \dots a_0^{\ell_m} v'_m a_0^{\ell_m}$ where each $v'_\alpha =_{G'} a_1^{\beta_\alpha}$ for some β_α and the product $v'_0 v'_1 \dots v'_m = 1$. It is easy to check that since w is reduced, the v'_α do not contain any letters a_0. Then

$$w = a^{\ell_0} t^{-1} v_1 t a^{\ell_1} t^{-1} v_2 t a^{\ell_2} \dots a^{\ell_{m-1}} t^{-1} v_m t a^{\ell_m}$$

where each $v_\alpha =_{G_p} a_0^{-\beta_\alpha}$ and the product $v_0 v_1 \dots v_m = 1$ (note that this is the same as saying that $\sum \beta_\alpha = 0$).

Since w' contains a letter $a_k \pm$, there exists an α^*, such that v'_{α^*} contains $a_{k-1}^{\pm}$. As a subword of w, the word v_{α^*} is geodesic, it has by induction hypothesis length at least $3 \cdot 2^k - 5$. Because $v_1 v_2 \dots v_m = 1$ the product of the other v_α also has length at least $3 \cdot 2^k - 5$, and furthermore, we have at least four t's and an a^{ℓ_α}, the bound follows.

For the second assertion of the lemma, we again apply induction. The base case is trivial. If $\ell(w) = 3 \cdot 2^{k+1} - 5$ then v_{α^*} has length $3 \cdot 2^k - 5$ and so has $\Pi_{\alpha \neq \alpha^*} v_\alpha = v_2$. Thus $v_1 =_G v_2 =_G a^{\pm^{k-1}p}$. So w contains exactly 4 letters t. Hence $w = t^{-1} v_1 t a^{\pm} t^{-1} v_1^{-1} t$, which implies $w =_G a^{\pm^k p}$. □

Furthermore we can bound the power of a which is represented by a word of given length avoiding high powers of t.

Lemma 5.7. *Let $k \geq 1$. Let v be a word of length less than $L \cdot 2^{k-1}$ in G representing an element a^n for some $n \in \mathbb{Z}$ such that $\psi^{-1}(v)$ does not contain the letter $a_k^{\pm}$. Then*

$$|n| < p^{p^{\cdot^{\cdot^{\cdot^{p^L}}}}}$$

where the number of p's is $k - 1$.

Proof. Without loss of generality we assume v to be reduced. Set $j := \max\{l : a_l \in \psi^{-1}(v)\}$. This implies $\psi^{-1}(v) \in G_0^j$. Applying successively Britton's Lemma to the word $\psi^{-1}(v) \circ a_0^{-n}$ we conclude that $\psi^{-1}(v)$ contains for each $\ell \in \{0, 1, \dots j\}$ a letter a_ℓ. Therefore $j < k$ and $\psi^{-1}(v)$ does not contain any a_ℓ with $\ell \geq k$.

We proceed by induction on k. Let $k = 1$. The word $\psi^{-1}(v)$ does not contain a letter $a_1^{\pm}$. Therefore $v = a^\alpha$ for some $|\alpha| < L$. Obviously $n = \alpha$ and we are done.

Let $k \geq 2$ and assume the statement to be true for $k - 1$. We only consider the case that n is positive, as the other case is symmetric. We may assume that v is such that n is maximal among all possible values for n over all choices of v as in the lemma. Note that then $l(v) = L \cdot 2^{k-1} - 1$, and furthermore, v is shortest possible among all v satisfying the assumptions of the lemma.

Now, as in the proof of Lemma 5.6 we obtain

$$v = a^{l_0} t^{-1} v_1 t a^{l_1} t^{-1} v_2 t \dots t^{-1} v_m t a^{l_m},$$

with $v_i =_G a^{-\beta_i}$ for some β_i such that $\sum \beta_i = 0$. But now we can calculate n in terms of l_i and β_i, namely

$$n = l_0 + \sum_{i=1}^{m} l_i p^{\sum_{j=1}^{i} \beta_j} \leq (\sum_{i=0}^{m} l_i) \cdot p^{\max_i \sum_{j=1}^{i} \beta_j} =: y.$$

Let c be such that $\max_i \sum_{j=1}^{i} \beta_j = \sum_{j=1}^{c} \beta_j$. By deleting all but four letters t and rearranging the letters a we obtain the word

$$v' = t^{-1} v_1 v_2 \dots v_c t a^{\sum_{i=0}^{m} l_i} t^{-1} v_{c+1} \dots v_m t.$$

Then $l(v') \leq l(v)$ and $v' =_G a^y$. Since v was chosen such that n is maximal, we obtain that $y = n$. Then $\ell(v) = \ell(v')$. So, we actually did not delete any t when creating v', and thus $v = v'$. Hence

$$v = t^{-1} v_1 t a^l t^{-1} v_2 t,$$

with $v_1^{-1} =_G v_2 =_G a^\alpha$ for some $\alpha \in \mathbb{Z}$ and $n = l \cdot p^\alpha$.

Assume that $l \geq 3$. Then we can build the word $v'' = t^{-1} v_1 a^{-1} t a^{l-2} t^{-1} a v_2 t$ which is of the same length as v and represents $a^{((l-2)p)p^\alpha}$ in contradiction to the maximality of n. Therefore $l \leq 2$. Since v is shortest possible under the assumptions of the lemma so are v_1 and v_2, and hence $\ell(v_1) = \ell(v_2)$. Since $\ell(v)$ is odd, it follows that $l = 1$ and

$$\ell(v_1) = \frac{L \cdot 2^{k-1} - 1 - 5}{2} < L \cdot 2^{k-2}.$$

By induction hypothesis

$$|\alpha| < p^{p^{\cdot^{\cdot^{p^L}}}}$$

where the number of p's is $k - 2$. and since $n = 1 \cdot p^\alpha$ we obtain the desired inequality. □

The two preceding lemmas imply Lemma 4.5, that is, that the w_k are geodesic:

Proof of Lemma 4.5. Lemma 5.7 implies that every word that represents w_k and has only letters a_j or a_j^{-1} with $j < k$ is at least as long as w_k. On the other hand, by Lemma 5.6 we know that any word containing a_k or a_k^{-1} is as least as long as w_k (and, if a word contains letters a_j for $j > k$ it is still longer). So the statement follows. □

6. The proof of Lemma 4.6

This final section is devoted to the proof of Lemma 4.6, which is the only ingredient missing for our proof of Theorem 4.2. We build on results from Section 5.

Proof of Lemma 4.6. First of all, observe that then by Lemma 5.6, every geodesic word representing a^n is a_k-less. So by Lemma 5.7, we know that

$$n < p^{p^{\cdot^{\cdot^{\cdot^{p^{12}}}}}} \tag{14}$$

where the number of p' s is $k-1$.

In order to prove the lemma, we use induction on k. For $k = 1$, we only have to check that $d(1, a^{p-n}) \geq 8 \geq 12 - 5 + \min\{n, 1\} - 0$ for all n with $n < 12$, by (14). This is true as by the choice of p, we have $p - n > 8$, and testing all words with at most 7 letters we see that none of them represents an a^x with $x \geq 8$.

So assume the lemma valid for $k-1$, our aim is to show it for k. Suppose otherwise, that is, assume there is a word v with $v =_G a^{{}^k p - n}$ and

$$\begin{aligned} \ell(v) &< 3 \cdot 2^{k+1} - 5 + \min\{d_n, 3 \cdot 2^k - 5\} - \min\{d_n, 2^{k-1}\} \\ &\leq 3 \cdot (2^{k+1} + 2^k) - 10 \\ &\leq 18 \cdot 2^{k-1} - 10. \end{aligned} \tag{15}$$

We claim that

$$\psi^{-1}(v) \text{ contains the letter } a_k. \tag{16}$$

In fact, otherwise we may apply Lemma 5.7 to v, with $L = 18$, to obtain that

$${}^k p - p^{p^{\cdot^{\cdot^{\cdot^{p^{12}}}}}} < {}^k p - n < p^{p^{\cdot^{\cdot^{\cdot^{p^{18}}}}}}$$

where on either side the number of p' s equals $k-1$, and the first inequality follows from (14). This, however, is impossible, as $p \geq 20$. We have thus proved (16).

Now, by Lemma 5.1, we can write $\psi^{-1}(v)$ as $a_0^{\ell_0} v_1 a_0^{\ell_1} v_2 a_0^{\ell_2} \ldots v_m a_0^{\ell_m}$, and thus,

$$v = a^{\ell_0} t^{-1} u_1 t a^{\ell_1} t^{-1} u_2 t a^{\ell_2} \ldots t^{-1} u_m t a^{\ell_m} \tag{17}$$

where $u_i =_G a^{-\alpha_i}$, for $i = 1, \ldots m$. Clearly,

$${}^k p - n = \ell_0 + \sum_{i=1}^{m} \ell_i p^{\sum_{j=1}^{i} \alpha_j}, \tag{18}$$

and the sum over all α_i equals 0. Note that since v is geodesic, we may assume that $\ell_i < p$ for $i = 1, \ldots m$.

Suppose $c \in \{1, \ldots, m\}$ is such that $\psi^{-1}(u_c)$ contains the letter $a_{k-1}^{\pm}$. Then by Lemma 5.6,

$$\ell(u_c) \geq 3 \cdot 2^k - 5. \tag{19}$$

So, as $3 \cdot (3 \cdot 2^k - 5) > \ell(v) - 5$, and moreover, since each u_c as above gives rise to two letters t, we conclude that there are less than 3 indices c such that $\psi^{-1}(u_c)$ contains the letter $a_{k-1}^{\pm}$. On the other hand, by (16), there is at least one such index, say c_1.

Moreover, since the expression in (17) contains m times a subword of the form $t^{-1} u_i t$, and also at least $m - 1$ letters a, we can use (19) to get that

$$m < \frac{\ell(v) - \ell(u_{c_1}) + 2}{4} < 3 \cdot 2^{k-1}. \tag{20}$$

Together, (18) and (20) imply that there is an index b such that

$$\ell_b \cdot p^{\sum_{i=1}^{b} \alpha_j} > \frac{p^{p^{\cdot^{\cdot^{\cdot^{p^{p-1}}}}}}}{3 \cdot 2^{k-1}}$$

where the number of p's equals $k-1$. Hence, since $p > 6$, and since $\ell_b < p$, we know that

$$p^{\sum_{i=1}^{b}\alpha_j} > \frac{p^{p^{\cdot^{\cdot^{\cdot^{p^{p-1}}}}}}}{p^k}$$

where again, the number of p's is $k-1$. Taking the logarithm, we obtain that

$$\sum_{i=1}^{b}\alpha_j > y := p^{p^{\cdot^{\cdot^{\cdot^{p^{p-1}}}}}} - k \tag{21}$$

where the number of p's is $k-2$. Because $\sum_{j=1}^{b}\alpha_j = -\sum_{j=b+1}^{m}\alpha_j$, this yields that

$$u_1u_2\dots u_b =_G u_{b+1}u_{b+2}\dots u_m =_G a^x,$$

where $x > y$. So, by Lemma 5.7, there is a second index c_2 such that $\psi^{-1}(u_{c_2})$ contains the letter $a^{\pm}_{k-1}$.

Consider the subword

$$z := t^{-1}u_{c_1}ta^{\ell_{c_1}}t^{-1}u_{c_1+1}\dots t^{-1}u_{c_2}t$$

of v. By the choice of the c_i,

$$\ell(z) \geq 2\cdot 3\cdot(2^k-5)+5. \tag{22}$$

So,

$$\ell(v)-\ell(z) \leq 3\cdot 2^k - 5. \tag{23}$$

Set

$$u := u^{-1}_{c_1-1}u^{-1}_{c_1-2}\dots u^{-1}_1 u^{-1}_m u^{-1}_{m-1}\dots u^{-1}_{c_2}$$

and consider the word

$$v' := a^{\ell_0}t^{-1}u_1ta^{\ell_1}\dots a^{\ell_{c_1-1}}t^{-1}uta^{\ell_{c_2}}t^{-1}u_{u_{c_2}}ta^{\ell_{c_2+1}}\dots t^{-1}u_mta^{\ell_m}.$$

Then $v' =_G a^q$ where

$$q = \ell_0 + \sum_{i=0}^{c_1-1}\ell_i p^{\sum_{j=1}^{i}\alpha_i} + \sum_{i=c_2+1}^{m}\ell_i p^{\sum_{j=1}^{i}\alpha_i}.$$

Here we used the fact that $\sum_{j=1}^{i}\alpha_j = -\sum_{j=i+1}^{m}\alpha_j$.

By (23) and by the definition of v', we know that $\ell(v') \leq 2\cdot(3\cdot 2^k-5)$, and moreover, $\psi^{-1}(v')$ does not contain any letter $a^{\pm}_k$, we obtain that

$$|q| < p^{p^{\cdot^{\cdot^{\cdot^{p^{12}}}}}}$$

where the number of p's is $k-1$. Set

$$s := \sum_{\gamma=c_1}^{c_2-1}\ell_\gamma p^{\sum_{j=1}^{\gamma}\alpha_j} = {}^kp - n - q.$$

Then

$${}^kp - 2p^{p^{\cdot^{\cdot^{\cdot^{p^{12}}}}}} < s < {}^kp + 2p^{p^{\cdot^{\cdot^{\cdot^{p^{12}}}}}}, \tag{24}$$

where the number of p's on each side is again $k-1$.

On the other hand, by (19) and since $\ell(v) \geq 3\cdot(2^{k+1}+2^k)-10$, we have that

$$\sum_{i=c_1+1}^{c_2-1}\ell(u_i) < 3\cdot 2^k - 5, \tag{25}$$

and, for each of these indices i, we know that u_i is a_{k-1}-free. Therefore each of the differences of the exponents of p in s is less than $p^{p^{\cdot^{\cdot^{\cdot^{p^{12}}}}}}$ where the number of p's is $k-2$. So by (21), we can write

$$s = \delta \cdot p^{p^{\cdot^{\cdot^{\cdot^{p^{p-2}}}}}}$$

where the number of p's is $k-1$, and δ is some integer. As the term after δ is greater than the interval from (24), we know that the only possible value for s is ${}^{k}p$. This yields that

$$s = {}^{k}p. \tag{26}$$

Thus ${}^{k}p = \sum_{i=c_1}^{c_2-1} \ell_i p^{\alpha_i}$. For elementary arithmetic reasons, and because all the α_i are different (since v is geodesic) and the ℓ_i are in $(0,p)$, this is only possible if $\ell_{c_1} = 1$ and $c_2 = c_1 + 1$. Hence z can be written as

$$z = t^{-1} u_{c_1} t a t^{-1} u_{c_2} t.$$

Taking the logarithm in (26), this implies that $\sum_{i=1}^{c_1} \alpha_i = {}^{k-1}p$. Hence,

$$u_{c_1} =_G a^{-{}^{k-1}p + \sum_{i=1}^{c_1-1} \alpha_i}$$

and

$$u_{c_2} =_G a^{{}^{k-1}p + \sum_{i=c_2+1}^{m} \alpha_i}.$$

Now we are able to prove the second part of the statement. So assume $n < \frac{p+7}{2}$. Then since by (22), by (18) and using the fact, that ${}^{k}p - n \equiv -n \mod p$ we know that $\ell_0 + \ell_m = -n$ or $\ell_0 + \ell_m = p - n$. In the first case the length of v is at least $3 \cdot 2^{k+1} - 5 + n$ and $d_n = n$, and so, we are done.

In the second case $\sum_{i=1}^{m-1} \ell_i p^{\alpha_i} = {}^{k}p - p$, which implies $m \geq 3$ and since by assumption, $p - n > n - 7$, we get that $m = 3$. One of the u_{c_i} now has to represent an element different from $a^{{}^{k-1}p}$ and therefore, by Lemma 5.6,

$$\sum_{i=1}^{3} \ell(u_i) \geq 3 \cdot 2^k - 5 + 3 \cdot 2^k - 4 + 1 = 3 \cdot 2^{k+1} - 8.$$

In total, counting the a's and t's involved, we obtain that

$$\ell(v) \geq 3 \cdot 2^{k+1} + p - n > 3 \cdot 2^{k+1} - 5 + n - 2.$$

This finishes the proof of the second assertion of Lemma 4.6.

We now apply the induction hypothesis with $n_1 := \sum_{i=1}^{c_1-1} \alpha_i$ in the role of n, which satisfies the assumptions as $a^{\sum_{i=1}^{c_1-1} \alpha_i} =_G u_1 u_2 \ldots u_{c_1-1}$. We then apply the induction hypothesis again with $n_2 := \sum_{i=c_2}^{m} \alpha_i$ in the role of n, which satisfies the assumptions as $a^{\sum_{i=c_2}^{m} \alpha_i} =_G u_{c_2} u_{c_2+1} \ldots u_m$. This gives for $j = 1, 2$

$$\ell(u_{c_j}) \geq 3 \cdot 2^k - 5 + \min\{d_{n_j}, 3 \cdot 2^{k-1} - 5\} - \min\{d_{n_j}, 2^{k-2}\}.$$

So, as v contains $3m - 1$ letters a and t outside the u_i, we obtain

$$\begin{aligned}
\ell(v) &\geq \ell(u_{c_1}) + \ell(u_{c_2}) + \ell(u_1 u_2 \ldots u_{c_1-1}) + \ell(u_{c_2} u_{c_2+1} \ldots u_m) + 3m - 1 \\
&\geq \ell(u_{c_1}) + \ell(u_{c_2}) + d_{n_1} + d_{n_2} + 3m - 1 \\
&\geq 3 \cdot 2^{k+1} + 3m - 11 \\
&\quad + \sum_{j=1,2} \left(\min\{d_{n_j}, 3 \cdot 2^{k-1} - 5\} - \min\{d_{n_j}, 2^{k-2}\} + d_{n_j}\right). \qquad (27)
\end{aligned}$$

Observe that by (15), and since the term in the sum above is always non-negative, we get that

$$d_n \geq 2^{k-1}. \tag{28}$$

We claim that for $j = 1, 2$

$$d_{n_j} \leq 3 \cdot 2^{k-1} - 5 \text{ or } d_{n_{3-j}} = 0. \tag{29}$$

Indeed, suppose $d_{n_1} > 3 \cdot 2^{k-1} - 5$. Then by comparing (15) with (27), we obtain that

$$\begin{aligned} 3 \cdot 2^k - 5 - 2^{k-1} \geq\ & \min\{d_n, 3 \cdot 2^k - 5\} - 2^{k-1} \\ &\geq 3m - 6 + \min\{d_{n_1}, 3 \cdot 2^{k-1} - 5\} - 2^{k-2} + d_{n_1} \\ &\quad + \min\{d_{n_2}, 3 \cdot 2^{k-1} - 5\} - d_{n_2} + d_{n_2} \\ &\geq 3m - 6 + 3 \cdot 2^{k-1} - 5 - 2^{k-2} + 3 \cdot 2^{k-1} - 5 + 1 \\ &\quad + \min\{d_{n_2}, 3 \cdot 2^{k-1} - 5\} \\ &\geq 3m - 5 + 3 \cdot 2^k - 10 - 2^{k-2} + \min\{d_{n_2}, 3 \cdot 2^{k-1} - 5\}. \end{aligned}$$

Therefore, since $m \geq 3$,

$$-2^{k-1} \geq -1 - 2^{k-2} + \min\{d_{n_2}, 3 \cdot 2^{k-1} - 5\},$$

implying that

$$1 \geq 2^{k-2} + \min\{d_{n_2}, 3 \cdot 2^{k-1} - 5\}.$$

Hence $d_{n_2} = 0$. In the same way we get that the assumption $d_{n_2} > 3 \cdot 2^{k-1} - 5$ implies that $d_{n_1} = 0$. This proves (29).

Let us define a new word $\tilde{v}$ which is obtained from v by replacing z with $t^{-1}\tilde{v}_1^{-1}\tilde{v}_2 t$, where the $\tilde{v}_i$ are geodesic words for a^{n_i}. That is,

$$\tilde{v} := a^{\ell_0} t^{-1} u_1 t \dots a^{ell_{c_1 - 1}} t^{-1} \tilde{v}_1^{-1} \tilde{v}_2 t a^{\ell_{c_2}} t^{-1} u_{c_2} \dots t^{-1} u_m t a^{\ell_m}.$$

Clearly, $\tilde{v}$ represents a^n.

First, suppose that both $\tilde{v}_i$ contain a letter t. Note that then we may assume that each of the $\tilde{v}_i$ starts with a t^{-1}. Hence, $d_n \leq \ell(\tilde{v}) - 2$. Observe that also, $d_{n_j} > 0$. Hence, by (29), $d_{n_j} \leq 3 \cdot 2^{k-1} - 5$.

By (15) and by (28),

$$\ell(v) < 3 \cdot 2^{k+1} - 5 + d_n - 2^{k-1}.$$

Moreover, since

$$\ell(z) = \ell(u_{c_1}) + \ell(u_{c_2}) + 5,$$

and by (29), we obtain

$$\begin{aligned} d_n &\leq \ell(\tilde{v}) - 2 \\ &\leq \ell(v) + \underbrace{d_{n_1} + d_{n_2} + 2}_{\leq \ell(t^{-1}\tilde{v_1}^{-1}\tilde{v}_2 t)} - \ell(z) - 2 \\ &< 3 \cdot 2^{k+1} - 5 + d_n - 2^{k-1} + d_{n_1} + d_{n_2} \\ &\quad - \sum_{j=1,2} \left(3 \cdot 2^k - 5 + d_{n_j} - \min\{d_{n_j}, 2^{k-2}\}\right) - 5 \\ &\leq d_n - 2^{k-1} + \sum_{j=1,2} 2^{k-2} \\ &\leq d_n, \end{aligned}$$

a contradiction.

So we may assume that one of $\tilde{v}_1, \tilde{v}_2$ does not contain a letter t, say $\tilde{v}_1$. Then we need not have that $d_n \le \ell(\tilde{v}) - 2$. On the other hand we can use the second hypothesis of the induction. Hence the last calculation becomes

$$\begin{aligned}
d_n &\le \ell(\tilde{v}) \\
&\le \ell(v) + d_{n_1} + d_{n_2} + 2 - \ell(z) \\
&< 3 \cdot 2^{k+1} - 5 + d_n - 2^{k-1} + d_{n_1} + d_{n_2} \\
&\quad - \sum_{j=1,2} \left(3 \cdot 2^k - 5 + d_{n_j}\right) - \min\{d_{n_2}, 2^{k-2}\} + \delta_{|n|, \frac{p+6}{2}} - 5 + 2 \\
&\le d_n - 2^{k-1} + 2^{k-2} + \delta_{|n|, \frac{p+6}{2}} + 2 \\
&\le d_n - 2^{k-2} + \delta_{|n_1|, \frac{p+6}{2}} + 2,
\end{aligned}$$

which yields a contradiction for $k > 3$. For $k = 3$ we deduce $n_1 = \frac{p+6}{2} > 7 = 3 \cdot 2^2 - 5$. So by (29), $d_{n_2} = 0$. So we can substitute the last two lines of the calculation above with

$$\begin{aligned}
d_n &< d_n - 2^2 + 1 + 2 \\
&\le d_n - 1,
\end{aligned}$$

which is also a contradiction.

So let $k = 2$. Then $\ell(v) < 3 \cdot 2^3 - 5 + 7 - 2 = 3 \cdot 2^3$, by (28). Therefore $m \le 3$. If $m = 3$ we have $\sum_{i=1}^{3} \alpha_i = 0$ and hence $\alpha_{c_1} \ne \alpha_{c_2}$. So $\alpha_{c_i} = \pm^2 p$ and $\alpha_{c_{3-i}} \ne {}^2 p$. By Lemma 5.6 we get $\sum_{i=1,2} \ell(u_{c_i}) \ge 3 \cdot 2^3 - 9$ and

$$\ell(v) \ge \sum_{i=1}^{3} \ell(u_i) + 3m - 1 \ge 3 \cdot 2^3 > \ell(v).$$

So we have $k = m = 2$. This implies $v = a^{\ell_0} t^{-1} u_{c_1} t a t^{-1} u_{c_2} a^{-n+\ell_0}$ and $\ell(v) = 3 \cdot 2^3 - 5 + n > 3 \cdot 2^3 - 5 + d_n - 1$, which is impossible by (15). □

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