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Hierarchical quadrature for multidimensional singular integrals - part II

by

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Abstract

In a previous part I, we introduced a method for the evaluation of singular integrals arising in the discretization of integral equations based on the repeated subdivision of domains. The integrals defined on these subdomains are classified such that a class of integrals can be expressed as a sum of regular integrals and representatives of other classes. A system of equations describes the relations between the classes. Furthermore the approximate value of the singular integrals only depends on the accuracy of the calculation of regular integrals. Part I left a gap on certain parameter configurations on which the mentioned system of equations is irregular. This paper shall close this gap. To this end, we introduce an alternative splitting strategy based on a modified Hadamard partie finie integral.

1 Introduction

Let us consider integrals as they arise in the discretization of integral equations. In simplified notation, terms of the form

$$\mathbb{I} = \int_{D^x} \int_{D^y} \|x - y\|^{\alpha} \mathrm{d}y\, \mathrm{d}x \quad \text{or} \quad \mathbb{I} = \int_{D^x} \int_{\partial D^y} \|x - y\|^{\alpha} \mathrm{d}y\, \mathrm{d}x \tag{1}$$

over domains D^x, $D^y \subset \mathbb{R}^n$ appear.

$\mathbb{I}$ is integrable for all $\alpha \in \mathbb{R}$, if the distance between the domains D^x and D^y is strictly positive. Likewise the integral is regular, if $\alpha \in 2\mathbb{N}_0$. In this case the integral can be evaluated exactly or treated by standard quadrature techniques [11], [3].

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Therefore we want to consider the case $\alpha \in \mathbb{R} \setminus 2\mathbb{N}_0$ in which D^x and D^y share at least one common point. If $-1 < \alpha < 0$, the kernel function is improperly integrable. If $\alpha \leq -1$ the integrals are strongly singular and $\mathbb{I}$ has to be interpreted as a partie finie integral in the sense of Hadamard [6], [4], [7]. Our assumptions are explained for

$$\mathbb{I} = \int_{D^x} \int_{D^y} \kappa(x,y) \, \mathrm{d}y \, \mathrm{d}x \tag{2}$$

with a kernel κ satisfying particular conditions.

In [8] we adapt the method given in [2] to arbitrary dimensions n for cubical domains and to $n \leq 3$ for simplicial ones. But it cannot be applied to certain hypersingular integrals. For example, if an n-dimensional element and an m-dimensional element share a d-dimensional face, we are restricted to $\alpha \neq -(n+m) + d^*$, $\forall d^* = d \ldots 0$. In the case of $n = m = 1$, [2] introduces a solution to the problem in the form of an alternative splitting strategy. This is based on the concept of Hadamard integrals and cannot be generalized to higher dimensions as it would require the use of curved subdomains.

With this paper we shall close this gap by replacing the Euclidean norm used in the definition of Hadamard integrals by the maximum norm. So the concepts given in [2] can be used for higher dimensions as well.

2 Basic definitions

The hierarchical quadrature mainly exploits two characteristics of the integrand $\kappa(x,y)$, denoted as translation invariance and homogeneity. Both are defined as follows.

Definition 2.1 (translational invariance)
A kernel $\kappa(x,y)$ is called *translational invariant*, if it satisfies

$$\kappa(x,y) = \kappa(x+c, y+c) \quad (c \in \mathbb{R}^n). \tag{3}$$

Definition 2.2 (homogeneity)
A kernel $\kappa(x,y)$ is called *homogeneous*, if it satisfies

$$\kappa(x,y) = s^g \kappa(sx, sy) \quad (s \in \mathbb{R}_{>0}), \tag{4}$$

where $g \in \mathbb{R}$ is the *degree of homogeneity*. Typically the parameters g and $-\alpha$ from (1) will coincide.

Furthermore, κ is assumed to be sufficiently smooth outside a neighborhood of the possible singularity at $x = y$. A useful characterization of the smoothness is given by the asymptotic smoothness (confer for instance [5]).

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Definition 2.3 (asymptotic smoothness)
The kernel $\kappa(x,y)$ is called *asymptotically smooth*, if there are constants $h_0, h_1 \in \mathbb{R}_{>0}$ such that

$$|\partial_x^\nu \partial_y^\mu \kappa(x,y)| \leq h_1 (h_0)^{|\nu+\mu|} (\mu+\nu)! \parallel x-y \parallel^{-g-|\nu+\mu|} \tag{5}$$

holds for all multi-indices $\nu \in \mathbb{N}_0^n$, $\mu \in \mathbb{N}_0^m$, and $g \in \mathbb{R}$ as mentioned in Definition 2.2.

Remark 2.4 (on the regularity of the grid)
We consider integrals as they arise in the discretization of integral equations. Therefore we assume that the domains D^x, D^y originate from a regular grid without hanging nodes. Two domains can either have a positive distance, share a common face (edge, vertex, ...) or are identical.

The next definition clarifies the designation of pairs of domains.

Definition 2.5 (regular and singular pairs of domains)
A pair of two domains is called *regular*, if the distance between the domains is strictly positive. Otherwise it is called *singular*. The dimension of the common face in the singular case is denoted by $d \geq 0$. The set of regular pairs is denoted by Γ_{reg} and the set of singular pairs sharing a common d-dimensional face is denoted by Γ_d.

Pairs of domains can be classified using equivalence classes. The underlying equivalence relation is defined next.

Definition 2.6 (equivalence of domains and of pairs of domains)
We call two domains ρ, $\sigma \subset \mathbb{R}^n$ *equivalent*, if a bijective transformation $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ exists satisfying

$$\Phi(\sigma) = \rho \tag{6}$$

where Φ is a mapping given by

$$x = \Phi(\hat{x}) = t + c\mathbf{I}\hat{x} \quad (x \in \rho, \ \hat{x} \in \sigma),$$

with a constant vector $t \in \mathbb{R}^n$, a constant $c \in \mathbb{R}$, and the identity matrix $\mathbf{I}$. The transformation Φ allows scaling and translation of elements but no rotations.

Furthermore, we call two pairs of domains $\rho_1 \times \rho_2$ and $\sigma_1 \times \sigma_2$ *equivalent*, if

$$\begin{aligned} \Phi(\sigma_1) &= \rho_1, \\ \Phi(\sigma_2) &= \rho_2 \end{aligned} \tag{7}$$

holds with the same mapping Φ.

Let us now recall the most important ideas of hierarchical quadrature.

3 Basic concepts of hierarchical quadrature

This section gives a short summarization of [8], collecting only the basic facts and ideas of hierarchical quadrature.

3

3.1 Hypercubical domains

Consider an arbitrary axis-aligned n-cube C. The n-dimensional reference cube $\gamma^n :=$ $[0,1]^n$ can be mapped to C by a linear transformation Φ^C. Due to this mapping we can transform the integral $\mathbb{I}$ to an integral over n- and m-dimensional reference elements and write

$$\mathbb{I} = \int_{C^x} \int_{C^y} \kappa(x,y)\,\mathrm{d}y\,\mathrm{d}x = c_1 \int_{\gamma^n} \int_{\gamma^m} \kappa_\gamma(\hat{x},\hat{y},0,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x} \tag{8}$$

with

$$\begin{aligned}
\kappa_\gamma(\hat{x},\hat{y},\upsilon^x,\upsilon^y) &:= \kappa\left(\Phi^{C^x}(\hat{x})+\upsilon^x, \Phi^{C^y}(\hat{y})+\upsilon^y\right) \\
&= \kappa(x+\upsilon^x, y+\upsilon^y) \qquad (\upsilon^x,\upsilon^y \in \mathbb{R}^n)
\end{aligned} \tag{9}$$

and c_1 being the product of the Jacobian determinants of $\Phi^{C^x} : \mathbb{R}^n \to \mathbb{R}^n$ and $\Phi^{C^y} : \mathbb{R}^m \to \mathbb{R}^n$, w.l.o.g. $n \geq m > 0$.

Remark 3.1 (on the subdivision of n-cubes)
Any n-cube C given by its vertices $\{v_1,\ldots,v_{2^n}\}$ can be subdivided into 2^n subcubes:

$$\begin{aligned}
C_1 &= conv\{v_1, v_{12}, \ldots, v_{1,2^n}\} \\
C_2 &= conv\{v_{21}, v_2, v_{23}, \ldots, v_{2,2^n}\} \\
&\vdots \\
C_{2^n-1} &= conv\{v_{2^n-1,1}, \ldots, v_{2^n-1,2^n-2}, v_{2^n-1}, v_{2^n-1,2^n}\} \\
C_{2^n} &= conv\{v_{2^n,1}, \ldots, v_{2^n-1,2^n}, v_{2^n}\}
\end{aligned}$$

with $v_{ij} := \frac{v_i + v_j}{2}$ denoting the midpoint between two vertices v_i and v_j.

Remark 3.2 (on equivalent elements)
Let $C^x \subset \mathbb{R}^n$ and $C^y \subset \mathbb{R}^m$ be two arbitrary hypercubes embedded in $\mathbb{R}^n$ sharing a common d-dimensional face, so w.l.o.g.: $n \geq m \geq d \geq 0$ and $m > 0$, with a labeling as follows:

$$\begin{aligned}
C^x &= conv\{v_1, \ldots, v_{2^d}, v^x_{2^d+1}, \ldots, v^x_{2^n}\}, \\
C^y &= conv\{v_1, \ldots, v_{2^d}, v^y_{2^d+1}, \ldots, v^y_{2^m}\}.
\end{aligned}$$

Furthermore let these elements be subdivided according to the rules mentioned in Remark 3.1. Then for the pairs $C^x_i \times C^y_i$, $i \in \{1,\ldots,2^d\}$, there exists a labeling such that these elements are equivalent to $C^x \times C^y$ with respect to Definition 2.6 using $c = \frac{1}{2}$. An example is given in Figure 1.

With the Remarks 3.1 and 3.2 in mind, we express the integral $\mathbb{I}$ as follows:
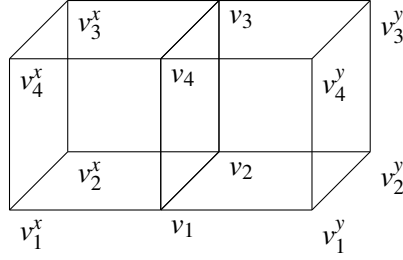
Figure 1: A possible labeling in the case of two cubes sharing a common 2-face.

$$\begin{aligned}
\mathbb{I} &= 2^d c \mathbb{I} && + c_1 \sum_{\gamma_i^n \times \gamma_j^m \in \Gamma_{\mathrm{reg}}} \int_{\gamma_i^n} \int_{\gamma_j^m} \kappa_\gamma(\hat{x}, \hat{y}, 0, 0)\, \mathrm{d}\hat{y}\, \mathrm{d}\hat{x} \\
& && + c_1 \sum_{d^*=0}^{d-1} \sum_{\gamma_i^n \times \gamma_j^m \in \Gamma_{d^*}} \int_{\gamma_i^n} \int_{\gamma_j^m} \kappa_\gamma(\hat{x}, \hat{y}, 0, 0)\, \mathrm{d}\hat{y}\, \mathrm{d}\hat{x} \\
&= \frac{c_1}{1 - 2^d c} \left(\sum_{\gamma_i^n \times \gamma_j^m \in \Gamma_{\mathrm{reg}}} \int_{\gamma_i^n} \int_{\gamma_j^m} \kappa_\gamma(\hat{x}, \hat{y}, 0, 0)\, \mathrm{d}\hat{y}\, \mathrm{d}\hat{x} \right. \\
& \qquad \left. + \sum_{d^*=0}^{d-1} \sum_{\gamma_i^n \times \gamma_j^m \in \Gamma_{d^*}} \int_{\gamma_i^n} \int_{\gamma_j^m} \kappa_\gamma(\hat{x}, \hat{y}, 0, 0)\, \mathrm{d}\hat{y}\, \mathrm{d}\hat{x} \right)
\end{aligned} \tag{10}$$

which is only valid for constants

$$c = \left(\frac{1}{2}\right)^{n+m+\alpha}, \qquad \alpha \neq -(n+m) + d.$$

The sums on the right-hand side of (10) contain only regular terms and singular ones of lower dimension. After a maximum of d applications of the method to the remaining singular integrals, all singular parts can be expressed in terms of regular integrals.

In a further step, we introduce equivalence classes induced by the relation given in Definition 2.6 to decrease the number of regular integrals and reuse already achieved results. The number N of equivalence classes is given by

$$N = \sum_{d^*=0}^{d} N_{d^*} = \sum_{d^*=0}^{d} 2^{d-d^*} \binom{d}{d^*}$$

and the solution can be described by a system of linear equations of the form $\mathbf{A}\hat{\mathbb{I}} = b$. Each row of the $N \times N$-matrix $\mathbf{A} = a_{ij}|_{i,j=1}^{N}$, which has triangular structure if an appropriate sorting of the equivalence classes is used, describes the representation of one integral

$\mathbb{I}_i$, $i = 1 \dots N$, defined on a representative of the class i in terms of all equivalence classes. The diagonal and the off-diagonal elements read

$$a_{ii} = 1 - 2^{d^*} c = 1 - 2^{d^* - n - m - \alpha},$$
$$a_{ij} = -n_{ij} c,$$

where n_{ij} reflects how often class j occurs in the representation of class i.

The vector $\hat{\mathbb{I}}$ stores the values of the remapped representatives $\mathbb{I}_i$ of the equivalence classes which have to be determined, and each entry of b collects the values of the sum of the regular integrals of the class in question. This method enables us to calculate the value of the integral $\mathbb{I}$ for all $\alpha \neq -(n+m) + d^*$, $\forall d^* = d \dots 0$.

3.2 Simplicial domains

The n-simplex T is the convex hull of a set of $n+1$ affinely independent points in Euclidean space of dimension n or higher. Working with simplices is not as straightforward as dealing with hypercubes as the splitting is more involved and the number of equivalence classes is higher. Compare [8] for further details and examples. As before, the singular integrals $\mathbb{I}$,

$$\mathbb{I} = \int_{T^x} \int_{T^y} \kappa(x, y) \, \mathrm{d}y \, \mathrm{d}x = c_1 \int_{\tau^n} \int_{\tau^m} \kappa_\tau(\hat{x}, \hat{y}, 0, 0) \, \mathrm{d}\hat{y} \, \mathrm{d}\hat{x}$$

with a kernel transformation as given in (9), may be mapped to the reference element τ by the same method shown in Section 3.1 and expressed in terms of regular integrals by means of a linear system. This system is again solvable with a unique solution for $\alpha \neq -(n+m) + d^*$, $\forall d^* = d \dots 0$.

4 Hadamard partie finie integrals

The methods described above cannot be applied to certain hypersingular integrals. The paper [2] introduces a solution for the case $n = m = 1$, which is based on the concept of Hadamard integrals and cannot be generalized to higher dimensions as it would require the use of curved subdomains.

In the following sections we are going to overcome this problem by replacing the Euclidean norm by the maximum norm in the definition of the Hadamard partie finie. In this way, the concepts given in [2] can be used for higher dimensions as well.

4.1 Concepts of Hadamard partie finie integrals

Let $f(x)$ be a locally integrable and homogeneous[1] function with degree g in $\mathbb{R} \setminus \{0\}$ and let f vanish for $x < 0$. Moreover let

$$\mathbb{I}(\varepsilon) = \int_\varepsilon^a f(x)\,\mathrm{d}x \qquad (a > \varepsilon > 0)\,.$$

Suppose furthermore, $\mathbb{I}(\varepsilon)$ can be written as

$$\mathbb{I}(\varepsilon) = \mathbb{I}_0(\varepsilon) + \mathbb{I}_\infty(\varepsilon), \tag{11}$$

with $\mathbb{I}_\infty(\varepsilon) = \sum_{i=0}^k \beta_i \phi_i(\varepsilon)$, $k := \max\{m \in \mathbb{Z}|m \leq -1-g\}$ using coefficients $\beta_i \in \mathbb{R}$ and functions $\phi_1(\varepsilon)\ldots\phi_n(\varepsilon)$ being taken from a fixed set of functions which consists usually of inverse powers of ε and logarithms yielding

$$\lim_{\varepsilon \to 0} \mathbb{I}_\infty(\varepsilon) = \infty \qquad (\forall \beta_i \neq 0)\,,$$

while

$$\lim_{\varepsilon \to 0} \mathbb{I}_0(\varepsilon) = h \in \mathbb{R}$$

exists. $\mathbb{I}_0(\varepsilon)$ and $\mathbb{I}_\infty(\varepsilon)$ are called the *finite* and the *infinite part* of the integral $\int_0^a f(x)\,\mathrm{d}x$. The *Hadamard partie finie* is then defined as [4]

$$\text{p.f.} \int_0^a f(x)\,\mathrm{d}x = h. \tag{12}$$

This definition can be easily extended to a multidimensional integral by

$$\text{p.f.} \int_\Omega f(x)\,\mathrm{d}x = \lim_{\varepsilon \to 0} \left\{ \int_{\Omega \setminus \|x\|_2 < \varepsilon} f(x)\,\mathrm{d}x - \sum_{i=0}^k \beta_i \phi_i(\varepsilon) \right\} \tag{13}$$

with $k := \max\{m \in \mathbb{Z}|m \leq -\dim(\Omega) - g\}$, β_i and $\phi_i(\varepsilon)$ as before. If the integral $\mathbb{I}$ exists in regular sense or as a Cauchy principal value, one can show the identity of the expressions.

4.2 Fixing the singularity by relative coordinates

In contrast to the notation used in (13), this paper is based on a double integral notation, derived from the finite element method. This entails that the singularity is not obviously

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[1] In general $f(x)$ has to be only a *pseudohomogeneous function* [7] which is a slightly weaker definition than postulated in Definition 2.2. But as we confined ourselves in this paper to homogeneous functions, we shall use this definition here as well.

located in the origin. Because of this the introduction of relative coordinates z as used in [10] seems favorable.

Let $D^x \subset \mathbb{R}^n$ and $D^y \subset \mathbb{R}^m$ be domains embedded in $\mathbb{R}^n$, $D^\circ \in \{C,T\}$, $\circ \in \{x,y\}$, with a common d-dimensional face, w.l.o.g., $n \geq m \geq d \geq 0$ and $m > 0$. Furthermore let $\Phi^{D^x} : \rho^n \to D^x$ and $\Phi^{D^y} : \rho^m \to D^y$ be mappings defined on the reference elements $\rho \in \{\gamma, \tau\}$ to the corresponding domains as introduced before. We now focus on $\Phi^{D^x}\left(\overline{\rho^n}|_{\hat{x}_i}\right) \cap \Phi^{D^y}\left(\overline{\rho^m}|_{\hat{y}_i}\right) = \sigma$ for $i \in \{1,\ldots,m\}$, $x_i \in \rho^n$ and $y_i \in \rho^m$. Depending on the shape of σ we distinguish the following cases:

1. σ is a line segment: One direction of z is given as $\hat{y}_i - \hat{x}_i$.

2. σ is a single point: One direction of z corresponds to $\hat{x}_i$ and one to $\hat{y}_i$, respectively.

The following example describes the procedure. The dimension p of the relative coordinates z is given by

$$p = n + m - d.$$

Example 4.1 (coordinates z in the case of two squares)
Let $D^x = [0,1] \times [0,1]$ and $D^y = [-1,0] \times [0,1]$ sharing a common one-dimensional face. With $\rho = \gamma^2 = [0,1] \times [0,1]$ we note

$$\Phi^{D^x}\left(\overline{\gamma^2}|_{\hat{x}_1}\right) \cap \Phi^{D^y}\left(\overline{\gamma^2}|_{\hat{y}_1}\right) = \{0\},$$

$$\Phi^{D^x}\left(\overline{\gamma^2}|_{\hat{x}_2}\right) \cap \Phi^{D^y}\left(\overline{\gamma^2}|_{\hat{y}_2}\right) = [0,1].$$

z is therefore given as

$$z = \{x_1, y_1, y_2 - x_2\}.$$

Remark 4.2 (on the ordering of entries in z)
The ordering of entries in z is arbitrary.

After fixing the singularity at the origin, the singularity can be enclosed by a p-dimensional ε-ball in order to apply the above and subsequent statements.

4.3 Replacing an ε-ball by an ε-cube

As mentioned, the definition given in (13) cannot be used to generalize the concepts given in [2] to higher dimensions as it would require the use of curved domains. But we shall show that it is possible to replace the ε-ball used in the definition of the multidimensional Hadamard integral by an axis-aligned hypercube. This implies the usage of the maximum norm instead the Euclidean norm in (13).

Let $B_\varepsilon(z)$ be an n-dimensional ε-ball around the point z and let furthermore $H_\varepsilon(z)$ be the smallest axis-aligned hypercube containing $B_\varepsilon(z)$. Using this notation, we rewrite (13) as an Hadamard integral:

$$\text{p.f.} \int_\Omega f(z)\,dz = \lim_{\varepsilon \to 0} \left\{ \int_{\Omega \backslash B_\varepsilon(0)} f(z)\,dz - \sum_{i=0}^k \beta_i \phi_i(\varepsilon) \right\},$$

$$= \lim_{\varepsilon \to 0} \left\{ \int_{\Omega \backslash H_\varepsilon(0)} f(z)\,dz + \int_{H_\varepsilon(0) \backslash B_\varepsilon(0)} f(z)\,dz - \sum_{i=0}^k \beta_i \phi_i(\varepsilon) \right\}.$$

We have to investigate the characteristics of the second integral on the right-hand side. We restrict ourselves to kernels of the shape $f(z) = \|z\|^\alpha$. Furthermore we use n-dimensional polar coordinates to determine the integral defined on the difference between the hypercube H_ε and the inner ball B_ε.

Remark 4.3 (on polar coordinates in higher dimensions)
The n-dimensional polar coordinates are given by [1] $(r, \phi, \vartheta_1 \ldots \vartheta_{n-2})$, a transformation

$$
\begin{aligned}
\xi_1 &= \cos(\varphi)\sin(\vartheta_1)\sin(\vartheta_2)\ldots\sin(\vartheta_{n-2}),\\
\xi_2 &= \sin(\varphi)\sin(\vartheta_1)\sin(\vartheta_2)\ldots\sin(\vartheta_{n-2}),\\
\xi_3 &= \cos(\vartheta_1)\sin(\vartheta_2)\ldots\sin(\vartheta_{n-2}),\\
&\vdots\\
\xi_{n-1} &= \cos(\vartheta_{n-3})\ldots\sin(\vartheta_{n-2}),\\
\xi_n &= \cos(\vartheta_{n-2}),
\end{aligned}
$$

and

$$x_i = c_i + r\xi_i,$$

with $r \in (0,R)$, $\varphi \in [-\pi, \pi)$, $\vartheta_i \in [0, \pi)\,\forall i = \{1, \ldots, n-2\}$. $c = (c_i)_{i=1}^n \in \mathbb{R}^n$ defines the center of the ball. The Jacobian is therefore given as

$$r^{n-1}\,dr\,d\varphi \prod_{j=1}^{n-2} \sin^j(\vartheta_j)\,d\vartheta_j.$$

Using the polar coordinates given above, the difference integral can be expressed as

$$\int_{H_\varepsilon(0) \backslash B_\varepsilon(0)} \|z\|^\alpha\,dz =$$
$$\int_{\varphi=0}^{2\pi} \int_{\vartheta_1=0}^{\pi} \cdots \int_{\vartheta_{n-2}=0}^{\pi} \int_{r=\varepsilon}^{R} r^\alpha r^{n-1}\,dr\,g(\varepsilon, \varphi, \vartheta_1, \cdots, \vartheta_{p-2}) \prod_{j=1}^{n-2} \sin^j(\vartheta_j)\,d\vartheta_j\,d\varphi \tag{14}$$

using an appropriate chosen function $R = R(\varepsilon, \varphi, \vartheta_1, \cdots, \vartheta_{n-2})$. For some examples of this function confer for instance [9][Anhang B]. In general, $R(\varepsilon, \varphi, \vartheta_1, \cdots, \vartheta_{n-2})$ has to fulfill the conditions

$$R(\varepsilon, \varphi, \vartheta_1, \cdots, \vartheta_{p-2}) = R(\varepsilon, \varphi - \pi, \pi - \vartheta_1, \cdots, \pi - \vartheta_{p-2}) \tag{15a}$$

and

$$R(\varepsilon, \varphi, \vartheta_1, \cdots, \vartheta_{p-2}) = \varepsilon \bar{R}(\varphi, \vartheta_1, \cdots, \vartheta_{p-2}) \tag{15b}$$

with ε from B_ε. The function g is of the shape

$$g(\varphi, \vartheta_1, \cdots, \vartheta_{p-2}) = \left(\sqrt{\bar{g}(\varphi, \vartheta_1, \cdots, \vartheta_{p-2})} \right)^\alpha$$

using a polynomial $\bar{g}$ in the variables $\varphi, \vartheta_1, \cdots, \vartheta_{p-2}$. The shape of $\bar{g}$ depends on the kernel in question and the ordering of the relative coordinates z. As R, g fulfills:

$$g(\varphi, \vartheta_1, \cdots, \vartheta_{p-2}) = g(\varphi - \pi, \pi - \vartheta_1, \cdots, \pi - \vartheta_{p-2}). \tag{16}$$

Now we are going to examine the structure of the innermost integral of (14) for various parameters α.

- The case of a regular or weakly singular kernel, $\alpha > -p$.
  It is sufficient to examine only the innermost integration. We note

$$\int_{r=\varepsilon}^{R(\varepsilon, \varphi, \vartheta_1, \cdots, \vartheta_{n-2})} r^{\alpha+n-1}\, \mathrm{d}r = \frac{\varepsilon^{\alpha+n}}{\alpha+n} \left(\bar{R}(\varphi, \vartheta_1, \cdots, \vartheta_{n-2}) - 1 \right)$$

  with $\bar{R}$ from (15b). This expression converges to 0 for $\varepsilon \to 0$.

- The case of a Cauchy singular value, $\alpha = -p$.
  To determine the integral in the context of a Cauchy singular value, (14) has to be split. To simplify the notation, we define

$$\bar{h}(\varphi, \vartheta_1, \cdots, \vartheta_{p-2}) := g(\varphi, \vartheta_1, \cdots, \vartheta_{p-2}) \prod_{j=1}^{p-2} \sin^j(\vartheta_j).$$

This yields with (16)

$$\begin{aligned}
\bar{h}(\varphi, \vartheta_1, \cdots, \vartheta_{p-2}) &= g(\varphi, \vartheta_1, \cdots, \vartheta_{p-2}) \prod_{j=1}^{p-2} \sin^j(\vartheta_j) \\
&= g(\varphi - \pi, \pi - \vartheta_1, \cdots, \pi - \vartheta_{p-2}) \prod_{j=1}^{p-2} \sin^j(\pi - \vartheta_j) \\
&= \bar{h}(\varphi - \pi, \pi - \vartheta_1, \cdots, \pi - \vartheta_{p-2}).
\end{aligned}$$

The polar coordinates $\xi := \xi(\varphi, \vartheta_1, \cdots, \vartheta_{p-2})$ introduced above, can be used to compress the notation:

$$h(\xi) = h(\xi(\varphi, \vartheta_1, \cdots, \vartheta_{p-2})) := \bar{h}(\varphi, \vartheta_1, \cdots, \vartheta_{p-2})$$

obviously, $h(\xi) = h(-\xi)$, and the following equation is valid:

$$\int_{\vartheta_{p-2}=0}^{\pi} \cdots \int_{\varphi=-\pi}^{\pi} \int_{r=\varepsilon}^{R(\varepsilon,\varphi,\ldots,\vartheta_{p-2})} r^{-1} \, \mathrm{d}r \, h(\xi) \, \mathrm{d}\varphi \ldots \mathrm{d}\vartheta_{p-2} \qquad (17)$$

$$= \int_{\vartheta_{p-2}=0}^{\pi} \cdots \int_{\varphi=-\pi}^{0} \int_{r=\varepsilon}^{R(\varepsilon,\varphi,\ldots,\vartheta_{p-2})} r^{-1} \, \mathrm{d}r \, h(\xi) \, \mathrm{d}\varphi \ldots \mathrm{d}\vartheta_{p-2}$$

$$+ \int_{\vartheta_{p-2}=0}^{\pi} \cdots \int_{\varphi=0}^{\pi} \int_{r=\varepsilon}^{R(\varepsilon,\varphi,\vartheta_1,\cdots,\vartheta_{p-2})} r^{-1} \, \mathrm{d}r \, h(\xi) \, \mathrm{d}\varphi \ldots \mathrm{d}\vartheta_{p-2}.$$

Substituting φ by $\varphi + \pi$, ϑ_i by $\pi - \vartheta_i$ and r by $-r$ in the first integral of the right-hand side and a permutation of the integral bounds in ϑ_i and r, we note for the first integral (17)

$$\int_{\vartheta_{p-2}=0}^{\pi} \cdots \int_{\varphi=-\pi}^{0} \int_{r=\varepsilon}^{R(\varepsilon,\varphi,\ldots,\vartheta_{p-2})} r^{-1} \, \mathrm{d}r \, h(\xi) \, \mathrm{d}\varphi \ldots \mathrm{d}\vartheta_{p-2} =$$

$$\int_{\vartheta_{p-2}=0}^{\pi} \cdots \int_{\varphi=0}^{\pi} \int_{r=-R(\varepsilon,\varphi,\ldots,\vartheta_{p-2})}^{-\varepsilon} r^{-1} \, \mathrm{d}r \, h(\xi) \, \mathrm{d}\varphi \ldots \mathrm{d}\vartheta_{p-2},$$

as $R(\varepsilon, \varphi - \pi, \ldots, \pi - \vartheta_{p-2}) = R(\varepsilon, \varphi, \ldots, \vartheta_{p-2})$ and $h(\xi) = h(-\xi)$. Integration w.r.t. r yields

$$\int_{\vartheta_{p-2}=0}^{\pi} \cdots \int_{\varphi=0}^{\pi} \underbrace{\left[\ln|r|_{-R(\varepsilon,\varphi,\ldots,\vartheta_{p-2})}^{-\varepsilon} + \ln|r|_{\varepsilon}^{R(\varepsilon,\varphi,\vartheta_1,\cdots,\vartheta_{p-2})} \right]}_{=0} h(\xi) \, \mathrm{d}\varphi \ldots \mathrm{d}\vartheta_{p-2} = 0.$$

- The case of a hypersingular kernel in the sense of Hadamard, $\alpha < -p$.

$$\int_{r=\varepsilon}^{R(\varepsilon,\varphi,\vartheta_1,\cdots,\vartheta_{n-2})} r^{\alpha+n-1} \, \mathrm{d}r = \frac{\varepsilon^{\alpha+n}}{\alpha+n} \left(\bar{R}(\varphi, \vartheta_1, \cdots, \vartheta_{n-2}) - 1 \right).$$

with $\bar{R}$ from (15b). The expression $\frac{\varepsilon^{\alpha+n}}{\alpha+n} (\bar{R}(\varphi, \vartheta_1, \cdots, \vartheta_{n-2}) - 1)$ is of the shape of a single term of $\mathbb{I}_\infty$ from (11) and converges to ∞ for $\varepsilon \to 0$ as $\alpha + n < 0$. Hence it has no influence on the value of the integral.

Using the comments above, we are able to rewrite (13) by

$$\mathrm{p.f.} \int_{\Omega} f(x) \, dx = \lim_{\varepsilon \to 0} \left\{ \int_{\Omega \backslash \|x\|_\infty < \varepsilon} f(x) \, \mathrm{d}x - \sum_{i=0}^{k} \bar{\beta}_i \phi_i(\varepsilon) \right\} \qquad (18)$$

using a constant $k := \max\{m \in \mathbb{Z} | m \leq -\dim(\Omega) - g\}$ and functions $\bar{\beta}_i \in \mathbb{R}$ and $\phi_i(\varepsilon)$ as given in 4.1.

5 A modified splitting strategy based on the Hadamard partie finie integral

With the replacement of the Euclidean norm by the maximum norm we are now able to develop a new splitting strategy which reflects the modified integral concept given in (18).

The following Section 5.1 describes the method in the two-dimensional case as introduced in [2] and can be seen as an introductory example. Section 5.2 extends the idea to hypercubes and Section 5.3 provides remarks on the treatment of higher dimensional simplices.

5.1 The two-dimensional case

This section describes the ideas given [2] in a short and compact fashion. For more details on the method and extended examples we therefore refer the reader to [2]. In the $1\mathrm{d}\times 1\mathrm{d}$ case we have

$$
\begin{aligned}
\mathbb{I}(\varepsilon) &= \int_0^1 \int_{\substack{0 \\ |x-y|\geq\varepsilon}}^1 \kappa(\hat{x},\hat{y},0,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x} \\
&= \int_0^{1-\varepsilon}\int_{\hat{x}+\varepsilon}^1 \kappa(\hat{x},\hat{y},0,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x} + \int_\varepsilon^1\int_0^{\hat{x}-\varepsilon}\kappa(\hat{x},\hat{y},0,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x}
\end{aligned}
\tag{19}
$$

with a kernel function $\kappa(\hat{x},\hat{y},0,0)$ as introduced in (8).

For $\varepsilon = 0$ this is equivalent to an integration defined on domains C_l and C_u as given below

$$
C_l := \{(\hat{x},\hat{y}) \in [0,1]^2 : \hat{y} < \hat{x}\}, \qquad C_u := \{(\hat{x},\hat{y}) \in [0,1]^2 : \hat{y} \geq \hat{x}\}.
\tag{20}
$$

In $\mathbb{R}^2$ the domains C_l and C_u describe an upper and a lower triangle. After multiple applications of the "red" splitting strategy on the singular elements, a sequence of elements can be described, as depicted in Figure 2. The regular elements of the subdivision are shaded in gray, the remaining singular elements are white.
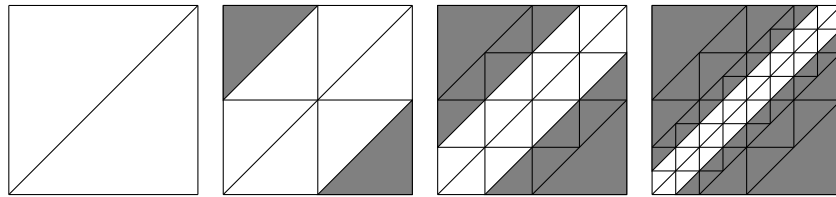

Figure 2: Sequence of regular (gray) and singular elements after multiple "red" subdivisions of the singular elements.

As introduced in Section 3.1, equivalence classes can be used to describe and classify the singular domains. For the domain C_l the following classes exist:

$$\mathbb{I}_0^l := \int_0^1 \int_0^{\hat{x}} \kappa(\hat{x},\hat{y},0,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x}, \qquad \mathbb{I}_1^l := \int_0^1 \int_{\hat{x}}^1 \kappa(\hat{x},\hat{y},1,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x},$$
$$\mathbb{I}_2^l := \int_0^1 \int_0^{\hat{x}} \kappa(\hat{x},\hat{y},1,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x}, \qquad \mathbb{I}_3^l := \int_0^1 \int_{\hat{x}}^1 \kappa(\hat{x},\hat{y},2,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x} \tag{21}$$

Only the elements $\mathbb{I}_0^l$ and $\mathbb{I}_1^l$ are singular, the elements $\mathbb{I}_2^l$ and $\mathbb{I}_3^l$ are regular. The classes are linked by the relations

$$\mathbb{I}_0^l = 2c\mathbb{I}_0^l + c\mathbb{I}_1^l + c\mathbb{I}_2^l, \tag{22a}$$

$$\mathbb{I}_1^l = c\mathbb{I}_1^l + c\mathbb{I}_2^l + 2c\mathbb{I}_3^l \tag{22b}$$

using a constant $c = \left(\frac{1}{2}\right)^{1+1+\alpha}$ as introduced before. Furthermore the domain C_u can be described by

$$\mathbb{I}_0^u := \int_0^1 \int_{\hat{x}}^1 \kappa(\hat{x},\hat{y},0,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x}, \qquad \mathbb{I}_1^u := \int_0^1 \int_0^{\hat{x}} \kappa(\hat{x},\hat{y},0,1)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x},$$
$$\mathbb{I}_2^u := \int_0^1 \int_{\hat{x}}^1 \kappa(\hat{x},\hat{y},0,1)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x}, \qquad \mathbb{I}_3^u := \int_0^1 \int_0^{\hat{x}} \kappa(\hat{x},\hat{y},0,2)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x}, \tag{23}$$

with singular elements $\mathbb{I}_0^u$, $\mathbb{I}_1^u$ and regular elements $\mathbb{I}_2^u$ and $\mathbb{I}_3^u$, connected by the relation

$$\mathbb{I}_0^u = 2c\mathbb{I}_0^u + c\mathbb{I}_1^u + c\mathbb{I}_2^u, \tag{24a}$$

$$\mathbb{I}_1^u = c\mathbb{I}_1^u + c\mathbb{I}_2^u + 2c\mathbb{I}_3^u. \tag{24b}$$

Remark 5.1 (on the notation)
By $\mathbb{I}_\bullet^\circ$, $\bullet \in \{0,\ldots,3\}$, $\circ \in \{u,l\}$, we identify the integral defined on a given domain as well as its value.

Using (22) and (24), we derive a system of linear equations:

$$\begin{pmatrix} 2c & c & & \\ & c & & \\ & & 2c & c \\ & & & c \end{pmatrix} \begin{pmatrix} \mathbb{I}_0^l \\ \mathbb{I}_1^l \\ \mathbb{I}_0^u \\ \mathbb{I}_1^u \end{pmatrix} + \begin{pmatrix} c\mathbb{I}_2^l \\ 2c\mathbb{I}_3^l + c\mathbb{I}_2^l \\ c\mathbb{I}_2^u \\ 2c\mathbb{I}_3^u + c\mathbb{I}_2^u \end{pmatrix} = \begin{pmatrix} \mathbb{I}_0^l \\ \mathbb{I}_1^l \\ \mathbb{I}_0^u \\ \mathbb{I}_1^u \end{pmatrix}. \tag{25}$$

A repeated application of this equation yields

$$\hat{\mathbb{I}} = \mathbf{A}\hat{\mathbb{I}} + b = \mathbf{A}(\mathbf{A}\hat{\mathbb{I}} + b) + b = \ldots = \mathbf{A}^k\hat{\mathbb{I}} + \sum_{\ell=0}^{k-1} \mathbf{A}^\ell b$$

from which we extract the partial sums

$$\hat{\mathbb{I}}^k = \sum_{\ell=0}^{k-1} \mathbf{A}^\ell b. \tag{26}$$

The right-hand side of (26) is formed by regular integrals only, corresponding to the regular elements after k subdivisions. Furthermore, the partial sums $\hat{\mathbb{I}}^k$ correspond to the Hadamard partie finie (19) using $\varepsilon = 2^{-k}$. The solution is therefore given by

$$\mathbb{I}(2^{-k}) := \hat{\mathbb{I}}_1^k + \hat{\mathbb{I}}_3^k. \tag{27}$$

Evaluation of the geometric series (26) yields $\hat{\mathbb{I}}^k = (\mathbf{I} - \mathbf{A})^{-1}(\mathbf{I} - \mathbf{A}^k)b$ using the identity $\mathbf{I} \in \mathbb{R}^{4\times 4}$. This expression converges in the case of a regular or weakly singular integrals to $(\mathbf{I} - \mathbf{A})^{-1}$, as $\mathbf{A}^k$ converges to zero and as a result we get

$$\hat{\mathbb{I}}^\infty = (\mathbf{I} - \mathbf{A})^{-1}b. \tag{28}$$

For the general case we highlight, that the equation noted above can be expressed via

$$\hat{\mathbb{I}}^\infty = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}b$$
$$= \begin{pmatrix} 1 & -1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-2c} & & & \\ & \frac{1}{1-c} & & \\ & & \frac{1}{1-2c} & \\ & & & \frac{1}{1-c} \end{pmatrix} \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} b \tag{29}$$
$$= \begin{pmatrix} \frac{b_1+b_2}{1-2c} - \frac{b_2}{1-c} \\ \frac{b_2}{1-c} \\ \frac{b_3+b_4}{1-2c} - \frac{b_4}{1-c} \\ \frac{b_4}{1-c} \end{pmatrix}. \tag{30}$$

Using this notation, it is possible to calculate even hyper-singular integrals. Only the case of a Cauchy singular value in the identical element ($\alpha = -2$) or in the common point ($\alpha = -1$) cannot be calculated, as in this case $(\mathbf{I} - \mathbf{A})$ is not regular anymore and therefore not invertible.

Following the principle of Hadamard partie finie integrals, divergent parts of the integral in question are collected in the expression $\mathbb{I}_\infty(\varepsilon)$ and discarded, if they are of the shape as described in (11). By the definition of a function $\varsigma(\lambda)$,

$$\varsigma(\lambda) := \begin{cases} \frac{1}{\lambda}, & \forall \lambda \neq 0, \\ 0, & \lambda = 0 \end{cases}$$

and the usage of the pseudo-inverse matrix

$$(\mathbf{I} - \mathbf{A})^+ = \mathbf{T} \begin{pmatrix} \varsigma(1-2c) & & & \\ & \varsigma(1-c) & & \\ & & \varsigma(1-2c) & \\ & & & \varsigma(1-c) \end{pmatrix} \mathbf{T}^{-1}$$

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the needed shape for the divergent parts is achieved. So the principle of Hadamard partie finie integrals can be adapted to (30) and we summarize

$$
\begin{aligned}
\hat{\mathbb{I}}^\infty &= (\mathbf{I} - \mathbf{A})^+ b \\
&= \begin{pmatrix} (b_1 + b_2)\varsigma(1-2c) - (b_2)\varsigma(1-c) \\ b_2\varsigma(1-c) \\ (b_3 + b_4)\varsigma(1-2c) - b_4\varsigma(1-c) \\ b_4\varsigma(1-c) \end{pmatrix}.
\end{aligned}
\tag{31}
$$

In the case of a regular $(\mathbf{I} - \mathbf{A})$ the upper equation is equivalent to (28) but is defined for all values of α and yields the desired results.

Remark 5.2 (on symmetric kernels)
In the case of a symmetric kernel κ_γ, $\mathbb{I}_0^l = \mathbb{I}_0^u$ and $\mathbb{I}_1^l = \mathbb{I}_1^u$ are valid and therefore (25) can be simplified to

$$
\begin{pmatrix} 1 - 2c & -c \\ 0 & 1 - c \end{pmatrix} \begin{pmatrix} \mathbb{I}_0^l \\ \mathbb{I}_1^l \end{pmatrix} = \begin{pmatrix} c\mathbb{I}_2^l \\ c\mathbb{I}_2^l + 2c\mathbb{I}_3^l \end{pmatrix}.
$$

The solution is given by $\mathbb{I} = 2\mathbb{I}_0^l$.

5.2 The higher dimensional case for cubes

Considering the ideas on Hadamard partie finie integrals given in Section 4.3 and the modified splitting strategy given before, it is now possible to extend the given method to higher dimensions.

For the sake of clarity and readability we first restrict ourselves to the case of two hypercubes of the same dimension and extend the method later. We consider the integral

$$
\mathbb{I} = \int_0^1 \cdots \int_0^1 \int_0^1 \cdots \int_0^1 \kappa_\gamma(\hat{x}, \hat{y}, 0, 0) \, d\hat{y} \, d\hat{x},
$$

$\hat{x} \in \mathbb{R}^n$, $\hat{y} \in \mathbb{R}^n$. In each space dimension a splitting as in (20) can be applied. This yields a sum of 2^n integrals. These subintegrals are subdivided in each space direction using a "red" refinement strategy. After separating regular and singular integrals, singular elements have to subdivided again, if they are not equivalent to the element they originate from.

The connection of different splitting rules regarding (21) and (23) shall be depicted by the symbol $\times$. The following Example 5.3 explains the notation and the method.

Example 5.3 (subdivision of a 2d$\times$2d Integral)
Let us consider the integrals $\mathbb{I}$ defined on $[0,1]^4$ using a kernel $\kappa_\gamma(\hat{x}, \hat{y}, 0, 0)$ as given in (9). The first step of subdivisions yields 4 integrals, representing the integration on the

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product of two triangles as depicted in (20):

$$
\begin{aligned}
\mathbb{I} &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \kappa_\gamma(\hat{x},\hat{y},0,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x} \\
&= \underbrace{\int_0^1 \int_0^1 \int_0^{\hat{x}_1} \int_0^{\hat{x}_2} \kappa_\gamma(\hat{x},\hat{y},0,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x}}_{\mathbb{I}_0^l \times \mathbb{I}_0^l} + \underbrace{\int_0^1 \int_0^1 \int_{\hat{x}_1}^1 \int_0^{\hat{x}_2} \kappa_\gamma(\hat{x},\hat{y},0,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x}}_{\mathbb{I}_0^u \times \mathbb{I}_0^l} \\
&\quad + \underbrace{\int_0^1 \int_0^1 \int_0^{\hat{x}_1} \int_{\hat{x}_2}^1 \kappa_\gamma(\hat{x},\hat{y},0,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x}}_{\mathbb{I}_0^l \times \mathbb{I}_0^u} + \underbrace{\int_0^1 \int_0^1 \int_{\hat{x}_1}^1 \int_{\hat{x}_2}^1 \kappa_\gamma(\hat{x},\hat{y},0,0)\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{x}}_{\mathbb{I}_0^u \times \mathbb{I}_0^u}
\end{aligned}
$$

using $\mathbb{I}_0^l$ and $\mathbb{I}_0^u$ as shown in (21) and (23). Each of these four given subintegrals is subdivided using the "red" refinement strategy. As an example we present this step using the integral $\mathbb{I}_0^l \times \mathbb{I}_0^u$. In the case of $\mathbb{I}_0^l$ a subdivision regarding (22a) and in the case of $\mathbb{I}_0^u$ a subdivision regarding (24a) is necessary and we obtain

$$
\begin{aligned}
\mathbb{I}_0^l \times \mathbb{I}_0^u \quad = \quad & 4c\mathbb{I}_0^l \times \mathbb{I}_0^u \quad + \quad 2c\mathbb{I}_0^l \times \mathbb{I}_1^u \quad + \quad 2c\mathbb{I}_0^l \times \mathbb{I}_2^u \\
+ \quad & 2c\mathbb{I}_1^l \times \mathbb{I}_0^u \quad + \quad c\mathbb{I}_1^l \times \mathbb{I}_1^u \quad + \quad c\mathbb{I}_1^l \times \mathbb{I}_2^u \\
+ \quad & 2c\mathbb{I}_2^l \times \mathbb{I}_0^u \quad + \quad c\mathbb{I}_2^l \times \mathbb{I}_1^u \quad + \quad c\mathbb{I}_2^l \times \mathbb{I}_2^u \quad .
\end{aligned}
$$

The general shape of the subdivisions is valid for the pairs $\mathbb{I}_0^l \times \mathbb{I}_0^l$, $\mathbb{I}_0^u \times \mathbb{I}_0^l$ and $\mathbb{I}_0^u \times \mathbb{I}_0^u$ as well.

All elements of the third row and third column of the right-hand side in the upper scheme are regular, all remaining pairs are singular and have to be further subdivided if they are not equivalent to the element they originate from.

For the pair of elements $\mathbb{I}_1^l \times \mathbb{I}_0^u$ the following is valid:

$$
\begin{aligned}
\mathbb{I}_1^l \times \mathbb{I}_0^u \quad = \quad & 2c\mathbb{I}_1^l \times \mathbb{I}_0^u \quad + \quad c\mathbb{I}_1^l \times \mathbb{I}_1^u \quad + \quad c\mathbb{I}_1^l \times \mathbb{I}_2^u \\
+ \quad & 2c\mathbb{I}_2^l \times \mathbb{I}_0^u \quad + \quad c\mathbb{I}_2^l \times \mathbb{I}_1^u \quad + \quad c\mathbb{I}_2^l \times \mathbb{I}_2^u \\
+ \quad & 4c\mathbb{I}_3^l \times \mathbb{I}_0^u \quad + \quad 2c\mathbb{I}_3^l \times \mathbb{I}_1^u \quad + \quad 2c\mathbb{I}_3^l \times \mathbb{I}_2^u
\end{aligned}
$$

with singular elements $\mathbb{I}_1^l \times \mathbb{I}_0^u$ and $\mathbb{I}_1^l \times \mathbb{I}_1^u$.

After such a sequential subdivision of all singular elements and the identification of equivalence classes, the relations between the representatives of the classes yields a system of linear equations whose entries are connected to the entries of (25) reading

$$
\bigotimes_{\mu=1}^n \hat{\mathbb{I}}_\mu = \bigotimes_{\mu=1}^n \left(\mathbf{A}_\mu \hat{\mathbb{I}}_\mu + b_\mu \right), \tag{32}
$$

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using the known constant $c = \left(\frac{1}{2}\right)^{n+n+\alpha}$,

$$
\mathbf{A}_1 = \begin{pmatrix} 2c & c & & \\ & c & & \\ & & 2c & c \\ & & & c \end{pmatrix}, \quad \mathbf{A}_\mu = \begin{pmatrix} 2 & 1 & & \\ & 1 & & \\ & & 2 & 1 \\ & & & 1 \end{pmatrix} \quad \forall \mu > 1, \quad \hat{\mathbb{I}}_\mu = \begin{pmatrix} \mathbb{I}_0^l \\ \mathbb{I}_1^l \\ \mathbb{I}_0^u \\ \mathbb{I}_1^u \end{pmatrix}
$$

and the terms b_μ given by

$$
b_1 = \begin{pmatrix} c\mathbb{I}_2^l \\ c\mathbb{I}_2^l + 2c\mathbb{I}_3^l \\ c\mathbb{I}_2^u \\ c\mathbb{I}_2^u + 2c\mathbb{I}_3^u \end{pmatrix} \quad \text{and} \quad b_\mu = \begin{pmatrix} \mathbb{I}_2^l \\ \mathbb{I}_2^l + 2\mathbb{I}_3^l \\ \mathbb{I}_2^u \\ \mathbb{I}_2^u + 2\mathbb{I}_3^u \end{pmatrix} \quad \forall \mu > 1,
$$

using the notation

$$
\hat{\mathbb{I}}_{\mu_1} \otimes \hat{\mathbb{I}}_{\mu_2} = \begin{pmatrix} \mathbb{I}_{\mu_1,0}^l \times \mathbb{I}_{\mu_2,0}^l \\ \mathbb{I}_{\mu_1,0}^l \times \mathbb{I}_{\mu_2,1}^l \\ \mathbb{I}_{\mu_1,0}^l \times \mathbb{I}_{\mu_2,0}^u \\ \vdots \end{pmatrix}.
$$

The product on the right-hand side can be split into a singular part, $\bigotimes_{\mu=1}^n \mathbf{A}_\mu \hat{\mathbb{I}}_\mu$, and a regular part, labeled as b_{reg} in the following. An integral can be considered as regular, if in at least one space dimension the elements have a positive distance. We rewrite (32) as

$$
\bigotimes_{\mu=1}^n \hat{\mathbb{I}}_\mu = \bigotimes_{\mu=1}^n \left(\mathbf{A}_\mu \hat{\mathbb{I}}_\mu\right) + b_{\text{reg}} = \bigotimes_{\mu=1}^n \mathbf{A}_\mu \left(\bigotimes_{\mu=1}^n \hat{\mathbb{I}}_\mu\right) + b_{\text{reg}}. \tag{33}
$$

This system is transformed analogously as (25). The repeated application of the equation yields

$$
\begin{aligned}
\bigotimes_{\mu=1}^n \hat{\mathbb{I}}_\mu &= \bigotimes_{\mu=1}^n \mathbf{A}_\mu \bigotimes_{\mu=1}^n \hat{\mathbb{I}}_\mu + b_{\text{reg}} = \bigotimes_{\mu=1}^n \mathbf{A}_\mu \left(\bigotimes_{\mu=1}^n \mathbf{A}_\mu \bigotimes_{\mu=1}^n \hat{\mathbb{I}}_\mu + b_{\text{reg}}\right) + b_{\text{reg}} \\
&= \cdots \\
&= \left(\bigotimes_{\mu=1}^n \mathbf{A}_\mu\right)^k \bigotimes_{\mu=1}^n \hat{\mathbb{I}}_\mu + \sum_{\ell=0}^{k-1} \left(\bigotimes_{\mu=1}^n \mathbf{A}_\mu\right)^\ell b_{\text{reg}} \\
&= \bigotimes_{\mu=1}^n \mathbf{A}_\mu^k \bigotimes_{\mu=1}^n \hat{\mathbb{I}}_\mu + \sum_{\ell=0}^{k-1} \bigotimes_{\mu=1}^n \mathbf{A}_\mu^\ell b_{\text{reg}}. \tag{34}
\end{aligned}
$$

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The partial sums

$$
{}^{\otimes}\hat{\mathbb{1}}^k := \sum_{\ell=0}^{k-1} \bigotimes_{\mu=1}^{n} \mathbf{A}_\mu^\ell b_{\text{reg}} = \left(\bigotimes_{\mu=1}^{n} \mathbf{I}_\mu - \bigotimes_{\mu=1}^{n} \mathbf{A}_\mu \right)^{-1} \left(\bigotimes_{\mu=1}^{n} \mathbf{I}_\mu - \bigotimes_{\mu=1}^{n} \mathbf{A}_\mu^k \right) b_{\text{reg}} \tag{35}
$$

are formed by regular integrals, as in the case of (26). After k steps they are equivalent to the modified Hadamard partie finie, described in Section 4.3, using $\varepsilon = 2^{-k}$. The final solution is therefore given as

$$
\mathbb{I}(2^{-k}) := \left({}^{\otimes}\hat{\mathbb{1}}^k \right)_{(1,1,\dots,1)} + \left({}^{\otimes}\hat{\mathbb{1}}^k \right)_{(1,1,\dots,3)} + \cdots + \left({}^{\otimes}\hat{\mathbb{1}}^k \right)_{(3,3,\dots,3)}.
$$

Figure 3 shows the projection of the $\mathbb{I}(2^{-k})$ using $k = 1, \dots, 5$ on the (y_1, y_2)-plane for a fixed $(x_1, x_2) = (0.7, 0.9)$ following Example 5.3.
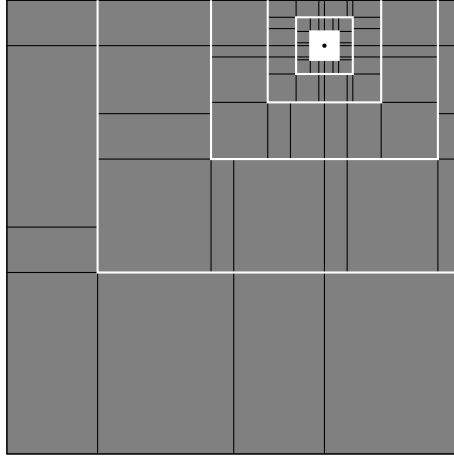

Figure 3: Series of regular elements (gray) in the variable $y \in [0,1]^2$ after repeated subdivisions (white lines with $\varepsilon = \frac{1}{2}, \dots, \frac{1}{32}$) using $(x_1, x_2) = (0.7, 0.9)$.

For regular or only weakly singular integrals the expression $\bigotimes_{\mu=1}^{n} \mathbf{A}_\mu^k$ converges to zero

and we consider again

$$
\begin{aligned}
{}^{\otimes}\hat{\mathbb{I}}^{\infty} &= \left(\bigotimes_{\mu=1}^{n} \mathbf{I}_{\mu} - \bigotimes_{\mu=1}^{n} \mathbf{A}_{\mu} \right)^{-1} b_{\mathrm{reg}} \\
&= \left(\bigotimes_{\mu=1}^{n} \mathbf{T}_{\mu} \left(\bigotimes_{\mu=1}^{n} \mathbf{I}_{\mu} - \bigotimes_{\mu=1}^{n} \mathbf{D}_{\mu} \right) \bigotimes_{\mu=1}^{n} \mathbf{T}_{\mu}^{-1} \right)^{-1} b_{\mathrm{reg}} \\
&= \left(\bigotimes_{\mu=1}^{n} \mathbf{T}_{\mu} \left(\bigotimes_{\mu=1}^{n} \mathbf{I}_{\mu} - \bigotimes_{\mu=1}^{n} \mathbf{D}_{\mu} \right)^{-1} \bigotimes_{\mu=1}^{n} \mathbf{T}_{\mu}^{-1} \right) b_{\mathrm{reg}} \qquad (36)
\end{aligned}
$$

using matrices $\mathbf{T}_{\mu}$, $\mathbf{T}_{\mu}^{-1}$, as given in (29), and

$$
\mathbf{D}_1 = \begin{pmatrix} 2c & & & \\ & c & & \\ & & 2c & \\ & & & c \end{pmatrix}, \qquad \mathbf{D}_{\mu} = \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 2 & \\ & & & 1 \end{pmatrix} \ \forall \mu > 1.
$$

The inverse $\left(\bigotimes_{\mu=1}^{n} \mathbf{I}_{\mu} - \bigotimes_{\mu=1}^{n} \mathbf{D}_{\mu} \right)^{-1}$ is given as a diagonal matrix with the entries $\frac{1}{1-2^{\ell}c}$, $\ell \in \{0,\dots,n\}$, and $c = \left(\frac{1}{2} \right)^{n+n+\alpha}$. By definition of a function

$$
\varsigma(\mathbf{A}) := \varsigma(a_{ij}) := \begin{cases} \frac{1}{a_{ij}}, & \forall a_{ij} \neq 0, \\ 0, & a_{ij} = 0, \end{cases} \qquad (37)
$$

it is again possible to introduce a pseudo-inverse matrix

$$
\left(\bigotimes_{\mu=1}^{n} \mathbf{I}_{\mu} - \bigotimes_{\mu=1}^{n} \mathbf{D}_{\mu} \right)^{+} := \varsigma \left(\bigotimes_{\mu=1}^{n} \mathbf{I}_{\mu} - \bigotimes_{\mu=1}^{n} \mathbf{D}_{\mu} \right).
$$

The diagonal entries of the pseudo-inverse matrix equal

$$
\varsigma \left(\bigotimes_{\mu=1}^{n} \mathbf{I}_{\mu} - \bigotimes_{\mu=1}^{n} \mathbf{D}_{\mu} \right)_{((i_1,i_1),\dots,(i_n,i_n))} = \begin{cases} \frac{1}{1-2^{\ell}c}, & \forall 1-2^{\ell}c \neq 0, \\ 0, & 1-2^{\ell}c = 0, \end{cases}
$$

$\ell \in \{0,\dots,n\}$ and $c = \left(\frac{1}{2} \right)^{n+n+\alpha}$. As in the two-dimensional case shown in (31), the solution ${}^{\otimes}\hat{\mathbb{I}}^{\infty}$ is given by

$$
{}^{\otimes}\hat{\mathbb{I}}^{\infty} = \left(\bigotimes_{\mu=1}^{n} \mathbf{I}_{\mu} - \bigotimes_{\mu=1}^{n} \mathbf{A}_{\mu} \right)^{+} b_{\mathrm{reg}} \qquad (38)
$$

for all parameters α. In the case of weakly singular and regular integrals this is equal to the solution of the system (32).

Example 5.4 (computation of a 2d×2d integral)

In extension of Example 5.3 we are going to determine the first entries of the elements of the system given in (32).

$$
\begin{aligned}
&&\bigotimes_{\mu=1}^{2}\hat{\mathbb{I}} &= \bigotimes_{\mu=1}^{2}\left(\mathbf{A}_\mu\hat{\mathbb{I}}_\mu+b_\mu\right)\\
\Leftrightarrow\quad && \hat{\mathbb{I}}_1\otimes\hat{\mathbb{I}}_2 &= \left(\mathbf{A}_1\hat{\mathbb{I}}_1+b_1\right)\otimes\left(\mathbf{A}_2\hat{\mathbb{I}}_2+b_2\right)\\
\Leftrightarrow\quad && \hat{\mathbb{I}}_1\otimes\hat{\mathbb{I}}_2 &= \mathbf{A}_1\hat{\mathbb{I}}_1\otimes\mathbf{A}_2\hat{\mathbb{I}}_2+\mathbf{A}_1\hat{\mathbb{I}}_1\otimes b_2+b_1\otimes\mathbf{A}_2\hat{\mathbb{I}}_2+b_1\otimes b_2\\
\Leftrightarrow\quad && \hat{\mathbb{I}}_1\otimes\hat{\mathbb{I}}_2 &= \left(\mathbf{A}_1\otimes\mathbf{A}_2\right)\left(\hat{\mathbb{I}}_1\otimes\hat{\mathbb{I}}_2\right)+\mathbf{A}_1\hat{\mathbb{I}}_1\otimes b_2+b_1\otimes\mathbf{A}_2\hat{\mathbb{I}}_2+b_1\otimes b_2\\
\Leftrightarrow\quad && \begin{pmatrix}\mathbb{I}_0^l\times\mathbb{I}_0^l\\ \mathbb{I}_0^l\times\mathbb{I}_1^l\\ \mathbb{I}_0^l\times\mathbb{I}_0^u\\ \vdots\end{pmatrix} &= \begin{pmatrix}4c & 2c & 0 & 0 & 2c & c & 0 & 0 & 0\\ 0 & 2c & 0 & 0 & 0 & c & 0 & 0 & 0 & \cdots\\ 0 & 0 & 4c & 2c & 0 & 0 & 2c & c & 0\\ & & & & \vdots\end{pmatrix}\begin{pmatrix}\mathbb{I}_0^l\times\mathbb{I}_0^l\\ \mathbb{I}_0^l\times\mathbb{I}_1^l\\ \mathbb{I}_0^l\times\mathbb{I}_0^u\\ \vdots\end{pmatrix}+b_{\mathrm{reg}}.
\end{aligned}
$$

The added b_{reg} is given in this example by three elements which read

$$
\begin{aligned}
b_{\mathrm{reg}} =&\mathbf{A}_1\hat{\mathbb{I}}_1\otimes b_2+b_1\otimes\mathbf{A}_2\hat{\mathbb{I}}_2+b_1\otimes b_2\\
=&\begin{pmatrix}2c\mathbb{I}_0^l\times\mathbb{I}_2^l+c\mathbb{I}_1^l\times\mathbb{I}_2^l\\ 2c\mathbb{I}_0^l\times\mathbb{I}_2^l+4c\mathbb{I}_0^l\times\mathbb{I}_3^l+c\mathbb{I}_1^l\times\mathbb{I}_2^l+2c\mathbb{I}_1^l\times\mathbb{I}_3^l\\ c\mathbb{I}_0^l\times\mathbb{I}_2^u+c\mathbb{I}_1^l\times\mathbb{I}_2^u\\ \vdots\end{pmatrix}+\begin{pmatrix}2c\mathbb{I}_2^l\times\mathbb{I}_0^l+c\mathbb{I}_2^l\times\mathbb{I}_1^l\\ c\mathbb{I}_2^l\times\mathbb{I}_1^l\\ 2c\mathbb{I}_2^l\times\mathbb{I}_0^u+c\mathbb{I}_2^l\times\mathbb{I}_1^u\\ \vdots\end{pmatrix}+\begin{pmatrix}c\mathbb{I}_2^l\times\mathbb{I}_2^l\\ c\mathbb{I}_2^l\times\mathbb{I}_2^l+2c\mathbb{I}_2^l\times\mathbb{I}_3^l\\ c\mathbb{I}_2^l\times\mathbb{I}_2^u\\ \vdots\end{pmatrix}.
\end{aligned}
$$

The constant c is again given as $2^{-(2+2+\alpha)}$. Applying (38) yields the solution of the system for all parameters $\alpha\in\mathbb{R}$ as

$$
\otimes\hat{\mathbb{I}}^\infty = \begin{pmatrix}\varsigma(1-4c) & \varsigma(1-4c)-\varsigma(1-2c) & 0 & 0 & \varsigma(1-4c)\varsigma(1-2c)\\ 0 & \varsigma(1-2c) & 0 & 0 & 0 & \cdots\\ 0 & 0 & \varsigma(1-4c) & \varsigma(1-4c)-\varsigma(1-2c) & 0\\ & & & \vdots\end{pmatrix}b_{\mathrm{reg}}
$$

using a function ς as introduced in (37).

Remark 5.5 (on not identical elements)

The method given above is based on the idea of representing the domain of integration as Cartesian product of line segments in direction of the unit vectors $\mathbf{n}$ of a Cartesian coordinate system.

In the case of not identical elements the projection of some coordinate directions show the appearance of a "common point". For the corresponding space directions the elements of (32) simplify to

$$
\begin{aligned}
\mathbf{A}_\mu &= \begin{cases}(c), & \mu=1,\\ (1), & \mu\neq 1,\end{cases}\\
\hat{\mathbb{I}}_\mu &= (\mathbb{I}_1^\bullet),\\
b_\mu &= \begin{cases}(c\mathbb{I}_2^\bullet+2c\mathbb{I}_3^\bullet), & \mu=1,\\ (\mathbb{I}_2^\bullet+2\mathbb{I}_3^\bullet), & \mu\neq 1\end{cases}
\end{aligned}
$$

using $\bullet \in \{l, u\}$ regarding the relation of the elements to each other. As a result we apply the following simplifications to the system given in (38) for the space direction in question:

$$\mathbf{D}_\mu \quad = \quad \begin{cases} (c), & \mu = 1, \\ (1), & \mu \neq 1, \end{cases}$$
$$\mathbf{T}_\mu = \mathbf{T}_\mu^{-1} \quad = \quad (1).$$

Remark 5.6 (on elements with different dimensions)
Following the notation throughout this paper, let $n = \dim(C^x)$, $m = \dim(C^y)$, d be the dimension of the common face of the elements in question and w.l.o.g. $n \geq m \geq d \geq 0$ and $m > 0$.

In the case of elements with different dimensions we are able to identify $n - m$ space directions i for which the kernel κ_γ is constant w.r.t. the variable y_i. The integrals in the corresponding space directions x_i are subdivided as shown in the following. The resulting elements have to be integrated in the systems (32) and (38) according to their index:

$$\mathbb{I}_0 = \int_0^1 \kappa_\gamma(\hat{x}, \hat{y}, 0, 0) \, d\hat{x}_i$$
$$= c \int_0^1 \kappa_\gamma(\hat{x}, \hat{y}, 0, 0) \, d\hat{x}_i + c \int_0^1 \kappa_\gamma(\hat{x}, \hat{y}, e_i, 0) \, d\hat{x}_i$$
$$= c \mathbb{I}_0 + c \mathbb{I}_1$$

using the i-th unit vector e_i of the Cartesian coordinate system and c given by the values of n, m, d and α. The elements of (32) and (38) are given by

$$\mathbf{A}_\mu \quad = \quad \begin{cases} (c), & \mu = 1, \\ (1), & \mu \neq 1, \end{cases}$$
$$\hat{\mathbb{I}}_\mu \quad = \quad (\mathbb{I}_0),$$
$$b_\mu \quad = \quad \begin{cases} (c\mathbb{I}_1), & \mu = 1, \\ (\mathbb{I}_1), & \mu \neq 1, \end{cases}$$
$$\mathbf{D}_\mu \quad = \quad \begin{cases} (c), & \mu = 1, \\ (1), & \mu \neq 1, \end{cases}$$
$$\mathbf{T}_\mu = \mathbf{T}_\mu^{-1} \quad = \quad (1).$$

5.3 The higher dimensional case for simplices

The method described for cubes in Section 5.2 can be applied to simplices as well. But, as mentioned in Section 3.2 and shown in [8], the subdivision of simplices becomes much more difficult than in the case of hypercubes. Therefore, we only show that the

ideas given before can be transferred to the case of simplices. Let

$$\mathbb{I} = \int_0^1 \int_0^{1-\hat{x}_1} \int_0^1 \frac{1}{\left\| \begin{pmatrix} \hat{x}_1 - \hat{y} \\ \hat{x}_2 \end{pmatrix} \right\|^2} \, \mathrm{d}\hat{y} \, \mathrm{d}\hat{x}_2 \, \mathrm{d}\hat{x}_1 \tag{39}$$

$$= \int_0^1 \int_0^{1-\hat{x}_1} \int_0^1 \kappa_\tau(\hat{x}, \hat{y}, 0, 0) \, \mathrm{d}\hat{y} \, \mathrm{d}\hat{x}$$

using $\hat{x} = (\hat{x}_1, \hat{x}_2)$ and the kernel function κ_τ as defined before.

This configuration equals the product of a triangle and a line segment, as shown in Figure 4. The given integral is not solvable by the method mentioned in Section 3.2 and given [8], as $\alpha = -(n+m) + d = -2$. The solution can be obtained using the method, based on the modified Hadamard partie finie integrals, given above.

In a first step, the domain is subdivided regarding (20):

$$\mathbb{I} = \int_0^1 \int_0^{1-\hat{x}_1} \int_0^{\hat{x}_1} \kappa_\tau(\hat{x}, \hat{y}, 0, 0) \, \mathrm{d}\hat{y} \, \mathrm{d}\hat{x} \tag{40a}$$

$$+ \int_0^1 \int_0^{1-\hat{x}_1} \int_{\hat{x}_1}^1 \kappa_\tau(\hat{x}, \hat{y}, 0, 0) \, \mathrm{d}\hat{y} \, \mathrm{d}\hat{x}. \tag{40b}$$

The two domains of integration represent the domains shown Figure 4 and shall be denoted as $\mathbb{I}_A$ and $\mathbb{I}_B$ in the following.

After this first step of subdivision, the resulting domains are again subdivided regarding (21) and (23) in $\hat{x}_1$-$\hat{y}$. Furthermore a subdivision in $\hat{x}_1$-$\hat{x}_2$ has to be applied. Figure 5 visualizes the projections of the applied subdivisions. The domains of integration given in (40a) and (40b) are treated independently. For (40a) the results are presented in the following. Besides the original element $\mathbb{I}_A$, we identify the two equivalence classes of singular elements for (40a) shown below

$$\mathbb{I}_{A1} = \int_0^1 \int_{1-\hat{x}_1}^1 \int_0^{\hat{x}_1} \kappa_\tau(\hat{x}, \hat{y}, 0, 0) \, \mathrm{d}\hat{y} \, \mathrm{d}\hat{x},$$

$$\mathbb{I}_{A2} = \int_0^1 \int_0^{1-\hat{x}_1} \int_{\hat{x}_1}^1 \kappa_\tau(\hat{x}, \hat{y}, \upsilon_{A2}, 0) \, \mathrm{d}\hat{y} \, \mathrm{d}\hat{x}$$

with $\upsilon_{A2} = (1, 0, 0)^T$.

As relation between the element classes we formulate a system of linear equations $\mathbf{A}\hat{\mathbb{I}}_A + b_{\mathrm{reg}} = \hat{\mathbb{I}}_A$, comparable to system (25),

$$\begin{pmatrix} 2c & c & c \\ & c & \\ & & c \end{pmatrix} \begin{pmatrix} \mathbb{I}_A \\ \mathbb{I}_{A1} \\ \mathbb{I}_{A2} \end{pmatrix} + \begin{pmatrix} cb_A \\ cb_{A1} \\ cb_{A2} \end{pmatrix} = \begin{pmatrix} \mathbb{I}_A \\ \mathbb{I}_{A1} \\ \mathbb{I}_{A2} \end{pmatrix},$$

with the representatives $\mathbb{I}_A$, $\mathbb{I}_{A1}$, $\mathbb{I}_{A2}$ of the equivalence classes, sums of scaled regular elements cb_A, cb_{A1}, cb_{A2} and the parameter $c = \left(\frac{1}{2}\right)^{n+m+\alpha}$, defined by $c = \left(\frac{1}{2}\right)^{2+1-2} = \frac{1}{2}$.
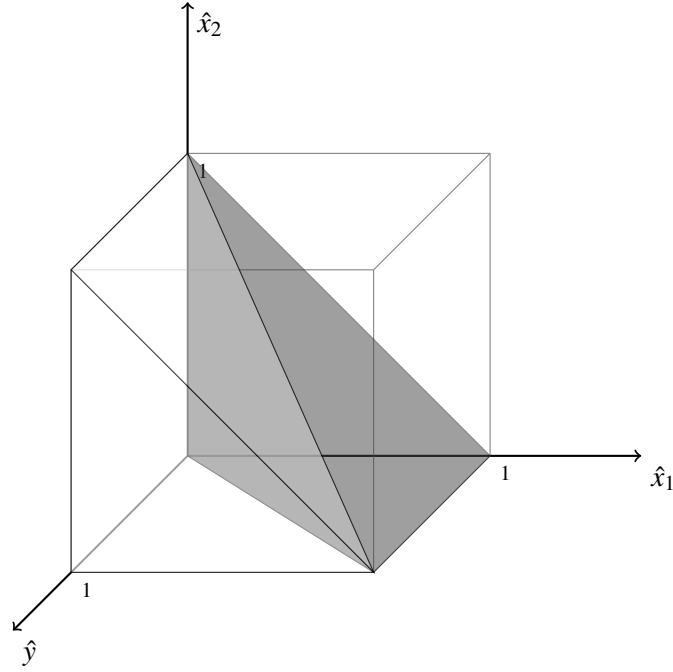
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Figure 4: The domains corresponding to the domains of integration of (39) after the first step of subdivision regarding (40a) (gray) and (40b) (white).
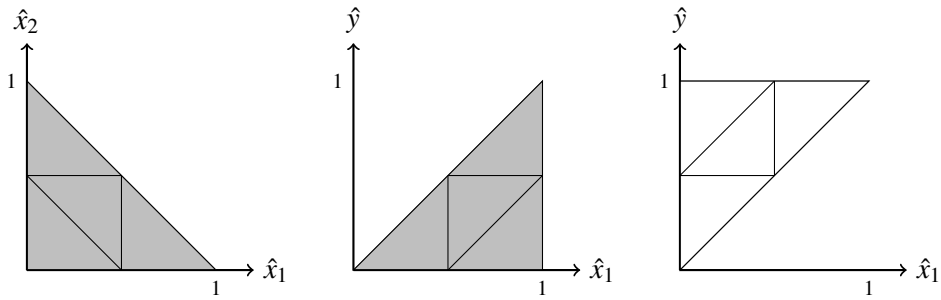

Figure 5: Projections after the second step of subdivisions for different pairs of space directions. The shading corresponds to Figure 4.

The repeated application of the system to itself as used in (26) and (35) yields again a partial sum $\hat{\mathbb{I}}_A^k$, whose entries are formed by regular elements. We note

$$\hat{\mathbb{I}}_A^k = \sum_{\ell=0}^{k-1} \mathbf{A}^\ell b = (\mathbf{I} - \mathbf{A})^{-1}(\mathbf{I} - \mathbf{A}^k)b_{\mathrm{reg}}. \tag{41}$$

$\mathbf{I}$ represents an appropriately chosen unit matrix.

In the case of regular or only weakly singular integrals, the expression $\mathbf{A}^k$ converges to zero and the solution is given by $\hat{\mathbb{I}}_A^\infty = (\mathbf{I} - \mathbf{A})^{-1}b$. In the general case, the idea noted in (29) or (30) can be used:

$$\hat{\mathbb{I}}_A^\infty = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}b_{\mathrm{reg}}$$

$$= \begin{pmatrix} 1 & -1 & -1 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-2c} & & \\ & \frac{1}{1-c} & \\ & & \frac{1}{1-c} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} c\,b_A \\ c\,b_{A1} \\ c\,b_{A2} \end{pmatrix}.$$

For an irregular $\mathbf{I} - \mathbf{A}$, the diagonal matrix $\mathbf{D}$ is not given. Because of this a pseudo-inverse matrix has to be introduced:

$$(\mathbf{I} - \mathbf{A})^+ = \mathbf{T} \begin{pmatrix} \varsigma(1 - 2c) & & \\ & \varsigma(1 - c) & \\ & & \varsigma(1 - c) \end{pmatrix} \mathbf{T}^{-1}.$$

The function ς is defined as

$$\varsigma(\lambda) := \begin{cases} \frac{1}{\lambda}, & \forall \lambda \neq 0, \\ 0, & \lambda = 0 \end{cases}$$

as already done in (37). The solution of the system of linear equations is given for all parameters α by

$$\hat{\mathbb{I}}_A^\infty = (\mathbf{I} - \mathbf{A})^+ b_{\mathrm{reg}}.$$

The value of the integral in question, $\mathbb{I}_A$, is represented by the first entry of the vector $\hat{\mathbb{I}}_A^\infty$. For the second part, $\mathbb{I}_B$, the same technique applies. Finally, the solution $\mathbb{I}$ of (39) is given as $\mathbb{I} = \mathbb{I}_A + \mathbb{I}_B$.

6 Conclusion and a remark on the connection between the methods of hierarchical integration

The methods presented in [8] and outlined in the Sections 3.1 and 3.2 for hypercubes and simplices describe methods for the distinct solution of integrals of the shape

$$\mathbb{I} = \int_{D^x} \int_{D^y} \kappa(x, y)\, \mathrm{d}y\, \mathrm{d}x$$

defined on hypercubes or simplices, using a translational invariant and homogeneous kernel function κ with a degree of homogeneity α. These methods exclude configurations with

$$\alpha = -(n+m)+d,$$

$n = \dim(D^x)$, $m = \dim(D^y)$ and d as given before.

The modified subdivision strategy for hypercubes and simplices, outlined in Sections 5.2 and 5.3 of this paper, describe alternative methods, which again yield distinct solutions, but do not restrict ourselves in the choice of the parameter α. The drawback of this method is the higher number of regular integrals.

The methods described in [8], outlined in this paper and the modifications undertaken in Sections 5.2 and 5.3 have in common, that they represent the integral in question as a distinct sum of integrals defined on distinct subdomains of the original domain of integration. From these sums the methods deduce systems of linear equations which again yield distinct solutions.

Therefore the methods represent, if they originate from identical specifications to areas and parameters, different representations of the same fact and therefore provide the same result.

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