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We find that the mixed maximally entangled states exist and prove that the form of the mixed maximally entangled states is unique in terms of the entanglement of formation. Moreover, even if the entanglement is quantified by other entanglement measures, this conclusion is still proven right. This result is a supplementary to the generally accepted fact that all maximally entangled states are pure. These states possess important properties of the pure maximally entangled states, for example, these states can be used as a resource for faithful teleportation and they can be distinguished perfectly by local operations and classical communication.

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I. INTRODUCTION

Quantum entanglement is central both to the foundations of quantum mechanics and to quantum information and computation [1–4]. Its importance has been demonstrated in various applications such as teleportation [5–9], superdense coding [10, 11], quantum computation [12] and E91 protocol of quantum cryptography [13], etc. The *maximally* entangled states are especially important for such quantum information processing tasks. Experimentalists try continuously to improve the quality of the entangled states by entanglement distillation and purification with which one can create a small set of highly entangled states from a large set of less entangled states (pure or mixed) [14]. Generally it is believed that the amount of entanglement in a mixed state can be increased by purifying it toward a pure maximally entangled state. That means the entanglement is increased by increasing the purity of the mixed state. The reason is simply because all known maximally entangled states are pure. Actually it has been proved [15] that all maximally entangled states are pure in bipartite $d \otimes d$ systems. The question is: Does there exist maximally entangled states which are mixed states. And further, if a mixed maximally entangled state exists, does it have any advantages and how to prepare it in a physical system?

In this paper, we introduce a class of mixed states in $d \otimes d'$ $(d' \geq 2d)$ Hilbert space which are maximally entangled. We prove that the form of the mixed maximally entangled states is unique. On the one hand, this result is conceptually new since really mixed maximally entangled states exist. On the other hand, those mixed maximally entangled states which are proved to have a unique form is actually equivalent to the pure maximally entangled state tensor product an ancillary state.

For $d \otimes d'$ $(d' \geq 2d)$ Hilbert space, a pure maximally entangled state is $|\phi\rangle = \sum_{i=1}^{d} \frac{1}{\sqrt{d}} |ii\rangle$. If a mixed state has the same entanglement quantified by a certain entanglement measure as this pure maximally entangled state, we call it the mixed maximally entangled state. The entanglement measure can be the well accepted entanglement of formation, or other entanglement measures such as concurrence, distillable entanglement, and the relative entropy of entanglement. The conclusion of this paper does not change.

One key application of the pure maximally entangled state is perfect teleportation. If a mixed state is used a quantum state will not be teleported faithfully. However, this is only true for the case of mixed state which is not maximally entangled. For mixed maximally entangled states presented in this paper, we will show that these states can be used as a resource for perfect teleportation. Also those mixed maximally entangled states can be distinguished perfectly by local operations and classical communication [16–20]. We also propose a preparation scheme to create experimentally this kind of states.

II. MIXED MAXIMALLY ENTANGLED STATE

We consider $d \otimes d'$ bipartite systems, assume $d' \geq d$ without loss of generality and take the well-accepted entanglement measure, entanglement of formation [21]. For a pure state $|\psi\rangle_{AB}$ shared by distant parties A and B, the entanglement of formation E is defined as the von Neumann entropy of either of the two subsystems: $E(|\psi\rangle_{AB}) = -\mathrm{Tr}(\rho_{A(B)} \log_2 \rho_{A(B)})$, where $\rho_{A(B)} = tr_{B(A)}(|\psi\rangle_{AB}\langle\psi|)$ is the reduced density matrix; as for a mixed state ρ_{AB}, the entanglement of formation is defined as the average entanglement of the pure states $E(\rho) = \inf \sum_i p_i E(|\psi_i\rangle)$, minimized over all pure state decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $\sum_i p_i = 1$. Hereafter we will use entanglement of formation to quantify the entanglement in a state unless otherwise stated.

To show the existence of the mixed maximally entangled state, we prove the following two lemmas:

Lemma 1. A pure $d \times d'$ bipartite state $|\psi\rangle$ is maximally entangled if and only if $E(|\psi\rangle) = \log_2 d$.

Proof. If a pure $d \times d'$ bipartite state $|\psi\rangle$ is maximally entangled, there must exist a local unitary operation $U_1\otimes U_2$ such that

$$(U_1\otimes U_2)|\psi\rangle=\sum_{i=1}^{d}\frac{1}{\sqrt{d}}|ii\rangle. \quad (1)$$

Due to the invariance of entanglement of formation under local unitary operation, we can get $E(|\psi\rangle)=E((U_1\otimes U_2)|\psi\rangle)=\log_2 d$ in terms of the definition of entanglement of formation.

We next prove the necessary condition. Suppose $E(|\psi\rangle)=\log_2 d$. In terms of the Schmidt decomposition theorem, we can rewrite this pure $d\times d'$ bipartite state $|\psi\rangle$ as

$$|\psi\rangle=\sum_{i=1}^{n}\sqrt{x_i}|i_A\rangle|i_B\rangle, \quad (2)$$

where x_i is nonnegative and satisfies the condition $\sum_{i=1}^{n}x_i=1$, $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ are orthonormal in Hilbert space H_A and H_B, respectively. In general, the Schmidt number n satisfies $n\leq d$. In terms of the definition of entanglement of formation, we have

$$E(|\psi\rangle)=-\sum_{i=1}^{n}x_i\log_2 x_i. \quad (3)$$

Under the constraint condition $\sum_{i=1}^{n}x_i=1$, we can obtain the stable point, $x_i=\frac{1}{n}$, $\forall i$, by the method of Lagrange multipliers. By further analysis, the stable point is actually the maximum point of function (3). Therefore, the maximum value of Eq. (3) $\log_2 n$ is obtained if and only if all x_is are equal, i.e., $x_i=\frac{1}{n}$, $\forall i$. As $E(|\psi\rangle)=\log_2 d$, $n=d$, $x_i=\frac{1}{d}$, $\forall i$. Thus, this state $|\psi\rangle$ is maximally entangled. □

Lemma 2. ρ is a mixed maximally entangled state with respect to the entanglement of formation E, if and only if all pure-state decompositions $\{p_i,\ |\psi_i\rangle\}$ of ρ satisfy the conditions that $E(|\psi_i\rangle)=\log_2 d$ for all i.

Proof. Firstly, we prove the sufficient condition. Assume ρ is a mixed maximally entangled state with respect to the entanglement formation E. Then, according to the definition of entanglement of formation for mixed maximally entangled states, we get its entanglement of formation, $E(\rho)=\log_2 d$. As the entanglement of formation of ρ is defined as

$$E(\rho)=\min\sum_i p_iE(|\psi_i\rangle)=\log_2 d, \quad (4)$$

where the minimum value is taken over all pure-state decompositions of ρ, $\rho_{AB}=\sum_i p_i|\psi_i\rangle\langle\psi_i|$. Moreover, for an arbitrary pure $d\times d'$ state $|\psi\rangle$, its entanglement of formation isn't greater than $\log_2 d$. Hence, all pure-state decompositions $\{p_i,\ |\psi_i\rangle\}$ of ρ must satisfy the conditions that $E(|\psi_i\rangle)=\log_2 d$ for all i, otherwise the equation (4) is violated.

If all pure-state decompositions $\{p_i,\ |\psi_i\rangle\}$ of ρ satisfy the conditions that $E(|\psi_i\rangle)=\log_2 d$ for all i, it is easy to obtain $E(\rho)=\log_2 d$ in the light of the definition of entanglement of formation. Therefore, the $d\times d'$ bipartite state ρ is maximally entangled. □

We also need the following lemma.

Lemma 3. $|\psi\rangle$ is a $d\otimes d'$ pure maximally entangled state if and only if for arbitrary given orthonormal complete basis $\{|i_A\rangle\}$ of the subsystem A, there exits an orthonormal basis $\{|i_B\rangle\}$ of the subsystem B such that $|\psi\rangle$ can be written in the following form,

$$|\psi\rangle=\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i_A\rangle\otimes|i_B\rangle. \quad (5)$$

Proof. If a pure state $|\psi\rangle$ is of the form (5), the entanglement of formation is $E(|\psi\rangle)=\log_2 d$. In terms of the Lemma 1, the state $|\psi\rangle$ is maximally entangled.

We next prove the sufficient condition. If the state $|\psi\rangle$ is maximally entangled, $|\psi\rangle$ can always be written as $|\psi\rangle=\frac{1}{\sqrt{d}}\sum_{j=1}^{d}|V_j\rangle\otimes|W_j\rangle$ according to Schmidt decomposition theorem and Lemma 1, where $|V_j\rangle$ and $|W_j\rangle$ are orthonormal basis vectors with respect to subsystems A and B, respectively. Due to the orthonormal completeness of $\{|i_A\rangle\}$ and $\{|V_j\rangle\}$, the orthonormal basis vector $|V_j\rangle$ can also be represented as $|V_j\rangle=\sum_{i=1}^{d}U_{ji}|i_A\rangle$, where U is a unitary matrix. The bipartite pure state $|\psi\rangle$ then can be written as $|\psi\rangle=\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i_A\rangle\otimes(\sum_{j=1}^{d}U_{ji}|W_j\rangle)=\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i_A\rangle\otimes|i_B\rangle$, where we have set $|i_B\rangle=\sum_{i=1}^{d}U_{ji}|W_j\rangle$ which is just an orthonormal basis vector. □

The main result of this paper is the following

Theorem 1. A $d\otimes d'$ $(d'\geq 2d)$ bipartite mixed state ρ is maximally entangled if and only if

$$\rho=\sum_m p_m|\psi_m\rangle\langle\psi_m|,\ \ \sum_m p_m=1, \quad (6)$$

where $|\psi_m\rangle=\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle\otimes|i_m\rangle$, $\{|i\rangle\}$ and $\{|i_m\rangle\}$ are orthonormal bases of the subsystems A and B respectively, satisfying $\langle j_n|i_m\rangle=\delta_{ij}\delta_{nm}$.

Proof. If a mixed state ρ has the pure-state decomposition (6), which is the spectral decomposition of ρ, according to the method [22], a general decomposition $\{q_n,|w_n\rangle\}$, $\rho=\sum_n q_n|w_n\rangle\langle w_n|$, is given by $\sqrt{q_n}|w_n\rangle=\sum_m U_{nm}\sqrt{p_m}|\psi_m\rangle$, n=1,…,l. Here U is an $l\times l$ unitary matrix, l is greater than or equal to the rank of ρ, and the following condition is satisfied, $\sum_m|U_{nm}|^2p_m/q_n=1$.

One can check that

$$\begin{aligned}\rho_A^{(n)} &= \mathrm{Tr}_B|w_n\rangle\langle w_n| \\ &= \mathrm{Tr}_B(\sum_{m,m'} U_{nm}U^*_{nm'}\sqrt{p_m}\sqrt{p_{m'}}|\psi_m\rangle\langle\psi_{m'}|/q_n) \\ &= \sum_{m,m'} \mathrm{Tr}_B(|\psi_m\rangle\langle\psi_{m'}|)U_{nm}U^*_{nm'}\sqrt{p_m}\sqrt{p_{m'}}/q_n \\ &= \frac{1}{d}\sum_{i=1}^{d}|i\rangle\langle i|\sum_{m,m'} U_{nm}U^*_{nm'}\sqrt{p_m}\sqrt{p_{m'}}\delta_{mm'}/q_n \\ &= \frac{1}{d}I.\end{aligned}$$

Then, we obtain $E(|w_n\rangle)=\log_2 d$ for an arbitrary pure-state decomposition $\{q_n, |w_n\rangle\}$ of ρ. Therefore from Lemma 1 and Lemma 2 we deduce that the mixed state ρ is maximally entangled.

If the $d\otimes d'$ bipartite mixed state ρ is maximally entangled, then in terms of Lemma 2 all states in the pure state decomposition of ρ are maximally entangled. Hence, the eigenvectors of ρ are also maximally entangled. In the light of Lemma 3 the eigenvectors of ρ are of the form, $|v_m\rangle = \sqrt{p_m}\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle\otimes|\phi_{im}\rangle$, where p_m is the mth eigenvalue, $\{|i\rangle\}$ and $\{|\phi_{im}\rangle\}$ are orthonormal bases of subsystems A and B respectively, i.e., $\langle\phi_{jm}|\phi_{im}\rangle=\delta_{ij}$. According to the method [22], then a general decomposition $\{q_n, |u_n\rangle\}$ of ρ is given by,

$$\begin{aligned}|u_n\rangle &= \sum_{m=1}^{k} U_{nm}|v_m\rangle/\sqrt{q_n} \\ &= \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle\otimes(\sum_{m=1}^{k} U_{nm}\sqrt{p_m}|\phi_{im}\rangle/\sqrt{q_n}) \\ &= \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle\otimes|\Phi_{in}\rangle \quad n=1,\ldots,l \end{aligned} \tag{7}$$

where $|\Phi_{in}\rangle=\sum_{m=1}^{k}U_{nm}\sqrt{p_m}|\phi_{im}\rangle/\sqrt{q_n}$, k is the rank of ρ_{AB}, and U is an $l\times l$ unitary matrix with $l\geq k$. Because of the maximal entanglement in $|u_n\rangle$, the state $|\Phi_{in}\rangle$ is orthonormal with respect to i according to Lemma 3. Then we can get

$$\begin{aligned}\langle\Phi_{in}|\Phi_{jn}\rangle &= \sum_{m,m'} U_{nm}U^*_{nm'}\sqrt{p_m p_{m'}}\langle\phi_{im'}|\phi_{jm}\rangle/q_n \\ &= \delta_{ij}.\end{aligned} \tag{8}$$

As long as we find an arbitrary $l\times l$ unitary matrix U with $l\geq k$, we will obtain a pure state decomposition $\{q_n, |u_n\rangle\}$ of ρ expressed as Eq.(7). Furthermore, in terms of Lemma 2 and Lemma 3, the corresponding state $|\Phi_{in}\rangle$ in Eq.(7) must satisfy Eq.(8) for any arbitrary $l\times l$ unitary matrix U. Due to the arbitrariness of the unitary U, one can obtain that $\langle\phi_{im}|\phi_{im'}\rangle=0$ for $m\neq m'$, and $\langle\phi_{im'}|\phi_{jm}\rangle=0$ for $j\neq i$ and $m\neq m'$ by choosing proper coefficients U_{nm} [23]. This conclusion gives rise to $\langle\phi_{jm'}|\phi_{im}\rangle=\delta_{ij}\delta_{mm'}$, which implies that the dimension d' of subsystem B must be greater or equal to kd.

Therefore, a bipartite mixed state ρ is maximally entangled if and only if it has the form Eq. (6). □

From the theorem we see that, if the rank of a mixed maximally entangled state ρ of $d\otimes d'$ system is k, d' must be greater or equal to kd. For the case $k=1$, ρ becomes a maximally entangled pure state. It is also evident that there do not exist mixed maximally entangled states in $d\otimes d$ systems [15].

We now give an example of mixed maximally entangled state of $2\otimes 4$ bipartite systems, $\rho=\frac{1}{2}(|\psi_1\rangle\langle\psi_1|+|\psi_2\rangle\langle\psi_2|)$, where $|\psi_1\rangle=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ and $|\psi_2\rangle=\frac{1}{\sqrt{2}}(|02\rangle+|13\rangle)$ are both 2×4 maximally entangled pure states. Suppose that $\{q_i, |\phi_i\rangle\}$ is an arbitrary pure-state decomposition of ρ, $\rho=\sum_i q_i|\phi_i\rangle\langle\phi_i|$. Then there must exist a unitary U such that the general decomposition $|\phi_i\rangle$ can be given by $|\phi_i\rangle=U_{i1}|\psi_1\rangle+U_{i2}|\psi_2\rangle$ with $|U_{i1}|^2+|U_{i2}|^2=1$ [22]. We have $\rho_A^{(i)}=tr_B(|\phi_i\rangle\langle\phi_i|)=\frac{1}{2}(|0\rangle\langle 0|+|1\rangle\langle 1|)$, and $E(|\phi_i\rangle)=1$. Therefore, ρ_{AB} is a mixed maximally entangled state by the Lemma 2.

Remark. As the classification and characterization of entanglement in multipartite states isn't fairly clear, this theorem is only valid for bipartite systems. If this mixed maximally entangled state is viewed as a multipartite state (e.g. the second Hilbert space is divided into two Hilbert spaces), then it is equivalent to a pure maximally entangled state tensor product a mixed state, $\rho=\sum_m p_m|\psi_m\rangle\langle\psi_m| = |\Phi^+\rangle\langle\Phi^+|\otimes\tilde{\rho}_a$, where $|\Phi^+\rangle=\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle\otimes|i\rangle$, $\tilde{\rho}_a\equiv\sum p_m|m\rangle_a\langle m|$ with $|m\rangle$ orthonormal eigenvectors corresponding to the eigenvalues p_m. However, we would like to emphasize that our main result in this paper is that for the first time, we identifies the unique form of the mixed maximally entangled state. Physically the result can be understood that particle A which is a qudit entangles with particle B which is still a qudit but with k different colors. Though we may not know the exact color of particle B, but always A and B is maximally entangled.

In terms of entanglement of formation, we obtain the unique form of the mixed maximally entangled state. In addition, if entanglement measures $E(|\psi\rangle)$ can distinguish pure maximally entangled states from non-maximally entangled states, the Lemma 1 and 3 are easily proved to be valid by changing the value of entanglement measure for maximally entangled states. On the basis of distinguishability for maximally entangled states, if the entanglement measures for mixed states are defined by the convex roof, the Lemma 2 must be true with the value of entanglement measure for maximally entangled states changed. In terms of Lemma 1, 2, 3, we can prove the Theorem 1 for such defined entanglement measures in a similar way. Therefore, as long as entanglement measures defined by the convex roof [24] can distinguish maximally entangled states from non-maximally entangled states, the Theorem 1 also holds for these entanglement measures such as concurrence [21].

Moreover, for other nonequivalent entanglement measures such as distillable entanglement and the relative entropy of entanglement [25–28], the theorem is also verified.

We will take the relative entropy of entanglement as an example. For the state $\rho = \frac{1}{4}[(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) + (|02\rangle + |13\rangle)(\langle 02| + \langle 13|)]$, we assume that there is a set of local operations performed on the second subsystem, a 4-dimensional system, and the corresponding operators are expressed as, $B_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$ and $B_2 = |2\rangle\langle 2| + |3\rangle\langle 3|$. Due to the monotonicity of the relative entropy of entanglement under local operations and classical communication (LOCC), that is, the relative entropy of entanglement cannot increase under LOCC operations, the relative entropy of this state satisfies the following inequality, $E_R(\rho) \geq \mathrm{Tr}[(I \otimes B_1^\dagger)\rho(I \otimes B_1)]E_R(\frac{(I\otimes B_1)\rho(I\otimes B_1^\dagger)}{\mathrm{Tr}(I\otimes B_1^\dagger)\rho(I\otimes B_1)}) + \mathrm{Tr}[(I \otimes B_2)\rho(I \otimes B_2^\dagger)]E_R(\frac{(I\otimes B_2)\rho(I\otimes B_2^\dagger)}{\mathrm{Tr}(I\otimes B_2)\rho(I\otimes B_2^\dagger)}) = 1/2[E_R((|00\rangle + |11\rangle)/\sqrt{2}) + E_R((|02\rangle + |13\rangle)/\sqrt{2})]$. In the light of the theorem in the reference [29] and the convexity of the relative entropy, we have $E_R(\rho) = 1$ which means this mixed state is maximally entangled. In a similar way, for the case with high dimension, this mixed state expressed in Eq. (6) of our paper can be proved maximally entangled. Next we prove the mixed maximally entangled state must be of the form of Eq. (6) for the relative entropy. The quantity of entanglement in maximally entangled states quantified by the relative entropy in $d \otimes d'$ ($d' \geq d$) Hilbert spaces is $\log_2 d$. We suppose that there exists a $d \otimes d'$ ($d' \geq d$) mixed maximally entangled state ρ'. Since the relative entropy is equal to the von Neumann reduced entropy for pure states and is convex, we have the following inequality: $E_R(\rho') \leq \sum_i p_i E_R(|\phi_i\rangle) = \sum_i p_i E_N(|\phi_i\rangle)$, where E_N denotes the von Neumann reduced entropy and $\{p_i, \phi_i\}$ is an arbitrary pure state decomposition of $\rho' = \sum_i p_i |\phi_i\rangle\langle\phi_i|$, $\sum_i p_i = 1$. Due to the maximally entangled states ρ', $E_R(\rho') = \log_2 d$, the inequality holds for any arbitrary pure-state decompositions of ρ', $\sum_i p_i E_N(|\phi_i\rangle) \geq \log_2 d$. Therefore, this inequality implies that the state ρ' is also maximally entangled for the entanglement of formation. In terms of the Theorem 1, the state ρ' must be of the form of Eq. (6).

For the distillable entanglement E_D which satisfies $E_D \leq E_F$ with E_F the entanglement of formation, we can also prove, in a similar way, the mixed maximally entangled states must be of the form of Eq. (6).

In fact, these mixed maximally entangled states can be used as a resource for perfect teleportation and can be distinguished perfectly by LOCC. In the following two sections, we provide a detailed protocol for perfect teleportation and perfect local distinguishability under LOCC.

III. TELEPORTATION WITH MIXED MAXIMALLY ENTANGLED STATE

Suppose Alice and Bob initially share a pair of particles, A_2 and B, in a mixed maximally entangled state of $2d \otimes d$ system, $\chi_{A_2B} = \frac{1}{2d}\sum_{i,j=0}^{d-1}(|i,i\rangle\langle j,j|) + |d+i,i\rangle\langle d+j,j|)$. Alice wants to send an unknown state of particle A_1, $|\psi\rangle_{A_1} = \sum_{i=0}^{d-1}\alpha_i|i\rangle_{A_1}$, to Bob by performing a complete von Neumann measurement on the joint system of particles A_1 and A_2 and informing Bob the result of measurement by classical communication.

We first define some operators so as to obtain the generalized Bell states [30]. Let h and g be $d \times d$ matrices such that $h|j\rangle = |(j+1) \mathrm{mod}\, d\rangle$, $g|j\rangle = \omega^j|j\rangle$, with $\omega = \exp\{-2\mathrm{i}\pi/d\}$. d^2 linear independent $d \times d$ matrices are defined as $U_{st} = h^t g^s$. One can check that $\{U_{st}\}$ satisfy the condition of *basis of the unitary operators* in the sense of [31], i.e., $\mathrm{Tr}(U_{st}U_{s't'}^\dagger) = d\delta_{tt'}\delta_{ss'}$, and $U_{st}U_{st}^\dagger = I_{d\times d}$. Therefore, we can construct $2d^2$ generalized Bell states:

$$\begin{aligned} |\Phi_{st}^1\rangle &= \frac{U_{st}\otimes I}{\sqrt{d}}\sum_{i=0}^{d-1}|i,i\rangle, \\ |\Phi_{st}^2\rangle &= \frac{U_{st}\otimes I}{\sqrt{d}}\sum_{i=0}^{d-1}|i,d+i\rangle. \end{aligned} \tag{9}$$

$\{|\Phi_{st}^1\rangle, |\Phi_{st}^2\rangle\}$ form a complete orthogonal normalized basis of $d \otimes 2d$ system.

The initial state of the three particles, A_1, A_2 and B, can be rewritten as: $|\psi\rangle_{A_1}\langle\psi| \otimes \chi_{A_2B} = \frac{1}{2d}\sum_{s,t,s',t'}\Big[|\Phi_{st}^1\rangle_{A_1A_2}\langle\Phi_{s't'}^1| \otimes U_{st}^\dagger|\psi\rangle_B\langle\psi|U_{s't'} + |\Phi_{st}^2\rangle_{A_1A_2}\langle\Phi_{s't'}^2| \otimes U_{st}^\dagger|\psi\rangle_B\langle\psi|U_{s't'}\Big]$.

To teleport the unknown state $|\psi\rangle_{A_1}$ to Bob, Alice can take a complete von Neumann measurement using the states $\{|\Phi_{st}^i\rangle\}$ on the joint system consisting of particles A_1 and A_2. Then she announces her measurement result, s and t, to Bob via classical communication. Bob can then transform the state of his particle into $|\psi\rangle_{A_1}$ by applying the corresponding operation U_{st}. This means the mixed maximally entangled state χ_{A_2B} can also be used as a resource for perfect teleportation.

This is different from the usual concept in which the general mixed state ρ in $d \otimes d$ system can not be used to teleport an unknown state $|\psi\rangle$ in d-dimension with unit fidelity [30, 32].

IV. LOCAL DISTINGUISHABILITY OF THE MIXED MAXIMALLY ENTANGLED STATES

Local distinguishability describes the property of states shared by distant parties which are discriminated by only local operations and classical communication. We start from a simple example: We have a set of mixed maximally entangled states $\{\rho(\Phi^\pm), \rho(\Psi^\pm)\}$, where $\rho(\Phi^{\pm}) = \frac{1}{2}(|\Phi_1^{\pm}\rangle\langle\Phi_1^{\pm}| + |\Phi_2^{\pm}\rangle\langle\Phi_2^{\pm}|)$, $\rho(\Psi^{\pm}) = \frac{1}{2}(|\Psi_1^{\pm}\rangle\langle\Psi_1^{\pm}| + |\Psi_2^{\pm}\rangle\langle\Psi_2^{\pm}|)$. Here states $|\Phi_1^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), |\Psi_1^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ are four Bell states, and correspondingly we denote $|\Phi_2^{\pm}\rangle = \frac{1}{\sqrt{2}}(|02\rangle \pm |13\rangle), |\Psi_1^{\pm}\rangle = \frac{1}{\sqrt{2}}(|03\rangle \pm |12\rangle)$. We can find that any two states from this set can be locally distinguished with certainty, this is similar to the case of four Bell states.

We then present a general result about the local distinguishability of the mixed maximally entangled states. We denote

$$\chi_{st} = \frac{1}{2}(|\Phi_{st}^1\rangle\langle|\Phi_{st}^1| + |\Phi_{st}^2\rangle\langle|\Phi_{st}^2|), \tag{10}$$

where $|\Phi_{st}^1\rangle$ and $|\Phi_{st}^2\rangle$ are defined in Eq. (9), then we have

Theorem 2. Any l mixed maximally entangled states of the set $\{\chi_{st}\}$ can be distinguished perfectly by local operations and classical communication in case $l(l-1) \leq 2d$, here d is prime.

The proof of this theorem is similar to the case of pure maximally entangled states as in [19].

Proof. According to Eq.(9), the mixed maximally entangled states (10) can be rewritten as

$$\begin{aligned}\chi_{st} &= \frac{1}{2}(|\Phi_{st}^1\rangle\langle|\Phi_{st}^1| + |\Phi_{st}^2\rangle\langle\Phi_{st}^2|)\\ &= \frac{U_{st}\otimes I}{2d}\sum_{i,j=0}^{d-1}(|i,i\rangle\langle j,j| + |i,d+i\rangle\langle j,d+j|)(U_{st}^{\dagger}\otimes I),\end{aligned}$$

where $U_{st} = h^s g^t$. h and g are defined in the last section, i.e., $h|j\rangle = |(j+1)\text{mod } d\rangle$, $g|j\rangle = \omega^j|j\rangle$, where $\omega = \exp\{2i\pi/d\}$. Therefore, h and g can be expressed in the following equation

$$h = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},\ g = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ 0 & \omega & 0 & \cdots & 0\\ 0 & 0 & \omega^2 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \omega^{d-1} \end{pmatrix},$$

and satisfy the conditions $gh = \omega hg$ and $g^{-1}h = \omega^{-1}hg^{-1}$.

For simplicity, we adopt the following denotation,

$$\rho^+ = \frac{1}{2d}\sum_{i,j=0}^{d-1}\left(|i,i\rangle\langle j,j| + |i,d+i\rangle\langle j,d+j|\right). \tag{11}$$

Then, χ_{st} takes the following expression $\chi_{st} = (U_{st}\otimes I)\rho^+(U_{st}^{\dagger}\otimes I)$. For an arbitrary $d\otimes d$ dimensional unitary matrix V and $2d\otimes 2d$ dimensional unitary matrix $V\oplus V$, we have

$$(I\otimes V)\sum_{i=0}^{d-1}|i,i\rangle = (V^T\otimes I)\sum_{i=0}^{d-1}|i,i\rangle, \tag{12}$$

$$[I\otimes(V\oplus V)]\sum_{i=0}^{d-1}|i,d+i\rangle = (V^T\otimes I)\sum_{i=0}^{d-1}|i,d+i\rangle, \tag{13}$$

where V^T denotes the transposition of V.

Suppose l mixed maximally entangled states are denoted by $\{(h^{s_i}g^{t_i}\otimes I)\rho^+(h^{s_i}g^{t_i}\otimes I)^{\dagger}\}_{i=1}^l$. In terms of the equation $h|j\rangle = |(j+1)\text{mod } d\rangle$, if the set $\{s_i\}_{i=1}^l$ has no equal s_i, we can locally distinguish these l mixed maximally entangled states simply by performing projecting-measurements in the computational basis $\{|i\rangle\langle i|\}$ on A side and $\{|i\rangle\langle i| + |d+i\rangle\langle d+i|\}$ on B side respectively, and subsequently by a classical communication.

In general, to locally distinguish these states, we first let A and B implement unitary operations U and V^T, respectively. This operation is equivalent to the transformation $Uh^{s_i}g^{t_i}V\otimes I$ on ρ^+. We next show that we can find these unitary operators U and V^T that can transform these l maximally entangled states to the set $\{(h^{s_i'}g^{t_i'}\otimes I)\rho^+(h^{s_i'}g^{t_i'}\otimes I)^{\dagger}\}_{i=1}^l$, where there are no equal s_i'. As we have proved, this set can be distinguished locally. Next we give these unitary operations.

We introduce d generalized Hadamard transformations, H_α, $(\alpha = 0, 1, \cdots, d-1)$, which are $d\otimes d$ dimensional unitary matrices with

$$(H_\alpha)_{jk} = \omega^{-jk}\omega^{-\alpha m_k}, m_k = k + (k+1) + \cdots + (d-1).$$

Deducing from the definition of H_α and the equation $g^{-1}h = \omega^{-1}hg^{-1}$, we have relations $H_\alpha h H_\alpha^{\dagger} = g^{-1}h^\alpha$, $H_\alpha g H_\alpha^{\dagger} = h$, and $H_\alpha h^{s_i}g^{t_i}H_\alpha^{\dagger} = \Gamma h^{\alpha s_i + t_i}g^{-s_i}$, where the whole phase factor $\Gamma = \omega^{-\alpha s_i(s_i+1)/2}$. Firstly, let A and B implement unitary operations H_α and $H_\alpha^*\oplus H_\alpha^*$, respectively. By applying Eqs. (12) and (13), the following equation holds

$$\begin{aligned}&[H_\alpha\otimes(H_\alpha^*\oplus H_\alpha^*)]\chi_{st}[H_\alpha^{\dagger}\otimes(H_\alpha^*\oplus H_\alpha^*)^{\dagger}]\\ &= (H_\alpha U_{st}H_\alpha^{\dagger})\rho^+(H_\alpha U_{st}H_\alpha^{\dagger})^{\dagger}.\end{aligned} \tag{14}$$

Given l maximally entangled states corresponding to $\{h^{s_i}g^{t_i}\}_{i=1}^l$, we can always transform them to the case $\{h^{s_i'}g^{t_i'}\}_{i=1}^l$, where the powers of h are different, by identity (do nothing) or $H_\alpha\otimes(H_\alpha^*\oplus H_\alpha^*)$, $(\alpha = 0, 1, \cdots, d-1)$. If not, then for each transformation at least two powers of h are equal. So at least we have $d+1$ equations altogether. But different combinations between l elements $\{h^{s_i}g^{t_i}\}_{i=1}^l$ is $\binom{l}{2} = l(l-1)/2$, which is less than or equal to d. This means two pairs, for example, (s_0, t_0) and (s_1, t_1) without loss of generality, appear twice in two different transformations, say α_0 and α_1, that is $s_0'(\alpha_0) = s_1'(\alpha_0)$ and $s_0'(\alpha_1) = s_1'(\alpha_1)$. Therefore, we obtain these equations,

$$\begin{aligned}\alpha_0 s_0 + t_0 &= \alpha_0 s_1 + t_1 \pmod d\\ \alpha_1 s_0 + t_0 &= \alpha_1 s_1 + t_1 \pmod d,\end{aligned}$$

Thus $(s_0, t_0) = (s_1, t_1)$, which contradicts our assumption that these l mixed maximally entangled states are orthogonal. This completes our proof. □

With mixed maximally entangled states, similar studies can be made for other information processing tasks such as superdense coding, quantum computation, cryptography, entanglement swapping and remote state preparation.

V. THE EXPERIMENTAL PREPARATION

The mixed maximally entangled state may be realized in an NMR system. We may choose material HC_2 in which the spin-1/2 H and two isotope C13s, which are regarded as a whole, provide a $2 \otimes 4$ system [33]. Suppose the system HC_2 is in the initial state, $|\psi\rangle = \sqrt{1/2}(|01\rangle - |10\rangle)$, and the two isotope C13s interact with the environment—a spin-1/2 particle. The spin-1/2 particle starts in the state $|0\rangle$ by applying an external magnetic field with direction along z-axis. We also assume that the interaction with the environment is described by the Hamiltonian, $H = c(\sqrt{q_1} M_1 \otimes \sigma_z + \sqrt{q_2} M_2 \otimes \sigma_x)$, where $M_1 = I_{4\times4}/\sqrt{2}$, $M_2 = (|0\rangle\langle 2| + |1\rangle\langle 3| + |3\rangle\langle 1| + |2\rangle\langle 0|)/\sqrt{2}$, and $q_1 + q_2 = 1$. c, q_1 and q_2 are real numbers and vary with the intensity of magnetic field in the environment. Then we choose an appropriate time and then obtain the Kraus operators $\sqrt{q_1} I \otimes M_1$ and $\sqrt{q_2} I \otimes M_2$ by performing a partial trace over the environment. The final state becomes $\rho' = \sum_{i=1}^{2} q_i (I \otimes M_i)|\psi\rangle\langle\psi|(I \otimes M_i^\dagger)$, which is a mixed maximally entangled state. However, the corresponding Hamiltonian between the spin-3/2 particle and an environment is not easy to manipulate. We hope the scheme to prepare the mixed maximally entangled states may help to provide some hints for the experimenters.

Due to the interactions with the environment in preparation and transmission, the entangled pure states usually become mixed ones and no longer entangled. However, the entanglement evolution [34, 35] for the mixed maximally entangled state under the influence of local preparation channel shows that the output state is always a maximally entangled state and still useful for many quantum information processing tasks like perfect teleportation. This result is easily obtained from the entanglement evolution equation $\$(\rho) = \sum_{i=1}^{2} q_i (I \otimes M_i)\rho(I \otimes M_i^\dagger)$, where $\rho = (1-p)|\psi_1\rangle\langle\psi_1| + p|\psi_2\rangle\langle\psi_2|$.

VI. CONCLUSIONS

We have found a novel quantum state — the mixed maximally entangled state and prove that the form of the mixed maximally entangled states is unique. A protocol is presented to teleport faithfully an unknown d-dimensional state by resource of a mixed maximally entangled state in $d \otimes 2d$. Furthermore, it is shown that any l mixed maximally entangled states of the set $\{(U_{st} \otimes I)\rho_{AB}(U_{st}^\dagger \otimes I)\}$ can be discriminated perfectly by local operations and classical communication. We also proposed a scheme to prepare these states in an NMR physical system.

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[22] According to the spectral decomposition theorem, we can rewrite a state ρ as $\rho = \sum_{i=1}^{n} p_i |v_i\rangle\langle v_i|$, where $|v_i\rangle$ are orthonormal eigenvectors corresponding to the nonzero eigenvalues p_i, n is the rank of ρ. Then a general pure-state decomposition $\{|w_i\rangle\}$ of ρ is given by $|w_i\rangle = \sum_{j=1}^{n} U_{ij}\sqrt{p_j}|v_j\rangle$, $i = 1 \cdots m$. Here U is an $m \times m$ unitary matrix, and m is greater than or equal to n. According to the unitarity of U, $\sum_i |w_i\rangle\langle w_i| = \rho$ can be verified. The states $|w_i\rangle$ are automatically subnormalized

so that $\langle w_i|w_i\rangle$ is equal to the probability of $|w_i\rangle$ in the decomposition.

[23] One can always find a unitary matrix U such that $U_{nn_1} = 1/\sqrt{2}$, $U_{nn_2} = \mathrm{i}/\sqrt{2}$ $(n_1 \neq n_2)$ for a fixed n (The elements with $m \neq n_1, n_2$ are zero). Then we have $\langle\phi_{in_1}|\phi_{jn_2}\rangle - \langle\phi_{in_2}|\phi_{jn_1}\rangle = 0$. We can also find one other unitary matrix U such that $U_{nn_1} = 1/\sqrt{2}$, $U_{nn_2} = 1/\sqrt{2}$ $(n_1 \neq n_2)$. Correspondingly, the equation $\langle\phi_{in_1}|\phi_{jn_2}\rangle + \langle\phi_{in_2}|\phi_{jn_1}\rangle = 0$ is satisfied. Hence $\langle\phi_{in_2}|\phi_{jn_1}\rangle = 0$. Then it is easy to obtain $\langle\phi_{im}|\phi_{jm'}\rangle = \delta_{ij}\delta_{mm'}$.

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