Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Schauder a priori estimates and regularity of solutions to degenerate-elliptic linear second-order partial differential equations

by

Paul Feehan and Camelia Pop

Preprint no.: 64 2012

SCHAUDER A PRIORI ESTIMATES AND REGULARITY OF SOLUTIONS TO DEGENERATE-ELLIPTIC LINEAR SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS

PAUL M. N. FEEHAN AND CAMELIA POP

Abstract. We establish Schauder a priori estimates and regularity for solutions to a class of degenerate-elliptic linear second-order partial differential equations. Furthermore, given a C^∞-smooth source function, we prove C^∞-regularity of solutions up to the portion of the boundary where the operator is degenerate. Degenerate-elliptic operators of the kind described in our article appear in a diverse range of applications, including as generators of affine diffusion processes employed in stochastic volatility models in mathematical finance [9, 27], generators of diffusion processes arising in mathematical biology [3, 11], and the study of porous media [6, 7].

Contents

Date: October 25, 2012.

2000 *Mathematics Subject Classification.* Primary 35J70; secondary 60J60.

Key words and phrases. Degenerate-elliptic partial differential operator, degenerate diffusion process, Hölder regularity, mathematical finance, a priori Schauder estimate.

PF was partially supported by NSF grant DMS-1059206 and the Max Planck Institut für Mathematik in der Naturwissenschaft.

1. Introduction

This article continues our development of regularity theory for solutions to the Dirichlet boundary value problem defined by a degenerate-elliptic operator. Degenerate-elliptic operators of the kind explored in our article can arise as generators of affine diffusion processes employed in stochastic volatility models in mathematical finance [9, 27], generators of diffusion processes arising in mathematical biology [3, 11], and the analysis of porous media [6, 7], to name just a few applications.

In [5], in addition to other results, Daskalopoulos and Feehan obtained existence of H^1 solutions to a variational equation defined by the Heston operator [27]. We recall that the Heston operator serves as a useful paradigm for degenerate-elliptic operators arising in mathematical finance. In [17], the authors proved global C^α_s-regularity of H^1 solutions to the variational equation defined by the Heston operator, while in [19], the authors established $\mathscr{H}^k$ as well as $C^{k,\alpha}_s$ and $C^{k,2+\alpha}_s$ regularity for those solutions, for all integers $k \geq 0$. However, our $C^{k,\alpha}_s$ and $C^{k,2+\alpha}_s$ regularity results in [19], although they provide an important stepping stone, are not optimal due to our reliance on variational methods. The purpose of the present article is prove analogues — for a broader class of degenerate-elliptic operators — of Schauder a priori estimates and regularity results for strictly elliptic operators in [26, Chapter 6]. When coupled with results of [5, 17, 19], we immediately obtain existence and $C^{k,2+\alpha}_s$ regularity for solutions to the Dirichlet boundary value problem, defined by a degenerate-elliptic operator, analogous to those expected from the Schauder approach for strictly elliptic operators in [26, Chapter 6]; uniqueness for broad class of linear second-order degenerate-elliptic operators, with the second-order (or Ventcel) boundary conditions of the kind implied by our choice of Daskalopoulos-Hamilton $C^{k,2+\alpha}_s$ Hölder spaces [6], is a consequence of the weak maximum principle discussed by the first author in [16].

To describe our results in more detail, suppose $\mathscr{O} \subsetneqq \mathbb{H}$ is a domain (possibly unbounded) in the open upper half-space $\mathbb{H} := \mathbb{R}^{d-1} \times \mathbb{R}_+$, where $d \geq 2$ and $\mathbb{R}_+ := (0,\infty)$, and $\partial_1\mathscr{O} := \partial\mathscr{O} \cap \mathbb{H}$ is the portion of the boundary $\partial\mathscr{O}$ of $\mathscr{O}$ which lies in $\mathbb{H}$, and $\partial_0\mathscr{O}$ is the interior of $\partial\mathbb{H} \cap \partial\mathscr{O}$, where $\partial\mathbb{H} = \mathbb{R}^{d-1} \times \{0\}$ is the boundary of $\bar{\mathbb{H}} := \mathbb{R}^{d-1} \times \bar{\mathbb{R}}_+$ and $\bar{\mathbb{R}}_+ := [0,\infty)$. We assume $\partial_0\mathscr{O}$ is non-empty and consider a linear second-order elliptic differential operator, L, on $\mathscr{O}$ which is degenerate along $\partial_0\mathscr{O}$. In this article, when the operator L is given by (1.3), we prove an a priori interior Schauder estimate and higher-order Hölder regularity up to the boundary portion, $\partial_0\mathscr{O}$ — as measured by certain weighted Hölder spaces, $C^{k,2+\alpha}_s(\bar{\mathscr{O}})$ (Definition 2.3) — for solutions to the elliptic boundary value problem,

$$Lu = f \quad \text{on } \mathscr{O}, \tag{1.1}$$

$$u = g \quad \text{on } \partial_1\mathscr{O}, \tag{1.2}$$

where $f : \mathscr{O} \to \mathbb{R}$ is a source function and the function $g : \partial_1\mathscr{O} \to \mathbb{R}$ prescribes a Dirichlet boundary condition. We denote $\underline{\mathscr{O}} := \mathscr{O} \cup \partial_0\mathscr{O}$ throughout our article, while $\bar{\mathscr{O}} = \mathscr{O} \cup \partial\mathscr{O}$ denotes the usual topological closure of $\mathscr{O}$ in $\mathbb{R}^d$. Furthermore, when $f \in C^\infty(\underline{\mathscr{O}})$, we will also show that

$u \in C^\infty(\underline{\mathscr{O}})$ (see Corollary 1.9). Since L becomes degenerate along $\partial_0\mathscr{O}$, such regularity results do not follow from the standard theory for strictly elliptic differential operators [26, 30].

Because the coefficient, b^d, will be assumed to obey a positive lower bound along $\partial_0\mathscr{O}$, *no boundary condition* is prescribed for the equation (1.1) along $\partial_0\mathscr{O}$. Indeed, one expects from [6] that the problem (1.1), (1.2) should be well-posed, given $f \in C^\alpha_s(\underline{\mathscr{O}})$ and $g \in C(\partial_1\mathscr{O})$ obeying mild pointwise growth conditions, when we seek solutions in $C^{2+\alpha}_s(\underline{\mathscr{O}}) \cap C_{\mathrm{loc}}(\bar{\mathscr{O}})$. The *degenerate-elliptic operator* considered in this article has the form[1]

$$Lv := -x_d \operatorname{tr}(aD^2v) - b \cdot Dv + cv \quad \text{on } \mathscr{O}, \quad v \in C^\infty(\mathscr{O}), \tag{1.3}$$

where $x = (x_1, \ldots, x_d)$ are the standard coordinates on $\mathbb{R}^d$. Occasionally we shall also need

$$L_0 v := (L - c)v = -x_d \operatorname{tr}(aD^2v) - b \cdot Dv \quad \text{on } \mathscr{O}, \quad v \in C^\infty(\mathscr{O}). \tag{1.4}$$

Throughout this article, we assume that the coefficient functions a, b, c of L (and L_0) are defined on $\bar{\mathscr{O}}$, the matrix (a^{ij}) is symmetric[2], and there is a positive constant b_0 such that

$$b^d \geq b_0 \quad \text{on } \partial_0\mathscr{O}.$$

We shall call L in (1.3) an *operator with constant coefficients* if the coefficients a, b, c are constant.

In [19], we proved existence and uniqueness of a solution, $u \in C^{2+\alpha_0}_s(\underline{\mathscr{O}}) \cap C(\bar{\mathscr{O}})$ for *some* $\alpha_0 = \alpha_0 \in (0,1)$, to (1.1), (1.2) when $\partial_1\mathscr{O}$ obeys a uniform exterior cone condition with cone K, and L is the elliptic Heston operator, and $f \in C^\infty(\underline{\mathscr{O}}) \cap C_b(\mathscr{O})$ and $g \in C^\infty(\overline{\partial_1\mathscr{O}})$. (The Hölder exponent, α_0, depends on the coefficients of L and the cone K.) In §1.1, we state the main results of our article and set them in context in §1.2, where we provide a survey of previous related research by other authors. In §1.3, we indicate some extensions of methods and results in our article which we plan to develop in subsequent articles. We provide a guide in §1.4 to the remainder of this article and point out some of the mathematical difficulties and issues of broader interest. We refer the reader to §1.5 for our notational conventions.

1.1. **Summary of main results.** We summarize our main results. Here, our use of the term "interior" is in the sense intended by [6], for example, $U \subset \mathscr{O}$ is an *interior subdomain* of a domain $\mathscr{O} \subsetneqq \mathbb{H}$ if $\bar{U} \subset \underline{\mathscr{O}}$ and by "interior regularity" of a function u on $\mathscr{O}$, we mean regularity of u up to $\partial_0\mathscr{O}$ — see Figure 1.1.

Our first main result is the following analogue of [6, Theorem I.1.3] (for a related degenerate-parabolic operator (1.25) and $d = 2$), [7, Theorem 3.1] (for a related degenerate-parabolic operator (1.25) with $d \geq 2$), and [26, Corollary 6.3 and Problem 6.1] (strictly elliptic operator). We refer the reader to Definitions 2.1, 2.2, and 2.3 for descriptions of the Daskalopoulos-Hamilton family of $C^{k,\alpha}_s$ and $C^{k,2+\alpha}_s$ Hölder norms and Banach spaces. For any $U \subsetneqq \mathbb{H}$, we denote

$$\|a\|_{C^{k,\alpha}_s(\bar{U})} := \sum_{i,j=1}^d \|a^{ij}\|_{C^{k,\alpha}_s(\bar{U})} \quad \text{and} \quad \|b\|_{C^{k,\alpha}_s(\bar{U})} := \sum_{i=1}^d \|b^i\|_{C^{k,\alpha}_s(\bar{U})}. \tag{1.5}$$

Theorem 1.1 (A priori interior Schauder estimate)**.** *For any $\alpha \in (0,1)$, integer $k \geq 0$, and positive constants b_0, λ_0, d_0, Λ, ν, there is a positive constant, $C = C(b_0, d, d_0, k, \alpha, \lambda_0, \Lambda, \nu)$,*

[1]The operator $-L$ is the generator of a degenerate-diffusion process with killing.

[2]The assumption of symmetry is just for convenience when applying changes of variables and is easily obtained by replacing a^{ij} with $\tilde{a}^{ij} := (a^{ij} + a^{ji})/2$.

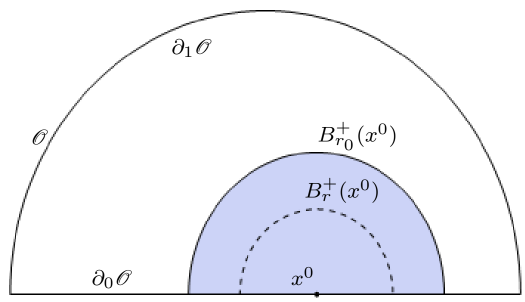

FIGURE 1.1. Boundaries and regions in Theorem 1.1 and Remark 1.2.

such that the following holds. Suppose [3] $\operatorname{height}(\mathscr{O}) \leq \nu$ *and the coefficients* a, b, c *of* L *in* (1.3) *belong to* $C^{k,\alpha}_s(\underline{\mathscr{O}})$ *and obey*

$$\|a\|_{C^{k,\alpha}_s(\bar{\mathscr{O}})} + \|b\|_{C^{k,\alpha}_s(\bar{\mathscr{O}})} + \|c\|_{C^{k,\alpha}_s(\bar{\mathscr{O}})} \leq \Lambda, \tag{1.6}$$

$$\langle a\xi, \xi\rangle \geq \lambda_0|\xi|^2 \quad \text{on } \bar{\mathscr{O}}, \quad \forall\, \xi \in \mathbb{R}^d, \tag{1.7}$$

$$b^d \geq b_0 \quad \text{on } \partial_0\mathscr{O}, \tag{1.8}$$

If $u \in C^{k,2+\alpha}_s(\underline{\mathscr{O}})$ *and* $\mathscr{O}' \subset \mathscr{O}$ *is a subdomain such that* $\operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}) \geq d_0$, *then*

$$\|u\|_{C^{k,2+\alpha}_s(\bar{\mathscr{O}}')} \leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar{\mathscr{O}})} + \|u\|_{C(\bar{\mathscr{O}})}\right). \tag{1.9}$$

Remark 1.2 (A priori interior Schauder estimate). The case where $k = 0$ and the domain is a half-ball, $\mathscr{O} = B^+_{r_0}(x^0)$, the coefficients, a, b, of L are constant, $c = 0$, and $u \in C^\infty(\bar{B}^+_{r_0}(x^0))$ is given by Corollary 6.8 Theorem 3.2 relaxes those conditions to allow $u \in C^{2+\alpha}_s(\underline{B}^+_{r_0}(x^0))$ and arbitrary $c \in \mathbb{R}$; Theorem 8.1 further relaxes the conditions on L to allow for variable coefficients, a, b, c, in $C^\alpha_s(\underline{B}^+_{r_0}(x^0))$; Theorem 8.3 relaxes the constraint $k = 0$ to allow for arbitrary integers $k \geq 0$; finally, Theorem 1.1 is proved in §8.1, where we relax the constraint that $\mathscr{O} = B^+_{r_0}(x^0)$ and allow for arbitrary domains $\mathscr{O} \subseteqq \mathbb{R}^{d-1} \times (0, \nu)$.

It is considerably more difficult to prove a global a priori estimate for a solution, $u \in C^{k,2+\alpha}_s(\bar{\mathscr{O}})$, when the intersection $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ is non-empty and we do not consider that problem in this article, but refer the reader to [19, §1.3] for a discussion of this issue. However, the global estimate in Corollary 1.3 has useful applications when $\partial_1\mathscr{O}$ does not meet $\partial_0\mathscr{O}$. For a constant $\nu > 0$, we define the *strip*,

$$S := \mathbb{R}^{d-1} \times (0, \nu), \tag{1.10}$$

[3] If we had allowed $\operatorname{height}(\mathscr{O}) = \infty$, we would need to modify our definition of Hölder norms to provide a weight for additional control when $x_d \to \infty$ because the coefficient matrix, $x_d a$, for D^2u would be unbounded due to (1.7). Weighted Hölder norms of this type were used by the authors in [18], for this reason, for the corresponding parabolic operator, $-\partial_t + L$, on $(0, T) \times \mathbb{H}$.

and note that $\partial_0 S = \mathbb{R}^{d-1} \times \{0\}$ and $\partial_1 S = \mathbb{R}^{d-1} \times \{\nu\}$.

Corollary 1.3 (A priori global Schauder estimate on a strip)**.** *For any $\alpha \in (0,1)$, positive constants λ_0, b_0, Λ, ν, and integer $k \geq 0$, there is a positive constant, $C = C(k,\alpha,\nu,d,\lambda_0,b_0,\Lambda)$, such that the following holds. Suppose the coefficients of L in* (1.3) *belong to $C^{k,\alpha}_s(\bar S)$, where $S = \mathbb{R}^{d-1} \times (0,\nu)$ as in* (1.10)*, and obey*

$$\|a\|_{C^{k,\alpha}_s(\bar S)} + \|b\|_{C^{k,\alpha}_s(\bar S)} + \|c\|_{C^{\alpha}_s(\bar S)} \leq \Lambda, \tag{1.11}$$

$$\langle a\xi, \xi\rangle \geq \lambda_0 |\xi|^2 \quad \text{on } \bar S, \quad \forall\, \xi \in \mathbb{R}^d, \tag{1.12}$$

$$b^d \geq b_0 \quad \text{on } \partial_0 S. \tag{1.13}$$

If $u \in C^{k,2+\alpha}_s(\bar S)$ and $u = 0$ on $\partial_1 S$, then

$$\|u\|_{C^{k,2+\alpha}_s(\bar S)} \leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar S)} + \|u\|_{C(\bar S)}\right), \tag{1.14}$$

and, when $c \geq 0$ on S,

$$\|u\|_{C^{k,2+\alpha}_s(\bar S)} \leq C \|Lu\|_{C^{k,\alpha}_s(\bar S)}, \tag{1.15}$$

Remark 1.4 (A priori global Schauder estimate on a strip)*.* For an operator, L, with constant coefficients, a, b, c, an a priori global Schauder estimate on a strip is proved as Corollary 7.2.

The Green's function for a operator L in (1.3) with constant coefficients can be extracted from Appendix B, where we construct explicit C^∞ solutions to $Lu = f$ on $\mathbb{H}$ and prove the following elliptic analogue of the existence result [6, Theorem I.1.2] for the initial value problem for a degenerate-parabolic model (1.25) on a half-space for the linearization of the porous medium equation (1.24).

Theorem 1.5 (Existence and uniqueness of a $C^\infty(\bar{\mathbb{H}})$ solution on the half-space when L has constant coefficients)**.** *Let L be an operator of the form* (1.3) *and require that the coefficients, a, b, c, are constant with $b^d > 0$ and $c > 0$. If $f \in C^\infty_0(\bar{\mathbb{H}})$, then there is a unique solution, $u \in C^\infty(\bar{\mathbb{H}})$, to $Lu = f$ on $\mathbb{H}$.*

Again, it is considerably more difficult to prove existence of a solution, u, in $C^{k,2+\alpha}_s(\bar{\mathscr{O}})$ or $C^{k,2+\alpha}_s(\underline{\mathscr{O}}) \cap C(\bar{\mathscr{O}})$, to (1.1), (1.2) when the intersection $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ is non-empty. We do not consider that problem in this article either and again refer the reader to [19, §1.3] for a discussion of this issue. However, in the case of a strip, $\partial_1\mathscr{O}$ does not meet $\partial_0\mathscr{O}$ and we have an existence result, Theorem 1.6, for an operator with variable coefficients. In §1.3, we discuss additional existence results which should also follow from Theorems 1.1 and 1.5 when $\partial_0\mathscr{O}$ is curved and $\partial_1\mathscr{O}$ is empty.

Theorem 1.6 (Existence and uniqueness of a $C^{k,2+\alpha}_s(\bar S)$ solution on a strip S)**.** *Let $\alpha \in (0,1)$, let $\nu > 0$ and $S = \mathbb{R}^{d-1} \times (0,\nu)$ be as in* (1.10)*, and let $k \geq 0$ be an integer. Let L be an operator as in* (1.3)*. If f and the coefficients of L in* (1.3) *belong to $C^{k,\alpha}_s(\bar S)$ and obey* (1.12) *and* (1.13) *for some positive constants b_0, λ_0, then there is a unique solution, $u \in C^{k,2+\alpha}_s(\bar S)$, to the boundary value problem,*

$$Lu = f \quad \text{on } S, \tag{1.16}$$

$$u = 0 \quad \text{on } \partial_1 S. \tag{1.17}$$

Remark 1.7 (Existence and uniqueness of a solution on a strip)*.* For an operator, L, with constant coefficients, a, b, c, existence and uniqueness of a solution on a strip is proved as Corollary B.4.

The preceding existence and uniqueness result on a strip leads to the following analogue of [26, Theorem 6.17] and is proved in §8.2.

Theorem 1.8 (Interior $C_s^{k,2+\alpha}$-regularity). *For any $\alpha \in (0,1)$ and integer $k \geq 0$, the following holds. Assume that the coefficients of L in (1.3) belong to $C_s^{k,\alpha}(\underline{\mathscr{O}})$ and obey (1.7) and (1.8) for some positive constants b_0, λ_0. If $u \in C^2(\mathscr{O})$ obeys* [4]

$$u \in C^1(\underline{\mathscr{O}}), \quad x_d D^2 u \in C(\underline{\mathscr{O}}), \quad \text{and} \quad Lu \in C_s^{k,\alpha}(\underline{\mathscr{O}}), \tag{1.18}$$

$$x_d D^2 u = 0 \quad \text{on } \partial_0 \mathscr{O}, \tag{1.19}$$

then $u \in C_s^{k,2+\alpha}(\underline{\mathscr{O}})$.

Given Theorem 1.8, one immediately obtains the following degenerate-elliptic analogue of the C^∞-regularity result for the degenerate-parabolic model for the linearization of the porous medium equation [6, Theorem I.1.1].

Corollary 1.9 (Interior C^∞-regularity). *Assume that the coefficients of L in (1.3) belong to $C^\infty(\underline{\mathscr{O}})$ and obey (1.7) and (1.8) for some positive constants b_0, λ_0. If $u \in C^2(\mathscr{O})$ obeys (1.18) for every integer $k \geq 0$, so $Lu \in C^\infty(\underline{\mathscr{O}})$, together with (1.19), then $u \in C^\infty(\underline{\mathscr{O}})$.*

Remark 1.10 (Regularity up to the "non-degenerate boundary"). Regarding the conclusion of Theorem 1.8, standard elliptic regularity results for linear, second-order, strictly elliptic operators [26, Theorems 6.19] also imply, when $k \geq 0$, that $u \in C^{k+2,\alpha}(\mathscr{O} \cup \partial_1 \mathscr{O})$ if u solves (1.1), (1.2) with $f \in C^{k,\alpha}(\mathscr{O} \cup \partial_1 \mathscr{O})$ and $g \in C^{k+2,\alpha}(\mathscr{O} \cup \partial_1 \mathscr{O})$, and $\partial_1 \mathscr{O}$ is $C^{k+2,\alpha}$. Because our focus in this article is on regularity of u up to the "degenerate boundary", $\partial_0 \mathscr{O}$, we shall omit further mention of such straightforward generalizations.

Finally, we refine our existence results in [19] when $d = 2$ for the *Heston* operator,

$$Av := -\frac{x_2}{2}\left(v_{x_1x_1} + 2\varrho\sigma v_{x_1x_2} + \sigma^2 v_{x_2x_2}\right) - (c_0 - q - x_2/2)v_{x_1} - \kappa(\theta - x_2)v_{x_2} + c_0 v, \tag{1.20}$$

where $q \geq 0, c_0 \geq 0, \kappa > 0, \theta > 0, \sigma > 0$, and $\varrho \in (-1,1)$ are constants (their financial interpretation is provided in [27]), and $v \in C^\infty(\mathbb{H})$. In particular, we give analogues of the existence results [26, Theorems 6.13 & 6.19] for the case of the Dirichlet boundary value problem for a strictly elliptic operator.

Theorem 1.11 (Existence and uniqueness of a $C_s^{k,2+\alpha}$ solution to a Dirichlet boundary value problem for the Heston operator). *Let $\alpha \in (0,1)$ and let $k \geq 0$ be an integer, let K be a finite right-circular cone, and require that $\partial_1 \mathscr{O}$ obeys a uniform exterior cone condition with cone K. If $f \in C_s^{k,\alpha}(\underline{\mathscr{O}}) \cap C_b(\mathscr{O})$ and*

$$\begin{cases} c_0 > 0 & \text{if } \operatorname{height}(\mathscr{O}) = \infty, \\ c_0 \geq 0 & \text{if } \operatorname{height}(\mathscr{O}) < \infty, \end{cases} \tag{1.21}$$

then there is a unique solution,

$$u \in C_s^{k,2+\alpha}(\underline{\mathscr{O}}) \cap C(\mathscr{O} \cup \partial_1 \mathscr{O}) \cap C_b(\mathscr{O}),$$

to the boundary value problem for the Heston operator,

$$Au = f \quad \text{on } \mathscr{O}, \tag{1.22}$$

$$u = 0 \quad \text{on } \partial_1 \mathscr{O}. \tag{1.23}$$

[4] We write $Du,\ x_d D^2 u \in C(\underline{\mathscr{O}})$ as an abbreviation for $u_{x_i},\ x_d u_{x_i x_j} \in C(\underline{\mathscr{O}})$, for $1 \leq i, j \leq d$ and write $x_d D^2 u = 0$ on $\partial_0 \mathscr{O}$ as an abbreviation for $\lim_{\mathbb{H} \ni x \to x^0} x_d D^2 u(x) = 0$ for all $x^0 \in \partial_0 \mathscr{O}$.

Remark 1.12 (Schauder a priori estimates and approach to existence of solutions). As we explain in [19, §1.3], the proof of existence of solutions, $u \in C_s^{k,2+\alpha}(\underline{\mathscr{O}}) \cap C(\bar{\mathscr{O}})$, to the boundary value problem, (1.1), (1.2), given $f \in C_s^{k,\alpha}(\underline{\mathscr{O}})$ and $g \in C(\bar{\mathscr{O}})$, is considerably more difficult when $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ is non-empty because, unlike in [6], one must consider a priori Schauder estimates and regularity near the "corner" points of the subdomain, $\mathscr{O} \subset \mathbb{H}$, where the "non-degenerate boundary", $\partial_1\mathscr{O}$, meets the "degenerate boundary", $\partial_0\mathscr{O}$.

Given an additional geometric hypothesis on $\mathscr{O}$ near points in $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$, the property that $u \in C(\mathscr{O} \cup \partial_1\mathscr{O}) \cap C_b(\mathscr{O})$ in the conclusion of Theorem 1.11 simplifies to $u \in C(\bar{\mathscr{O}})$.

Corollary 1.13 (Existence and uniqueness of a globally continuous $C_s^{k,2+\alpha}$ solution to a Dirichlet boundary value problem for the Heston operator)**.** *If in addition to the hypotheses of Theorem 1.11 the domain, $\mathscr{O}$, satisfies a uniform exterior and interior cone condition on $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ with cone K in the sense of* [19]*, then $u \in C_s^{k,2+\alpha}(\underline{\mathscr{O}}) \cap C(\bar{\mathscr{O}})$.*

Remark 1.14 (Existence of solutions to a Dirichlet boundary value problem). Theorem 1.11 and Corollary 1.13 should generalize to from the Heston operator A in (1.20) to an operator L in (1.3) with $C_s^{k,2+\alpha}$ coefficients and $d \geq 2$.

1.2. **Survey of previous related research.** We provide a brief survey of some related research by other authors on Schauder a priori estimates and regularity theory for solutions to degenerate-elliptic and degenerate-parabolic partial differential equations most closely related to the results described in our article.

The principal feature which distinguishes the equation (1.1), when the operator L is given by (1.3), from the linear, second-order, strictly elliptic operators in [26] and their boundary value problems, is the degeneracy of L due to the factor, x_d, in the coefficient matrix for D^2u and, because $b_0 > 0$ in (1.3), the fact that boundary conditions may be omitted along $x_d = 0$ when we seek solutions, u, with sufficient regularity up to $x_d = 0$.

The literature on degenerate elliptic and parabolic equations is vast, with the well-known articles of Fabes, Kenig, and Serapioni [12, 13], Fichera [21, 22], Kohn and Nirenberg [29], Murthy and Stampacchia [33, 34] and the monographs of Levendorskiĭ [31] and Oleĭnik and Radkevič [35, 36, 37], being merely the tip of the iceberg.

As far as the authors can tell, however, there has been relatively little prior work on a priori Schauder estimates and higher-order Hölder regularity of solutions up to the portion of the domain boundary where the operator becomes degenerate. In this context, the work of Daskalopoulos, Hamilton, and Rhee [6, 7, 38] and of Koch stands out in recent years because of their introduction of the cycloidal metric on the upper-half space, weighted Hölder norms, and weighted Sobolev norms which provide the key ingredients required to unlock the existence, uniqueness, and higher-order regularity theory for solutions to the porous medium equation (1.24) and the degenerate-parabolic model equation (1.25) on the upper half-space given by the linearization of the porous medium equation in suitable coordinates.

Daskalopoulos and Hamilton [6] proved existence and uniqueness of C^∞ solutions, u, to the Cauchy problem for the porous medium equation [6, p. 899] (when $d = 2$),

$$-u_t + \sum_{i=1}^d (u^m)_{x_i x_i} = 0 \quad \text{on } (0,T) \times \mathbb{R}^d, \quad u(\cdot, 0) = g \quad \text{on } \mathbb{R}^d, \tag{1.24}$$

with constant $m > 1$ and initial data, $g \geq 0$, compactly supported in $\mathbb{R}^d$, together with C^∞-regularity of its free boundary, $\partial\{u > 0\}$, provided the initial pressure function is non-degenerate (that is, $Du^{m-1} \geq a > 0$) on boundary of its support at $t = 0$. Their analysis is based on their development of existence, uniqueness, and regularity results for the linearization of the porous medium equation near the free boundary and, in particular, their *model linear degenerate operator* [6, p. 901] (generalized from $d = 2$ in their article),

$$Lu = -x_d \sum_{i=1}^{d} u_{x_i x_i} - \beta u_{x_d}, \quad u \in C^\infty(\mathbb{H}), \tag{1.25}$$

where β is a positive constant, analogous to the combination of parameters, $2\kappa\theta/\sigma^2$, in (1.20), following a suitable change of coordinates [6, p. 941].

The same model linear degenerate operator (for $d \geq 2$), was studied independently by Koch [28, Equation (4.43)] and, in a Habilitation thesis, he obtained existence, uniqueness, and regularity results for solutions to (1.24) which complement those of Daskalopoulos and Hamilton [6]. Koch employs weighted Sobolev space methods, Moser iteration, and pointwise estimates for the fundamental solution. However, by adapting the approach of Daskalopoulos and Hamilton [6], we avoid having to rely on difficult pointwise estimates for the Green's function for the operator L in (1.3). Although tantalizingly explicit — see [9, 14, 15, 20, 27] for the Green's function and fundamental solution of the elliptic and parabolic Heston operator (1.20) and Appendix B — these kernel functions appear quite intractable for the analysis required to emulate the role of potential theory for the Laplace operator in the traditional development of Schauder theory in [26].

While the Daskalopoulos-Hamilton Schauder theory for degenerate-parabolic operators has been adopted so far by relatively few other researchers, it has also been employed by De Simone, Giacomelli, Knüpfer, and Otto in [8, 25, 24] and by Epstein and Mazzeo in [10].

1.3. **Extensions and future work.** We defer to a subsequent article the development of a priori global Schauder $C^{k,2+\alpha}_s(\bar{\mathscr{O}})$ estimates, existence, and regularity theory for solutions u to the elliptic boundary value problem (1.1), (1.2) when f and the coefficients, a, b, c, of L in (1.3) belong to $C^{k,\alpha}_s(\bar{\mathscr{O}})$, the boundary data function g belongs to $C^{k,2+\alpha}_s(\bar{\mathscr{O}})$, and $\mathscr{O}$ has boundary portion $\partial_1\mathscr{O}$ of class $C^{k+2,\alpha}$ and $C^{k,2+\alpha}$-transverse to $\partial_0\mathscr{O}$. For reasons we summarize in [19, §1.3], the development of global Schauder a priori estimates, regularity, and existence theory appears very difficult when the intersection $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ is non-empty.

However, if $\mathscr{O} \subset \mathbb{R}^d$ is a bounded domain and L is an elliptic linear second-order partial differential operator which is equivalent to an operator L^{x_0} of the form (1.3) in local coordinates near *every* point $x^0 \in \partial\mathscr{O}$, then Theorem 1.1 will quickly lead to a *global* $C^{k,2+\alpha}_s(\bar{\mathscr{O}})$ a priori estimate for u if $\partial\mathscr{O}$ is of class $C^{k,2+\alpha}_s$. Moreover, for $g \in C^{k,2+\alpha}_s(\bar{\mathscr{O}})$, the method of proof of [30, Theorem 6.5.3] (or indeed [6, Theorem II.1.1]) should adapt to give existence of a solution $u \in C^{k,2+\alpha}_s(\bar{\mathscr{O}})$ to (1.1), (1.2).

As we noted in Remark 1.14, we expect our existence results (Theorem 1.11 and Corollary 1.11) for solutions to the Dirichlet boundary value problem (1.22), (1.23) to generalize from the case of the Heston operator, A, in (1.20) on subdomains of the half-plane to operators of the form L in (1.3) on subdomains of the half-space with $C^{k,\alpha}_s(\underline{\mathscr{O}})$ coefficients, a, b, c. These generalizations may be developed in two ways. First, the proof of Theorem 1.11 relies on existence and regularity theory for solutions to a variational equation defined by (1.22) and a choice of suitable weighted Sobolev spaces in [5, 17, 19]; we expect that analysis to extend without difficulty to operators of the form L in (1.3). Second, we expect the a priori interior Schauder estimates that we develop in this article, which are in the style of [26, Corollary 6.3], to extend to more refined and sharper a priori "global" interior Schauder estimates, in the style of [26, Theorem 6.2, Lemmas 6.20 and 6.21]. Aside from facilitating "rearrangement arguments", we expect such "global" a priori interior Schauder estimates — relying on a choice of suitable weighted Hölder spaces similar to those employed in [26, Chapter 6] — to permit the use of the continuity method to prove existence of solutions within a self-contained Schauder framework and this theme will be developed by the authors in a subsequent article.

While our a priori Schauder estimates rely on the specific form of the degeneracy factor, x_d, of the operator L in (1.3) on a subdomain of the half-space, we obtained weak and strong maximum principles for a much broader class of degenerate operators in [16]. Therefore, we plan to extend the a priori Schauder estimates and regularity theory for degenerate-elliptic operators such as

$$Lv = -\vartheta\operatorname{tr}(aD^2v) - b\cdot Dv + cv \quad \text{on } \mathscr{O}, \quad v\in C^\infty(\mathscr{O}),$$

where $\vartheta \in C^\alpha_{\mathrm{loc}}(\bar{\mathscr{O}})$ and $\vartheta > 0$ on a subdomain $\mathscr{O} \subset \mathbb{R}^d$ with non-empty boundary portion $\partial_0\mathscr{O} = \operatorname{int}(\{x\in\partial\mathscr{O} : \vartheta(x)=0\})$.

1.4. **Outline and mathematical highlights of the article.** For the convenience of the reader, we provide a brief outline of the article. In §2 , we review the construction of the Daskalopoulos-Hamilton-Hölder families of norms and Banach spaces [6].

In §3, we derive a priori local C^0 estimates for derivatives of solutions, u, to $Lu = 0$ on half-balls $B^+_{r_0}(x_0) \subset \mathbb{H}$ centered at points $x^0 \in \partial\mathbb{H}$, when L has constant coefficients. However, our method of proof differs significantly from that of Daskalopoulos and Hamilton [6], who apply a comparison principle for a certain non-linear parabolic operator and which directly uses the fact that this operator is parabolic. We were not able to replace their "parabolic" comparison argument by one which is suitable for the elliptic operators we consider in this article. Instead, we employ a simpler approach using a version of Brandt's finite-difference method [4] to estimate derivatives in directions parallel to $\partial\mathbb{H}$ and methods of ordinary differential equations to estimate derivatives in directions normal to $\partial\mathbb{H}$.

In §4, we adapt and slightly streamline the arguments of Daskalopoulos and Hamilton in [6] for their model degenerate-parabolic operator (1.25) to the case of our degenerate-elliptic operator (1.3) and derive a C^0 a priori estimate of the remainder of the first-order Taylor polynomial of a function u on a half-ball, $B^+_{r_0}(x_0)$.

In §5, we obtain a priori local interior Schauder estimates for a function u on a ball $B_{r_0}(x^0) \Subset \mathbb{H}$, where we keep track of the distance between the ball center, $x^0 \in \mathbb{H}$, and $\partial\mathbb{H}$, again when L has constant coefficients.

In §6, we apply the results of the previous sections to prove our main $C^{2+\alpha}_s$ a priori interior local Schauder estimate (Theorem 3.2) for an operator L with constant coefficients on a half-ball, $B^+_{r_0}(x_0)$.

In §7, we prove $C^{k,2+\alpha}_s$ a priori interior local Schauder estimate (Theorem 7.1) and a global a priori global Schauder estimate on a strip (Corollary 7.2), both when L has constant coefficients.

In §8, we relax the assumption in the preceding sections that the coefficients of the operator L in (1.3) are constant and prove a $C^{2+\alpha}_s$ a priori interior local Schauder estimate (Theorem 8.1) for a function, u, on a half-ball, $B^+_{r_0}(x^0)$ when L has variable coefficients. We then prove a $C^{k,2+\alpha}_s$ a priori local interior Schauder estimate for arbitrary $k \in \mathbb{N}$ (Theorem 8.3) and complete the proofs of Theorem 1.1 and Corollary 1.3. Next, we prove our global $C^{k,2+\alpha}_s(\bar{S})$ existence result on strips, Theorem 1.6, and complete the proofs of our main $C^{k,2+\alpha}_s$ regularity result, Theorem 1.8, and the $C^{k,2+\alpha}_s(\underline{\mathscr{O}})$ existence results, Theorem 1.11 and Corollary 1.13, for solutions to a Dirichlet boundary value problem for the Heston operator.

We collect some additional useful results and their proofs in several appendices to this article. In Appendix A, we prove a comparison principle for operators which include those of the form L in (1.3) with $c \geq 0$ (rather than $c \geq c_0$ for a positive constant c_0) when the domain, $\mathscr{O}$, is unbounded but has finite height, extending one of the comparison principles in [16]. In Appendix B, we prove Theorem 1.5. In Appendix C, we summarize the interpolation inequalities and boundary properties of functions in weighted Hölder spaces proved in [6] and [18].

1.5. **Notation and conventions.** In the definition and naming of function spaces, including spaces of continuous functions and Hölder spaces, we follow Adams [2] and alert the reader to occasional differences in definitions between [2] and standard references such as Gilbarg and Trudinger [26] or Krylov [30].

We let $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ denote the set of non-negative integers. If $S \subset \mathbb{R}^d$, we let $\bar{S}$ denote its closure with respect to the Euclidean topology and denote $\partial S := \bar{S} \setminus S$. For $r > 0$ and $x^0 \in \mathbb{R}^d$, we let $B_r(x^0) := \{x \in \mathbb{R}^d : |x - x^0| < r\}$ denote the open ball with center x^0 and radius r. We denote $B_r^+(x^0) := B_r(x^0) \cap \mathbb{H}$ when $x^0 \in \partial\mathbb{H}$. When x^0 is the origin, $O \in \mathbb{R}^d$, we denote $B_r(x^0)$ and $B_r^+(x^0)$ by B_r and B_r^+ for brevity.

If $V \subset U \subset \mathbb{R}^d$ are open subsets, we write $V \Subset U$ when U is bounded with closure $\bar{U} \subset V$. By $\operatorname{supp}\zeta$, for any $\zeta \in C(\mathbb{R}^d)$, we mean the *closure* in $\mathbb{R}^d$ of the set of points where $\zeta \neq 0$.

We use $C = C(*, \dots, *)$ to denote a constant which depends at most on the quantities appearing on the parentheses. In a given context, a constant denoted by C may have different values depending on the same set of arguments and may increase from one inequality to the next.

2. Preliminaries

In this section, we review the construction of the Daskalopoulos-Hamilton-Hölder families of norms and Banach spaces [6].

We first recall the definition of the *cycloidal distance function*, $s(\cdot, \cdot)$, on $\bar{\mathbb{H}}$ by

$$s(x^1, x^2) := \frac{|x^1 - x^2|}{\sqrt{x_d^1 + x_d^2 + |x^1 - x^2|}}, \quad \forall x^1, x^2 \in \bar{\mathbb{H}}, \tag{2.1}$$

where $x^i = (x_1^i, \dots, x_d^i)$, for $i = 1, 2$, and $|x^1 - x^2|$ denotes the usual Euclidean distance between points $x^1, x^2 \in \mathbb{R}^d$. Analogues of the cycloidal distance function (2.1) between points (t^1, x^1), $(t^2, x^2) \in [0, \infty) \times \bar{\mathbb{H}}$, in the context of parabolic differential equations, were introduced by Daskalopoulos and Hamilton in [6, p. 901] and Koch in [28, p. 11] for the study of the porous medium equation.

Observe that, by (2.1),

$$s(x, x^0) \leq |x - x^0|^{1/2}, \quad \forall x, x^0 \in \bar{\mathbb{H}}. \tag{2.2}$$

The reverse inequality take its simplest form when $x^0 \in \partial\mathbb{H}$, so $x_d^0 = 0$, in which case the inequalities $x_d \leq |x - x^0|$ and

$$|x - x^0| = s(x, x^0)\sqrt{x_d + |x - x^0|} \leq s(x, x^0)\sqrt{2|x - x^0|},$$

give

$$|x - x^0| \leq 2s(x, x^0)^2, \quad \forall x \in \bar{\mathbb{H}},\ x^0 \in \partial\mathbb{H}. \tag{2.3}$$

Following [2, §1.26], for a domain $U \subset \mathbb{H}$, we let $C(U)$ denote the vector space of continuous functions on U and let $C(\bar{U})$ denote the Banach space of functions in $C(U)$ which are bounded and uniformly continuous on U, and thus have unique bounded, continuous extensions to $\bar{U}$, with norm

$$\|u\|_{C(\bar{U})} := \sup_U |u|.$$

Noting that U may be *unbounded*, we let $C_{\mathrm{loc}}(\bar{U})$ denote the linear subspace of functions $u \in C(U)$ such that $u \in C(\bar{V})$ for every precompact open subset $V \Subset \bar{U}$. We let $C_b(U) := C(U) \cap L^\infty(U)$.

Daskalopoulos and Hamilton provide the

Definition 2.1 (C^α_s norm and Banach space)**.** [6, p. 901] Given $\alpha \in (0,1)$ and a domain $U \subset \mathbb{H}$, we say that $u \in C^\alpha_s(\bar{U})$ if $u \in C(\bar{U})$ and

$$\|u\|_{C^\alpha_s(\bar{U})} < \infty,$$

where

$$\|u\|_{C^\alpha_s(\bar{U})} := [u]_{C^\alpha_s(\bar{U})} + \|u\|_{C(\bar{U})}, \tag{2.4}$$

and

$$[u]_{C^\alpha_s(\bar{U})} := \sup_{\substack{x^1,x^2\in U\\ x^1\neq x^2}} \frac{|u(x^1)-u(x^2)|}{s(x^1,x^2)^\alpha}. \tag{2.5}$$

We say that $u \in C^\alpha_s(\underline{U})$ if $u \in C^\alpha_s(\bar{V})$ for all precompact open subsets $V \Subset \underline{U}$, recalling that $\underline{U} := U \cup \partial_0 U$. We let $C^\alpha_{s,\mathrm{loc}}(\bar{U})$ denote the linear subspace of functions $u \in C^\alpha_s(U)$ such that $u \in C^\alpha_s(\bar{V})$ for every precompact open subset $V \Subset \bar{U}$.

It is known that $C^\alpha_s(\bar{U})$ is a Banach space [6, §I.1] with respect to the norm (2.4).

We shall need the following higher-order weighted Hölder $C^{k,\alpha}_s$ and $C^{k,2+\alpha}_s$ norms and Banach spaces pioneered by Daskalopoulos and Hamilton [6]. We record their definition here for later reference.

Definition 2.2 ($C^{k,\alpha}_s$ norms and Banach spaces)**.** [6, p. 902] Given an integer $k \geq 0$, $\alpha \in (0,1)$, and a domain $U \subset \mathbb{H}$, we say that $u \in C^{k,\alpha}_s(\bar{U})$ if $u \in C^k(\bar{U})$ and

$$\|u\|_{C^{k,\alpha}_s(\bar{U})} < \infty,$$

where

$$\|u\|_{C^{k,\alpha}_s(\bar{U})} := \sum_{|\beta|\leq k} \|D^\beta u\|_{C^\alpha_s(\bar{U})}, \tag{2.6}$$

where $\beta := (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$ and

$$\|u\|_{C^\alpha_s(\bar{U})} := \|u\|_{C(\bar{U})} + [u]_{C^\alpha_s(\bar{U})}.$$

When $k = 0$, we denote $C^{0,\alpha}_s(\bar{U}) = C^\alpha_s(\bar{U})$.

Definition 2.3 ($C^{k,2+\alpha}_s$ norms and Banach spaces)**.** [6, pp. 901–902] Given an integer $k \geq 0$, a constant $\alpha \in (0,1)$, and a domain $U \subset \mathbb{H}$, we say that $u \in C^{k,2+\alpha}_s(\bar{U})$ if $u \in C^{k+1,\alpha}_s(\bar{U})$, the derivatives, $D^\beta u$, $\beta \in \mathbb{N}^d$ with $|\beta| = k+2$, of order $k+2$ are continuous on U, and the functions, $x_d D^\beta u$, $\beta \in \mathbb{N}^d$ with $|\beta| = k+2$, extend continuously up to the boundary, ∂U, and those extensions belong to $C^\alpha_s(\bar{U})$. We define

$$\|u\|_{C^{k,2+\alpha}_s(\bar{U})} := \|u\|_{C^{k+1,\alpha}_s(\bar{U})} + \sum_{|\beta|=k+2} \|x_d D^\beta u\|_{C^\alpha_s(\bar{U})}. \tag{2.7}$$

We say that[5] $u \in C^{k,2+\alpha}_s(\underline{U})$ if $u \in C^{k,2+\alpha}_s(\bar{V})$ for all precompact open subsets $V \Subset \underline{U}$. When $k=0$, we denote $C^{0,2+\alpha}_s(\bar{U}) = C^{2+\alpha}_s(\bar{U})$.

For any non-negative integer k, we let $C^k_0(\underline{U})$ denote the linear subspace of functions $u \in C^k(U)$ such that $u \in C^k(\bar{V})$ for every precompact open subset $V \Subset \underline{U}$ and define $C^\infty_0(\underline{U}) := \cap_{k\geq 0} C^k_0(\underline{U})$. Note that we also have $C^\infty_0(\underline{U}) = \cap_{k\geq 0} C^{k,\alpha}_s(\underline{U}) = \cap_{k\geq 0} C^{k,2+\alpha}_s(\underline{U})$.

3. Interior local estimates of derivatives

As in [6], we begin with the derivation of local estimates of derivatives of solutions on half-balls $B^+_r(x_0)$ centered at points $x^0 \in \partial\mathbb{H}$, but the method of proof differs significantly from the method of proof in [6, §I.4 & I.5]. In [6], Daskalopoulos and Hamilton apply a comparison principle to a suitably chosen function, defined in terms of the derivatives (see the definitions of Y at the beginning of [6, §I.5] and of X in the proof of [6, Corollary I.5.3]). Their comparison principle directly uses the fact that the operator is parabolic, and we were not able to replace the "parabolic" comparison argument by one which is suitable for the elliptic operators we consider in this article. (The Daskalopoulos-Hamilton approach can be viewed as a variant of the Bernstein method — see the proof [30, Theorem 8.4.4] in the case of the heat operator and [30, Theorem 2.5.2] in the case of the Laplace operator.)

Instead, we apply a combination of finite-difference arguments, methods of ordinary differential equations, and, in this section, restrict to the homogeneous of version of the equation (1.1) with $f=0$. We adapt Brandt's finite-difference method [4] (see also [26, §3.4]) to obtain a priori local estimates for $D^\beta u$, where $\beta \in \mathbb{N}^d$ is any multi-index with non-negative integer entries of the form $\beta = (\beta^1, \dots, \beta^{d-1}, 0)$. The method of Brandt also uses a comparison principle, but it is applied to finite differences, instead of functions of derivatives of u, such as X and Y in [6, §I.5]. Brandt's approach is also mentioned by Gilbarg and Trudinger in [26, p. 47] as an alternative to the usual methods for proving a priori interior Schauder estimates such as [26, Corollary 6.3]. We are able to apply the finite-difference estimates method not only on balls $B_r(x^0) \Subset \mathbb{H}$ as in [4], but also on half-balls $B^+_r(x_0) \subset \mathbb{H}$ centered at points $x^0 \in \partial\mathbb{H}$ because the degeneracy of the elliptic operator L in (1.3) along $\partial_0 B^+_r(x_0)$ and the fact that $b^d > 0$ along $\partial_0 B^+_r(x_0)$ (see (3.2)) implies that no boundary condition need be imposed along $\partial_0 B^+_r(x_0)$.

In §3.1 we summarize the interior local Schauder estimate and regularity results we will prove in sections 3, 4, 5, and 6. In §3.2, we develop C^0 interior local estimates for derivatives $D^\beta u$ when $\beta_d = 0$ and in §3.3, we extend those estimates to case $\beta_d > 0$.

3.1. A priori interior local Schauder estimate and regularity statements in the case of constant coefficients.

Throughout sections 3, 4, 5, and 6, we further assume the

Hypothesis 3.1 (Constant coefficients and positivity)**.** The coefficients, a, b, c, of the operator L in (1.3) are constant; there is a positive constant, λ_0, such that [6]

$$\langle a\xi, \xi\rangle \geq \lambda_0 |\xi|^2, \quad \forall, \xi \in \mathbb{R}^d; \tag{3.1}$$

and [7]

$$b^d = b_0 > 0. \tag{3.2}$$

[5]In [6, pp. 901–902], when defining the spaces $C^{k,\alpha}_s(\mathscr{A})$ and $C^{k,2+\alpha}_s(\mathscr{A})$, it is assumed that $\mathscr{A}$ is a compact subset of the *closed* upper half-space, $\bar{\mathbb{H}}$.

[6]Condition (3.1) is first used in the proof of Lemma 5.1

[7]Condition (3.2) is required by our maximum principle (Lemma A.1 and Corollary A.2). Our maximum principle is in turn required in §3; sections 4 and 5 depend on §3; and sections 6, 7, and 8 each depend on sections 5 and 6.

The condition (3.2) is first required in the proof of Lemma 5.1. When the coefficients of L are constant, we denote

$$\Lambda = \sum_{i,j=1}^{d} |a^{ij}| + \max_i |b^i| + |c|. \tag{3.3}$$

Our main goal in sections 3, 4, 5, and 6 is to prove the following versions of Theorems 1.1 and 1.8 when $k = 0$ and L has constant coefficients and the domain is a half-ball, $B^+_{r_0}(x^0)$ with $x^0 \in \partial\mathbb{H}$.

Theorem 3.2 (A priori interior local Schauder estimate when L has constant coefficients)**.** *For any $\alpha \in (0,1)$ and constants r and r_0 with $0 < r < r_0$, there is a positive constant, $C = C(\alpha, r, r_0, d, \lambda_0, b_0, \Lambda)$, such that the following holds. If $x^0 \in \partial\mathbb{H}$ and $u \in C^{2+\alpha}_s(\underline{B}^+_{r_0}(x^0))$, then*

$$\|u\|_{C^{2+\alpha}_s(\bar{B}^+_r)} \leq C \left(\|Lu\|_{C^{\alpha}_s(\bar{B}^+_{r_0}(x^0))} + \|u\|_{C(\bar{B}^+_{r_0}(x^0))} \right). \tag{3.4}$$

Our goal in the remainder of this section is to derive a priori estimates for Du and $x_d D^2 u$ on half-balls, $B^+_r(x^0)$, centered at points $x^0 \in \partial\mathbb{H}$. Because our operator, L, is invariant with respect to translations in the variables $(x_1, \ldots, x_{d-1})$ when the coefficients, a, b, c, are constant, we can assume without loss of generality that x^0 is the origin, $O \in \mathbb{R}^d$, and write $B^+_{r_0}(x^0) = B^+_{r_0}$ and $B^+_r(x^0) = B^+_r$ in our proof of Theorem 3.2.

3.2. **Interior local estimates for derivatives parallel to the degenerate boundary.** To derive a priori local estimates for $D^\beta u$, for $\beta \in \mathbb{N}^d$ with $\beta_d = 0$, it will be useful to consider the following transformation,

$$u(x) =: v(y), \quad x \in \mathbb{H}, \tag{3.5}$$

where $y = \phi(x) := x + \xi x_d$ and $\xi = (\xi_1, \ldots, \xi_{d-1}, \xi_d) \in \mathbb{R}^d$. We choose ξ such that

$$\xi_i := -b^i/b^d, \quad \forall i \neq d, \quad \xi_d = 0, \tag{3.6}$$

where we have used assumption (3.2) that $b^d > 0$. Note that ϕ is a diffeomorphism on $\bar{\mathbb{H}}$ which restricts to the identity map on $\partial\mathbb{H}$. We now consider the operator $\widetilde{L}_0$ defined by

$$L_0 u(x) =: \widetilde{L}_0 v(y), \quad x \in \mathbb{H},$$

and by direct calculations we obtain

$$\widetilde{L}_0 v = -y_d \tilde{a}^{ij} v_{y_i y_j} - \tilde{b}^i v_{y_i} \quad \text{on } \mathbb{H}, \tag{3.7}$$

where

$$\begin{aligned}
\tilde{a}^{ij} &:= a^{ij} + \frac{1}{2}\left(\xi_j a^{id} + \xi_i a^{jd}\right) + \xi_i \xi_j a^{dd}, \quad \forall i, j \neq d,\\
\tilde{a}^{id} = \tilde{a}^{di} &:= a^{id} + \xi_i a^{dd}, \quad \forall i \neq d,\\
\tilde{a}^{dd} &:= a^{dd},\\
\tilde{b}^i &:= b^i + \xi_i b^d, \quad \forall i \neq d,\\
\tilde{b}^d &= b^d.
\end{aligned} \tag{3.8}$$

The purpose of the transformation (3.5) is to ensure that the coefficients $\tilde{b}^i$ of the partial derivatives with respect to y_i in the definition (3.7) of the operator $\widetilde{L}_0$ are zero when $i \neq d$. The matrix $\tilde{a}$ is symmetric and positive definite, but now the constant of strict ellipticity depends on b^i/b^d, that is, on b^d and Λ, and on the constant of strict ellipticity, λ_0, of the matrix a.

Lemma 3.3 (Local estimates for first-order derivatives of v parallel to $\partial\mathbb{H}$)**.** *Let $0<r<r_0$, and let $v\in C^2(B^+_{r_0})\cap C(\bar B^+_{r_0})$ obey*

$$\widetilde{L}_0 v = 0 \quad \text{on } B^+_{r_0},$$

and assume that v satisfies

$$Dv,\ y_d D^2 v \in C(\underline{B}^+_{r_0}) \quad \text{and} \quad y_d D^2 v = 0 \quad \text{on } \partial_0 B^+_{r_0}. \tag{3.9}$$

Then there is a positive constant, $C=C(r,r_0,d,\lambda_0,b_0,\Lambda)$, such that

$$\|v_{y_k}\|_{C(\bar B^+_r)} \leq C\|v\|_{C(\bar B^+_{r_0})}, \quad \forall k\neq d.$$

Proof. We adapt the finite-difference argument employed by Brandt (1969) in [4] to prove the local estimates for derivatives, v_{y_k}, when $k\neq d$. We let $r_2:=(r+r_0)/2$ and $r_3:=\min\{(r_0-r)/2,\ 1/2\}$, and consider the $(d+1)$-dimensional cylinder,

$$\mathcal{C} := \left\{(y,y_{d+1})\in\mathbb{H}\times\mathbb{R}_+ : y\in B^+_{r_2},\ 0<y_{d+1}<r_3\right\}.$$

We consider the auxiliary function,

$$\phi(y,y_{d+1}) := \frac{1}{2}\left(v(y+y_{d+1}e_k) - v(y-y_{d+1}e_k)\right), \quad \forall (y,y_{d+1})\in\mathcal{C},$$

where $\mathcal{C}$ is defined above, and $e_k\in\mathbb{R}^d$ is the vector whose coordinates are all zero except for the k-th coordinate, which is 1. We choose a constant $c_0>0$ small enough, say $c_0=\lambda_0/2$, such that the differential operator,

$$\widetilde{L}^1_0 := \widetilde{L}_0 - c_0 y_d\partial_{y_k y_k} + c_0 y_d \partial_{y_{d+1}y_{d+1}},$$

is elliptic on $\mathbb{H}\times\mathbb{R}_+$. By the definition of the function ϕ, we notice that

$$\widetilde{L}^1_0\phi = 0 \quad \text{on } \mathcal{C},$$

because $\widetilde{L}_0 v = 0$ on $B^+_{r_0}$. For $y^0\in\bar B^+_r$, we consider the auxiliary function defined on $\mathcal{C}$,

$$\psi := C_1\|v\|_{C(\bar B^+_{r_0})}\left[y_{d+1}(1-y_{d+1}) + C_2\left(\sum_{i=1}^{d-1}(y_i-y^0_i)^2 + y_d^2(y_d-y_d^0) + y_{d+1}^2\right)\right],$$

where the positive constants C_1, C_2 will be suitably chosen below. We want to choose C_2 sufficiently small that

$$\widetilde{L}^1_0\psi \geq 0 \quad \text{on } \mathcal{C}.$$

By direct calculation, we obtain

$$\begin{aligned}
\psi_{y_i} &= 2C_1C_2\|v\|_{C(\bar B^+_{r_0})}(y_i-y_i^0), \quad i=1,2,\ldots,d-1,\\
\psi_{y_iy_i} &= 2C_1C_2\|v\|_{C(\bar B^+_{r_0})}, \quad i=1,2,\ldots,d-1,\\
\psi_{y_{d+1}y_{d+1}} &= 2C_1\left(C_2-1\right)\|v\|_{C(\bar B^+_{r_0})},\\
\psi_{y_d} &= 2x_dC_1C_2\|v\|_{C(\bar B^+_{r_0})}\left(\frac{3}{2}y_d - y_d^0\right),\\
\psi_{y_dy_d} &= 2C_1C_2\|v\|_{C(\bar B^+_{r_0})}\left(3y_d-y_d^0\right),
\end{aligned}$$

and so,

$$\begin{aligned}
\widetilde{L}_0^1\psi &= -y_d\tilde{a}^{ij}v_{y_iy_j} - \tilde{b}^i v_{y_i} - c_0 y_d\left(\psi_{y_{d+1}y_{d+1}} - \psi_{y_ky_k}\right)\\
&= -2y_dC_1\|v\|_{C(\bar{B}^+_{r_0})}\left[C_2\left(\left(\sum_{i=1}^{d-1}\tilde{a}^{ii}\right) + \tilde{a}^{dd}(3y_d - y^0_d) - c_0 + \tilde{b}^d\left(\frac{3}{2}y_d - y^0_d\right)\right) + c_0(C_2-1)\right]\\
&\geq -2y_dC_1\|v\|_{C(\bar{B}^+_{r_0})}\left[C_2\left(\left(\sum_{i=1}^{d-1}\tilde{a}^{ii}\right) + 3r\tilde{a}^{dd} + 2r\tilde{b}^d\right) - c_0\right]\quad\text{on } \mathcal{C},
\end{aligned}$$

using the facts that the $\tilde{a}^{ii}$, for $i = 1, \ldots, d$, and $\tilde{b}^d$ are positive constants, while $\tilde{b}^i = 0$, $i \neq d$, by the transformation (3.5), and $y_d < r$. We choose the constant C_2 such that

$$C_2 \leq c_0\left(\sum_{i=1}^{d-1}\tilde{a}^{ii} + 3r\tilde{a}^{dd} + 2r\tilde{b}^d\right)^{-1},$$

so that we have

$$\widetilde{L}_0^1\psi \geq 0 \quad\text{on } \mathcal{C}.$$

Because $\widetilde{L}_0^1\phi = 0$ on $\mathcal{C}$, the preceding inequality yields

$$\widetilde{L}_0^1\left(\pm\phi - \psi\right) \leq 0 \quad\text{on } \mathcal{C}.$$

By the definition of the auxiliary function, ψ, and using the fact that $y^0 \in \bar{B}^+_r$ and $0 < y_{d+1} < 1/2$, we may choose a positive constant, $C_1 = C_1(r, r_0, C_2)$, large enough that

$$\pm\,\phi - \psi \leq 0 \quad\text{on } \partial_1\mathcal{C}. \tag{3.10}$$

The portion $\partial_1\mathcal{C}$ of the boundary of $\mathcal{C}$ consists of the sets

$$\{y_{d+1} = 0,\ y \in B^+_{r_2}\},\quad \{y_{d+1} = r_3,\ y \in B^+_{r_2}\},\quad\text{and}\quad \{y_{d+1} \in (0, r_3),\ y \in \partial_1 B^+_{r_2}\}.$$

To establish inequality (3.10) along the portion $\{y_{d+1} = 0\}$ of the boundary, $\partial_1\mathcal{C}$, note that $\phi = 0$, and so (3.10) holds on this portion of the boundary since $\psi \geq 0$. For the second portion of the boundary, $\partial_1\mathcal{C}$, using the fact that $r_3 \leq 1/2$, we notice that

$$y_{d+1}(1 - y_{d+1}) + C_2\left(\sum_{i=1}^{d-1}(y_i - y^0_i)^2 + y_d^2(y_d - y^0_d) + y^2_{d+1}\right) \geq r_3/2 \quad\text{on } \{y_{d+1} = r_3,\ y \in B^+_{r_2}\}.$$

For the third portion of the boundary, using the fact that $y^0 \in B^+_r$ and $y \in B^+_{r_2}$ and $r < r_2$, we see that on $\{y_{d+1} \in (0, r_3),\ y \in \partial_1 B^+_{r_2}\}$ we have

$$y_{d+1}(1 - y_{d+1}) + C_2\left(\sum_{i=1}^{d-1}(y_i - y^0_i)^2 + y_d^2(y_d - y^0_d) + y^2_{d+1}\right) \geq C_2(d-1)(r_2 - r)^2.$$

Therefore, we can find a constant $C_3 = C_3(r, r_0, d)$ such that

$$\psi \geq C_1C_3\|v\|_{C(B^+_r)} \quad\text{on } \{y_{d+1} = r_3,\ y \in B^+_{r_2}\} \cup \{y_{d+1} \in (0, r_3),\ y \in \partial_1 B^+_{r_2}\}$$

We may choose the constant $C_1 = C_1(r, r_0, d)$ large enough so that $C_1C_3 \geq 1$, and using the definition of ϕ, we have

$$\psi \geq |\phi| \quad\text{on } \{y_{d+1} = r_3,\ y \in B^+_{r_2}\} \cup \{y_{d+1} \in (0, r_3),\ y \in \partial_1 B^+_{r_2}\}.$$

Now, inequality (3.10) follows. By (3.9) we have $\phi \in C(\bar{\mathcal{C}})$, and $D\phi$, $y_dD^2\phi \in C(\mathcal{C} \cup \partial_0\mathcal{C})$, and $y_dD^2\phi = 0$ on $\partial_0\mathcal{C}$, where $\partial_0\mathcal{C}$ is the interior of $\{y_d = 0\} \cap \bar{\mathcal{C}}$. Since $\psi \in C^\infty(\bar{\mathcal{C}})$, we may apply the comparison principle [16, Theorem 5.1] to ϕ and ψ on the domain $\mathcal{C}$. We find that $\pm\phi - \psi \leq 0$ on $\mathcal{C}$, and so by the definition of the function ϕ, we have, for all $y^0 \in B_r^+$ and $y_{d+1} \in (0, r_3)$,

$$\frac{1}{2y_{d+1}}|v(y^0 + y_{d+1}e_k) - v(y_0 - y_{d+1}e_k)| \leq C_1 \|v\|_{C(\bar{B}_{r_0}^+)} \left(1 - y_{d+1} + C_2 y_{d+1}\right).$$

The preceding inequality yields

$$|v_{y_k}(y^0)| \leq C_1 \|v\|_{C(\bar{B}_{r_0}^+)}, \quad \forall y^0 \in \bar{B}_r^+,$$

for a constant $C_1 = C_1(r, r_0, d, \lambda_0, b_0, \Lambda)$, and this concludes the proof. □

Lemma 3.4 (Local estimates for higher-order derivatives of v parallel to $\partial\mathbb{H}$)**.** *Let $k \in \mathbb{N}$ and $0 < r < r_0$. Then there is a constant $C = C(k, r, r_0, d, \lambda_0, b_0, \Lambda)$, such that for any $v \in C^\infty(\bar{B}_{r_0}^+)$ obeying*

$$\widetilde{L}_0 v = 0 \quad \text{on} \quad B_{r_0}^+,$$

we have

$$\|D^\beta v\|_{C(\bar{B}_r^+)} \leq C \|v\|_{C(\bar{B}_{r_0}^+)}, \tag{3.11}$$

for all multi-indices $\beta = (\beta_1, \ldots, \beta_{d-1}, 0) \in \mathbb{N}^d$ such that $|\beta| \leq k$.

Proof. Lemma 3.3 establishes the result when $|\beta| = 1$. We prove the higher-order derivative estimates parallel to $\partial\mathbb{H}$ by induction. We assume the induction hypothesis: For any $0 < r < r_0$, there is a constant $C_1 = C_1(k-1, r, r_0, d, \lambda_0, b_0, \Lambda)$, such that

$$\|D^{\beta'} v\|_{C(\bar{B}_r^+)} \leq C_1 \|v\|_{C(\bar{B}_{r_0}^+)},$$

for all multi-indices $\beta' = (\beta'_1, \ldots, \beta'_{d-1}, 0) \in \mathbb{N}^d$ such that $|\beta'| \leq k-1$. Since $\widetilde{L}_0 v = 0$ on $B_{r_0}^+$, we also have that $\widetilde{L}_0 D^\beta v = 0$ on $B_{r_0}^+$, for all multi-indices β with $\beta_d = 0$. We fix such a multi-index β. Let $k \in \mathbb{N}$ be such that $\beta_k \neq 0$, and set $\beta' := \beta - e_k$. We set $r_2 := (r + r_0)/2$ and apply Lemma 3.3 to $D^{\beta'} v$ with $0 < r < r_2$ to obtain

$$\|D^\beta v\|_{C(\bar{B}_r^+)} \leq C_2 \|D^{\beta'} v\|_{C(\bar{B}_{r_2}^+)},$$

for some positive constant $C_2 = C_2(r, r_2, d, \lambda_0, b_0, \Lambda)$. The conclusion now follows from the preceding estimate and the induction hypothesis applied to $D^{\beta'} v$ with $0 < r_2 < r_0$, since $|\beta'| \leq k - 1$. □

From (3.5), we have

$$D^\beta u(x) = D^\beta v(y), \quad y = x + \xi x_d, \quad x \in \mathbb{H},$$

for all $\beta \in \mathbb{N}^d$ such that $\beta_d = 0$. Therefore, Lemmas 3.3 and 3.4 give us the following estimates for $D^\beta u$.

Lemma 3.5 (Local estimates of higher-order derivatives of u parallel to $\partial\mathbb{H}$)**.** *Let $k \in \mathbb{N}$ and $r_0 > 0$. Then there are positive constants, $r_1 = r_1(r_0, b_0, \Lambda) < r_0$ and $C = C(r_0, d, k, \lambda_0, b_0, \Lambda)$, such that for any function $u \in C^\infty(\bar{B}_{r_0}^+)$ solving*

$$L_0 u = 0 \quad \text{on } B_{r_0}^+, \tag{3.12}$$

we have, for all $\beta \in \mathbb{N}^d$ with $\beta_d = 0$ and $|\beta| \leq k$,

$$\|D^\beta u\|_{C(\bar{B}_{r_1}^+)} \leq C \|u\|_{C(\bar{B}_{r_0}^+)}.$$

Proof. Let $\phi : \mathbb{H} \to \mathbb{H}$ be the affine transformation $y = \phi(x) = x + \xi x_d$, $x \in \mathbb{H}$, where $\xi \in \mathbb{R}^d$ is defined by (3.6). Let $s_0 = s_0(r_0, b_0, \Lambda) > 0$ be small enough such that $B^+_{s_0} \subset \phi(B^+_{r_0})$. Then, $v \in C^\infty(\bar{B}^+_{s_0})$ and $\widetilde{L}_0 v = 0$ on $B^+_{s_0}$, since $u \in C^\infty(\bar{B}^+_{r_0})$, and $L_0 u = 0$ on $B^+_{r_0}$. Let $s_1 = s_0/2$ and apply Lemma 3.4 to v with r replaced by s_1 and r_0 replaced by s_0. For any $k \in \mathbb{N}$, there is a positive constant $C = C(k, r_0, d, \lambda_0, b_0, \Lambda)$, such that for all $\beta \in \mathbb{N}^d$ with $\beta_d = 0$, we have

$$\|D^\beta v\|_{C(\bar{B}^+_{s_1})} \leq C \|v\|_{C(\bar{B}^+_{s_0})}. \tag{3.13}$$

We now choose $r_1 = r_1(s_1, b_0, \Lambda)$ small enough such that $\phi(B^+_{r_1}) \subset B^+_{s_1}$. Using the fact that $D^\beta u(x) = D^\beta v(\phi(x))$, we obtain

$$\begin{aligned}
\|D^\beta u\|_{C(\bar{B}^+_{r_1})} &\leq \|D^\beta v\|_{C(\bar{B}^+_{s_1})} \quad \text{(by the facts that } \phi(B^+_{r_1}) \subset B^+_{s_1} \text{ and } u(x) = v(\phi(x)))\\
&\leq C\|v\|_{C(\bar{B}^+_{s_0})} \quad \text{(by (3.13))}\\
&\leq C\|u\|_{C(\bar{B}^+_{r_0})} \quad \text{(by the facts that } B^+_{s_0} \subset \phi(B^+_{r_0}) \text{ and } u(x) = v(\phi(x))).
\end{aligned}$$

This concludes the proof. □

3.3. **Interior local estimates for derivatives normal to the degenerate boundary.** We again shall use the affine transformation (3.5) of coordinates, but now with a different choice of the vector ξ, that is

$$\xi^i := -a^{id}/a^{dd}, \quad \forall i \neq d, \quad \xi_d = 0, \tag{3.14}$$

and, given a function u on $\mathbb{H}$, we define the function w by

$$u(x) =: w(y), \quad y = x + \xi x_d, \quad x \in \mathbb{H}. \tag{3.15}$$

Then, by analogy with (3.7), we obtain

$$\bar{L}_0 w := y_d \bar{a}^{ij} w_{y_i y_j} + \bar{b}^i w_{y_i} \quad \text{on } \mathbb{H},$$

where we notice that $\bar{a}^{id} = 0$ by (3.8) and the choice of the vector ξ. Also, we have that $D^\beta u(x) = D^\beta w(y)$, for all $\beta \in \mathbb{N}^d$ with $\beta_d = 0$. Thus, Lemma 3.5 applies to w, and we obtain a priori local estimates for all derivatives of w parallel to $\partial\mathbb{H}$.

Next, we derive an a priori local estimate for w_{y_d}.

Lemma 3.6 (Local estimate for w_{y_d})**.** *Let $0 < r < r_0$. Then there is a positive constant, $C = C(r, r_0, d, \lambda_0, b_0, \Lambda)$, such that for any function $w \in C^\infty(\bar{B}^+_{r_0})$ obeying*

$$\bar{L}_0 w = 0 \quad \text{on } B^+_{r_0}, \tag{3.16}$$

we have

$$\|w_{y_d}\|_{C(\bar{B}^+_r)} \leq C\|w\|_{C(\bar{B}^+_{r_0})}.$$

Proof. Because $\bar{a}^{id} = 0$, for all $i \neq d$, we can rewrite the equation $\bar{L}_0 w = 0$ on $B^+_{r_0}$ as

$$y_d w_{y_d y_d} + \theta w_{y_d} = f \quad \text{on } B^+_{r_0},$$

where, for simplicity, we denote $\theta := \bar{b}^d/\bar{a}^{dd} > 0$, and define f by

$$f := y_d \sum_{i,j=1}^{d-1} \frac{\bar{a}^{ij}}{\bar{a}^{dd}} w_{y_i y_j} + \sum_{i=1}^{d-1} \frac{\bar{b}^i}{\bar{a}^{dd}} w_{y_i} \quad \text{on } \bar{B}^+_{r_0}.$$

We can estimate $\|f\|_{C(\bar{B}_r^+)}$ in terms of $\|w\|_{C(\bar{B}_{r_0}^+)}$ by applying Lemma 3.5 to control the supremum norms of w_{y_i} and $w_{y_iy_j}$ on $\bar{B}_r^+$, for all $i,j \neq d$. The preceding ordinary differential equation can be rewritten as

$$\left(y_d^\theta w_{y_d}\right)_{y_d} = y_d^{\theta-1} f \quad \text{on } B_{r_0}^+$$

and, integrating with respect to y_d, we obtain

$$y_d^\theta w_{y_d}(y) = \int_0^{y_d} f(y',s) s^{\theta-1}\,ds, \quad y \in B_{r_0}^+,$$

where denote $y = (y', y_d)$, and use the fact that $\theta > 0$, and $w_{y_d} \in C(\bar{B}_{r_0}^+)$. Thus, we have

$$|y_d^\theta w_{y_d}(y)| \leq \|f(y',\cdot)\|_{C([0,y_d])} \int_0^{y_d} s^{\theta-1}\,ds = \frac{1}{\theta} y_d^\theta \|f(y',\cdot)\|_{C([0,y_d])}, \quad y \in B_{r_0}^+,$$

from where it follows, by the definition of f, that

$$|w_{y_d}(y)| \leq C \sum_{\substack{\beta \in \mathbb{N}^d \\ \beta_d = 0;\ |\beta| \leq 2}} \|D^\beta w(y',\cdot)\|_{C([0,y_d])}, \quad y \in B_{r_0}^+,$$

for some constant $C = C(\lambda_0, b_0, \Lambda)$. Now applying Lemma 3.5 to estimate $D^\beta w$ on B_r^+, for all $0 < r < r_0$, and for all $\beta \in \mathbb{N}^d$ with $\beta_d = 0$ and $|\beta| \leq 2$, we obtain the supremum estimate for w_{y_d} on $\bar{B}_r^+$ in terms of the supremum estimate of w on $\bar{B}_{r_0}^+$. □

Lemma 3.7 (Local estimates for $D^\beta D_{y_d} w$ with $\beta_d = 0$)**.** *Let $k \in \mathbb{N}$, and let $0 < r < r_0$. Then there is a constant, $C = C(k, r, r_0, d, \lambda_0, b_0, \Lambda)$, such that for any function $w \in C^\infty(\bar{B}_{r_0}^+)$ obeying* (3.16) *we have*

$$\|D^\beta D_{y_d} w\|_{C(\bar{B}_r^+)} \leq C \|w\|_{C(\bar{B}_{r_0}^+)},$$

for all $\beta \in \mathbb{N}^d$ with $\beta_d = 0$ and $|\beta| \leq k$.

Proof. Since $\bar{L}_0 w = 0$ on $B_{r_0}^+$, we also have $\bar{L}_0 D^\beta w = 0$ on $B_{r_0}^+$, for all $\beta \in \mathbb{N}^d$ with $\beta_d = 0$. Lemma 3.6 then applies with r replaced by $r_2 = (r + r_0)/2$, and gives us

$$\|D^\beta D_{y_d} w\|_{C(\bar{B}_r^+)} \leq C_0 \|D^\beta w\|_{C(\bar{B}_{r_2}^+)},$$

where $C_0 = C_0(r, r_0, d, \lambda_0, b_0, \Lambda)$ is a positive constant. Next, we apply Lemma 3.5 to estimate $D^\beta w$ and give a constant $C_1 = C_1(k, r_2, r_0, d, \lambda_0, b_0, \Lambda)$ such that

$$\|D^\beta w\|_{C(\bar{B}_{r_2}^+)} \leq C_1 \|w\|_{C(\bar{B}_{r_0}^+)}.$$

Now combining the preceding two inequalities, we obtain the a priori local estimate for $D^\beta D_{y_d} w$. □

Lemma 3.8 (Local estimate for $w_{y_dy_d}$)**.** *Let $k \in \mathbb{N}$ and $0 < r < r_0$. Then there is a positive constant, $C = C(r, r_0, d, \lambda_0, b_0, \Lambda)$, such that for any function $w \in C^\infty(\bar{B}_{r_0}^+)$ obeying* (3.16) *we have*

$$\|w_{y_dy_d}\|_{C(\bar{B}_r^+)} \leq C \|w\|_{C(\bar{B}_{r_0}^+)}.$$

Proof. By taking another derivative with respect to y_d in the equation $\bar{L}_0 w = 0$ on $B_{r_0}^+$, we see that w_{y_d} is a solution to

$$y_d \sum_{i,j=1}^d \bar{a}^{ij} (w_{y_d})_{y_iy_j} + \sum_{i=1}^{d-1} \left(\bar{b}^i + 2\bar{a}^{id}\right) (w_{y_d})_{y_i} + \left(\bar{b}^d + \bar{a}^{dd}\right) (w_{y_d})_{y_d} = -\sum_{i,j=1}^{d-1} \bar{a}^{ij} w_{y_iy_j}.$$

Applying the method of proof of Lemma 3.6 with $\theta := \left(\bar{b}^d + \bar{a}^{dd}\right)/\bar{a}^{dd}$ and

$$f := -\sum_{i,j=1}^{d-1} \bar{a}^{ij} w_{y_iy_j} - y_d \sum_{i,j=1}^{d-1} \bar{a}^{ij} w_{y_dy_iy_j} - \sum_{i=1}^{d-1} \left(\bar{b}^i + 2\bar{a}^{id}\right) w_{y_dy_i},$$

we obtain

$$\|w_{y_dy_d}\|_{C(\bar{B}_r^+)} \leq C \sum_{\substack{\beta\in\mathbb{N}^d \\ \beta_d=0,1;\ |\beta|\leq 3}} \|D^\beta w\|_{C(\bar{B}_r^+)},$$

where $C = C(d, \lambda_0, b_0, \Lambda)$ is a positive constant. We can estimate the supremum norms of $D^\beta w$ on B_r^+ , for all $\beta \in \mathbb{N}^d$ with $\beta_d = 0, 1$, in terms of the supremum norm of w on $B_{r_0}^+$ with the aid of Lemmas 3.5 and 3.7. Now, the supremum estimate for $w_{y_dy_d}$ on B_r^+ follows immediately. □

From the definition (3.15) of w, using the fact that $\xi_d = 0$, we have

$$\begin{aligned}
u_{x_d}(x) &= \sum_{k=1}^{d-1} \xi_k w_{y_k}(y) + w_{y_d}(y),\\
u_{x_i}(x) &= u_{y_i}(y), \quad \forall i \neq d,\\
u_{x_dx_d}(x) &= \sum_{k,l=1}^{d-1} \xi_k\xi_l w_{y_ky_l}(y) + 2\sum_{k=1}^{d-1} \xi_k w_{y_ky_d}(y) + w_{y_dy_d}(y),\\
u_{x_ix_d}(x) &= \sum_{k=1}^{d-1} \xi_k w_{y_iy_k}(y) + w_{y_iy_d}(y), \quad \forall i \neq d,\\
u_{x_ix_j}(x) &= w_{y_iy_j}(y), \quad \forall i, j \neq d,
\end{aligned} \tag{3.17}$$

for $x \in \mathbb{H}$. Using the preceding identities together with the estimates of Lemmas 3.6, 3.7 and 3.8, we obtain

Lemma 3.9 (Local estimates for second-order derivatives of u)**.** *Let $r_0 > 0$. Then there are positive constants, $r_1 = r_1(r_0, \lambda_0, b_0, \Lambda) < r_0$ and $C = C(r_0, d, \lambda_0, b_0, \Lambda)$, such that for all $u \in C^\infty(\bar{B}_{r_0}^+)$ obeying* (3.12)*, we have*

$$\|D^\beta u\|_{C(\bar{B}_{r_1}^+)} \leq C\|u\|_{C(\bar{B}_{r_0}^+)},$$

for all $\beta \in \mathbb{N}^d$ with $|\beta| \leq 2$.

Proof. Let $\phi : \mathbb{H} \to \mathbb{H}$ be the affine transformation $y = \phi(x) = x + \xi x_d$, $x \in \mathbb{H}$, where $\xi \in \mathbb{R}^d$ is defined by (3.14). Let $s_0 = s_0(r_0, \lambda_0, \Lambda) > 0$ be small enough such that $B_{s_0}^+ \subset \phi(B_{r_0}^+)$. Let $s_1 = s_1(s_0, b_0, \Lambda) < s_0$ denote the constant r_1 given by Lemma 3.5 applied with r_0 replaced by s_0. Then, the function w defined by (3.15) has the property that $w \in C^\infty(\bar{B}_{s_0}^+)$ and $\widetilde{L}_0 w = 0$ on $B_{s_0}^+$, since $u \in C^\infty(\bar{B}_{r_0}^+)$ and $L_0 u = 0$ on $B_{r_0}^+$. We apply Lemma 3.6, if $\beta = e_d$, Lemma 3.7, if $\beta = e_i + e_d$ and $i \neq d$, and Lemma 3.8, if $\beta = 2e_d$, to the function w with r replaced by s_1 and r_0 replaced by s_0. We apply Lemma 3.5, if $\beta = e_i$ or $\beta = e_i + e_j$, for all $i, j \neq d$, to the function w with r_1 replaced by s_1 and r_0 replaced by s_0. Then, for any $k \in \mathbb{N}$, there is a positive constant, $C = C(k, r_0, d, \lambda_0, b_0, \Lambda)$, such that for all $\beta \in \mathbb{N}^d$ with $|\beta| \leq 2$, we have

$$\|D^\beta w\|_{C(\bar{B}_{s_1}^+)} \leq C\|w\|_{C(\bar{B}_{s_0}^+)}. \tag{3.18}$$

We now choose $r_1 = r_1(r_0, \lambda_0, b_0, \Lambda)$ small enough such that $\phi(B^+_{r_1}) \subset B^+_{s_1}$. Using (3.17), we obtain

$$\begin{aligned}\|D^\beta u\|_{C(\bar{B}^+_{r_1})} &\leq \|D^\beta w\|_{C(\bar{B}^+_{s_1})} \quad \text{(by the facts that } \phi(B^+_{r_1}) \subset B^+_{s_1} \text{ and } u(x) = w(\phi(x)))\\ &\leq C\|w\|_{C(\bar{B}^+_{s_0})} \quad \text{(by (3.18))}\\ &\leq C\|u\|_{C(\bar{B}^+_{r_0})} \quad \text{(by the facts that } B^+_{s_0} \subset \phi(B^+_{r_0}) \text{ and } u(x) = w(\phi(x))).\end{aligned}$$

This conclude the proof. □

4. Polynomial approximation and Taylor remainder estimates

We adapt and slightly streamline the arguments of Daskalopoulos and Hamilton in [6, §I.6 & I.7] for their model degenerate-parabolic operators acting on functions $u(t, x)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$, to the case of our degenerate-elliptic operators acting on functions $u(x)$, $x \in \mathbb{R}^d$. The goal of this section is to derive an estimate of the remainder of the first-order Taylor polynomial of a function u on half-balls centered at points in $\partial\mathbb{H}$ (Corollary 4.7). This result, when combined with the interior Schauder estimates of section §5, will lead to the full Schauder estimate for a solution on a half-ball centered at point in $\partial\mathbb{H}$ (Theorem 3.2). *Throughout this section, we continue to assume Hypothesis 3.1 and so the coefficients, a, b, c, of the operator L in* (1.3) *and the coefficients, a, b, of the operator L_0 in* (1.4) *are constant.*

We let $T^P_k v$ denote the *Taylor polynomial of degree* k of a smooth function v, centered at a point $P \in \mathbb{R}^d$, and let $R^P_k := v - T^P_k$ denote the *remainder*. We then have the following analogue of [6, Theorem I.6.1].

Proposition 4.1 (Polynomial approximation)**.** *There is a positive constant $C = C(d, \lambda_0, b_0, \Lambda)$, such that for any $r_0 > 0$, and any function $u \in C^\infty(\bar{B}^+_{r_0})$, there is a polynomial p of degree* 1*, such that for any $r \in (0, r_0)$ we have*

$$\|u - p\|_{C(\bar{B}^+_r)} \leq C\left(\frac{r^2}{r_0^2}\|u\|_{C(\bar{B}^+_{r_0})} + r_0\|L_0 u\|_{C(\bar{B}^+_{r_0})}\right). \tag{4.1}$$

Proof. We first consider the case when $r_0 = 1$ and then when $r_0 > 0$ is arbitrary.

Step 1 ($r_0 = 1$)**.** We let $f := L_0 u$ and we choose a smooth, non-negative, cutoff function, ψ, such that

$$\psi \restriction_{B^+_{1/2}} \equiv 1 \quad \text{and} \quad \psi \restriction_{\mathbb{H}\setminus B^+_1} \equiv 0.$$

We fix some $\nu > 1$, and let $S = \mathbb{R}^{d-1} \times (0, \nu)$ as in (1.10). By Theorem B.3, there is a unique solution, $u_1 \in C^\infty(\bar{S})$, to

$$\begin{cases} L_0 u_1 = \psi f & \text{on } S,\\ u_1(\cdot, \nu) = 0 & \text{on } \mathbb{R}^{d-1}.\end{cases}$$

Then, by setting $u_2 := u - u_1$, we see that $u_2 \in C^\infty(\bar{B}^+_{r_0})$ and satisfies $L_0 u_2 = (1 - \psi)f$ on $B^+_{r_0}$. Notice that the definition of the functions u_1 and u_2 differs from that of their analogues, h and $f - h$, in the proof of [6, Theorem I.6.1]. The reason for this change is that the zeroth-order coefficient in the definition of L_0 is zero, and so uniqueness of $C^\infty(\bar{\mathbb{H}})$ solutions to the equation $L_0 u = f$ on $\mathbb{H}$ does not hold since we may add any constant to a solution, u. Since $u = u_1 + u_2$, we have

$$\|u - T^0_1 u_2\|_{C(\bar{B}^+_r)} \leq \|u_2 - T^0_1 u_2\|_{C(\bar{B}^+_r)} + \|u_1\|_{C(\bar{B}^+_r)}. \tag{4.2}$$

By the Mean Value Theorem, we know that

$$\|u_2 - T^0_1 u_2\|_{C(\bar{B}^+_r)} \leq Cr^2 \|D^2 u_2\|_{C(\bar{B}^+_r)},$$

where $C = C(d)$. Because $L_0 u_2 = 0$ on $B^+_{1/2}$, we may apply Lemma 3.9 to u_2 with $r = 1/2$. Then there are constants, $r_1 = r_1(d)$ and $C = C(d, \lambda_0, b_0, \Lambda)$, such that for any $r \in (0, r_1)$ we have

$$\|D^2 u_2\|_{C(\bar{B}^+_r)} \leq C\|u_2\|_{C(\bar{B}^+_{1/2})},$$

from where it follows that

$$\|u_2 - T^0_1 u_2\|_{C(\bar{B}^+_r)} \leq Cr^2 \|u_2\|_{C(\bar{B}^+_1)}.$$

Corollary A.2 gives the estimate

$$\|u_1\|_{C(\mathbb{R}^{d-1}\times(0,\nu))} \leq C\|\psi f\|_{C(\mathbb{R}^{d-1}\times(0,\nu))} \leq \|f\|_{C(\bar{B}^+_1)}, \tag{4.3}$$

where the second inequality follows because the support of ψ is contained in $\bar{B}^+_1$. Since $u = u_1 + u_2$, we have

$$\|u_2\|_{C(\bar{B}^+_1)} \leq \|u\|_{C(\bar{B}^+_1)} + \|u_1\|_{C(\bar{B}^+_1)},$$

and so, combining the preceding two inequalities,

$$\|u_2\|_{C(\bar{B}^+_1)} \leq \|u\|_{C(\bar{B}^+_1)} + \|f\|_{C(\bar{B}^+_1)}.$$

Thus, we have proved that

$$\|u_2 - T^0_1 u_2\|_{C(\bar{B}^+_r)} \leq Cr^2 \|u\|_{C(\bar{B}^+_1)} + C\|f\|_{C(\bar{B}^+_1)}, \quad \forall r \in (0, r_1).$$

When $r \in [r_1, 1)$, we have, for all $x \in \bar{B}^+_r$,

$$\begin{aligned} |u_2(x) - T^0_1 u_2(x)| &\leq d|Du_2(0)|r + |u_2(x)| + |u_2(0)| \\ &\leq Cr^2 \|u\|_{C(\bar{B}^+_1)} \quad \text{(by Lemma 3.9 and the fact that } r_1 \leq r), \end{aligned}$$

where $C = C(d)$ is a positive constant. Combining the cases $0 < r < r_1$ and $r_1 \leq r < 1$, we obtain

$$\|u_2 - T^0_1 u_2\|_{C(\bar{B}^+_r)} \leq Cr^2 \|u\|_{C(\bar{B}^+_1)} + C\|f\|_{C(\bar{B}^+_1)}, \quad \forall r \in (0, 1),$$

for a constant $C = C(d, \lambda_0, b_0, \Lambda)$. The preceding estimate together with the identity $u = u_1 + u_2$ and (4.3) show that

$$\|u - T^0_1 u_2\|_{C(\bar{B}^+_r)} \leq C\left(r^2 \|u\|_{C(\bar{B}^+_s)} + \|L_0 u\|_{C(\bar{B}^+_s)}\right),$$

and so, the conclusion (4.1) follows with $p = T^0_1 u_2$, in the special case when $r_0 = 1$.

Step 2 (Arbitrary $r_0 > 0$)**.** When $r_0 > 0$ is arbitrary, we use rescaling. We let $\tilde{u}(x) := u(r_0 x)$, for all $x \in B^+_1$, and we see that $(L_0\tilde{u})(x) = r_0 (L_0 u)(r_0 x)$. Notice that the rescaling property $(L_0\tilde{u})(x) = r_0 (L_0 u)(r_0 x)$ does not hold in this form if the zeroth-order coefficient of L_0 is non-zero.

We apply the preceding step to $\tilde{u}$ with r replaced by r/r_0. Then, there is a polynomial $\tilde{p}$ such that

$$\|\tilde{u} - \tilde{p}\|_{C(\bar{B}^+_{r/r_0})} \leq C\left(\frac{r^2}{r_0^2}\|\tilde{u}\|_{C(\bar{B}^+_1)} + \|L_0\tilde{u}\|_{C(\bar{B}^+_1)}\right),$$

which is equivalent to

$$\|u - p\|_{C(\bar{B}^+_r)} \leq C\left(\frac{r^2}{r_0^2}\|u\|_{C(\bar{B}^+_{r_0})} + r_0\|L_0 u\|_{C(\bar{B}^+_{r_0})}\right),$$

where we set $p(x) := \tilde{p}(x/r_0)$. We notice that the polynomial p depends on r_0, but not on r.

The proof of Proposition 4.1 is now complete. □

Proposition 4.1 is used to obtain the following analogue of [6, Theorem I.7.1].

Proposition 4.2. *For any $\alpha \in (0,1)$, there is a positive constant $S = S(d, \lambda_0, b_0, \Lambda)$, such that for any $u \in C^\infty(\bar{B}_1^+)$ with $T_1^0 u = 0$, we have*

$$\sup_{0<r\leq 1} \frac{\|u\|_{C(\bar{B}_r^+)}}{r^{1+\alpha}} \leq S\left(\|u\|_{C(\bar{B}_1^+)} + \sup_{0<r\leq 1} \frac{\|L_0 u\|_{C(\bar{B}_r^+)}}{r^\alpha}\right). \tag{4.4}$$

Proof. Because $T_1^0 u = 0$ and $u \in C^\infty(\bar{B}_1^+)$, it follows that the quantity on the left-hand side of the inequality (4.4) is finite. In addition, $T_1^0 u = 0$ implies that $L_0 u(0) = 0$, and so we also have

$$\sup_{0<r\leq 1} \frac{\|L_0 u\|_{C(\bar{B}_r^+)}}{r^\alpha} < \infty.$$

Let $r_* \in (0,1]$ be such that

$$\sup_{0<r\leq 1} \frac{\|u\|_{C(\bar{B}_r^+)}}{r^{1+\alpha}} = \frac{\|u\|_{C(\bar{B}_{r_*}^+)}}{r_*^{1+\alpha}},$$

and we define for simplicity

$$Q := \|u\|_{C(\bar{B}_1^+)} + \sup_{0<r\leq 1} \frac{\|L_0 u\|_{C(\bar{B}_r^+)}}{r^\alpha}. \tag{4.5}$$

We let S (depending on u) be such that

$$\frac{\|u\|_{C(\bar{B}_{r_*}^+)}}{r_*^{1+\alpha}} = SQ. \tag{4.6}$$

It is sufficient to find an upper bound on S, independent of u, to give the conclusion (4.4).

Let q and s be positive constants such that $0 < q < r_* < s \leq 1$. We apply Proposition 4.1 to u with r replaced by q and r_* and r_0 replaced by s. Then, we can find a degree-one polynomial, p, such that

$$\|u - p\|_{C(\bar{B}_q^+)} \leq C\left(\frac{q^2}{s^2}\|u\|_{C(\bar{B}_s^+)} + s\|L_0 u\|_{C(\bar{B}_s^+)}\right), \tag{4.7}$$

$$\|u - p\|_{C(\bar{B}_{r_*}^+)} \leq C\left[\frac{r_*^2}{s^2}\|u\|_{C(\bar{B}_s^+)} + s\|L_0 u\|_{C(\bar{B}_s^+)}\right]. \tag{4.8}$$

But

$$\|p\|_{C(\bar{B}_{r_*}^+)} \leq C\frac{r_*}{q}\|p\|_{C(\bar{B}_q^+)},$$

for some positive constant C, depending only on d. We can then estimate

$$\|p\|_{C(\bar{B}_{r_*}^+)} \leq C\frac{r_*}{q}\left(\|u - p\|_{C(\bar{B}_q^+)} + \|u\|_{C(\bar{B}_q^+)}\right),$$

and using (4.7) and the fact that $q < r_*$, we obtain

$$\|p\|_{C(\bar{B}_{r_*}^+)} \leq C\frac{r_*}{q}\left[\frac{r_*^2}{s^2}\|u\|_{C(\bar{B}_s^+)} + s\|L_0 u\|_{C(\bar{B}_s^+)} + \|u\|_{C(\bar{B}_q^+)}\right]. \tag{4.9}$$

From

$$\|u\|_{C(\bar{B}_{r_*}^+)} \leq \|u - p\|_{C(\bar{B}_{r_*}^+)} + \|p\|_{C(\bar{B}_{r_*}^+)},$$

and (4.8) and (4.9), we see that

$$\begin{aligned}\|u\|_{C(\bar{B}^+_{r_*})} &\leq C\left[\frac{r_*^2}{s^2}\|u\|_{C(\bar{B}^+_s)} + s\|L_0u\|_{C(\bar{B}^+_s)} + \frac{r_*^3}{qs^2}\|u\|_{C(\bar{B}^+_s)} + \frac{r_*s}{q}\|L_0u\|_{C(\bar{B}^+_s)} + \frac{r_*}{q}\|u\|_{C(\bar{B}^+_q)}\right]\\ &\leq C\left[\frac{r_*^2}{s^2}\|u\|_{C(\bar{B}^+_s)} + \frac{r_*}{q}\|u\|_{C(\bar{B}^+_q)} + \frac{r_*s}{q}\|L_0u\|_{C(\bar{B}^+_s)}\right],\end{aligned}$$

where we have used the fact that $q < r_* < s$ to obtain the last inequality. We divide by $r_*^{1+\alpha}$ and find that

$$\frac{\|u\|_{C(\bar{B}^+_{r_*})}}{r_*^{1+\alpha}} \leq C\left[\left(\frac{r_*}{s}\right)^{1-\alpha}\frac{\|u\|_{C(\bar{B}^+_s)}}{s^{1+\alpha}} + \left(\frac{q}{r_*}\right)^{\alpha}\frac{\|u\|_{C(\bar{B}^+_q)}}{q^{1+\alpha}} + \frac{s}{q}\left(\frac{s}{r_*}\right)^{\alpha}\frac{\|L_0u\|_{C(\bar{B}^+_s)}}{s^{\alpha}}.\right]$$

From the preceding inequality and definitions (4.5) of Q and (4.6) of S, we deduce that

$$SQ \leq C\left[\left(\frac{r_*}{s}\right)^{1-\alpha} + \left(\frac{q}{r_*}\right)^{\alpha}\right]SQ + C\frac{s}{q}\left(\frac{s}{r_*}\right)^{\alpha}Q,$$

By choosing r_*/s and q/r_* small enough, we obtain a bound on S depending only on $C = C(d,\lambda_0,b_0,\Lambda)$. Hence, the estimate (4.4) now follows. □

We apply Proposition 4.2 to $R^0_1u := u - T^0_1u$. Note that $L_0T^0_1u = (L_0u)(0)$ and so

$$L_0\left(u - T^0_1u\right) = L_0u - (L_0u)(0) = R^0_0L_0u,$$

because $x_dD^2u = 0$ on $\partial\mathbb{H}$ and the zeroth-order coefficient of L_0 is zero. Thus, Proposition 4.2 yields the following analogue of [6, Corollary I.7.2].

Corollary 4.3. *For any $\alpha \in (0,1)$, there is a positive constant $S = S(d,\lambda_0,b_0,\Lambda)$, such that for any $u \in C^\infty(\bar{B}^+_1)$ we have*

$$\sup_{0<r\leq 1}\frac{\|R^0_1u\|_{C(\bar{B}^+_r)}}{r^{1+\alpha}} \leq S\left(\|R^0_1u\|_{C(\bar{B}^+_1)} + \sup_{0<r\leq 1}\frac{\|R^0_0L_0u\|_{C(\bar{B}^+_r)}}{r^{\alpha}}\right).$$

Using the inequality (2.3),

$$|x| \leq 2s(x,0)^2, \quad \forall x \in \mathbb{H},$$

where we recall that the cycloidal distance function, $s(x^1,x^2)$ for all $x^1,x^2 \in \bar{\mathbb{H}}$, is given by (2.1), we see that there is a positive constant, $C = C(\alpha,d)$, such that

$$\begin{aligned}\sup_{0<r\leq 1}\frac{\|R^0_0L_0u\|_{C(\bar{B}^+_r)}}{r^{\alpha}} &\leq C\sup_{0<r\leq 1}\left\|\frac{L_0u(x) - L_0u(0)}{s^{2\alpha}(x,0)}\right\|_{C(\bar{B}^+_r)}\\ &\leq C\left[L_0u\right]_{C^{2\alpha}_s(\bar{B}^+_1)}.\end{aligned}$$

Therefore, Corollary 4.3 gives us the following partial analogue of [6, Corollary I.7.5].

Corollary 4.4. *For any $\alpha \in (0,1)$, there is a positive constant $C = C(\alpha,d,\lambda_0,b_0,\Lambda)$, such that for any $u \in C^\infty(\bar{B}^+_1)$ and $0 < r \leq 1$, we have*

$$\|R^0_1u\|_{C(\bar{B}^+_r)} \leq Cr^{1+\alpha/2}\left(\|R^0_1u\|_{C(\bar{B}^+_1)} + [L_0u]_{C^{\alpha}_s(\bar{B}^+_1)}\right).$$

Next, we improve the estimate in Corollary 4.4 with the following analogue of [6, Theorem I.7.3 & I.7.6].

Proposition 4.5. *For any $\alpha \in (0,1)$, there is a positive constant $C = C(\alpha, d, \lambda_0, b_0, \Lambda)$, such that for any $u \in C^\infty(\bar{B}_1^+)$ we have*

$$\|T_1^0 u\|_{C(\bar{B}_1^+)} \leq C\left(\|u\|_{C(\bar{B}_1^+)} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+)}\right).$$

Proof. Because T_1^0 is a degree-one polynomial, there is a positive constant $C = C(d)$ such that

$$\|T_1^0 u\|_{C(\bar{B}_1^+)} \leq \frac{C}{r}\|T_1^0 u\|_{C(\bar{B}_r^+)}, \quad \forall r \in (0,1].$$

By Corollary 4.4, we have for all $r \in (0,1]$,

$$\|R_1^0 u\|_{C(\bar{B}_r^+)} \leq C r^{1+\alpha/2}\left(\|u\|_{C(\bar{B}_1^+)} + \|T_1^0 u\|_{C(\bar{B}_1^+)} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+)}\right).$$

By combining the preceding two inequalities, we find that

$$\begin{aligned}\|T_1^0 u\|_{C(\bar{B}_1^+)} &\leq \frac{C}{r}\left(\|u\|_{C(\bar{B}_1^+)} + \|R_1^0 u\|_{C(\bar{B}_1^+)}\right)\\ &\leq \left(\frac{C}{r} + Cr^{\alpha/2}\right)\|u\|_{C(\bar{B}_1^+)} + Cr^{\alpha/2}\|T_1^0 u\|_{C(\bar{B}_1^+)} + Cr^{\alpha/2}\,[L_0 u]_{C_s^\alpha(\bar{B}_1^+)}.\end{aligned}$$

By choosing r small enough so that $Cr^{\alpha/2} \leq 1/2$, we obtain the conclusion. □

Proposition 4.5 implies the following special case ($r = 1$) of [6, Corollary I.7.8].

Corollary 4.6. *For any $\alpha \in (0,1)$, there is a positive constant $C = C(\alpha, d, \lambda_0, b_0, \Lambda)$, such that for any $u \in C^\infty(\bar{B}_1^+)$ we have*

$$\|R_1^0 u\|_{C(\bar{B}_1^+)} \leq C\left(\|u\|_{C(\bar{B}_1^+)} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+)}\right).$$

Corollaries 4.4 and 4.6 yield the following analogue of [6, Corollary I.7.8].

Corollary 4.7. *For any $\alpha \in (0,1)$, there is a positive constant $C = C(\alpha, d, \lambda_0, b_0, \Lambda)$, such that for any $u \in C^\infty(\bar{B}_1^+)$ and $0 < r \leq 1$, we have*

$$\|R_1^0 u\|_{C(\bar{B}_r^+)} \leq C r^{1+\alpha/2}\left(\|u\|_{C(\bar{B}_1^+)} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+)}\right).$$

5. Schauder estimates away from the degenerate boundary

In this section, we use a scaling argument to obtain elliptic Schauder estimates away from the degenerate boundary analogous to the parabolic versions of those estimates in [6, §I.8]. Our argument is shorter because we only aim to obtain the estimates in Lemma 5.1 and Corollary 5.3. Even though these estimates are weaker than their analogues [6, Corollary I.8.7] and [6, Corollary I.8.8], respectively, they are sufficient to obtain the full Schauder estimate (3.4) in Theorem 3.2. The estimate (3.4) is proved using a combination of the Schauder estimate on balls $B_r(x_0) \Subset \mathbb{H}$ which we prove in this section, and the results of §6. The proof of Proposition 6.1 uses Corollary 5.3, which is derived from Lemma 5.1. We have encountered a similar situation in the proof of Hölder continuity along $\partial\mathbb{H}$ of a weak solution to the Heston elliptic equation in [17, Theorem 1.11]. *Throughout this section, we continue to assume Hypothesis 3.1 and so the coefficients, a, b, c, of the operator L in* (1.3) *and the coefficients, a, b, of the operator L_0 in* (1.4) *are constant.*

For any $r > 0$, we let Q_r denote the point $re_d \in \mathbb{H}$. We have the following analogue of [6, Corollary I.8.7].

Lemma 5.1. *For any $\alpha \in (0,1)$ and positive constants μ and λ such that $0<\mu<\lambda<1$, there is a positive constant $C=C(\alpha,\mu,\lambda,d,\lambda_0,\Lambda)$, such that the following holds. For any function $u \in C^\infty(\bar{B}_{\lambda r}(Q_r))$, we have*

$$\begin{aligned}[Du]_{C^\alpha_s(\bar{B}_{\mu r}(Q_r))} + \left[x_d D^2 u\right]_{C^\alpha_s(\bar{B}_{\mu r}(Q_r))} \leq C \Bigg[& \frac{1}{r^{1+\alpha/2}} \|u\|_{C(\bar{B}_{\lambda r}(Q_r))} \\ & + \frac{1}{r^{\alpha/2}} \|L_0 u\|_{C(\bar{B}_{\lambda r}(Q_r))} + [L_0 u]_{C^\alpha_s(\bar{B}_{\lambda r}(Q_r))} \Bigg].\end{aligned} \tag{5.1}$$

Remark 5.2. The estimate in [6, Corollary I.8.7] does not contain the term $\|L_0 u\|_{C(\bar{B}_{\lambda r}(Q_r))}$ appearing on the right-hand side of our interior estimate (5.1). However, our estimate is sufficient to give the Schauder estimate (3.4) in our Theorem 3.2.

Proof. The result follows by rescaling. We denote $x=(ry', r+ry_d) \in \mathbb{H}$, where we recall that we denote $y=(y',y_d) \in \mathbb{H} = \mathbb{R}^{d-1} \times \mathbb{R}_+$, and define

$$v(y) = u(x), \quad \forall y \in B_\lambda.$$

By the hypothesis $u \in C^\infty(\bar{B}_{\lambda r}(Q_r))$, it follows that $v \in C^\infty(B_\lambda)$ and v is a solution to the strictly elliptic equation

$$\frac{1+y_d}{2} a^{ij} v_{y_i y_j}(y) + b^i v_{y_i}(y) = r\tilde{f}(y), \quad \forall y=(y',y_d) \in B_\lambda,$$

where $\tilde{f}(y) := f(ry', r+ry_d)$, for all $y \in B_\lambda$, and $f := L_0 u$. By the interior Schauder estimates [30, Theorem 7.1.1], there is a constant $C=C(\alpha,\mu,\lambda,d,\lambda_0,\Lambda)$, such that

$$\|D^2 v\|_{C^\alpha(\bar{B}_\mu)} \leq C \left(\|v\|_{C(B_\lambda)} + r\|\tilde{f}\|_{C^\alpha(\bar{B}_\lambda)} \right). \tag{5.2}$$

By direct calculation, we obtain

$$\begin{aligned} \|v\|_{C(\bar{B}_\lambda)} &= \|u\|_{C(\bar{B}_{\lambda r}(Q_r))}, \\ \|\tilde{f}\|_{C(\bar{B}_\lambda)} &= \|f\|_{C(\bar{B}_{\lambda r}(Q_r))}, \\ [\tilde{f}]_{C^\alpha(\bar{B}_\lambda)} &\leq C r^{\alpha/2} [f]_{C^\alpha_s(\bar{B}_{\lambda r}(Q_r))}, \end{aligned} \tag{5.3}$$

where $C=C(\alpha)$. To see the last inequality, recall that $x=(ry', r+ry_d)$, for all $(y',y_d) \in B_\lambda$. For any $y^i \in B_\lambda$, for $i=1,2$, we have

$$\frac{|\tilde{f}(y^1) - \tilde{f}(y^2)|}{|y^1-y^2|^\alpha} = \frac{|f(x^1)-f(x^2)|}{s(x^1,x^2)^\alpha} \frac{s(x^1,x^2)^\alpha}{|y^1-y^2|^\alpha}.$$

By (2.1), we notice that

$$\frac{s(x^1,x^2)}{|y^1-y^2|} = \frac{r|y^1-y^2|}{\sqrt{r(2+y^1_d+y^2_d+|y^1-y^2|)}} \frac{1}{|y^1-y^2|} \leq \sqrt{\frac{r}{2}},$$

and so, by letting $C=2^{-\alpha/2}$, we obtain

$$[\tilde{f}]_{C^\alpha(\bar{B}_\lambda)} \leq C r^{\alpha/2} [f]_{C^\alpha_s(\bar{B}_{\lambda r}(Q_r))}.$$

We also have

$$\left[x_d D^2 u\right]_{C^\alpha_s(\bar{B}_{\mu r}(Q_r))} \leq C r^{-(1+\alpha/2)} \|D^2 v\|_{C^\alpha(\bar{B}_\mu)}, \tag{5.4}$$

for a constant $C = C(\alpha, d)$. To establish (5.4), we only need to consider quotients of the form

$$\frac{|x_d^1 D^2u(x^1) - x_d^2 D^2u(x^2)|}{s(x^1,x^2)^\alpha},$$

where $x^1, x^2 \in B_{\mu r}$, and all their coordinates coincide, except for the i-th one, where $i = 1, \ldots, d$. We only consider the case when $i = d$, as all the other cases, $i = 1, \ldots, d-1$, follow in the same way. Recall that we denote $x = (ry', r + ry_d)$, for all $y \in B_\mu$. We obtain

$$\frac{|x_d^1 D^2u(x^1) - x_d^2 D^2u(x^2)|}{s(x^1,x^2)^\alpha} \leq \frac{|x_d^1 - x_d^2|}{s(x^1,x^2)^\alpha}|D^2u(x^1)| + x_d^2 \frac{|D^2u(x^1) - D^2u(x^2)|}{s(x^1,x^2)^\alpha}.$$

Using the definition of the cycloidal metric (2.1), and the fact that $D^2u(x) = 1/r^2 D^2v(y)$, for all $x \in B_{\mu r}$, we see that

$$\begin{aligned}
\frac{|x_d^1 D^2u(x^1) - x_d^2 D^2u(x^2)|}{s(x^1,x^2)^\alpha} &\leq \left(2x_d^1 + 2x_d^2\right)^{\alpha/2} |x_d^1 - x_d^2|^{1-\alpha} \frac{1}{r^2}|D^2v(y^1)| \\
&\quad + x_d^2 \frac{1}{r^2} \frac{|D^2v(y^1) - D^2v(y^2)|}{|y^1 - y^2|^\alpha} \frac{|y^1 - y^2|^\alpha}{s(x^1,x^2)^\alpha} \\
&\leq 2^\alpha r^{-(1+\alpha/2)} \|D^2v\|_{C(\bar{B}_\mu)} \\
&\quad + r^{-1} \left[D^2v\right]_{C^\alpha(\bar{B}_\mu)} \frac{|y^1 - y^2|^\alpha}{s(x^1,x^2)^\alpha},
\end{aligned}$$

where we used the fact that $x_d^i \leq r$, for all $x^1, x^2 \in B_{\mu r}$. We also have by (2.1),

$$\frac{|y^1 - y^2|}{s(x^1,x^2)} = \frac{r^{-1}|x_d^1 - x_d^2|}{|x_d^1 - x_d^2|} \sqrt{x_d^1 + x_d^2 + |x_d^1 - x_d^2|} \leq Cr^{-1/2},$$

which implies that

$$\frac{|x_d^1 D^2u(x^1) - x_d^2 D^2u(x^2)|}{s(x^1,x^2)^\alpha} \leq Cr^{-(1+\alpha/2)} \|D^2v\|_{C_s^\alpha(\bar{B}_\mu)},$$

for a constant $C = C(\alpha)$. Now, the inequality (5.4) follows immediately.

Using the preceding estimates (5.3) and (5.4), it follows by (5.2) that

$$\left[x_d D^2u\right]_{C_s^\alpha(\bar{B}_{\mu r}(Q_r))} \leq C \left(r^{-(1+\alpha/2)} \|u\|_{C(\bar{B}_{\lambda r}(Q_r))} + r^{-\alpha/2} \|L_0 u\|_{C(\bar{B}_{\lambda r}(Q_r))} + [L_0 u]_{C_s^\alpha(\bar{B}_{\lambda r}(Q_r))}\right),$$

where we substituted $L_0 u$ for f.

To obtain the estimate for the Hölder seminorm of Du, we proceed by analogy with the argument for $x_d D^2u$. □

We have the following analogue of [6, Corollary I.8.8].

Corollary 5.3. *For any $\alpha \in (0,1)$ and positive constants μ and λ such that $0 < \mu < \lambda < 1$, there is a constant $C = C(\alpha, \mu, \lambda, d, \lambda_0, \Lambda)$, such that for any function $u \in C^\infty(B_r(Q_r))$ we have*

$$\|R_2^{Q_r} u\|_{C(\bar{B}_{\mu r}(Q_r))} \leq C \left(\|u\|_{C(\bar{B}_{\lambda r}(Q_r))} + r^{1+\alpha/2} \left[L_0 u\right]_{C_s^\alpha(\bar{B}_{\lambda r}(Q_r))} + r\|L_0 u\|_{C(\bar{B}_{\lambda r}(Q_r))}\right).$$

Proof. As in case of the inequality preceding [6, Corollary I.8.8], we have

$$\|R_2^{Q_r} u\|_{C(\bar{B}_{\mu r}(Q_r))} \leq Cr^{1+\alpha/2} \left[x_d D^2u\right]_{C_s^\alpha(\bar{B}_{\mu r}(Q_r))},$$

for a constant $C = C(d)$. Thus, the conclusion follows from Lemma 5.1 and the preceding inequality. □

6. Schauder estimates near the degenerate boundary

In this section, we use the results of the previous sections to prove our main a priori interior local Schauder estimate (Theorem 3.2) for the operator L on half-balls centered at points in the "degenerate boundary", $\partial\mathbb{H}$. *Throughout this section, we continue to assume Hypothesis 3.1, and so the coefficients, a, b, c, of the operator L in* (1.3) *and the coefficients, a, b, of the operator L_0 in* (1.4) *are constant.*

We begin with an analogue of [6, Theorem I.9.1].

Proposition 6.1. *For any $\alpha \in (0,1)$, there is a constant $C = C(\alpha, d, \lambda_0, b_0, \Lambda)$, such that the following holds. For any function $u \in C^\infty(\bar{B}_1^+)$ and any $0 < r \leq 1/2$, we have*

$$|D^2 u(Q_r)| \leq C r^{\alpha/2-1}\left(\|u\|_{C(\bar{B}_1^+)} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+)}\right).$$

Proof. We choose $\mu = 1/4$ and $\lambda = 1/2$ in Corollary 5.3. We consider the points $Q_r := re_d$ and $P := O \in \mathbb{R}^d$. Let $p := T_2^{Q_r} u - T_1^P u$, where we recall that $T_2^{Q_r} u$ is the second-degree Taylor polynomial of u at Q_r, and $T_1^P u$ is the first-degree Taylor polynomial of u at P. Then, we also have that $p := R_1^P u - R_2^{Q_r} u$, where we recall that $R_2^{Q_r} u$ is the remainder of the second-degree Taylor polynomial of u at Q_r, and $R_1^P u$ is the remainder of the first-degree Taylor polynomial of u at P. There is a positive constant, $C = C(d, \mu)$, such that

$$|D^2 p| \leq \frac{C}{r^2}\|p\|_{C(\bar{B}_{\mu r}(Q_r))},$$

which implies, from the definition of p, that

$$|D^2 u(Q_r)| \leq \frac{C}{r^2}\|R_1^P u - R_2^{Q_r} u\|_{C(\bar{B}_{\mu r}(Q_r))}. \tag{6.1}$$

Corollary 4.7 applied to $R_1^P u$ gives

$$\|R_1^P u\|_{C(\bar{B}_r^+)} \leq C r^{1+\alpha/2}\left(\|u\|_{C(\bar{B}_1^+)} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+)}\right), \tag{6.2}$$

and the interior Schauder estimate in Corollary 5.3 applied to $R_1^P u$ yields

$$\begin{aligned}\|R_2^{Q_r} R_1^P u\|_{C(\bar{B}_{\mu r}(Q_r))} \leq C \Big(&\|R_1^P u\|_{C(\bar{B}_{\lambda r}(Q_r))} \\ &+ r\|L_0 R_1^P u\|_{C(\bar{B}_{\lambda r}(Q_r))} + r^{1+\alpha/2}\left[L_0 R_1^P u\right]_{C_s^\alpha(\bar{B}_{\lambda r}(Q_r))}\Big),\end{aligned}$$

for a constant $C = C(\alpha, \mu, \lambda, d, \lambda_0, \Lambda)$. We notice that $L_0 R_1^P u = L_0 u - (L_0 u)(P)$, from where it follows that

$$\|L_0 R_1^P u\|_{C(\bar{B}_{\lambda r}(Q_r))} \leq C r^{\alpha/2}\,[L_0 u]_{C_s^\alpha(\bar{B}_1^+)}, \tag{6.3}$$

using the fact (2.2) that $s(x^1, x^2) \leq |x^1 - x^2|^{1/2}$, for all $x^1, x^2 \in \bar{\mathbb{H}}$, and also that

$$\left[L_0 R_1^P u\right]_{C_s^\alpha(\bar{B}_{\lambda r}(Q_r))} = [L_0 u]_{C_s^\alpha(\bar{B}_{\lambda r}(Q_r))}. \tag{6.4}$$

The preceding three inequalities, together with (6.2), give us

$$\|R_2^{Q_r} R_1^P u\|_{C(\bar{B}_{\mu r}(Q_r))} \leq C r^{1+\alpha/2}\left(\|u\|_{C(\bar{B}_1^+)} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+)}\right),$$

where we used the fact that $B_{\mu r}(Q_r) \subset B_1$, when $0 < r \leq 1$, for all $0 < \mu < 1$. Notice that $R_2^{Q_r} u = R_2^{Q_r} R_1^P u$, and so the preceding estimate becomes

$$\|R_2^{Q_r} u\|_{C(\bar{B}_{\mu r}(Q_r))} \leq C r^{1+\alpha/2}\left(\|u\|_{C(\bar{B}_1^+)} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+)}\right).$$

The conclusion now follows from the preceding estimate, and inequalities (6.2) and (6.1). ☐

Note that the definition (2.1) of the cycloidal metric gives

$$s((x',x_d),(x',0)) = \sqrt{x_d/2}, \quad \forall (x',x_d) \in \mathbb{H},$$

and hence, via Proposition 6.1, we obtain the following analogues of [6, Theorems I.9.3 & I.9.4].

Corollary 6.2. *For any $\alpha \in (0,1)$, there is a constant $C = C(\alpha, d, \lambda_0, b_0, \Lambda)$, such that for all $0 < x_d \leq 1/2$ and $x' \in \mathbb{R}^{d-1}$, and any function $u \in C^\infty(\bar{B}_1^+(x',0))$, we have*

$$|x_d D^2 u(x',x_d)| \leq Cs\left((x',x_d),(x',0)\right)^\alpha \left(\|u\|_{C(\bar{B}_1^+(x',0))} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+(x',0))}\right).$$

Corollary 6.3. *For any $\alpha \in (0,1)$, there is a constant $C = C(\alpha, d, \lambda_0, b_0, \Lambda)$, such that for all $0 < x_d \leq 1/2$ and $x' \in \mathbb{R}^{d-1}$, and any function $u \in C^\infty(\bar{B}_1^+(x',0))$, we have*

$$|Du(x',x_d) - Du(x',0)| \leq Cs\left((x',x_d),(x',0)\right)^\alpha \left(\|u\|_{C(\bar{B}_1^+(x',0))} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+(x',0))}\right).$$

Proof. Following the proof of [6, Theorem I.9.4], using Proposition 6.1 and translation-invariance with respect to $x' \in \mathbb{R}^{d-1}$ to obtain the second inequality, we have

$$\begin{aligned}
|Du(x',x_d) - Du(x',0)| &\leq \int_0^{x_d} |Du_{x_d}(x',t)|dt \\
&\leq C\left(\|u\|_{C(\bar{B}_1^+(x',0))} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+(x',0))}\right)\int_0^{x_d} t^{\alpha/2-1}dt \\
&= C\left(\|u\|_{C(\bar{B}_1^+(x',0))} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+(x',0))}\right) x_d^{\alpha/2}.
\end{aligned}$$

Using the fact that $s((x',x_d),(x',0)) = \sqrt{x_d/2}$, we obtain the conclusion. ☐

Next, we use Lemma 5.1 (for estimates away from $\partial\mathbb{H}$) and the Taylor remainder estimates in Corollary 4.7 (for estimates near $\partial\mathbb{H}$) to prove the following analogue of [6, Theorems I.9.5].

Proposition 6.4. *Let $\alpha \in (0,1)$, and $0 < r \leq 1/4$, and $0 < \mu < 1$. Then there is a constant, $C = C(\alpha, \mu, d, \lambda_0, \Lambda)$, such that for any function $u \in C^\infty(\bar{B}_1^+)$, we have*

$$[Du]_{C_s^\alpha(\bar{B}_{\mu r}(Q_r))} + \left[x_d D^2 u\right]_{C_s^\alpha(\bar{B}_{\mu r}(Q_r))} \leq C\left(\|u\|_{C(\bar{B}_1^+)} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+)}\right).$$

Proof. For simplicity, we denote $v := R_1^P u = u - T_1^P u$. We notice that

$$[Du]_{C_s^\alpha(\bar{B}_{\mu r}(Q_r))} = [Dv]_{C_s^\alpha(\bar{B}_{\mu r}(Q_r))} \quad \text{and} \quad [x_d D^2 u]_{C_s^\alpha(\bar{B}_{\mu r}(Q_r))} = [x_d D^2 v]_{C_s^\alpha(\bar{B}_{\mu r}(Q_r))},$$

and hence we only need to estimate $[Dv]_{C_s^\alpha(\bar{B}_{\mu r}(Q_r))}$ and $[x_d D^2 v]_{C_s^\alpha(\bar{B}_{\mu r}(Q_r))}$. The proof is similar to the proof of Proposition 6.1. The interior Schauder estimates in Lemma 5.1 applied to v with $\lambda = (1+\mu)/2$ yield

$$\begin{aligned}
[Dv]_{C_s^\alpha(\bar{B}_{\mu r}(Q_r))} + \left[x_d D^2 v\right]_{C_s^\alpha(\bar{B}_{\mu r}(Q_r))} \leq C\Bigg[&\frac{1}{r^{1+\alpha/2}}\|v\|_{C(\bar{B}_{\lambda r}(Q_r))} \\
&+ \frac{1}{r^{\alpha/2}}\|L_0 v\|_{C(\bar{B}_{\lambda r}(Q_r))} + [L_0 v]_{C_s^\alpha(\bar{B}_{\lambda r}(Q_r))}\Bigg],
\end{aligned}$$

for some constant $C = C(\alpha, \mu, d, \lambda_0, \Lambda)$. The conclusion now follows from the preceding estimate and inequalities (6.2) applied on $B_{\lambda r}(Q_r)$ instead of B_r^+ (notice that $B_{\lambda r}(Q_r) \subset B_{1/2}^+$, since $0 < r \leq 1/4$), together with (6.3) and (6.4). ☐

Next, we have the following analogue of [6, Theorems I.9.7 & I.9.8].

Proposition 6.5. *For $\alpha \in (0,1)$, there are constants $\gamma = \gamma(d) \in (0,1)$, and $C = C(\alpha, d, \lambda_0, b_0, \Lambda)$, such that for any function $u \in C^\infty(\bar{B}_1^+)$, we have*

$$[Du]_{C_s^\alpha(\bar{B}_\gamma^+)} + \left[x_d D^2 u\right]_{C_s^\alpha(\bar{B}_\gamma^+)} \leq C \left(\|u\|_{C(\bar{B}_1^+)} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+)}\right). \tag{6.5}$$

Proof. We combine the arguments of the proofs of [6, Theorems I.9.7 & I.9.8]. Let $x^i \in B_\gamma^+$, for $i = 1, 2$, where γ will be fixed below. We may assume without loss of generality that $x_d^1 \geq x_d^2$. We consider two cases.

Case 1 (x^1 and x^2 close together relative to $\operatorname{dist}(\cdot, \partial\mathbb{H})$)**.** If $|x_1 - x_1| \leq x_d^1/4$, then $x^2 \in B_{x_d^1/4}(x^1)$, and the estimate (6.5) follows if we assume $0 < \gamma \leq 1/2$ and apply Proposition 6.4 with $\mu = 1/4$ and $r = x_d^1$.

Case 2 (x^1 and x^2 farther apart relative to $\operatorname{dist}(\cdot, \partial\mathbb{H})$)**.** We next consider the case when

$$|x^1 - x^2| > x_d^1/4. \tag{6.6}$$

Writing $x = (\bar{x}, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$, we define the points,

$$x^3 := (\bar{x}^1, 0) \quad \text{and} \quad x^4 := (\bar{x}^2, 0),$$
$$x^5 := (\bar{x}^1, r) \quad \text{and} \quad x^6 := (\bar{x}^2, r),$$

where the positive constant r will be chosen below. Notice that when (6.6) holds, we have

$$s(x^1, x^2) \geq \frac{1}{8}\sqrt{x_d^1},$$

by the definition (2.1) of the cycloidal distance function. By the definition of the points x^i, for $i = 3, 4$, and the fact that $s((x', x_d), (x', 0)) = \sqrt{x_d/2}$, we see that

$$\begin{aligned} &s(x^1, x^2) \geq 8 s(x^1, x^3), \\ &s(x^1, x^2) \geq 8 s(x^2, x^4) \quad (\text{since } x_d^1 \geq x_d^2). \end{aligned} \tag{6.7}$$

Let v denote Du or $x_d D^2 u$, and consider the difference

$$\begin{aligned} v(x^1) - v(x^2) = &\left(v(x^1) - v(x^3)\right) + \left(v(x^3) - v(x^5)\right) + \left(v(x^5) - v(x^6)\right) \\ &+ \left(v(x^6) - v(x^4)\right) + \left(v(x^4) - v(x^2)\right). \end{aligned} \tag{6.8}$$

Using the distance inequalities (6.7), we find that

$$\begin{aligned} \frac{\left|v(x^1) - v(x^3)\right|}{s(x^1, x^2)^\alpha} &\leq 8^\alpha \frac{\left|v(x^1) - v(x^3)\right|}{s(x^1, x^3)^\alpha}, \\ \frac{\left|v(x^2) - v(x^4)\right|}{s(x^1, x^2)^\alpha} &\leq 8^\alpha \frac{\left|v(x^2) - v(x^4)\right|}{s(x^2, x^4)^\alpha}. \end{aligned}$$

By Corollary 6.2, if $v = x_d D^2 u$, and Corollary 6.3, if $v = Du$, we obtain

$$\frac{\left|v(x^1) - v(x^3)\right|}{s(x^1, x^2)^\alpha} + \frac{\left|v(x^2) - v(x^4)\right|}{s(x^1, x^2)^\alpha} \leq C \left(\|u\|_{C(\bar{B}_1^+)} + [L_0 u]_{C_s^\alpha(\bar{B}_1^+)}\right), \tag{6.9}$$

for a constant $C = C(\alpha, d, \lambda_0, b_0, \Lambda)$.

We now let $r := B s^2(x^1, x^2)$, where the constant B will be chosen below. Using the fact that $s(x^3, x^5) = \sqrt{r/2}$ and definition of x^i, for $i = 3, 5$, we obtain

$$\frac{\left|v(x^3) - v(x^5)\right|}{s(x^1, x^2)^\alpha} = (B/2)^{\alpha/2} \frac{\left|v(x^3) - v(x^5)\right|}{s(x^3, x^5)^\alpha}.$$

Because $x^i \in B^+_\gamma$, for $i = 1, 2$, and due the inequality (2.2), we can choose the constant $B := 1/(4\gamma)$ such that

$$r = Bs^2(x^1, x^2) \leq B|x^1 - x^2| \leq B\gamma \leq 1/4.$$

We apply Corollary 6.2, when $v = x_d D^2 u$, and Corollary 6.3, when $v = Du$, to obtain

$$\frac{\left|v(x^3) - v(x^5)\right|}{s(x^1, x^2)^\alpha} \leq C\,(B/2)^{\alpha/2} \left(\|u\|_{C(\bar{B}^+_1)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_1)}\right). \tag{6.10}$$

The inequality,

$$\frac{\left|v(x^4) - v(x^6)\right|}{s(x^1, x^2)^\alpha} \leq C\,(B/2)^{\alpha/2} \left(\|u\|_{C(\bar{B}^+_1)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_1)}\right), \tag{6.11}$$

follows by the same argument used to obtain the estimate (6.10).

Using (6.6) and the assumption $x^1_d \geq x^2_d$, we see that

$$|\bar{x}^1 - \bar{x}^2| \leq |x^1 - x^2| \leq \frac{3}{2} \frac{|x^1 - x^2|^2}{x^1_d + x^2_d + |x^1 - x^2|} = \frac{3}{2} s^2(x^1, x^2).$$

Recalling that $B = 1/(4\gamma)$ and $r = Bs^2(x^1, x^2)$, we have

$$|\bar{x}^1 - \bar{x}^2| \leq \frac{3}{2B} Bs^2(x^1, x^2) \leq 6\gamma r.$$

Next, we choose $\gamma = 1/24$, and so

$$|\bar{x}^1 - \bar{x}^2| \leq r/4.$$

for all $x^i = (x^i_1, \cdots, x^i_d) \in B^+_\gamma$, for $i = 1, 2$. Because $|\bar{x}^1 - \bar{x}^2| \leq r/4$, we may apply Proposition 6.4, with $\mu = 1/4$, to obtain

$$\frac{\left|v(x^5) - v(x^6)\right|}{s(x^5, x^6)^\alpha} \leq C \left(\|u\|_{C(\bar{B}^+_1)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_1)}\right).$$

Again using the definition $r := Bs^2(x^1, x^2)$, we notice that

$$s(x^5, x^6) \leq \frac{|\bar{x}^1 - \bar{x}^2|}{\sqrt{2r}} \leq \frac{\sqrt{3}}{4} s(x^1, x^2),$$

and so the preceding two inequalities yield

$$\frac{\left|v(x^5) - v(x^6)\right|}{s^\alpha(x^1, x^2)} \leq C \left(\|u\|_{C(\bar{B}^+_1)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_1)}\right). \tag{6.12}$$

Combining the estimates (6.9), (6.10), (6.11) and (6.12) gives us the estimate (6.5), when condition (6.6) holds.

The conclusion now follows from the two cases we considered. □

By analogy with [6, Corollary I.9.9], we have

Proposition 6.6. *For any $\alpha \in (0, 1)$, there are positive constants $\gamma = \gamma(d) \in (0, 1)$, and $C = C(\alpha, d, \lambda_0, b_0, \Lambda)$, such that the following holds. If $u \in C^\infty(\bar{B}^+_1)$, then*

$$\|u\|_{C^{2+\alpha}_s(\bar{B}^+_\gamma)} \leq C \left(\|u\|_{C(\bar{B}^+_1)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_1)}\right). \tag{6.13}$$

Proof. Let $\gamma = \gamma(d) \in (0,1)$ be as in Proposition 6.5. The bound on $x_d D^2u$ follows from Corollary 6.2. Proposition 6.5 gives us the estimate (6.13) for the $C^\alpha_s(\bar{B}^+_\gamma)$ Hölder seminorms of Du and $x_d D^2u$. We only need to show the bound on of Du, namely that there is a constant $C = C(\alpha, d, \lambda_0, b_0, \Lambda)$, such that

$$\|Du\|_{C(\bar{B}^+_\gamma)} \leq C \left(\|u\|_{C(\bar{B}^+_1)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_1)}\right). \tag{6.14}$$

We follow the argument of [6, p. 932]. Let $x^0 \in \bar{B}^+_\gamma$ be such that $|Du(x^0)| = \|Du\|_{C(\bar{B}^+_\gamma)}$. Then by Proposition 6.5 we have, for all $x \in \bar{B}^+_\gamma$,

$$|Du(x) - Du(x^0)| \leq C_0 \left(\|u\|_{C(\bar{B}^+_1)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_1)}\right),$$

for a constant $C_0 = C_0(\alpha, d, \lambda_0, b_0, \Lambda)$.

Let $N \geq 2$ be a positive integer such that

$$\|Du\|_{C(\bar{B}^+_\gamma)} \geq NC_0 \left(\|u\|_{C(\bar{B}^+_1)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_1)}\right).$$

Estimate (6.14) will follow if we can find an upper bound on N, independent of u. The preceding two inequalities give

$$|Du(x)| \geq (N-1)C_0 \left(\|u\|_{C(\bar{B}^+_1)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_1)}\right), \quad \forall x \in \bar{B}^+_\gamma,$$

and the Mean Value Theorem yields

$$|u(x) - u(x^0)| \geq |x - x^0|(N-1)C_0 \left(\|u\|_{C(\bar{B}^+_1)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_1)}\right), \quad \forall x, x^0 \in \bar{B}^+_\gamma.$$

Choosing $x \in B^+_\gamma$ such that $|x - x^0| \geq \gamma/2$, we obtain a contradiction with (6.5) if N is too large. Thus, (6.14) follows. □

We have the following corollary of Proposition 6.6:

Corollary 6.7. *For any $\alpha \in (0,1)$, there are positive constants, $\gamma = \gamma(d) \in (0,1)$ and $C = C(\alpha, d, \lambda_0, b_0, \Lambda)$, such that for any $r > 0$ the following holds. If $u \in C^\infty(\bar{B}^+_r)$, then*

$$\|u\|_{C^{2+\alpha}_s(\bar{B}^+_{\gamma r})} \leq Cr^{-(1+\alpha/2)} \left(\|u\|_{C(\bar{B}^+_r)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_r)}\right). \tag{6.15}$$

Proof. Let $\gamma = \gamma(d) \in (0,1)$ be as in Proposition 6.5. We set $v(x) := u(rx)$, for all $x \in B^+_1$. The estimates of Proposition 6.6 applied to v give us

$$\|u\|_{C^{2+\alpha}_s(\bar{B}^+_{\gamma r})} \leq C \left(\|u\|_{C(\bar{B}^+_r)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_r)}\right),$$

where $C = C(\alpha, r, d, \lambda_0, b_0, \Lambda)$. The dependency of the constant C on r follows as in the proof of Lemma 5.1, and so we obtain (6.15). □

We now generalize Corollary 6.7 to allow for any $\gamma \in (0,1)$ and make explicit the dependency of the constant C appearing in (6.15) on r and γ.

Corollary 6.8. *If $\alpha \in (0,1)$, then there are positive constants, $p = p(\alpha)$ and $C = C(\alpha, d, \lambda_0, b_0, \Lambda)$, such that, for any $r > 0$ and $\gamma \in (0,1)$, the following holds. If $u \in C^\infty(\bar{B}^+_r)$, then*

$$\|u\|_{C^{2+\alpha}_s(\bar{B}^+_{\gamma r})} \leq C((1-\gamma)r)^{-p} \left(\|u\|_{C(\bar{B}^+_r)} + [L_0 u]_{C^\alpha_s(\bar{B}^+_r)}\right). \tag{6.16}$$

Proof. Let $\gamma_d \in (0,1)$ be the constant appearing in the hypotheses of Corollary 6.7. We consider two cases, $0 < \gamma \leq \gamma_d$ and $\gamma > \gamma_d$, and clearly we only need to consider the second case. Our proof of (6.16) now follows standard covering argument. Let $t := (1-\gamma)r/2$ and divide the half-ball, $B^+_{\gamma r}$, into the two regions,

$$U_1 := B^+_{\gamma r} \cap \left(\mathbb{R}^{d-1} \times (0, \gamma_d t/2)\right) \quad \text{and} \quad U_2 := B^+_{\gamma r} \setminus U_1.$$

We cover U_1 by a finite number of half-balls, $B^+_{\gamma_d t}(x^n)$, centered at points $x^n \in \partial_0 B^+_r(x^0)$ and we apply the estimate (6.15) to obtain

$$\begin{aligned}\|u\|_{C^{2+\alpha}_s(\bar B^+_{\gamma_d t}(x^n))} &\leq C t^{-(1+\alpha/2)} \left(\|u\|_{C(\bar B^+_r)} + [L_0 u]_{C^\alpha_s(\bar B^+_r)}\right)\\ &\leq C((1-\gamma)r)^{-(1+\alpha/2)} \left(\|u\|_{C(\bar B^+_r)} + [L_0 u]_{C^\alpha_s(\bar B^+_r)}\right) \quad (\text{using } t = (1-\gamma)r/2)\,.\end{aligned}$$

In the region U_2, the operator L_0 is strictly elliptic (because $x_d \geq \gamma_d t/2 > 0$), and so we may apply [26, Corollary 6.3]. Using the elliptic analogue of the parabolic estimate [18, Proposition 3.13], there is a positive constant $p = p(\alpha)$ such that

$$\|u\|_{C^{2+\alpha}_s(\bar U_2)} \leq C((1-\gamma)r)^{-p} \left(\|u\|_{C(\bar B^+_r)} + [L_0 u]_{C^\alpha_s(\bar B^+_r)}\right).$$

Estimate (6.16) follows by combining the preceding two inequalities. □

Proof of Theorem 3.2. We combine the localization procedure in the proof of [30, Theorem 8.11.1] with Corollary 6.8. We divide the proof in two steps. Set $R := (r + r_0)/2$.

Step 1 (A priori estimate for $u \in C^\infty(\underline{B}^+_{r_0})$)**.** Consider the sequence of radii, $\{r_n\}_{n\geq 1} \subset [r, R)$, defined by $r_1 := r$ and

$$r_n := r + (R-r)\sum_{k=1}^{n-1} \frac{1}{2^n}, \quad \forall n \geq 2. \tag{6.17}$$

Denote $B_n := B^+_{r_n}(x^0)$, for all $n \geq 1$. Let $\{\varphi_n\}_{n\geq 1}$ be a sequence of $C^\infty_0(\bar{\mathbb{H}})$ cutoff functions such that, for all $n \geq 1$, we have $0 \leq \varphi_n \leq 1$ with $\varphi_n = 1$ on B_n and $\varphi_n = 0$ outside B_{n+1}. Let

$$\alpha_n := \|u\varphi_n\|_{C^{2+\alpha}_s(\bar B_n)}, \quad \forall n \geq 1. \tag{6.18}$$

By applying the estimate (6.16) to $u\varphi_n$ with $r = r_{n+1}$ and $\gamma = r_n/r_{n+1}$, we obtain

$$\begin{aligned}\alpha_n &\leq C(r_{n+1} - r_n)^{-p} \left(\|u\varphi_n\|_{C(\bar B_{n+1})} + [L_0 u]_{C^\alpha_s(\bar B_{n+1})}\right)\\ &\leq C(R-r)^{-p} 2^{(n-1)p} \left(\|u\varphi_n\|_{C(\bar B_{n+1})} + \|Lu\|_{C^\alpha_s(\bar B_{n+1})} + \|u\varphi_{n+1}\|_{C^\alpha_s(\bar B_{n+1})}\right),\end{aligned}$$

where the last inequality follows from the fact that $L = L_0 + c$ by (1.3) and (1.4) and employing (6.17). The interpolation inequalities (Lemma C.2) give, for any $\varepsilon > 0$,

$$\|u\varphi_{n+1}\|_{C^\alpha_s(\bar B_{n+1})} \leq \varepsilon \alpha_{n+1} + C\varepsilon^{-m} \|u\varphi_{n+1}\|_{C(B_{n+1})} \quad (\text{by } (6.18)),$$

where $C = C(d, \alpha, R)$ and $m = m(d, \alpha)$ are positive constants independent of ε. Choosing $\varepsilon := \delta C^{-1} 2^{-(n-1)p}(R-r)^p$, we obtain, for all $\delta > 0$,

$$\begin{aligned}\alpha_n \leq \delta\alpha_{n+1} &+ C\left((R-r)^{-p}2^{(n-1)p} + \delta^{-m}(R-r)^{-p(m+1)}2^{(n-1)(m+1)p}\right)\\ &\times \left(\|Lu\|_{C^\alpha_s(\bar B^+_R(x^0))} + \|u\varphi_{n+1}\|_{C^\alpha_s(\bar B^+_R(x^0))}\right),\end{aligned}$$

and now the estimate (3.4) follows as in the proofs of [30, Theorem 8.11.1] or [18, Theorem 3.8].

Step 2 (A priori estimate for $u \in C^{2+\alpha}_s(\underline{B}^+_{r_0}(x^0)))$**.** Choose a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty(\bar{B}^+_R(x^0))$ such that $u_n \to u$ in $C^{2+\alpha}_s(\bar{B}^+_R(x^0))$ as $n \to \infty$. Applying the estimate (3.4) to each u_n and then taking the limit as $n \to \infty$, yields the a priori estimate (3.4) for $u \in C^{2+\alpha}_s(\underline{B}^+_{r_0}(x^0))$.

This concludes the proof of Theorem 3.2. □

7. Higher-order a priori Schauder estimates for operators with constant coefficients

In this section, we prove a higher-order version of Theorem 3.2, our basic a priori local interior Schauder estimate, and a global a priori global Schauder estimate on a strip (Corollary 7.2), both when L has constant coefficients. *Throughout this section, we continue to assume Hypothesis 3.1 and so the coefficients, a, b, c, of the operator L in* (1.3) *and the coefficients, a, b, of the operator L_0 in* (1.4) *are constant.*

Theorem 7.1 (Higher-order a priori local interior Schauder estimate when L has constant coefficients)**.** *Assume the hypotheses of Theorem 3.2 and let $k \in \mathbb{N}$. If $u \in C^{k,2+\alpha}_s(\underline{B}^+_{r_0}(x^0))$, then*

$$\|u\|_{C^{k,2+\alpha}_s(\bar{B}^+_r)} \leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar{B}^+_{r_0}(x^0))} + \|u\|_{C(\bar{B}^+_{r_0}(x^0))}\right), \tag{7.1}$$

where C now also depends on k.

Proof. Choose $r_1 := (r+r_0)/2 \in (r, r_0)$. For any multi-index $\beta \in \mathbb{N}^d$ with $|\beta| := \beta_1 + \cdots + \beta_d \leq k$, direct calculation yields

$$D^\beta Lv = L_{(\beta_d)} D^\beta v - \beta_d \sum_{i,j\neq d} a^{ij} D^{\beta_0 + (\beta_d - 1)e_d} v_{x_i x_j}, \quad v \in C^\infty(\mathbb{H}), \tag{7.2}$$

where we write $\beta_0 := \beta - \beta_d e_d$ and, for $l \in \mathbb{N}$,

$$L_{(l)} v := -x_d a^{ij} v_{x_i x_j} - \sum_{i\neq d}\left(b^i + 2l a^{id}\right) v_{x_i} + \left(b^d + l a^{dd}\right) v_{x_d} + cv.$$

Note that $L_{(0)} = L$. To prove (7.1), we see by Definition 2.3 that it suffices to establish

$$\|D^\beta u\|_{C^{2+\alpha}_s(\bar{B}^+_r)} \leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar{B}^+_{r_0}(x^0))} + \|u\|_{C(\bar{B}^+_{r_0}(x^0))}\right), \tag{7.3}$$

for any multi-index $\beta \in \mathbb{N}^d$ with $|\beta| \leq k$, where C has the dependencies given in our hypotheses.

Theorem 3.2 yields (7.1) when $k = 0$. Therefore, as an induction hypothesis for k, we assume that (7.1) holds with k replaced by any $l \in \mathbb{N}$ in the range $0 \leq l \leq k-1$ and we seek to prove (7.3) and hence (7.1) by induction on l when $|\beta| = k$.

We first consider the case $\beta_d = 0$, so $LD^\beta v = D^\beta Lv$. Then

$$\begin{aligned}
\|D^\beta u\|_{C^{2+\alpha}_s(\bar{B}^+_r)} &\leq C\left(\|LD^\beta u\|_{C^\alpha_s(\bar{B}^+_{r_1}(x^0))} + \|D^\beta u\|_{C(\bar{B}^+_{r_1}(x^0))}\right) \quad \text{(by (3.4)} \\
&\leq C\left(\|D^\beta Lu\|_{C^\alpha_s(\bar{B}^+_{r_1}(x^0))} + \|D^\beta u\|_{C(\bar{B}^+_{r_1}(x^0))}\right) \quad \text{(by (7.2))} \\
&\leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar{B}^+_{r_1}(x^0))} + \|u\|_{C^{k,\alpha}_s(\bar{B}^+_{r_1}(x^0))}\right) \quad \text{(by Definition 2.2)} \\
&\leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar{B}^+_{r_1}(x^0))} + \|u\|_{C^{k-1,2+\alpha}_s(\bar{B}^+_{r_1}(x^0))}\right) \quad \text{(by Definition 2.3)} \\
&\leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar{B}^+_{r_1}(x^0))} + \|Lu\|_{C^{k-1,\alpha}_s(\bar{B}^+_{r_0}(x^0))} + \|u\|_{C(\bar{B}^+_{r_0}(x^0))}\right),
\end{aligned}$$

where the final inequality follows by induction on l and the a priori Schauder estimate (7.1) with k replaced by $l=k-1$ (and r replaced by r_1). Since $r_1<r_0$, we can combine terms and obtain (7.3) in the case $\beta_d=0$.

Now we consider the case $0\leq\beta_d\leq k$ and argue by induction on β_d. As an induction hypothesis for β_d, we assume that (7.3) holds when $0\leq\beta_d\leq k-1$. For β_d in the range $1\leq\beta_d\leq k$ (and thus $|\beta_0|\leq k-1$), we have

$$\begin{aligned}
&\|D^\beta u\|_{C^{2+\alpha}_s(\bar B^+_r)}\\
&\quad\leq C\left(\|L_{(\beta_d)}D^\beta u\|_{C^\alpha_s(\bar B^+_{r_1}(x^0))}+\|D^\beta u\|_{C(\bar B^+_{r_1}(x^0))}\right)\quad\text{(by (3.4))}\\
&\quad\leq C\left(\|D^\beta Lu\|_{C^\alpha_s(\bar B^+_{r_1}(x^0))}+\sum_{i,j\neq d}\|D^{\beta_0+(\beta_d-1)e_d}u_{x_ix_j}\|_{C^\alpha_s(\bar B^+_{r_1}(x^0))}+\|D^\beta u\|_{C(\bar B^+_{r_1}(x^0))}\right)\\
&\quad\leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))}+\max_{i\neq d}\|D^{\beta_0+e_i+(\beta_d-1)e_d}u\|_{C^{1,\alpha}_s(\bar B^+_{r_1}(x^0))}+\|u\|_{C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))}\right),
\end{aligned}$$

where the penultimate inequality follows from (7.2) and the final inequality by Definition 2.2 of our Hölder norms. Because $C^{2+\alpha}_s(\bar B^+_{r_1}(x^0))\hookrightarrow C^{1,\alpha}_s(\bar B^+_{r_1}(x^0))$ by Definitions 2.2 and 2.3, we see that

$$\begin{aligned}
\|D^\beta u\|_{C^{2+\alpha}_s(\bar B^+_r)}&\leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))}+\max_{i\neq d}\|D^{\beta_0+e_i+(\beta_d-1)e_d}u\|_{C^{2+\alpha}_s(\bar B^+_{r_1}(x^0))}+\|u\|_{C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))}\right)\\
&\leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))}+\|Lu\|_{C^{k,\alpha}_s(\bar B^+_{r_0}(x^0))}+\|u\|_{C(\bar B^+_{r_0}(x^0))}+\|u\|_{C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))}\right)\\
&\qquad\text{(by induction on }\beta_d\text{ and (7.3) since }\beta_d-1\leq k-1)\\
&\leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar B^+_{r_0}(x^0))}+\|u\|_{C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))}+\|u\|_{C(\bar B^+_{r_0}(x^0))}\right)\quad\text{(since }r_1<r_0)\\
&\leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar B^+_{r_0}(x^0))}+\|u\|_{C^{k-1,2+\alpha}(\bar B^+_{r_1}(x^0))}+\|u\|_{C(\bar B^+_{r_0}(x^0))}\right)\\
&\leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))}+\|Lu\|_{C^{k-1,\alpha}_s(\bar B^+_{r_0}(x^0))}+\|u\|_{C(\bar B^+_{r_0}(x^0))}\right),
\end{aligned}$$

where the penultimate inequality follows from the embedding $C^{k-1,2+\alpha}_s(\bar B^+_{r_1}(x^0))\hookrightarrow C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))$ implied by Definitions 2.2 and 2.3 and the final inequality follows by induction on l and the a priori Schauder estimate (7.1) with k replaced by $l=k-1$ (and r replaced by r_1). Again, since $r_1<r_0$, we can combine terms and obtain (7.3) in this case too. □

Let $\nu>0$ and let $S=\mathbb{R}^{d-1}\times(0,\nu)$, as in (1.10). Theorem 7.1 together with a priori estimates for strictly elliptic operators in [26, §6] now imply the following global Schauder estimate on strips.

Corollary 7.2 (A priori global Schauder estimate on a strip when L has constant coefficients)**.** *For any $\alpha\in(0,1)$, constant $\nu>0$, and $k\in\mathbb{N}$, there is a positive constant, $C=C(k,\alpha,\nu,d,\lambda_0,b_0,\Lambda)$, such that the following holds. If $u\in C^{k,2+\alpha}_s(\bar S)$ and $u=0$ on $\partial_1 S$, then*

$$\|u\|_{C^{k,2+\alpha}_s(\bar S)}\leq C\left(\|Lu\|_{C^{k,\alpha}_s(\bar S)}+\|u\|_{C(\bar S)}\right). \tag{7.4}$$

and, when $c\geq 0$,

$$\|u\|_{C^{k,2+\alpha}_s(\bar S)}\leq C\|Lu\|_{C^{k,\alpha}_s(\bar S)}. \tag{7.5}$$

Proof. Let $r := \nu/2$, and let $\{x^n\}_{n\in\mathbb{N}} \subset \partial\mathbb{H}$ be a sequence of points such that

$$\mathbb{R}^{d-1} \times (0, r/4) \subset \bigcup_{n\in\mathbb{N}} B^+_{r/2}(x^n).$$

Using the a priori interior local Schauder estimate (3.4) on each half-ball $B^+_r(x^n)$, we obtain

$$\|u\|_{C^{k,2+\alpha}_s(\bar{B}^+_{r/2}(x^n))} \leq C \left(\|Lu\|_{C^{k,\alpha}_s(\bar{B}^+_r(x^n))} + \|u\|_{C(\bar{B}^+_r(x^n))} \right).$$

By applying a standard covering argument to the strip $S_0 := \mathbb{R}^{d-1} \times (0, r)$, we find that

$$\|u\|_{C^{k,2+\alpha}_s(\bar{S}_0)} \leq C \left(\|Lu\|_{C^{k,\alpha}_s(\bar{S})} + \|u\|_{C(\bar{S})} \right).$$

By [26, Lemma 6.5 & Problem 6.2] and a similar covering argument, there is a constant $\delta > 0$ such that, if $S_1 := \mathbb{R}^{d-1} \times (\nu - \delta, \nu)$, we have

$$\|u\|_{C^{k,2+\alpha}_s(\bar{S}_1)} \leq C \left(\|Lu\|_{C^{k,\alpha}_s(\bar{S})} + \|u\|_{C(\bar{S})} \right).$$

Setting $S_2 := \mathbb{R}^{d-1} \times (r/4, \nu - \delta/2)$ and now applying [26, Corollary 6.3 & Problem 6.1] and a covering argument, we obtain

$$\|u\|_{C^{k,2+\alpha}_s(\bar{S}_2)} \leq C \left(\|Lu\|_{C^{k,\alpha}_s(\bar{S})} + \|u\|_{C(\bar{S})} \right).$$

By combining the preceding three estimates, we obtain (7.4) and by appealing to Corollary A.2, we obtain (7.5). ☐

8. A priori Schauder estimates, global existence, and regularity for operators with variable coefficients

In §8.1, we relax the condition in Hypothesis 3.1 that the coefficients, a, b, c, of the operator L in (1.3) are constant, which we assumed in sections 3, 4, 5, and 6, to prove a generalization (Theorem 8.1) of our $C^{2+\alpha}_s$ a priori Schauder estimate (Theorem 3.2) from the case of constant coefficients, a, b, c, to the case of variable coefficients. We then prove Theorem 8.3, extending the preceding $C^{2+\alpha}_s$ a priori Schauder estimate to a $C^{k,2+\alpha}_s$ a priori Schauder estimate for arbitrary $k \in \mathbb{N}$. This allows us to complete the proofs of Theorem 1.1 and Corollary 1.3. In § refsubsec:Regularity, we prove our global $C^{k,2+\alpha}_s(\bar{S})$ existence result on strips, S, and hence a $C^{k,2+\alpha}_s(\underline{B}^+_{r_0}(x^0))$-regularity result, Theorem 8.4, on half-balls, $B^+_{r_0}(x^0)$. We conclude the section with the proofs of Theorems 1.8 and 1.11, and Corollary 1.13.

8.1. A priori Schauder estimates for operators with variable coefficients.

We begin with a generalization of Theorem 3.2 to the case of variable coefficients.

Theorem 8.1 (A priori interior local Schauder estimate when L has variable coefficients)**.** *Let $\alpha \in (0, 1)$ and let r_0, λ_0, b_0, Λ be positive constants. Suppose that the coefficients a^{ij}, b^i, and c of L in* (1.3) *belong to $C^\alpha_s(\underline{B}^+_{r_0}(x^0))$, where $x^0 \in \partial\mathbb{H}$, and obey*

$$\|a\|_{C^\alpha_s(\bar{B}^+_{r_0}(x^0))} + \|b\|_{C^\alpha_s(\bar{B}^+_{r_0}(x^0))} + \|c\|_{C^\alpha_s(\bar{B}^+_{r_0}(x^0))} \leq \Lambda, \tag{8.1}$$

$$b^d \geq b_0 \quad \text{on } \partial_0 B^+_{r_0}(x^0), \tag{8.2}$$

$$\langle a\xi, \xi\rangle \geq \lambda_0 |\xi|^2 \quad \text{on } \underline{B}^+_{r_0}(x^0), \quad \forall\, \xi \in \mathbb{R}^d, \tag{8.3}$$

Then, for all $r \in (0, r_0)$, *there is a positive constant* $C = C(\alpha, r, r_0, d, \lambda_0, b_0, \Lambda)$ *such that, for any function*[8] $u \in C_s^{2+\alpha}(\underline{B}_{r_0}^+(x^0))$, *we have*

$$\|u\|_{C_s^{2+\alpha}(\bar{B}_r^+(x^0))} \leq C \left(\|Lu\|_{C_s^\alpha(\bar{B}_{r_0}^+(x^0))} + \|u\|_{C(\bar{B}_{r_0}^+(x^0))} \right). \tag{8.4}$$

Proof. We use the a priori interior local Schauder estimate (3.4) for the operator with constant coefficients (given by Theorem 3.2) and the interpolation inequalities for the Hölder norms defined by the cycloidal metric (Lemma C.2), the method of freezing coefficients as in the proofs of [9] [30, Theorem 7.1.1] (elliptic case), [30, Theorem 8.11.1] (parabolic case), and, in particular, [18, Theorem 3.8] for the parabolic version of our elliptic operator (1.3) to obtain (8.4). □

We now generalize Corollary 7.2 to the case of variable coefficients when u has compact support in a strip.

Proposition 8.2 (Higher-order a priori global Schauder estimate for compactly supported functions on a strip when L has variable coefficients)**.** *Let* $\alpha \in (0,1)$ *and* ν, λ_0, b_0, Λ *be positive constants and* $k \in \mathbb{N}$. *Suppose* $S = \mathbb{R}^{d-1} \times (0, \nu)$ *as in* (1.10) *and the coefficients* a, b, c *of* L *in* (1.3) *belong to* $C_s^{k,\alpha}(\bar{S})$ *and obey* (1.11), (1.12), *and* (1.13). *Then there are positive constants,* $C = C(k, \alpha, \nu, d, \lambda_0, b_0, \Lambda)$ *and* $\delta = \delta(k, \alpha, \nu, d, \lambda_0, b_0, \Lambda) < \nu/2$, *such that the following holds. If* $u \in C_s^{k,2+\alpha}(\bar{S})$ *has* compact support *in* $\bar{S}$ *with* $\operatorname{diam}(\operatorname{supp} u) \leq \delta$ *and* $u = 0$ *on* $\partial_1 S$, *then*

$$\|u\|_{C_s^{k,2+\alpha}(\bar{S})} \leq C \left(\|Lu\|_{C_s^{k,\alpha}(\bar{S})} + \|u\|_{C(\bar{S})} \right), \tag{8.5}$$

and, when $c \geq 0$ *on* S,

$$\|u\|_{C_s^{k,2+\alpha}(\bar{S})} \leq C \|Lu\|_{C_s^{k,\alpha}(\bar{S})}. \tag{8.6}$$

Proof. Fix $x^0 \in S \cap \operatorname{supp} u$ and let L_{x^0} denote the operator with constant coefficients $a(x^0)$, $b(x^0)$, $c(x^0)$. By applying (7.4) for the operator L_{x_0} with constant coefficients, we obtain

$$\|u\|_{C_s^{k,2+\alpha}(\bar{S})} \leq C_0 \left(\|L_{x_0} u\|_{C_s^{k,\alpha}(\bar{S}))} + \|u\|_{C(\bar{S})} \right),$$

and hence

$$\|u\|_{C_s^{k,2+\alpha}(\bar{S})} \leq C_0 \left(\|Lu\|_{C_s^{k,\alpha}(\bar{S})} + \|(L - L_{x_0})u\|_{C_s^{k,\alpha}(\bar{S})} + \|u\|_{C(\bar{S})} \right), \tag{8.7}$$

where C_0 has the dependencies stated for the constant C in the estimate (7.4).

For any $x^1, x^2 \in \operatorname{supp} u$, the cycloidal distance-function bound (2.2) and our hypothesis on $\operatorname{supp} u$ imply that $s(x^1, x^2) \leq |x^1 - x^2|^{1/2} \leq \delta^{1/2}$, for some $\delta \in (0, \nu/2)$ to be selected later. We first consider the case $\operatorname{supp} u \subset B_{2\delta}^+(y^0)$ for some $y^0 \in \partial_0 S$. We further restrict to the case $k = 0$ initially. Observe that

$$(L - L_{x_0})u = -x_d \operatorname{tr}((a - a(x^0)) D^2 u) - (b - b(x^0)) \cdot Du + (c - c(x^0))u.$$

We consider in turn each of the three terms appearing in our expression for $(L - L_{x_0})u$. From Definition 2.2,

$$\|(b - b(x^0)) \cdot Du\|_{C_s^\alpha(\bar{S})} = \|(b - b(x^0)) \cdot Du\|_{C(\bar{S})} + [(b - b(x^0)) \cdot Du]_{C_s^\alpha(\bar{S})}.$$

---

[8] It is enough to require $u \in C_s^{2+\alpha}(\underline{B}_{r_0}^+(x^0))$ since the estimate trivially holds if $\|Lu\|_{C_s^\alpha(\bar{B}_{r_0}^+(x^0))}$ and $\|u\|_{C(\bar{B}_{r_0}^+(x^0))}$ are not finite.

[9] This method is also employed in the proof of [26, Theorem 6.2], but Gilbarg and Trudinger employ a family of "global" interior Hölder norms (which we do not develop in this article) which allows a rearrangement argument.

The coefficient bounds (8.14) ensure that

$$\|(b-b(x^0))\cdot Du\|_{C(\bar S)} \le \left(\|b\|_{C(\bar S)} + |b(x^0)|\right)\|Du\|_{C(\bar S)} \le 2\Lambda\|Du\|_{C(\bar S)},$$

while the interpolation inequality (C.3) yields, for some $m = m(d,\alpha)$ and $C_1 = C_1(d,\alpha,\delta)$ (because $\operatorname{diam}(\operatorname{supp} u) = \delta$) and any $\varepsilon \in (0,1)$,

$$\|Du\|_{C(\bar S)} \le \varepsilon\|u\|_{C_s^{2+\alpha}(\bar S)} + C_1\varepsilon^{-m}\|u\|_{C(\bar S)}, \tag{8.8}$$

and thus, combining (8.8) with the preceding inequality, yields

$$\|(b-b(x^0))\cdot Du\|_{C(\bar S)} \le 2\varepsilon\Lambda\|u\|_{C_s^{2+\alpha}(\bar S)} + C_1\Lambda\varepsilon^{-m}\|u\|_{C(\bar S)}. \tag{8.9}$$

Writing, for $x^1,\ x^2 \in S\cap \operatorname{supp} u$,

$$\begin{aligned}&\frac{(b(x^1)-b(x^0))\cdot Du(x^1) - (b(x^2)-b(x^0))\cdot Du(x^2)}{s(x^1,x^2)^\alpha}\\ &= \frac{(b(x^1)-b(x^2))}{s(x^1,x^2)^\alpha}\cdot Du(x^1) + (b(x^1)-b(x^0))\cdot\frac{\left(Du(x^1)-Du(x^2)\right)}{s(x^1,x^2)^\alpha},\end{aligned}$$

we obtain

$$[(b-b(x^0))\cdot Du]_{C_s^\alpha(\bar S)} \le [b]_{C_s^\alpha(\bar S)}\left(\|Du\|_{C(\bar S)} + s(x^1,x^0)^\alpha[Du]_{C_s^\alpha(\bar S)}\right).$$

Since $\operatorname{diam}(\operatorname{supp} u) = \delta$ and $x^0, x^1 \in \operatorname{supp} u$, by combining the preceding inequality with the coefficient bounds (8.14) and the interpolation inequality (8.8), we see that

$$[(b-b(x^0))\cdot Du]_{C_s^\alpha(\bar S)} \le \Lambda\left(\varepsilon\|u\|_{C_s^{2+\alpha}(\bar S)} + C_1\varepsilon^{-m}\|u\|_{C(\bar S)} + \delta^{\alpha/2}\|u\|_{C_s^{2+\alpha}(\bar S)}\right). \tag{8.10}$$

Therefore, by combining (8.9) and (8.10), we obtain

$$\|(b-b(x^0))\cdot Du\|_{C_s^\alpha(\bar S)} \le \Lambda(3\varepsilon+\delta^{\alpha/2})\|u\|_{C_s^{2+\alpha}(\bar S)} + 2C_1\Lambda\varepsilon^{-m}\|u\|_{C(\bar S)}. \tag{8.11}$$

An identical analysis, just replacing the coefficient vector b by the matrix a, and Du by x_dD^2u, and the interpolation inequality (C.4) by (C.5), yields

$$\|\operatorname{tr}(x_d(a-a(x^0))D^2u)\|_{C_s^\alpha(\bar S)} \le \Lambda(3\varepsilon+\delta^{\alpha/2})\|u\|_{C_s^{2+\alpha}(\bar S)} + 2C_1\Lambda\varepsilon^{-m}\|u\|_{C(\bar S)}. \tag{8.12}$$

Similarly, replacing the coefficient vector b by the function c, and Du by u, and the interpolation inequality (C.4) by (C.2), yields

$$\|(c-c(x^0))u\|_{C_s^\alpha(\bar S)} \le \Lambda(3\varepsilon+\delta^{\alpha/2})\|u\|_{C_s^{2+\alpha}(\bar S)} + 2C_1\Lambda\varepsilon^{-m}\|u\|_{C(\bar S)}. \tag{8.13}$$

We combine (8.11), (8.12), and (8.13) to give

$$\|(L-L_{x_0})u\|_{C_s^\alpha(\bar S)} \le 3\Lambda(3\varepsilon+\delta^{\alpha/2})\|u\|_{C_s^{2+\alpha}(\bar S)} + 6C_1\Lambda\varepsilon^{-m}\|u\|_{C(\bar S)}.$$

We now choose $\varepsilon > 0$ such that $9C_0\Lambda\varepsilon = 1/4$ and choose $\delta \in (0,\nu/2)$ (which we fix for the remainder of the proof) such that $3C_0\Lambda\delta^{\alpha/2} \le 1/4$ and combine the preceding inequality with (8.7) to give

$$\|u\|_{C_s^{2+\alpha}(\bar S)} \le C_0\|Lu\|_{C_s^\alpha(\bar S)} + \frac{1}{2}\|u\|_{C_s^{2+\alpha}(\bar S)} + C_2\|u\|_{C(\bar S)},$$

for some constant C_2 with at most the dependencies stated for C stated in our hypotheses. Rearrangement and the maximum principle estimate (Corollary A.2) for $\|u\|_{C(\bar S)}$ now give the conclusions (8.5) and (8.6) when $c \ge 0$ on S in the case $k = 0$.

Next, suppose $k \geq 1$ and let $\beta \in \mathbb{N}^d$ be a multi-index with $|\beta| \leq k$. Because

$$D^\beta(vw) = \sum_{\substack{\beta'+\beta''=\beta \\ \beta',\beta''\in\mathbb{N}^d}} D^{\beta'} v D^{\beta''} w.$$

for any $v, w \in C^{k,\alpha}_s(\bar{S})$, we may apply the preceding analysis virtually unchanged with $v = a - a(x^0)$, $b - b(x^0)$, or $c - c(x^0)$ and $w = x_d D^2 u$, Du, or u, respectively, for each $\beta, \beta', \beta'' \in \mathbb{N}^d$ with $|\beta| \leq k$ and $\beta' + \beta'' = \beta$. This completes the proof when $\operatorname{supp} u \subset B^+_\delta(y^0)$ for some $y^0 \in \partial_0 S$.

Because $\operatorname{supp} u \subset B_{\delta/2}(x^*)$, for some $x^* \in \bar{S}$, the case $\operatorname{dist}(x^*, \partial_0 S) \leq \delta$ is covered by our analysis for half-balls, $B^+_{2\delta}(y^0)$, with $y^0 \in \partial_0 S$. If $\operatorname{dist}(x^*, \partial_0 S) \geq \delta/2$, then the operator L is strictly elliptic since $x_d \geq \delta/2$ and [26, Theorem 6.6 and Problem 6.2] imply that

$$\|u\|_{C^{k,2+\alpha}_s(\bar{S})} \leq C_0 \left(\|Lu\|_{C^{k,\alpha}_s(\bar{S}))} + \|u\|_{C(\bar{S})} \right),$$

which is just (8.5). Combining the preceding inequality with the maximum principle estimate (Corollary A.2) for $\|u\|_{C(\bar{S})}$ again gives the conclusion (8.6) when $c \geq 0$ on S. □

Finally, we use Proposition 8.2 to generalize Theorem 7.1 to the case of variable coefficients to obtain the following analogue of [6, Theorem I.1.3] (for a related degenerate-parabolic operator with constant coefficients) and [26, Corollary 6.3 & Problem 6.1].

Theorem 8.3 (Higher-order a priori interior local Schauder estimate when L has variable coefficients)**.** *Let $\alpha \in (0,1)$ and let r_0, λ_0, b_0, Λ be positive constants and let $k \in \mathbb{N}$. Suppose the coefficients a, b, c of L in* (1.3) *belong to $C^{k,\alpha}_s(\underline{B}^+_{r_0}(x^0))$, where $x^0 \in \partial\mathbb{H}$, and obey* (8.2), (8.3), *and*

$$\|a\|_{C^{k,\alpha}_s(\bar{B}^+_{r_0}(x^0))} + \|b\|_{C^{k,\alpha}_s(\bar{B}^+_{r_0}(x^0))} + \|c\|_{C^{k,\alpha}_s(\bar{B}^+_{r_0}(x^0))} \leq \Lambda. \tag{8.14}$$

Then, for any $r \in (0, r_0)$, there is a positive constant, $C = C(k, \alpha, r, r_0, d, \lambda_0, b_0, \Lambda)$ such that the following holds. If $u \in C^{k,2+\alpha}_s(\underline{B}^+_{r_0}(x^0))$ then

$$\|u\|_{C^{k,2+\alpha}_s(\bar{B}^+_r(x^0))} \leq C \left(\|Lu\|_{C^{k,\alpha}_s(\bar{B}^+_{r_0}(x^0))} + \|u\|_{C(\bar{B}^+_{r_0}(x^0))} \right). \tag{8.15}$$

Proof. We apply an induction argument. When $k = 0$, the estimate (8.15) follows from Theorem 8.1 and so we may assume without loss of generality that $k \geq 1$. By induction, we may assume that the estimate (8.15) holds with constant $C = C(l, *)$ when k is replaced by $l \in \mathbb{N}$ in the range $0 \leq l \leq k-1$.

Let $r_1 := (r + r_0)/2$ and choose a cutoff function $\varphi \in C^\infty_0(\bar{\mathbb{H}})$ such that $0 \leq \varphi \leq 1$ on $\bar{\mathbb{H}}$ and $\varphi = 1$ on $\bar{B}^+_r(x^0)$ while $\operatorname{supp} \varphi \subset \bar{B}^+_{r_1}(x^0)$ and note that, for $u \in C^{k,2+\alpha}_s(\underline{B}^+_{r_0}(x^0))$ and thus $u_0 := \varphi u \in C^{k,2+\alpha}_s(\bar{S})$, we have

$$\|u\|_{C^{k,2+\alpha}_s(\bar{B}^+_r(x^0))} \leq \|u_0\|_{C^{k,2+\alpha}_s(\bar{S})},$$

where $S = \mathbb{R}^{d-1} \times (0, r_0)$ is the strip as in (1.10), and $u_0 = 0$ on $\partial_1 S$. By Proposition 8.2, we obtain

$$\|u_0\|_{C^{k,2+\alpha}_s(\bar{S})} \leq C_0 \left(\|Lu_0\|_{C^{k,\alpha}_s(\bar{S})} + \|u_0\|_{C(\bar{S})} \right),$$

where we use C_0 to denote the constant C in (8.5), and hence, combining the preceding two inequalities,

$$\|u\|_{C^{k,2+\alpha}_s(\bar{B}^+_r(x^0))} \leq C_0 \left(\|L(\varphi u)\|_{C^{k,\alpha}_s(\bar{B}^+_{r_1}(x^0))} + \|u\|_{C(\bar{B}^+_{r_0}(x^0))} \right). \tag{8.16}$$

Notice that $L(\varphi u) = \varphi L u + [L,\varphi]u$ and that $[L,\varphi]$ is a first-order partial differential operator,

$$\begin{aligned}[L,\varphi]u &= L(\varphi u) - \varphi L u\\ &= -\operatorname{tr}(x_d a D^2(\varphi u)) - b\cdot D(\varphi u) + \varphi \operatorname{tr}(x_d a D^2 u) + \varphi b\cdot Du,\end{aligned}$$

and so

$$[L,\varphi]u = -\operatorname{tr}(x_d a((D^2\varphi)u + D\varphi\times Du)) - (b\cdot D\varphi)u, \tag{8.17}$$

where $D\varphi\times Du$ denotes the $d\times d$ matrix with entries $\varphi_{x_i}u_{x_j}$. Observe that

$$\|L(\varphi u)\|_{C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))} \le \|[L,\varphi]u\|_{C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))} + \|\varphi L u\|_{C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))}.$$

Because of the structure (8.17) of $[L,\varphi]$ (with factor x_d in the coefficients of the first-order derivatives) and the fact that $C^{k,\alpha}_s(\bar B^+_{r_1}(x^0)) \subset C^{k-1,2+\alpha}_s(\bar B^+_{r_1}(x^0))$ (by Definitions 2.2 and 2.3), we obtain

$$\|[L,\varphi]u\|_{C^{k,\alpha}_s(\bar B^+_{r_1}(x^0))} \le C\|u\|_{C^{k-1,2+\alpha}_s(\bar B^+_{r_1}(x^0))},$$

where C has at most the dependencies stated for the constant in the estimate (8.15). By our induction hypothesis, we can apply the local Schauder estimate (8.15) with k replaced by $l=k-1$ to give

$$\|u\|_{C^{k-1,2+\alpha}_s(\bar B^+_{r_1}(x^0))} \le C\left(\|Lu\|_{C^{k-1,\alpha}_s(\bar B^+_{r_0}(x^0))} + \|u\|_{C(\bar B^+_{r_0}(x^0))}\right).$$

Combining the preceding three bounds with (8.16) yields the inequality,

$$\|u\|_{C^{k,2+\alpha}_s(\bar B^+_{r}(x^0))} \le C\left(\|Lu\|_{C^{k,\alpha}_s(\bar B^+_{r_0}(x^0))} + \|u\|_{C(\bar B^+_{r_0}(x^0))}\right), \tag{8.18}$$

and this is (8.15). □

We can now prove the generalization of Corollary 7.2 to the case of variable coefficients.

Proof of Corollary 1.3. The proof is virtually identical to that of Corollary 7.2 except that we replace appeals to Theorems 3.2 and 7.1 (for the case of constant coefficients) by appeals to Theorems 8.1 and 8.3 (for the case of variable coefficients). □

We can now give the

Proof of Theorem 1.1. Since we can apply Theorem 8.3 to half-balls, $B^+_{r_0}(x^0) = B_{r_0}(x^0)\cap\mathbb{H}\subset\mathscr{O}$ when $x_0\in\partial\mathscr{O}$, and the standard a priori interior Schauder estimate for strictly elliptic operators [26, Corollary 6.3 & Problem 6.1] to balls, $B_{r_0}(x^0)\Subset\mathscr{O}$ when $x_0\in\mathscr{O}$, the remainder of the argument is very similar to the proof of Corollary 7.2, when the domain, $\mathscr{O}$, is an infinite strip. □

8.2. **Regularity.** We begin with the proof of our global existence result on strips.

Proof of Theorem 1.6. The proof follows by the method of continuity. We denote

$$\mathscr{L}_0 := -x_d\sum_{i=1}^d \partial_{x_ix_i} - \partial_{x_d}.$$

Then Corollary B.4 implies that, for any $f\in C^{k,\alpha}_s(\bar S)$, there is a unique solution $u\in C^{k,2+\alpha}_s(\bar S)$. We consider $L_t := (1-t)\mathscr{L}_0 + tL$, for all $t\in[0,1]$. Given Corollary 1.3 and the existence and uniqueness of solutions in $C^{k,2+\alpha}_s(\bar S)$ for the operator $\mathscr{L}_0$, the method of continuity [26, Theorem 5.2] applies and gives the result. □

We have the following analogue of [26, Theorem 6.17], albeit with a quite different proof.

Theorem 8.4 (Higher-order interior local regularity of solutions when L has variable coefficients)**.** *Assume the hypotheses of Theorem 8.3 for the operator L in* (1.3)*. If $u \in C^2(B^+_{r_0}(x^0))$ obeys*

$$u,\ Du,\ x_d D^2 u \in C(\underline{B}^+_{r_0}(x^0)) \quad \text{and} \quad Lu \in C^{k,\alpha}_s(\underline{B}^+_{r_0}(x^0)), \tag{8.19}$$

$$x_d D^2 u = 0 \quad \text{on } \partial_0 B^+_{r_0}(x^0), \tag{8.20}$$

then $u \in C^{k,2+\alpha}_s(\underline{B}^+_{r_0}(x^0))$.

Proof. Let $r \in (0, r_0)$, and $r_1 := (r + r_0)/2$, and $r_2 := (r_1 + r)/2$, so $r < r_1 < r_2 < r_0$. Let $\varphi \in C^\infty_0(\bar{\mathbb{H}})$ be a cutoff function such that $0 \le \varphi \le 1$ on $\mathbb{H}$ with $\varphi = 1$ on $\bar{B}^+_{r_1}(x^0)$ and $\operatorname{supp}\varphi \subset \bar{B}^+_{r_2}(x^0)$. Denoting $S = \mathbb{R}^{d-1} \times (0, r_0)$ as in (1.10) and $u_0 := u\varphi$ on $\bar{S}$, we see that $u_0 \in C^2(S)$ is a solution to (1.16), (1.17) with f replaced by

$$f_0 := \varphi L u + [L, \varphi] u \quad \text{on } \bar{S}.$$

By hypothesis, $\varphi L u \in C^{k,\alpha}_s(\underline{B}^+_{r_0}(x^0))$, while Lemma C.3 and (8.19) and (8.20) ensure that $[L, \varphi]u \in C^\alpha_s(\bar{S})$, so

$$f_0 \in C^\alpha_s(\bar{S}),$$

while the conditions (8.19) and (8.20) on u imply that u_0 obeys

$$u_0,\ Du_0,\ x_d D^2 u_0 \in C(\underline{S}) \quad \text{and} \quad x_d D^2 u_0 = 0 \quad \text{on } \partial_0 S.$$

Corollary B.4 implies that there is a unique solution $v \in C^{2+\alpha}_s(\bar{S})$ to (1.16), (1.17) with f replaced by f_0 and the maximum principle, Lemma A.1, implies that $u_0 = v$. Thus, $u \in C^{2+\alpha}_s(\bar{B}^+_r(x^0))$.

When $k \ge 1$, we argue by induction and suppose that $u \in C^{k-1,2+\alpha}_s(\bar{B}^+_r(x^0))$ as our induction hypothesis. But then $[L, \varphi]u \in C^{k,\alpha}_s(\bar{B}^+_r(x^0))$ by the proof of Theorem 8.3 and so $f_0 \in C^{k,\alpha}_s(\bar{S})$. Now Corollary B.4 implies that $v \in C^{k,2+\alpha}_s(\bar{S})$ by the preceding argument for $k = 0$, and thus $u \in C^{k,2+\alpha}_s(\bar{B}^+_r(x^0))$ since $u_0 = v$ on $\bar{S}$. □

We can now complete the

Proof of Theorem 1.8. This is an immediate consequence of Theorem 8.4 and [26, Theorem 6.17] since we can apply those regularity results to any half-ball, $B^+_{r_0}(x^0) = B_{r_0}(x^0) \cap \mathbb{H} \Subset \underline{\mathscr{O}}$ when $x^0 \in \partial\mathbb{H}$, or ball, $B_{r_0}(x^0) \Subset \mathscr{O}$ when $x^0 \in \mathbb{H}$, respectively. □

Finally, we complete the proofs of Theorem 1.11 and Corollary 1.13.

Proof of Theorem 1.11. The argument is very similar to the proof of Corollary B.4, so we just highlight the differences. Because $f \in C^{k,\alpha}_s(\underline{\mathscr{O}}) \cap C_b(\mathscr{O})$, we can apply the regularizing procedure described in [6, §I.11] and [6, Theorem I.11.3] to construct a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset C^\infty_0(\bar{\mathbb{H}})$ such that $f_n \to f$ in $C^{k,\alpha}_s(\bar{U}) \cap C_b(\mathscr{O})$ as $n \to \infty$, for all $U \Subset \underline{\mathscr{O}}$, and

$$\|f_n\|_{C^{k,\alpha}_s(\bar{U}')} \le C' \|f\|_{C^{k,\alpha}_s(\bar{U})}, \quad \forall n \in \mathbb{N},$$

where $U' \Subset \underline{U}$ and $U \Subset \underline{\mathscr{O}}$ and C' may depend on U and U', and

$$\|f_n\|_{C(\bar{\mathscr{O}})} \le C \|f\|_{C(\bar{\mathscr{O}})}, \quad \forall n \in \mathbb{N},$$

for some positive constant, $C = C(d)$.

Let $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty(\underline{\mathscr{O}}) \cap C(\mathscr{O} \cup \partial_1\mathscr{O}) \cap C_b(\mathscr{O})$ be the corresponding (unique) sequence of solutions to (1.22), (1.23), with f replaced by f_n, provided by [19, Theorem 1.11]. The maximum principle estimate (Corollary A.2 for the case $c_0 = 0$ and [16, Proposition 2.19 & Theorem 5.3] for the case $c_0 > 0$) implies that

$$\|u_n\|_{C(\bar{\mathscr{O}})} \leq C_0 \|f_n\|_{C(\bar{\mathscr{O}})}, \quad \forall n \in \mathbb{N},$$

for a constant C_0 depending on the coefficients of A and ν when $\operatorname{height}(\mathscr{O}) = \nu$ and $c_0 = 0$ or $C_0 = 1/c_0$ when $c_0 > 0$ and $\operatorname{height}(\mathscr{O}) = \infty$. The remainder of the argument is now the same as the proof of Corollary B.4. □

Proof of Corollary 1.13. The conclusion follows immediately from Theorem 1.11 and [19, Corollary 1.13], since the latter result ensures that $u \in C(\bar{\mathscr{O}})$. □

Appendix A. Maximum principle for degenerate-elliptic operators on domains of finite height

In this appendix, we prove a comparison principle for operators which include those of the form L in (1.3) with $c \geq 0$ when the domain, $\mathscr{O}$, is unbounded. Notice that when c does not have a uniform positive lower bound, the weak maximum principle [16, Theorem 5.3] does not immediately apply when $\mathscr{O}$ is unbounded.

Lemma A.1 (Comparison principle on a strip)**.** *Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain of finite height. Let*[10]

$$Lv := -\operatorname{tr}(aD^2 v) - b \cdot Dv + cv \quad \text{on } \mathscr{O}, \quad v \in C^\infty(\mathscr{O}),$$

require that its coefficients, $a : \bar{\mathscr{O}} \to \mathbb{R}^{d\times d}$, and $b : \bar{\mathscr{O}} \to \mathbb{R}^d$, and $c : \bar{\mathscr{O}} \to \mathbb{R}$ obey

$$\begin{aligned} a(x) &= 0 \quad \text{on } \partial_0\mathscr{O}, \\ \langle a\xi, \xi\rangle &\geq 0 \quad \text{on } \mathscr{O}, \quad \forall \xi \in \mathbb{R}^d, \\ \inf_{\mathscr{O}} b^d &> 0 \quad \text{on } \mathscr{O}, \\ \operatorname{tr}(a(x)) + \langle x, b(x)\rangle &\leq K(1+|x|^2), \quad \forall x \in \mathscr{O}, \\ \sup_{\mathscr{O}} a^{dd} &< \infty \quad \text{on } \mathscr{O}, \\ c &\geq 0 \quad \text{on } \mathscr{O}, \end{aligned}$$

for some positive constant K. Suppose that $u \in C^2(\mathscr{O}) \cap C_{\mathrm{loc}}(\bar{\mathscr{O}})$, and $\sup_{\mathscr{O}} u < \infty$, and $Du,\ \operatorname{tr}(aD^2u) \in C(\underline{\mathscr{O}})$, and

$$\operatorname{tr}(aD^2 u) = 0 \quad \text{on } \partial_0\mathscr{O}.$$

If $Lu \leq 0$ on $\mathscr{O}$ and $u \leq 0$ on $\partial_1\mathscr{O}$, then $u \leq 0$ on $\bar{\mathscr{O}}$.

Proof. Define constants $b_0 > 0$ and $\Lambda > 0$ by

$$\Lambda := \sup_{\mathscr{O}} a^{dd} \quad \text{and} \quad b_0 := \inf_{\mathscr{O}} b^d. \tag{A.1}$$

Let σ be a positive constant, to be fixed shortly, and define $v \in C^2(\mathscr{O}) \cap C_{\mathrm{loc}}(\bar{\mathscr{O}})$ with $\sup_{\mathscr{O}} v < \infty$ by the transformation

$$u(x', x_d) = e^{-\sigma x_d} v(x', x_d), \quad \forall (x', x_d) \in \bar{\mathscr{O}}, \tag{A.2}$$

noting that $\sup_{\mathscr{O}} v < \infty$ since $\sup_{\mathscr{O}} v < \infty$ and $\operatorname{height}(\mathscr{O}) < \infty$ by hypothesis. By direct calculation, we find that

$$Lu = e^{-\sigma x_d}\left(-a^{ij} v_{x_i x_j} - \left(b^i - 2\sigma a^{id}\right) v_{x_i} + \left(c + \sigma b^d - \sigma^2 a^{dd}\right) v\right).$$

[10]Note the more general definition of the coefficient $a(x)$ in Lemma A.1 and Corollary A.2.

We now define coefficients $\tilde{a}, \tilde{b}, \tilde{c}$ of an operator $\widetilde{L}$ by

$$\begin{aligned}\widetilde{L}v &:= -\tilde{a}^{ij} v_{x_i x_j} - \tilde{b}^i v_{x_i} + \tilde{c} v\\ &:= -a^{ij} v_{x_i x_j} - \left(b^i - 2\sigma a^{id}\right) v_{x_i} + \left(c + \sigma b^d - \sigma^2 a^{dd}\right) v,\end{aligned}$$

and we notice, by our hypotheses on u and definition of v, that $\widetilde{L}v \leq 0$ on $\mathscr{O}$ and $v \leq 0$ on $\partial_1 \mathscr{O}$. Since $a = 0$ on $\partial_0 \mathscr{O}$, we have

$$\tilde{b}^d = b^d \geq b_0 > 0 \quad \text{on } \partial_0 \mathscr{O}.$$

We now choose $\sigma := b_0/(2\Lambda)$, so that, using $c \geq 0$ and $a^{dd} \leq \Lambda$,

$$\tilde{c} \equiv c + \sigma b^d - \sigma^2 a^{dd} \geq \sigma b_0 - \sigma^2 \Lambda = \sigma b_0/2 > 0 \quad \text{on } \mathscr{O}.$$

Then, [16, Theorem 5.3] applies to v, and now the conclusion follows immediately for u also. □

Corollary A.2 (Maximum principle estimate)**.** *Let $\nu > 0$ and let $\mathscr{O} \subsetneqq \mathbb{R}^{d-1} \times (0, \nu)$ be a domain, let L be as in Lemma A.1, and let $f \in C_b(\mathscr{O})$, and $g \in C_b(\partial_1 \mathscr{O})$. If u obeys the regularity properties on $\bar{\mathscr{O}}$ in the hypotheses of Lemma A.1 and*

$$\begin{aligned} Lu &= f \quad \text{on } \mathscr{O},\\ u &= g \quad \text{on } \partial_1 \mathscr{O},\end{aligned}$$

then there is a positive constant, $C = C(\nu, b_0, \Lambda)$ with b_0, Λ as in (A.1)*, such that*

$$\|u\|_{C(\bar{\mathscr{O}})} \leq C \left(\|f\|_{C(\bar{\mathscr{O}})} + \|g\|_{C(\overline{\partial_1 \mathscr{O}})} \right).$$

Proof. We define $v = e^{\sigma x_d} u$ as in (A.2), where σ is chosen as in the proof of Lemma A.1. Then,

$$\begin{aligned} \widetilde{L}v &= \tilde{f} \quad \text{on } \mathscr{O},\\ v &= \tilde{g} \quad \text{on } \partial_1 \mathscr{O},\end{aligned}$$

where $\tilde{f} := e^{\sigma x_d} f$ on $\mathscr{O}$ and $\tilde{g} := e^{\sigma x_d} g$ on $\partial_1 \mathscr{O}$. Because $\tilde{c} \geq b_0^2/(4\Lambda) > 0$ on $\mathscr{O}$ from the proof of Lemma A.1, we can apply [16, Proposition 2.19] to give

$$\|v\|_{C(\bar{\mathscr{O}})} \leq \frac{1}{\tilde{c}} \left(\|\tilde{f}\|_{C(\bar{\mathscr{O}})} + \|\tilde{g}\|_{C(\overline{\partial_1 \mathscr{O}})} \right).$$

The conclusion follows since $x_d \in [0, \nu]$ for all $x \in \bar{\mathscr{O}}$ and we can take $C := e^{\sigma\nu}/\tilde{c} = 4\Lambda e^{\sigma\nu}/b_0^2$. □

Appendix B. Existence of solutions for degenerate-elliptic operators with constant coefficients on half-spaces and strips

In this section, we prove existence of smooth solutions to $Lu = f$ on the half-space $\mathbb{H}$ or on strips $S = \mathbb{R}^{d-1} \times (0, \nu)$ as in (1.10), for some $\nu > 0$, when the source function f is assumed to be smooth with compact support in $\bar{\mathbb{H}}$ or in $\mathbb{R}^{d-1} \times [0, \nu)$, respectively, and under the assumption of Hypothesis 3.1, that *the coefficients, a, b, c, of the operator L in* (1.3) *and so the coefficients, a, b, of the operator L_0 in* (1.4) *are constant.* The method of proof is similar to that of [6, Theorem I.1.2] and it is based on taking the Fourier transform in the first $(d-1)$-variables. The problem is then reduced to the study of the Kummer ordinary differential equations whose solutions can be expressed in terms of the confluent hypergeometric functions, M and U [1, §13].

We begin by reviewing the properties of the confluent hypergeometric functions which will be used in the proofs of Theorems 1.5 and B.3.

Lemma B.1 (Properties of the confluent hypergeometric functions)**.** [1] *Let $a \in \mathbb{C}$ be such that its real part is positive, $\Re(a) > 0$, and let b be a positive constant. Then the following holds, for all $y > 0$.*

(1) *Asymptotic behavior as $y \to +\infty$:*

$$M(a,b,y) = \frac{\Gamma(b)}{\Gamma(a)} y^{a-b} e^y \left(1 + O(y^{-1})\right) \quad [1, \S\, 13.1.4], \tag{B.1}$$

$$U(a,b,y) = y^{-a} \left(1 + O(y^{-1})\right) \quad [1, \S\, 13.1.8]. \tag{B.2}$$

(2) *Asymptotic behavior as $y \to 0$:*

$$M(a,b,0) = 1 \quad [1, \S\, 13.1.2] \quad \text{and} \quad M(a,b,y) = 1 + O(y) \quad [1, \S\, 13.5.5], \tag{B.3}$$

and

$$\begin{aligned}
U(a,b,y) &= \frac{\Gamma(b-1)}{\Gamma(a)} y^{1-b} + O(y^{b-2}) \quad \text{if } b > 2, \quad [1, \S\, 13.5.6],\\
U(a,b,y) &= \frac{\Gamma(b-1)}{\Gamma(a)} y^{1-b} + O(|\log y|) \quad \text{if } b = 2, \quad [1, \S\, 13.5.7],\\
U(a,b,y) &= \frac{\Gamma(b-1)}{\Gamma(a)} y^{1-b} + O(1) \quad \text{if } 1 < b < 2, \quad [1, \S\, 13.5.8],\\
U(a,b,y) &= -\frac{1}{\Gamma(a)} (\log y + \psi(a) + 2\gamma) + O(y|\log y|) \quad \text{if } b = 1, \quad [1, \S\, 13.5.9],\\
U(a,b,y) &= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} + O(y^{1-b}) \quad \text{if } 0 < b < 1, \quad [1, \S\, 13.5.10],\\
U(a,b,0) &= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \quad \text{if } 0 < b < 1, \quad [1, \S\, 13.1.2 \;\&\; 13.1.3],
\end{aligned} \tag{B.4}$$

where $\psi(a) = \Gamma'(a)/\Gamma(a)$ and $\gamma \in \mathbb{R}$ is Euler's constant [1, § 6.1.3].

(3) *Differential properties:*

$$M'(a,b,y) = \frac{a}{b} M(a+1, b+1, y) \quad [1, \S\, 13.4.8], \tag{B.5}$$

$$U'(a,b,y) = -aU(a+1, b+1, y) \quad [1, \S\, 13.4.21]. \tag{B.6}$$

(4) *Recurrence relations:*

$$(b-1)M(a-1, b-1, y) = (b-1-y)M(a,b,y) + yM'(a,b,y) \quad [1, \S\, 13.4.14], \tag{B.7}$$

$$U(a-1, b-1, y) = (1-b+y)U(a,b,y) - yU'(a,b,y) \quad [1, \S\, 13.4.27]. \tag{B.8}$$

We can now give the

Proof of Theorem 1.5. Uniqueness of the solution, $u \in C^\infty(\bar{\mathbb{H}})$, follows from the maximum principle [16, Theorem 5.3]. By simple changes of variables described in the proof of [18, Proposition A.1], which leave invariant any strip of the form $\mathbb{R}^{d-1} \times (0, \nu)$, for $\nu > 0$, we may assume without loss of generality that $a^{ij} = \delta^{ij}$ in (1.3) and so the differential operator L has the form

$$Lv = -x_d v_{x_i x_i} - b^i v_{x_i} + cv, \quad \forall v \in C^2(\mathbb{H}),$$

where b^d and c are again positive constants.

We adapt the method of proof of [6, Theorem I.1.2]. We fix $f \in C_0^\infty(\bar{\mathbb{H}})$. If $u \in C^\infty(\bar{\mathbb{H}})$ is a solution to $Lu = f$ on $\mathbb{H}$, then we expect that its Fourier transform in the $x' = (x_1, \dots, x_{d-1})$-variables,

$$\tilde{u}(\xi; x_d) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^{d-1}} u(x', x_d) e^{-ix'\xi}\, dx', \quad \forall \xi \in \mathbb{R}^{d-1},$$

is a solution, for each $\xi \in \mathbb{R}^{d-1}$, to the ordinary differential equation,

$$-x_d \tilde{u}_{x_d x_d}(\xi; x_d) - b^d \tilde{u}_{x_d}(\xi; x_d) + \left(c + i \sum_{k=1}^{d-1} b^k \xi_k + |\xi|^2 x_d \right) \tilde{u}(\xi; x_d) = \tilde{f}(\xi; x_d), \tag{B.9}$$

for all $x_d \in (0, \infty)$, where $\tilde{f}(\xi; x_d)$ is the Fourier transform of $f(x', x_d)$ with respect to $x' \in \mathbb{R}^d$. We show that the ordinary differential equation (B.9) has a smooth enough solution, $\tilde{u}$, in a sense to be specified, such that its inverse Fourier transform,

$$u(x', x_d) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^{d-1}} \tilde{u}(\xi; x_d) e^{ix'\xi}\, d\xi, \tag{B.10}$$

is a $C^\infty(\bar{\mathbb{H}})$ solution to the equation $Lu = f$ on $\mathbb{H}$.

Defining the function $v(\xi; y)$, for $y = 2|\xi| x_d$ and each $\xi \in \mathbb{R}^{d-1} \setminus \{0\}$, by

$$\tilde{u}(\xi; x_d) =: e^{-|\xi| x_d} v(\xi; 2|\xi| x_d), \quad \forall \xi \in \mathbb{R}^{d-1} \setminus \{0\}, \quad \forall x_d \in \mathbb{R}_+, \tag{B.11}$$

we see that v is a solution to the Kummer ordinary differential equation,

$$-y v_{yy}(\xi; y) - (b - y) v_y(\xi; y) + a(\xi) v(\xi; y) = g(\xi; y), \quad \forall y \in \mathbb{R}_+, \tag{B.12}$$

where we denote

$$\begin{aligned} b &:= \frac{b^d}{2}, \\ a(\xi) &:= \frac{c + b^d|\xi| + i \sum_{k=1}^{d-1} b^k \xi_k}{2|\xi|}, \\ g(\xi; y) &:= \frac{e^{y/2}}{2|\xi|} \tilde{f}\left(\xi; \frac{y}{2|\xi|} \right). \end{aligned} \tag{B.13}$$

Because $b^d > 0$ and $c > 0$ by hypothesis, we see that $b > 0$ and $\Re(a(\xi)) > 0$ when $\xi \neq 0$. Since f has compact support in $\bar{\mathbb{H}}$, the function $g(\xi; \cdot)$ also has compact support in $\bar{\mathbb{R}}_+$.

It suffices to study the solutions, $v(\xi; \cdot)$, to the Kummer equations for $\xi \in \mathbb{R}^{d-1} \setminus \{0\}$, and so without loss of generality, we will assume in the sequel that $\xi \neq 0$. The remainder of the proof of Theorem 1.5 is completed in two steps.

Step 1 (Solution to the Kummer ordinary differential equation)**.** The general solution to the Kummer ordinary differential equation (B.12) can be written in the form $v = v^h + v^p$, where

$$\begin{aligned} v^h(\xi; y) &:= c_1 M(a(\xi), b; y) + c_2 U(a(\xi), b; y), \\ v^p(\xi; y) &:= -M(a(\xi), b; y) \int_y^\infty g(\xi; z) \frac{U(a(\xi), b; z)}{W(a(\xi), b; z)}\, dz - U(a(\xi), b; y) \int_0^y g(\xi; z) \frac{M(a(\xi), b; z)}{W(a(\xi), b; z)}\, dz, \end{aligned}$$

with $c_1, c_2 \in \mathbb{R}$, and

$$W(a(\xi), b; y) := -\frac{\Gamma(b) y^{-b} e^y}{\Gamma(a(\xi))}, \quad \forall \xi \in \mathbb{R}^{d-1} \setminus \{0\}, \ \forall y \in \bar{\mathbb{R}}_+, \tag{B.14}$$

is the Wronskian of the Kummer function, $M(a(\xi), b, y)$, and the Tricomi function, $U(a(\xi), b, y)$, [1, § 13.1.22]. We want to find a solution, $v \in C^\infty(\bar{\mathbb{R}}_+)$, to (B.12).

From (B.1), we see that the function $M(a(\xi), b; y)$ is unbounded as y tends to $+\infty$, and so we choose the constant $c_1 = 0$, because we only consider bounded solutions. At $y = 0$, we obtain from (B.6) and (B.4) that $U'(a(\xi), b, y)$ is unbounded, since $b > 0$, and so we choose the constant $c_2 = 0$, because we only consider solutions to the Kummer equation which are smooth on $\bar{\mathbb{R}}_+$. Thus, we obtain

$$\begin{aligned} v(\xi; y) = &-M(a(\xi), b; y) \int_y^\infty g(\xi; z) \frac{U(a(\xi), b; z)}{W(a(\xi), b; z)}\, dz \\ &- U(a(\xi), b; y) \int_0^y g(\xi; z) \frac{M(a(\xi), b; z)}{W(a(\xi), b; z)}\, dz. \end{aligned} \tag{B.15}$$

Given v defined as above, and $\tilde{u}$ defined as in (B.11), we will prove the following properties of the solution, $\tilde{u}$, to verify that u defined by (B.10) is a $C^\infty(\bar{\mathbb{H}})$ solution to $Lu = f$ on $\mathbb{H}$, as asserted by Theorem 1.5.

Lemma B.2 (Properties of $\tilde{u}$)**.** *If $f \in C_0^\infty(\bar{\mathbb{H}})$, then the function $\tilde{u}$ defined by* (B.11) *has the following properties.*

(1) *For all $\xi \in \mathbb{R}^{d-1} \setminus \{0\}$, we have*

$$\lim_{x_d \nearrow \infty} \tilde{u}(\xi; x_d) = 0. \tag{B.16}$$

(2) *The function $\tilde{u}(\xi; \cdot)$ belongs to $C^\infty(\bar{\mathbb{R}}_+)$, for all $\xi \in \mathbb{R}^{d-1} \setminus \{0\}$.*

(3) *The function $\tilde{u}(\xi; \cdot)$ obeys*

$$|\tilde{u}(\xi; x_d)| < \frac{1}{c} \sup_{y \geq 0} |\tilde{f}(\xi; y)|, \quad \forall \xi \in \mathbb{R}^{d-1} \setminus \{0\},\ \forall x_d \in \bar{\mathbb{R}}_+, \tag{B.17}$$

where c is the zeroth-order coefficient of L in (1.3)*.*

(4) *The function $\tilde{u}(\cdot; x_d)$ decays faster than any polynomial in ξ, for all $x_d \in \bar{\mathbb{R}}_+$.*

(5) *The functions $D^k_{x_d} \tilde{u}$ decay faster than any polynomial in ξ, for all $k \in \mathbb{N}$.*

Step 2 (Existence of a solution, $u \in C^\infty(\bar{\mathbb{H}})$, to $Lu = f$ on $\mathbb{H}$)**.** From Lemma B.2, Items (2) and (4), we see that the function u defined by (B.10) has an arbitrary number of derivatives in the first $(d-1)$-variables which are continuous on $\bar{\mathbb{H}}$. From Lemma B.2, Item (2), we see that u also admits an arbitrary number of derivatives in the x_d-variable, and they are continuous on $\bar{\mathbb{H}}$. Now we consider $D^{\beta_d e_d} u$, for $\beta_d \in \mathbb{N}$, which satisfies

$$D^{\beta_d e_d} u(x', x_d) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^{d-1}} D^{\beta_d e_d} \tilde{u}(\xi; x_d) e^{i x' \xi}\, d\xi, \quad \forall x_d \in \mathbb{R}_+.$$

By Lemma B.2, Item (5), the function $D^{\beta_d e_d} \tilde{u}$ decays faster than any polynomial in ξ, and so $D^\lambda D^{\beta_d e_d} u$ exists and is continuous on $\bar{\mathbb{H}}$, for all $\lambda \in \mathbb{N}^d$ with $\lambda_d = 0$. Thus, u belongs to $C^\infty(\bar{\mathbb{H}})$. Since $\tilde{u}$ solves (B.9), we find that u solves $Lu = f$ on $\mathbb{H}$ by taking the inverse Fourier transform of $\tilde{u}(\xi; x_d)$ in $\xi \in \mathbb{R}^{d-1}$. From [16, Theorem 5.3], it follows that u is the unique $C^\infty(\bar{\mathbb{H}})$ solution to $Lu = f$ on $\mathbb{H}$.

Aside from the proof of Lemma B.2, given below, this completes the proof of Theorem 1.5. □

It remains to prove Lemma B.2.

Proof of Lemma B.2. We organize the proof into several steps.

Step 1 (Proof of Item (1))**.** First, we verify that the function v defined in Equation (B.15) is well-defined. We write $v = v^1 + v^2$, where we set

$$v^1(\xi;y) := -M(a(\xi),b;y)\int_y^\infty g(\xi;z)\frac{U(a(\xi),b;z)}{W(a(\xi),b;z)}\,dz,$$
$$v^2(\xi;y) := -U(a(\xi),b;y)\int_0^y g(\xi;z)\frac{M(a(\xi),b;z)}{W(a(\xi),b;z)}\,dz.$$

Recall that $g(\xi;\cdot)$ has compact support in $\bar{\mathbb{R}}_+$, and so for the function $v^1(\xi;y)$, we only need to verify that it is continuous up to $y=0$. From the property (B.3) in Lemma B.1, we know that $M(a(\xi),b;y)$ is continuous in y up to $y=0$ with $M(a(\xi),b;0)=1$. Identities (B.4) and definition (B.14) of the Wronskian imply that

$$\frac{U(a(\xi),b;y)}{W(a(\xi),b;y)} \sim \begin{cases} \max\{y,\ y^{2(b-2)}\}, & \text{if } b>2,\\ y\log y, & \text{if } b=2,\\ y, & \text{if } 1\le b<2,\\ y^b, & \text{if } 0<b<1,\end{cases}$$

and so this function is integrable near $y=0$. Since $g(\xi;\cdot)$ has compact support in $\bar{\mathbb{R}}_+$, we see that $v^1(\xi,\cdot)\in C(\bar{\mathbb{R}}_+)$ and

$$\lim_{y\nearrow\infty} v^1(\xi;y) = 0, \quad \forall \xi\in\mathbb{R}^{d-1}\setminus\{0\}.$$

Next, we consider the behavior of the function $v^2(\xi;\cdot)$. Near $y=0$, the property (B.3) and definition (B.14) of the Wronskian yield

$$\frac{M(a(\xi),b;y)}{W(a(\xi),b;y)} \sim y^b.$$

Combining this result the asymptotic behavior (B.4) of U as $y\to 0$, we find that the limit of $v^2(\xi;y)$, as y tends to 0, exists. The limit of the integral,

$$\int_0^y g(\xi;z)\frac{M(a(\xi),b;z)}{W(a(\xi),b;z)}\,dz,$$

as $y\to\infty$ obviously exists because the function $g(\xi;\cdot)$ has compact support in $\bar{\mathbb{R}}_+$. Moreover, using the asymptotic behavior (B.2) of $U(a(\xi),b;y)$ as $y\to+\infty$, we obtain

$$\lim_{y\nearrow\infty} v^2(\xi;y) = 0, \quad \forall \xi\in\mathbb{R}^{d-1}\setminus\{0\}.$$

Since $v=v^1+v^2$, we obtain the limit property (B.16) for $\tilde{u}$ as $y\to+\infty$ using (B.11).

Step 2 (Proof of Item (2))**.** The argument employed in Step 1 shows that $\tilde{u}(\xi;\cdot)\in C(\bar{\mathbb{R}}_+)$, for all $\xi\in\mathbb{R}^{d-1}\setminus\{0\}$. Next, we want to show that $D^k_{x_d}\tilde{u}(\xi;\cdot)\in C(\bar{\mathbb{R}}_+)$, for all $k\in\mathbb{N}$ and $\xi\in\mathbb{R}^{d-1}\setminus\{0\}$, but for this it is suffices to show that $D^k_y v(\xi;\cdot)\in C(\bar{\mathbb{R}}_+)$, for all $k\in\mathbb{N}$, by (B.11).

We first consider the case $k=1$. A direct calculation shows that

$$v_y(\xi;y) = -M_y(a(\xi),b;y)\int_y^\infty g(\xi;z)\frac{U(a(\xi),b;z)}{W(a(\xi),b;z)}\,dz$$
$$- U_y(a(\xi),b;y)\int_0^y g(\xi;z)\frac{M(a(\xi),b;z)}{W(a(\xi),b;z)}\,dz.$$

Using identities (B.5) and (B.6), we obtain

$$\begin{aligned} v_y(\xi;y) = &-\frac{a(\xi)}{b}M(a(\xi)+1,b+1;y)\int_y^\infty g(\xi;z)\frac{U(a(\xi),b;z)}{W(a(\xi),b;z)}\,dz \\ &+a(\xi)U(a(\xi)+1,b+1;y)\int_0^y g(\xi;z)\frac{M(a(\xi),b;z)}{W(a(\xi),b;z)}\,dz, \end{aligned} \tag{B.18}$$

and the same argument as in the beginning of the proof of Lemma B.2, gives us immediately that $v_y(\xi;\cdot)\in C(\bar{\mathbb{R}}_+)$. Hence, $v(\xi;\cdot)\in C^1(\bar{\mathbb{R}}_+)$, for all $\xi\in\mathbb{R}^{d-1}\setminus\{0\}$.

We next show that $v_y(\xi;\cdot)$ in (B.18) is the unique $C^1(\bar{\mathbb{R}}_+)$ solution to the Kummer equation,

$$-yw_{yy}(\xi;y)-(b+1-y)w_y(\xi;y)+(a(\xi)+1)w(\xi;y)=g_y(\xi;y),\quad \forall y\in\mathbb{R}_+.$$

Our goal is to show that $v_y=w$, where we define

$$\begin{aligned} w(\xi;y) := &-M(a(\xi)+1,b+1;y)\int_y^\infty g_z(\xi;z)\frac{U(a(\xi)+1,b+1;z)}{W(a(\xi)+1,b+1;z)}\,dz \\ &-U(a(\xi)+1,b+1;y)\int_0^y g_z(\xi;z)\frac{M(a(\xi)+1,b+1;z)}{W(a(\xi)+1,b+1;z)}\,dz, \end{aligned}$$

for $y\in\mathbb{R}_+$, $\xi\in\mathbb{R}^{d-1}\setminus\{0\}$. Integrating by parts in the expression of w, we obtain

$$\begin{aligned} w(\xi;y) = &\,M(a(\xi)+1,b+1;y)\int_y^\infty g(\xi;z)\frac{U_zW-UW_z}{W^2}(a(\xi)+1,b+1;z)\,dz \\ &+U(a(\xi)+1,b+1;y)\int_0^y g(\xi;z)\frac{M_zW-MW_z}{W^2}(a(\xi)+1,b+1;z)\,dz. \end{aligned}$$

The expression of v_y in (B.18) coincides with that of w if

$$\begin{aligned} -\frac{U_zW-UW_z}{W^2}(a(\xi)+1,b+1;z) &= \frac{a(\xi)}{b}\frac{U}{W}(a(\xi),b;z), \\ -\frac{M_zW-MW_z}{W^2}(a(\xi)+1,b+1;z) &= -a(\xi)\frac{M}{W}(a(\xi),b;z). \end{aligned}$$

But the preceding two identities follow from the definition of the Wronskian, W, in (B.14), from (B.5) and (B.6), and from the recursion relations (B.7) and (B.8). Hence, the function v_y is the unique $C^1(\bar{\mathbb{R}}_+)$ solution to the corresponding Kummer equation, which obviously implies that $v_{yy}\in C(\bar{\mathbb{R}}_+)$.

Inductively, it follows that, for any $k\in\mathbb{N}$, the derivative $D^k_y v$ exists and is the unique $C^1(\bar{\mathbb{R}}_+)$ solution to the Kummer equation,

$$-y(D^k_y v)_{yy}(\xi;y)-(b+k-y)(D^k_y v)_y(\xi;y)+(a(\xi)+k)D^k_y v(\xi;y)=D^k_y g(\xi;y),\quad \forall y\in\mathbb{R}_+.$$

Thus, $v(\xi;\cdot)\in C^\infty(\bar{\mathbb{R}}_+)$, and so $\tilde{u}(\xi;\cdot)\in C^\infty(\bar{\mathbb{R}}_+)$ by (B.11), and $D^k\tilde{u}(\xi;\cdot)$ satisfies the ordinary differential equation, for all $x_d\in(0,\infty)$ and $k\in\mathbb{N}$,

$$\begin{aligned} &-x_d(D^k_{x_d}\tilde{u})_{x_dx_d}(\xi;x_d)-(b+k-2|\xi|x_d)(D^k_{x_d}\tilde{u})_{x_d}(\xi;x_d)+(a(\xi)+k)D^k_{x_d}\tilde{u}(\xi;x_d) \\ &\quad =(2|\xi|)^{k+1}D^k_y g(\xi;2|\xi|x_d). \end{aligned} \tag{B.19}$$

Notice that the right-hand side in the preceding equation is a function with compact support in $\bar{\mathbb{R}}_+$.

Step 3 (Proof of Items (3) and (4))**.** We adapt the argument of [6, Theorem I.1.2]. We fix $\xi \neq 0$. We write $\tilde{u}(\xi;x_d) = p(\xi;x_d) + iq(\xi;x_d)$ and $\tilde{f}(\xi;x_d) = \tilde{g}(\xi;x_d) + i\tilde{h}(\xi;x_d)$. Then, equation (B.9) becomes

$$\begin{cases} -x_d p_{x_d x_d}(\xi;x_d) - b^d p_{x_d}(\xi;x_d) + (c + x_d|\xi|^2)p(\xi;x_d) - b\xi q(\xi;x_d) = \tilde{g}(\xi;x_d), \\ -x_d q_{x_d x_d}(\xi;x_d) - b^d q_{x_d}(\xi;x_d) + (c + x_d|\xi|^2)q(\xi;x_d) + b\xi p(\xi;x_d) = \tilde{h}(\xi;x_d), \end{cases}$$

where $b\xi$ denotes the inner product of $(b^1,\dots,b^{d-1})$ with ξ. Defining $F(\xi;x_d) := |\tilde{u}(\xi;x_d)|^2 = p^2(\xi;x_d) + q^2(\xi;x_d)$, we obtain (where now we omit the $(\xi;x_d)$-variables)

$$\begin{aligned} F_{x_d} &= 2pp_{x_d} + 2qq_{x_d}, \\ F_{x_d x_d} &= 2p_{x_d}^2 + 2pp_{x_d x_d} + 2q_{x_d}^2 + 2qq_{x_d x_d}, \end{aligned}$$

which gives us

$$\begin{aligned} &x_d F_{x_d x_d} + b^d F_{x_d} - 2cF \\ &\quad = 2p(x_d p_{x_d x_d} + b^d p_{x_d} - cp) + 2q(x_d q_{x_d x_d} + b^d q_{x_d} - cq) + 2x_d(p_{x_d}^2 + q_{x_d}^2) \\ &\quad \geq 2p\left(-\tilde{g} + x_d|\xi|^2 p - b\xi q\right) + 2q\left(-\tilde{h} + x_d|\xi|^2 q + b\xi p\right) \\ &\quad \geq -2p\tilde{g} - 2q\tilde{h} \\ &\quad \geq -cF - \frac{1}{c}\left(\tilde{g}^2 + \tilde{h}^2\right), \end{aligned}$$

where we recall that $c > 0$ by hypothesis, and so it follows that

$$x_d F_{x_d x_d}(\xi;x_d) + b^d F_{x_d}(\xi;x_d) - cF(\xi;x_d) \geq -\frac{1}{c}\sup_{x_d \in \bar{\mathbb{R}}_+} |\tilde{f}(\xi;x_d)|^2, \quad \forall x_d \in \mathbb{R}_+.$$

Now let

$$G(\xi;x_d) := F(\xi;x_d) - \frac{1}{c^2}\sup_{x_d \in \bar{\mathbb{R}}_+} |\tilde{f}(\xi;x_d)|^2.$$

Then,

$$\begin{cases} x_d G_{x_d x_d}(\xi;x_d) + b^d G_{x_d}(\xi;x_d) - cG(\xi;x_d) \geq 0, \\ \lim_{x_d \nearrow \infty} G(\xi;x_d) \leq 0, \end{cases}$$

where we used (B.16) to determine the behavior of $G(\xi;x_d)$ as x_d approaches ∞. Therefore, the function $G(\xi;x_d)$ is bounded, and the maximum principle [16, Theorem 5.3] then implies that $G(\xi,x_d) \leq 0$, for all $x_d \in \bar{\mathbb{R}}_+$, and so

$$|\tilde{u}(\xi;x_d)|^2 \leq \frac{1}{c^2}\sup_{y \in \bar{\mathbb{R}}_+} |\tilde{f}(\xi;y)|^2, \quad \forall x_d \in \bar{\mathbb{R}}_+,$$

which is equivalent to (B.17).

Since f belongs to $C_0^\infty(\bar{\mathbb{H}})$, the function $\sup_{y \in \bar{\mathbb{R}}_+} |\tilde{f}(\xi;y)|$ decays faster than any polynomial in ξ by [23, Theorem 8.22 (e)]. Therefore, from (B.17) we see that the function $\tilde{u}$ also decays faster than any polynomial in ξ.

Step 4 (Proof of Item (5))**.** By (B.19) and the fact that the right-hand side in (B.19) is a function with compact support in $\bar{\mathbb{R}}_+$, we see that the preceding steps can be applied to $D_{x_d}^k \tilde{u}$ instead of $\tilde{u}$, for all $k \in \mathbb{N}$. Therefore, we obtain that the functions $D_{x_d}^k \tilde{u}$ decay faster than any polynomial in ξ, for all $k \in \mathbb{N}$.

This completes the proof of Lemma B.2. □

We now prove the existence and uniqueness of smooth solutions on strips in the half-space. We fix $\nu > 0$ and recall from (1.10) that $S = \mathbb{R}^{d-1} \times (0, \nu)$, so that $\partial_0 S = \mathbb{R}^{d-1} \times \{0\}$ and $\partial_1 S = \mathbb{R}^{d-1} \times \{\nu\}$. We have the following elliptic analogue of [6, Theorem I.1.2] in the parabolic case, but for finite-height strips rather than the half-space.

Theorem B.3 (Existence and uniqueness of a $C^\infty(\bar{S})$ solution on a strip when L has constant coefficients)**.** *Let L be an operator of the form* (1.3) *and require that the coefficients, a, b, c, are constant with $b^d > 0$ and $c \geq 0$. Then, for any function, $f \in C_0^\infty(\bar{S})$, there is a unique solution, $u \in C^\infty(\bar{S})$, to*

$$\begin{cases} Lu = f & \text{on } S, \\ u = 0 & \text{on } \partial_1 S. \end{cases} \tag{B.20}$$

Proof. The method of proof is the same as that of Theorem 1.5, so we only highlight the main differences. Uniqueness of the solution, $u \in C^\infty(\bar{S})$, follows from the maximum principle, Lemma A.1, for L. By analogy with (B.9), for each $\xi \in \mathbb{R}^{d-1} \setminus \{0\}$, we construct the function $\tilde{u}(\xi; \cdot)$ to be the unique solution in $C^\infty([0, x_d^0])$ to

$$-x_d \tilde{u}_{x_d x_d}(\xi; x_d) - b^d \tilde{u}_{x_d}(\xi; x_d) + \left(c + i \sum_{k=1}^{d-1} b^k \xi_k + |\xi|^2 x_d \right) \tilde{u}(\xi; x_d) = \tilde{f}(\xi; x_d), \quad \forall x_d \in (0, \nu),$$
$$\tilde{u}(\xi; \nu) = 0,$$

by defining the new function, $v(\xi; \cdot)$, by (B.11) and proving that $v(\xi; \cdot)$ is the unique solution in $C^\infty([0, x_d^0])$ to the Kummer equation,

$$-y v_{yy}(\xi; y) - (b - y) v_y(\xi; y) + a(\xi) v(\xi; y) = g(\xi; y), \quad \forall y \in (0, 2|\xi|\nu),$$
$$v(\xi; 2|\xi|\nu) = 0,$$

for each $\xi \in \mathbb{R}^{d-1} \setminus \{0\}$, where the coefficients b and $a(\xi)$, and the function g are defined in the same way as in (B.13). The arguments employed in the proof of Theorem B.3 show now that the unique solution in $C^\infty(\bar{S})$ to the preceding ordinary differential equation is given by

$$\begin{aligned} v(\xi; y) := CM(a(\xi), b, y) - M(a(\xi), b; y) \int_y^\infty g(\xi; z) \frac{U(a(\xi), b; z)}{W(a(\xi), b; z)} \, dz \\ - U(a(\xi), b; y) \int_0^y g(\xi; z) \frac{M(a(\xi), b; z)}{W(a(\xi), b; z)} \, dz, \end{aligned}$$

where the constant C is chosen such that the boundary condition, $v(\xi, 2|\xi|\nu) = 0$, is satisfied. The only remaining modification that we need lies in Step 3 of the proof of Lemma B.2. The reason why this part of the proof does not adapt immediately is because we used the fact that the zeroth-order coefficient, c, of L in (1.3) is strictly positive to derive (B.17), while now we assume $c \geq 0$. To circumvent this issue, we apply the method of proof of Step 3 of Lemma B.2 not to F, but to $e^{-\sigma x_d} F$, where we choose the positive constant, σ, small enough. Notice that this is the same as the approach we employed in the proof of Lemma A.1 to overcome the fact that $c = 0$. □

Corollary B.4 (Existence and uniqueness of a $C_s^{k,2+\alpha}$ solution on a strip when L has constant coefficients)**.** *Let $\alpha \in (0,1)$ and $k \in \mathbb{N}$. Let L be an operator as in* (1.3) *and require that the coefficients, a, b, c, are constant with $b^d > 0$ and $c \geq 0$. If $f \in C_s^{k,\alpha}(\bar{S})$, then there is a unique solution $u \in C_s^{k,2+\alpha}(\bar{S})$ to the boundary problem* (B.20).

Proof. Uniqueness of the solution, $u \in C_s^{k,2+\alpha}(\bar S)$, follows from the maximum principle, Lemma A.1, for L since any $u \in C_s^{k,2+\alpha}(\bar S)$ has the property that Du and $x_d D^2 u$ are continuous on $\bar S$ by Definition 2.3 and that $x_d D^2 u = 0$ on $\partial_0 S$ by Lemma C.1. Let $\{f_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\bar S)$ be a sequence such that $f_n \to f$ in $C_s^{k,2+\alpha}(\bar S)$ as $n \to \infty$ and

$$\|f_n\|_{C_s^{k,\alpha}(\bar S)} \leq C\|f\|_{C_s^{k,\alpha}(\bar S)}.$$

Such a sequence can be constructed using [6, Theorem I.11.3]. Let $u_n \in C^\infty(\bar S)$ be the unique solution to (B.20), with f replaced by f_n, given by Theorem B.3. In particular, each solution satisfies the global Schauder estimate (1.15) which, when combined with the preceding inequality, gives

$$\|u_n\|_{C_s^{k,2+\alpha}(\bar S)} \leq C\|f\|_{C_s^{k,\alpha}(\bar S)}, \quad \forall n \in \mathbb{N}.$$

By applying the Arzelà-Ascoli Theorem, we can extract a subsequence, which we continue to denote by $\{u_n\}_{n\in\mathbb{N}}$, which converges in $C_s^{k,2+\alpha}(\bar S)$ to a limit function $u \in C_s^{k,2+\alpha}(\bar S)$ as $n \to \infty$. Since $\{f_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}$, $\{Du_n\}_{n\in\mathbb{N}}$ and $\{x_d D^2 u_n\}_{n\in\mathbb{N}}$ also converge uniformly on compact subsets of $\bar S$ to f, u, Du and $x_d D^2 u$, respectively, as $n \to \infty$, we see that u solves (B.20). □

Appendix C. Interpolation inequalities and boundary properties of functions in weighted Hölder spaces

A parabolic version of following result is included in [6, Proposition I.12.1] when $d = 2$ and proved in [18] when $d \geq 2$ for parabolic weighted Hölder spaces. For completeness, we restate the result here for the elliptic weighted Hölder spaces used in this article.

Lemma C.1 (Boundary properties of functions in weighted Hölder spaces)**.** [18, Lemma 3.1] *If $u \in C_s^{2+\alpha}(\underline{\mathbb{H}})$ then, for all $x^0 \in \partial\mathbb{H}$,*

$$\lim_{\mathbb{H}\ni x\to x^0} x_d D^2 u(x) = 0. \tag{C.1}$$

In [18], we also proved the following interpolation inequalities parabolic weighted Hölder spaces analogous to those for standard parabolic Hölder spaces [30, 32]. For completeness, we restate these interpolation inequalities below for elliptic weighted Hölder spaces, analogous to those for standard elliptic Hölder spaces in [26, Lemmas 6.32 & 6.35], [30, Theorem 3.2.1].

Lemma C.2 (Interpolation inequalities for weighted Hölder spaces)**.** [18, Lemma 3.2] *Let $\alpha \in (0,1)$ and $r_0 > 0$. Then there are positive constants, $m = m(d,\alpha)$ and $C = C(d, r_0, \alpha)$, such that the following holds. If $u \in C_s^{2+\alpha}(\bar B_{r_0}^+(x^0))$, where $x^0 \in \partial\mathbb{H}$, and $\varepsilon \in (0,1)$, then*

$$\|u\|_{C_s^\alpha(\bar B_{r_0}^+(x^0))} \leq \varepsilon\|u\|_{C_s^{2+\alpha}(\bar B_{r_0}^+(x^0))} + C\varepsilon^{-m}\|u\|_{C(\bar B_{r_0}^+(x^0))}, \tag{C.2}$$

$$\|Du\|_{C(\bar B_{r_0}^+(x^0))} \leq \varepsilon\|u\|_{C_s^{2+\alpha}(\bar B_{r_0}^+(x^0))} + C\varepsilon^{-m}\|u\|_{C(\bar B_{r_0}^+(x^0))}, \tag{C.3}$$

$$\|x_d Du\|_{C_s^\alpha(\bar B_{r_0}^+(x^0))} \leq \varepsilon\|u\|_{C_s^{2+\alpha}(\bar B_{r_0}^+(x^0))} + C\varepsilon^{-m}\|u\|_{C(\bar B_{r_0}^+(x^0))}, \tag{C.4}$$

$$\|x_d D^2 u\|_{C(\bar B_{r_0}^+(x^0))} \leq \varepsilon\|u\|_{C_s^{2+\alpha}(\bar B_{r_0}^+(x^0))} + C\varepsilon^{-m}\|u\|_{C(\bar B_{r_0}^+(x^0))}. \tag{C.5}$$

We add here the following

Lemma C.3 (Hölder continuity for $x_d Du$)**.** *Let $r > 0$, and assume that $u \in C^2(B_r)$ is such that Du and $x_d D^2 u$ belong to $C(\bar B_r^+)$. Then, $x_d Du \in C_s^\alpha(\bar B_r^+)$.*

Proof. For this we only need to show that for any $x^1, x^2 \in B^+_{r_2}$ such that all their coordinates coincide, except for the i-th one, we have

$$\frac{|x^1_d Du(x^1) - x^2_d Du(x^2)|}{s(x^1,x^2)^\alpha} \leq C,$$

for some positive constant, C. We show this for the case $i = d$, and all the other cases, $i = 1,\dots,d-1$, follow by a similar argument. We have

$$\begin{aligned}\frac{|x^1_d Du(x^1) - x^2_d Du(x^2)|}{s(x^1,x^2)^\alpha} &\leq \frac{|x^1_d - x^2_d|}{s(x^1,x^2)^\alpha}|Du(x^1)| + x^2_d \frac{|Du(x^1) - Du(x^2)|}{s(x^1,x^2)^\alpha} \\ &\leq \left(\|Du\|_{C(\bar{B}^+_{r_2})} + x^2_d |D^2 u(x^3)|\right) \frac{|x^1 - x^2|}{s(x^1,x^2)^\alpha},\end{aligned}$$

where $x^3 \in B_{r_3}$ is a point on the line connecting x^1 and x^2, and we apply the Mean Value Theorem. We may assume without loss of generality that $x^2_d < x^1_d$, and because $x^3_d \geq x^2_d$, we have that $x^2_d |D^2u(x^3)| \leq \|x_d D^2 u\|_{C(\bar{B}^+_{r_2})}$. Using the definition (2.1) of the cycloidal metric, we obtain

$$\frac{|x^1_d Du(x^1) - x^2_d Du(x^2)|}{s(x^1,x^2)^\alpha} \leq \left(\|Du\|_{C(\bar{B}^+_{r_2})} + \|x_d D^2 u\|_{C(\bar{B}^+_{r_2})}\right) r_2^{1-\alpha/2}.$$

Therefore, $x_d Du$ belongs to $C^\alpha_s(\bar{B}^+_{r_2})$, for all $\alpha \in (0,1)$. This completes the proof of Lemma C.3 □

References

[1] M. Abramovitz and I. A. Stegun, *Handbook of mathematical functions*, Dover, New York, 1972.

[2] R. A. Adams, *Sobolev spaces*, Academic Press, Orlando, FL, 1975.

[3] S. R. Athreya, M. T. Barlow, R. F. Bass, and E. A. Perkins, *Degenerate stochastic differential equations and super-Markov chains*, Probab. Theory Related Fields **123** (2002), 484–520.

[4] A. Brandt, *Interior estimates for second-order elliptic differential (or finite-difference) equations via the maximum principle*, Israel J. Math. **7** (1969), 95–121.

[5] P. Daskalopoulos and P. M. N. Feehan, *Existence, uniqueness, and global regularity for variational inequalities and obstacle problems for degenerate elliptic partial differential operators in mathematical finance*, `arxiv.org/abs/1109.1075v1`.

[6] P. Daskalopoulos and R. Hamilton, *C^∞-regularity of the free boundary for the porous medium equation*, J. Amer. Math. Soc. **11** (1998), 899–965.

[7] P. Daskalopoulos and E. Rhee, *Free-boundary regularity for generalized porous medium equations*, Commun. Pure Appl. Anal. **2** (2003), 481–494.

[8] A. De Simone, H. Knüpfer, and F. Otto, *2-d stability of the Néel wall*, Calc. Var. Partial Differential Equations **27** (2006), 233–253.

[9] D. Duffie, J. Pan, and K. Singleton, *Transform analysis and asset pricing for affine jump diffusions*, Econometrica **68** (2000), 1343–1376.

[10] C. L. Epstein and R. Mazzeo, *Wright-Fisher diffusion in one dimension*, SIAM J. Math. Anal. **42** (2010), 568–608.

[11] ———, *Degenerate diffusion operators arising in population biology*, (2011), 341 pages, `arxiv.org/abs/1110.0032`.

[12] E. B. Fabes, *Properties of nonnegative solutions of degenerate elliptic equations*, Proceedings of the international conference on partial differential equations dedicated to Luigi Amerio on his 70th birthday (Milan/Como, 1982), vol. 52, 1982, pp. 11–21 (1985).

[13] E. B. Fabes, C. E. Kenig, and R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), 77–116.

[14] P. M. N. Feehan, *The Heston partial differential equation, boundary value problems, and confluent hypergeometric functions*, unpublished manuscript available on request.

[15] ———, *Integral transform solutions to the Heston PDE*, unpublished manuscript available on request.

[16] ———, *Partial differential operators with non-negative characteristic form, maximum principles, and uniqueness for boundary value and obstacle problems*, arxiv.org/abs/1204.6613v1.

[17] P. M. N. Feehan and C. A. Pop, *Degenerate elliptic operators in mathematical finance and Hölder continuity for solutions to variational equations and inequalities*, arxiv.org/abs/1110.5594v2.

[18] ———, *Degenerate-parabolic partial differential equations with unbounded coefficients, martingale problems, and a mimicking theorem for Itô processes*, arxiv.org/abs/1112.4824v1.

[19] ———, *Higher-order regularity for solutions to degenerate elliptic variational equations in mathematical finance*, arxiv.org/abs/1208.2658.

[20] ———, *A priori Schauder estimates and regularity of solutions to degenerate-elliptic linear second-order partial differential equations*, submitted.

[21] G. Fichera, *Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine*, Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8) **5** (1956), 1–30.

[22] ———, *On a unified theory of boundary value problems for elliptic-parabolic equations of second order*, Boundary problems in differential equations, Univ. of Wisconsin Press, Madison, 1960, pp. 97–120.

[23] G. B. Folland, *Real analysis*, second ed., Wiley, New York, 1999.

[24] L. Giacomelli and H. Knüpfer, *A free boundary problem of fourth order: classical solutions in weighted Hölder spaces*, Comm. Partial Differential Equations **35** (2010), 2059–2091.

[25] L. Giacomelli, H. Knüpfer, and F. Otto, *Smooth zero-contact-angle solutions to a thin-film equation around the steady state*, J. Differential Equations **245** (2008), 1454–1506.

[26] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, second ed., Springer, New York, 1983.

[27] S. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of Financial Studies **6** (1993), 327–343.

[28] H. Koch, *Non-Euclidean singular integrals and the porous medium equation*, Habilitation Thesis, University of Heidelberg, 1999, www.mathematik.uni-dortmund.de/lsi/koch/publications.html.

[29] J. J. Kohn and L. Nirenberg, *Degenerate elliptic-parabolic equations of second order*, Comm. Pure Appl. Math. **20** (1967), 797–872.

[30] N. V. Krylov, *Lectures on elliptic and parabolic equations in Hölder spaces*, American Mathematical Society, Providence, RI, 1996.

[31] S. Z. Levendorskiĭ, *Degenerate elliptic equations*, Kluwer, Dordrecht, 1993.

[32] G. M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[33] M. K. V. Murthy and G. Stampacchia, *Boundary value problems for some degenerate elliptic operators*, Ann. Mat. Pura Appl. **80** (1968), 1–122.

[34] ———, *Errata corrige: "Boundary value problems for some degenerate-elliptic operators"*, Ann. Mat. Pura Appl. (4) **90** (1971), 413–414.

[35] O. A. Oleĭnik and E. V. Radkevič, *Second order equations with nonnegative characteristic form*, Plenum Press, New York, 1973.

[36] E. V. Radkevič, *Equations with nonnegative characteristic form. I*, J. Math. Sci. **158** (2009), 297–452.

[37] ———, *Equations with nonnegative characteristic form. II*, J. Math. Sci. **158** (2009), 453–604.

[38] E. Rhee, *Free boundary regularity in quasi linear degenerated diffusion equations*, Ph.D. thesis, University of California, Irvine, 2000.

(PF) Department of Mathematics, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019

E-mail address: feehan@math.rutgers.edu

(PF) Max Planck Institut für Mathematik in der Naturwissenschaft, Inselstrasse 22, 04103 Leipzig, Germany

(CP) Department of Mathematics, University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104-6395

E-mail address: cpop@math.upenn.edu