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by

Fatihcan M. Atay and Lavinia Roncoroni

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Fatihcan M. Atay & Lavinia Roncoroni

Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany

Abstract

We analyze the lumpability of linear differential equations on Banach spaces, namely, the possibility of projecting the dynamics by a linear reduction operator onto a smaller state space on which a self-contained dynamical description exists. We first consider systems whose evolution is described by bounded linear operators, and extend previous results by relaxing some of the hypotheses. The lumpability condition is then expressed as the invariance of the kernel of the reduction operator under the evolution operator. Next, as the main contribution of the paper, we consider dynamics defined by unbounded operators. We use methods from the theory of strongly continuous semigroups to obtain conditions on the reduction operator for lumpability. We indicate several applications to particular systems, including delay differential equations.

1 Introduction

Consider a linear dynamical system defined on a Banach space X :

$$\begin{aligned} A : \mathcal{D}(A) \subseteq X &\rightarrow X, \\ \begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0. \end{cases} \end{aligned} \tag{1}$$

We assume that the dynamics (1) is well-defined, in the sense that for every $x_0 \in \mathcal{D}(A)$ there exists a unique classical solution $x(t) \in C^1([0, +\infty), \mathcal{D}(A))$ that depends continuously on the initial condition x_0 . In addition, consider a linear bounded map

$$M : X \rightarrow Y, \quad y = Mx,$$

where Y is another Banach space. We view the operator M as a *reduction* of the state space: it is surjective but not an isomorphism. The question is whether the new variables y also satisfy a well-posed and self-contained dynamics on Y , say

$$\dot{y}(t) = \hat{A}y(t)$$

for some linear operator \hat{A} . If this is the case, then we refer to M as a *reduction* or *lumping* operator, following the definition of Wei and Kuo [16].

Definition 1. The system (1) is said to be *lumpable* by the operator M if there exists a linear operator $\hat{A}: Y \rightarrow Y$ such that the following diagram commutes

$$\begin{array}{ccc}
 Y & \xrightarrow{\hat{A}} & Y \\
 \uparrow M & & \uparrow M \\
 X & \xrightarrow{A} & X
 \end{array} \tag{2}$$

that is,

$$MA = \hat{A}M.$$

The term *lumping* originates from chemical reaction systems, where the aim is to aggregate all the species involved in the reaction into a few groups, called *lumps* of chemical reagents [16]. In general, the operator M can also represent other types of reduction, for example projections. Diagram (2) can also be interpreted in the context of *multi-level systems*, where X and Y represent, respectively, micro (lower) and macro (upper) levels, each with its own autonomous description. We will investigate the general theoretical conditions on M to generate a new autonomous level of description.

Before dealing with operators on a generic Banach space we briefly look at the situation in the n -dimensional Euclidean space, which was the original setting in [16], namely,

$$\begin{cases} \dot{x}(t) = Ax(t) & A: \mathbb{C}^n \rightarrow \mathbb{C}^n \\ x(0) = x_0 \end{cases}$$

where the matrix M has full row rank and

$$\text{Rank}(M) = k < n,$$

so that M represents a reduction of the state space dimension. In the notation of diagram (2) with $X = \mathbb{C}^n$ and $Y = \mathbb{C}^k$, the operator \hat{A} that makes the diagram commute will be a k -dimensional matrix. In this finite dimensional setting the following result is known (see, for example, [7]).

Proposition 1. *The following statements are equivalent:*

1. $MA = \hat{A}M$,
2. $\text{Ker}(M)$ is A -invariant,
3. $\text{Ker}(M) \subseteq \text{Ker}(MA)$.

The lumpability of finite dimensional systems has been studied by, e.g., Li and Rabitz in application to chemical kinetics (see [7, 13]), and by Gurvits and Ledoux in the setting Markov chains [4]. Our aim in this paper is to extend these results to infinite dimensional systems involving both bounded and unbounded operators. Previous work in this area was carried out by Coxson [3], and Zoltan and Toth [11].

We will obtain more general conditions for lumpability of linear systems: In the last section of this paper we will only require that the dynamics is well-posed, in the sense of the Hille and Yosida Theorem. Our approach is essentially based on the theory of strongly continuous semigroups in Banach spaces.

Observation 1 (Lumpability and Observability). It is possible to view the action of the lumping operator M as yielding a *system observable* $y = Mx$ or the output of a linear time-invariant control system [3]:

$$\begin{cases} \dot{x}(t) = Ax(t) & A : \mathbb{C}^n \rightarrow \mathbb{C}^n \\ y(t) = Mx(t). \end{cases}$$

Recall that the system is called *observable* if every initial condition $x_0 \in \mathbb{C}^n$ can be uniquely reconstructed from the system output y . This happens if and only if the *observability matrix*

$$\mathcal{O} := \begin{pmatrix} M \\ MA \\ \vdots \\ MA^{n-1} \end{pmatrix}$$

has full rank n . It is easy to see that if the system is lumpable by M , then

$$\text{Rank}(\mathcal{O}) = \text{Rank}(M) = k < n,$$

so that the system is not observable. In fact one can think of lumpability as the opposite of observability, because if the new level is closed there is no possibility to reconstruct from it the information contained in the lower level [6].

The conditions on M given by Proposition 1 actually impose strong limitations to the possibility of lumping. For example if one is interested in a linear combination of the components in the vector x , like an average, the only possibility is to take M as a left eigenvector of A , $MA = \lambda M$ for some $\lambda \in \mathbb{C}$. As we will see in the following sections, the choice of M is more varied in the infinite dimensional setting even if the restrictions on M to have a lumping are similar to the case of matrices.

2 Dynamical systems with bounded operators

Consider system (1) when the operator A belongs to the Banach algebra $\mathcal{B}(X)$ of linear bounded operators from an infinite dimensional Banach space X to itself. Recall that the topology of $\mathcal{B}(X)$ is induced by the norm

$$\|A\| := \sup_{\|x\| \leq 1} \|Ax\|.$$

Since A is a bounded operator, the system (1) is well defined and the solutions are given by $x(t) = e^{At}x(0)$ for the family of exponential operators

$$e^{At} := \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

It follows from the properties of a Banach algebra that this series is absolutely convergent, and hence convergent in the topology of $\mathcal{B}(X)$. We consider the diagram (2) where $M : X \rightarrow Y$ is a linear, bounded and surjective operator between Banach spaces X, Y . In order to obtain a lumping it is necessary that the kernel of M is invariant under A :

Theorem 2. *There exists a linear, bounded operator $\hat{A} \in \mathcal{B}(Y)$ such that $MA = \hat{A}M$ if and only if $\text{Ker}(M) \subseteq \text{Ker}(MA)$.*

Observation 2. This theorem was proved by Coxson [3] using the pseudoinverse of a bounded operator under the additional assumption that the kernel of M be topologically complemented in X . In other words, it was assumed that there exists a closed subspace $N \subset X$ such that

$$N \cap \text{Ker}(M) = 0, \quad N + \text{Ker}(M) = X. \quad (3)$$

This is the condition for the existence of a bounded right inverse of M [2], i.e., a bounded linear operator T such that

$$T : Y \rightarrow X, \quad M \circ T = I_Y.$$

However in a generic Banach space not every closed subspace is complemented. As an example one can take the Banach space $X = L^1(S^1)$ of integrable functions on the unit circle: Here, the closed subspace of all functions f whose Fourier coefficients $\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$ vanish for $n < 0$ is not complemented in X (for a proof, see e.g. [12]). Moreover, it is known that every Banach space that is not isomorphic to a Hilbert space has at least one non-complemented closed subspace [8]. Consequently, the hypothesis that $\text{Ker}(M)$ be topologically complemented is not easy to satisfy in applications. For this reason we present an alternative proof that does not require condition (3) but uses only the continuity and surjectivity of M .

Proof. If there exists an operator \hat{A} such that $MA = \hat{A}M$, then the kernel of M is invariant under A , since

$$Mx = 0 \quad \Rightarrow \quad MAx = \hat{A}Mx = 0.$$

We need to prove the inverse implication. Let us define the quotient space

$$\frac{X}{\text{Ker}(M)}$$

as the set of the equivalence classes $[x]$, $x \in X$, given by the relation

$$y \in [x] \quad \Leftrightarrow \quad x - y \in \text{Ker}(M).$$

Since the kernel is closed, the quotient is still a Banach space with the norm

$$\|[x]\| := \inf_{m \in \text{Ker}(M)} \|x - m\|. \quad (4)$$

Consider the following diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{M} & Y \\
 & \searrow \pi & \uparrow \widetilde{M} \\
 & & \frac{X}{\text{Ker}(M)}
 \end{array} \tag{5}$$

where π is the quotient projection $\pi(x) = [x]$. Since we can choose $m = 0$ in (4), the quotient projection satisfies $\|\pi(x)\| \leq \|x\|$. Define a new operator on the quotient:

$$\widetilde{M} : \frac{X}{\text{Ker}(M)} \rightarrow Y, \quad \widetilde{M}[x] := Mx.$$

This definition is well-posed in the sense that it does not depend on the choice of the particular element in the equivalence class. Furthermore, since $[x] = [x - m] \forall m \in \text{Ker}(M)$, we can write:

$$\begin{aligned}
 \|\widetilde{M}[x]\| &= \inf_{m \in \text{Ker}(M)} \|\widetilde{M}[x - m]\| = \inf_{m \in \text{Ker}(M)} \|M(x - m)\| \\
 &\leq \inf_{m \in \text{Ker}(M)} \|M\| \|x - m\| = \|M\| \|[x]\|,
 \end{aligned}$$

which implies that \widetilde{M} is bounded. We observe the following facts:

1. Since M is surjective, \widetilde{M} is also a surjective operator between Banach spaces. The open mapping theorem implies that \widetilde{M} is an open map on the quotient.
2. \widetilde{M} is injective; in fact, by definition

$$\widetilde{M}[x] = 0 \Leftrightarrow x \in \text{Ker}(M),$$

and the zero element in the quotient space is exactly $[0] := \text{Ker}(M)$.

3. \widetilde{M} is an open and continuous bijection (that is, a homeomorphism), so the inverse

$$\widetilde{M}^{-1} : Y \rightarrow \frac{X}{\text{Ker}(M)}$$

is a bounded linear operator.

Now consider the diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{MA} & Y \\
 & \searrow \pi & \uparrow \widetilde{MA} \\
 & & \frac{X}{\text{Ker}(M)}
 \end{array} \tag{6}$$

where the operator $\widetilde{MA} : \frac{X}{\text{Ker}(M)} \rightarrow Y$ is defined by

$$\widetilde{MA}[x] := MAx.$$

This is well defined; in fact if $y \in [x]$ then $y - x = m$ for some element $m \in \text{Ker}(M)$ and the invariance of $\text{Ker}(M)$ under A implies $MAx = MAy$. The operator is not \widetilde{MA} a homeomorphism but it is still bounded (which can be shown in the same way as for \widetilde{M}). We can now define the linear operator \widehat{A} on Y :

$$\widehat{A}y := MAx, \quad y = Mx.$$

Again the definition is well posed thanks to the invariance hypothesis and to the surjectivity of M . It only remains to show that \widehat{A} is bounded. But, referring to the diagrams (5) and (6), we can write

$$\widehat{A}y = MAx = \widetilde{MA}[x] = (\widetilde{MA} \circ \widetilde{M}^{-1})y,$$

which shows that \widehat{A} is a composition of linear bounded operators, and hence bounded. \square

Observation 3. A basic kind of lumping can be applied by the quotient projection itself. Consider a closed subset $\mathcal{C} \subset X$ such that $A\mathcal{C} \subseteq \mathcal{C}$, and take as the upper level space exactly the quotient $\frac{X}{\mathcal{C}}$:

$$\begin{array}{ccc} \frac{X}{\mathcal{C}} & \xrightarrow{\widehat{A}} & \frac{X}{\mathcal{C}} \\ \pi \uparrow & & \uparrow \pi \\ X & \xrightarrow{A} & X \end{array} \quad (7)$$

By the invariance of \mathcal{C} we can define the bounded linear operator $\widehat{A}[x] := [Ax]$. Diagram (7) then commutes because for $x \in X$,

$$\pi Ax = [Ax] = \widehat{A}[x] = \widehat{A}\pi x.$$

Example 1 (Convolution). Consider the convolution operator A on $L^1(\mathbb{R}^N)$:

$$Af(x) = h * f(x) := \int_{\mathbb{R}^N} h(x-y)f(y) dy dx,$$

for some given function $h \in L^1(\mathbb{R}^N)$. Since $L^1(\mathbb{R}^N)$ is a Banach algebra with the convolution product, A belongs to $\mathcal{B}(L^1(\mathbb{R}^N))$. Define the continuous functional M by

$$Mf := \int_{\mathbb{R}^N} f(x) dx,$$

which is surjective from $\mathbf{L}^1(\mathbb{R}^N)$ onto the real line. Using Fubini's theorem and the invariance of the Lebesgue measure under translations, we can write

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h(x-y)f(y) dydx = \left(\int_{\mathbb{R}^N} h(x) dx \right) \left(\int_{\mathbb{R}^N} f(x) dx \right).$$

This implies that $\text{Ker}(M)$ is invariant under A , so that M represents a lumping of the system associated with the convolution operator. The operator \hat{A} on the upper level is simply

$$\hat{A}x := \lambda x, \quad \lambda = \int_{\mathbb{R}^N} h(x) dx.$$

Observation 4. As in the finite dimensional case, the systems analyzed in this section can be seen as control systems with output given by $y = Mx$. In this setting the system

$$\begin{cases} \dot{x}(t) = Ax(t), & A \in \mathcal{B}(X) \\ y(t) = Mx(t) \end{cases} \quad (8)$$

is said to be completely observable if and only if

$$\bigcap_{k=0}^{+\infty} \text{Ker}(MA^k) = \{0\}$$

(see [14]). If the system is lumpable by M , by definition

$$\bigcap_{k=0}^{+\infty} \text{Ker}(MA^k) = \text{Ker}(M) \neq \{0\},$$

so that it is non-observable [3].

3 Unbounded operators

We now turn to the case when the operator that generates the dynamics on the lower level is unbounded. Indeed, in many applications one needs to deal with dynamical systems defined on a proper subset of the Banach space X , such as partial or delay differential equations. We consider the abstract Cauchy problem:

$$\begin{cases} \dot{u}(t) = Au(t) \\ u(0) = u_0 \end{cases} \quad u_0 \in X \quad (9)$$

where $A : \mathcal{D}(A) \subset X \rightarrow X$ is a linear unbounded operator. The existence and uniqueness of a smooth solution is no longer automatically guaranteed as in the bounded case. Such problems have been extensively studied since the work of Hille and Yosida in the 1950s. We briefly recall the essential elements of the theory.

3.1 Background in semigroup theory

Let X be a Banach space. A one-parameter family of bounded operators $\{T(t)\}_{t \geq 0}$ in $\mathcal{B}(X)$ is called a *strongly continuous semigroup* if

1. $T(0) = I$,
2. $T(t+s) = T(t)T(s) \quad \forall t, s \geq 0$,
3. The map $t \mapsto T(t)x \in X$ is continuous for every $x \in X$.

The third property is called *strong continuity* because it can be seen as the continuity of the map $t \mapsto T(t) \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ is endowed with the *strong operator topology*, in which a sequence $\{T_n\} \subset \mathcal{B}(X)$ is said to be *strongly convergent* to $T \in \mathcal{B}(X)$ if

$$\lim_{n \rightarrow +\infty} \|T_n x - T x\| = 0 \quad \forall x \in X.$$

The *generator* of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ is defined to be the closed and densely defined operator $A : \mathcal{D}(A) \subset X \rightarrow X$ such that

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{h \rightarrow 0^+} \frac{1}{h} (T(h)x - x) \in X \right\},$$

$$Ax := \lim_{h \rightarrow 0^+} \frac{1}{h} (T(h)x - x).$$

The following results are then basic in semigroup theory [5, 9, 10].

Theorem 3. *The dynamics (9) is well-posed if and only if A is the generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on X , and in that case for every $u_0 \in \mathcal{D}(A)$ the unique classical solution of (9) is given by*

$$t \mapsto T(t)u_0.$$

Theorem 4. *If A is the generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ then the following hold*

1. $x \in \mathcal{D}(A) \Rightarrow T(t)x \in \mathcal{D}(A)$, and

$$\frac{d}{dt} T(t)x = T(t)Ax = AT(t)x \quad \forall t \geq 0.$$

2. $\forall x \in \mathcal{D}(A)$,

$$T(t)x - x = \int_0^t T(s)Ax \, ds.$$

3. $\forall x \in X$,

$$T(t)x - x = A \int_0^t T(s)x \, ds.$$

A strongly continuous semigroup is characterized by a real number ω called the *growth bound of the semigroup*, defined as

$$\omega := \inf \{ \omega_0 \in \mathbb{R} \text{ such that there exists } C > 0 \text{ with } \|T(t)\| \leq Ce^{\omega_0 t} \quad \forall t > 0 \}.$$

The growth bound is linked to the spectral properties of the generator A ; in fact, it is possible to show that

$$\sup_{\lambda \in \sigma(A)} \{\operatorname{Re}(\lambda)\} \leq \omega,$$

where $\sigma(A)$ is the spectrum of A . Another important property is the possibility to write the integral operator of a generator as the Laplace transform of the associated semigroup:

Proposition 5 (Integral representation of the resolvent operator). *Let A be the generator of $\{T(t)\}_{t \geq 0}$ and $\rho(A)$ be the complementary set of $\sigma(A)$. Then the following hold.*

1. *If $\operatorname{Re}(\lambda) > \omega$, then $\lambda \in \rho(A)$ and*

$$(\lambda I - A)^{-1}x := \int_0^\infty e^{-\lambda s} T(s)x \, ds \quad \forall x \in X.$$

2. *If the integral*

$$\mathcal{R}(\lambda) := \int_0^\infty e^{-\lambda s} T(s)x \, ds$$

exists for every $x \in X$, then $\lambda \in \rho(A)$ and $(\lambda I - A)^{-1} = \mathcal{R}(\lambda)$.

Since not every operator generates a semigroup, it is useful to have conditions for a linear operator to be a generator.

Theorem 6 (Feller-Miyadera-Philips, 1952). *Let $(A, \mathcal{D}(A))$ be a linear operator on a Banach space X and $\omega \in \mathbb{R}, C \geq 1$ constants. Then the following statements are equivalent.*

1. *A generates a strongly continuous semigroup satisfying $\|T(t)x\| \leq Ce^{\omega t}$ for every $t \geq 0$.*
2. *A is closed, densely defined and for every $\lambda \in \mathbb{C}$ such that $\operatorname{Re}(\lambda) > \omega$ one has $\lambda \in \rho(A)$ and $\|(\lambda I - A)^{-n}\| \leq \frac{C}{(\operatorname{Re}(\lambda) - \omega)^n} \quad \forall n \in \mathbb{N}$.*

Example 2. As a simple example of a closed and densely defined operator that is not a generator, consider the Banach space $C[0, 1]$ and the differentiation operator

$$Af := f', \quad f \in \mathcal{D}(A) := C^1[0, 1].$$

This operator cannot be the generator of a strongly continuous semigroup because its spectrum is the whole complex plane, $\sigma(A) = \mathbb{C}$, so that the second condition in Theorem 6 does not hold.

Finally, we report the statement of the following theorem about closed invariant subspaces, that will be fundamental in the analysis of lumpability with unbounded operators (for more details see [17]).

Theorem 7 ($T(t)$ -invariance of a closed subspace). *Let A be the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ having growth bound ω . Consider a closed subspace $\mathcal{V} \subset \mathbf{X}$ such that*

$$A(\mathcal{D}(A) \cap \mathcal{V}) \subseteq \mathcal{V}.$$

Let $A|_{\mathcal{V}} : \mathcal{D}(A) \cap \mathcal{V} \rightarrow \mathcal{V}$ be the infinitesimal generator restricted to \mathcal{V} . Then the following conditions are equivalent:

1.

$$\mathcal{V} \text{ is invariant under } T(t),$$

2.

$$\text{there exists } \lambda > \omega \text{ such that } \lambda \in \rho(A) \cap \rho(A|_{\mathcal{V}}). \quad (10)$$

Theorem (7) follows essentially from the fact that condition (10) implies the invariance of \mathcal{V} under the resolvent operators of A

$$\mathcal{R}(\lambda, A) := (\lambda I - A)^{-1}$$

for every complex number λ with real part greater than ω (see [17]). This kind of invariance implies the invariance under the semigroup, which follows from an asymptotic exponential formula of the kind

$$T(t)x = \lim_{n \rightarrow +\infty} \left[\frac{n}{t} \mathcal{R}\left(\frac{n}{t}, A\right) \right]^n x$$

(see e.g. [10, Section 1.8]).

3.2 Lumpability for unbounded operators

We are now ready to address the problem of lumpability in the unbounded case. The aim is to obtain the commutativity of the following diagram.

$$\begin{array}{ccc} M(\mathcal{D}(A)) \subset Y & \xrightarrow{\hat{A}} & Y \\ \uparrow M & & \uparrow M \\ \mathcal{D}(A) \subset X & \xrightarrow{A} & X \end{array}$$

We assume that the linear operator $M : X \rightarrow Y$ is bounded and surjective, while A and \hat{A} will be defined on a proper subset of X and Y , respectively.

Suppose that the dynamics on the lower level is well defined, so that A generates a strongly continuous semigroup on X , which we denote by $\{T(t)\}_{t \geq 0}$. We want the

new operator \hat{A} to be again the generator of a strongly continuous semigroup in order to obtain a well-defined dynamics on the upper level. Thus, for the commutativity of the diagram, we need the lumping relation $MA = \hat{A}M$ to hold on $\mathcal{D}(A)$.

Theorem 8. *The following conditions are equivalent:*

1. $\text{Ker}(M)$ is invariant under $T(t)$ for every $t \geq 0$;
2. There exists a linear operator \hat{A} on $M(\mathcal{D}(A))$ such that \hat{A} generates a strongly continuous semigroup on Y , and $\hat{A}M = MA$ (i.e. system (9) is lumpable by the operator M).

Proof. 1. \Rightarrow 2.: let us suppose that $\text{Ker}(M)$ is invariant under $T(t)$ for every $t \geq 0$. We first build a family of linear operators on Y in the following way: for $t \geq 0$,

$$\hat{T}(t)y := MT(t)x, \quad y = Mx. \quad (11)$$

For each $t \geq 0$, $\hat{T}(t)$ is well defined due to the invariance of the kernel, and applying the same arguments as in the continuous case we can show that it is bounded. Moreover, the family (11) is a strongly continuous semigroup on Y because:

1.

$$\hat{T}(0)y = \hat{T}(0)Mx = MT(0)x = Mx = y;$$

2. $\forall t, s \geq 0$,

$$\begin{aligned} \hat{T}(t+s)y &= MT(t+s)x = MT(t)T(s)x \\ &= \hat{T}(t)MT(s)x = \hat{T}(t)\hat{T}(s)Mx = \hat{T}(t)\hat{T}(s)y; \end{aligned}$$

3.

$$\begin{aligned} \lim_{h \rightarrow 0^+} \hat{T}(h)y - y &= \lim_{h \rightarrow 0^+} \|MT(h)x - Mx\| \\ &\leq \lim_{h \rightarrow 0^+} \|M\| \|T(h)x - x\| = 0. \end{aligned}$$

Let \hat{A} be the generator of the new semigroup $\hat{T}(t)$. Consider an element $y = Mx$ in $M(\mathcal{D}(A))$. By the definition of a generator and the continuity of M on X ,

$$\begin{aligned} \hat{A}y &= \lim_{h \rightarrow 0^+} \frac{1}{h} (\hat{T}(h)y - y) = \lim_{h \rightarrow 0^+} \frac{1}{h} (MT(h)x - Mx) \\ &= M \left(\lim_{h \rightarrow 0^+} \frac{1}{h} (T(h)x - x) \right) = MAx. \end{aligned}$$

This implies that \hat{A} is defined on $M(\mathcal{D}(A))$, which is a dense subset of Y since A is densely defined and M is bounded and surjective. On this subset the lumping relation holds also between the two generators:

$$\hat{A}Mx = MAx.$$

We have thus obtained the inclusion $M(\mathcal{D}(A)) \subset \mathcal{D}(\hat{A})$. We next show that the domain of \hat{A} is exactly $M(\mathcal{D}(A))$. For this purpose, we take $\lambda \in \mathbb{C}$ that belongs both to the resolvent set of A and \hat{A} and use the integral representation of the resolvent operator. Given an arbitrary element y for which \hat{A} is defined, there exists $s = Mx \in Y$ such that $y = (\lambda I - \hat{A})^{-1}s$. Hence we can write:

$$\begin{aligned} y &= \int_0^{+\infty} e^{-\lambda t} \hat{T}(t) s \, dt = \int_0^{+\infty} e^{-\lambda t} \hat{T}(t) Mx \, dt \\ &= \int_0^{+\infty} e^{-\lambda t} MT(t)x \, dt = M \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt \\ &= M(\lambda I - A)^{-1}x \\ &= Mz, \end{aligned}$$

where z belongs to $\mathcal{D}(A)$. This implies that $\mathcal{D}(A) = M(\mathcal{D}(A))$.

2. \Rightarrow 1.: let us show that the invariance of the $\text{Ker}(M)$ under the semigroup is a necessary condition to have a well-defined dynamics on Y . Suppose that the operator $\hat{A}y := MAx$ defined on $M(\mathcal{D}(A))$ generates a strongly continuous semigroup on Y . Let us consider the following maps from \mathbb{R}^+ to Y :

1. $t \mapsto \hat{T}(t)y$,
2. $t \mapsto MT(t)x$,

where $y = Mx$, $x \in \mathcal{D}(A)$. These two maps are both solutions of the abstract Cauchy problem

$$\begin{cases} \dot{v}(t) = \hat{A}v(t), \\ v(0) = y. \end{cases} \quad (12)$$

In fact, the first map is a solution by definition, while for the second map we have

$$\frac{d}{dt} MT(t)x = M \frac{d}{dt} T(t)x = MAT(t)x = \hat{A}MT(t)x$$

and $MT(0)x = Mx = y$, where we have used the continuity of M to interchange with the differentiation. Since the solution of the Cauchy problem (12) is unique, for all $t > 0$ we have

$$\hat{T}(t)Mx = MT(t)x,$$

and this equality holds for every $x \in \mathcal{D}(A)$. The operators MT and $\hat{T}M$ are equal on a dense subspace of Y , so they coincide on the whole space. The invariance of $\text{Ker}(M)$ under the semigroup follows then from the relation $MT = \hat{T}M$, which proves the statement above. \square

In the lumping analysis we need to take into account that if a closed subspace is invariant under $T(t)$ for all $t \geq 0$, then by definition it is invariant under the infinitesimal generator A ; however, the converse is not true. This can be seen by a simple example: Let X be the Banach space $C_0(\mathbb{R})$ of all continuous functions on the real line that tend to zero at infinity, endowed with the supremum norm. The operator

$$Af = f', \quad \mathcal{D}(A) = \{f \in C_0^1(\mathbb{R}) : f' \in X\},$$

generates the strongly continuous semigroup of left translations $T(t)f(s) = f(s+t)$. Clearly, the closed subspace $\mathcal{C} = \{f \in X : f(s) = 0, \forall s \leq 0\}$ is invariant under A but not invariant under translations.

It is typically the case in applications that one knows the generator A but not the associated semigroup. Therefore, it is necessary to find conditions on M that give the invariance of its kernel under the semigroup without knowing the semigroup itself. The next result gives conditions on the operator A to obtain lumpability of system (9).

Theorem 9. *System (9) is lumpable by the linear, bounded and surjective operator $M : X \rightarrow Y$ if and only if the following two conditions hold:*

1. $A(\text{Ker}(M) \cap \mathcal{D}(A)) \subset \text{Ker}(M)$;
2. there exists $\lambda > \omega$ such that $(\lambda I - A)$ is surjective from $\text{Ker}(M) \cap \mathcal{D}(A)$ to $\text{Ker}(M)$.

Proof. If (9) is lumpable by M , by definition there exists a linear operator \hat{A} such that $MA = \hat{A}M$ on $\mathcal{D}(A)$ and \hat{A} generates a strongly continuous semigroup on Y . By Theorem 8 we can say that $\text{Ker}(M)$ is $T(t)$ -invariant. Then we use Theorem 7 to obtain conditions 1 and 2 of the proposition.

Conversely, condition 1 gives that $\text{ker}(M)$ is invariant under A . Since the injectivity of $(\lambda I - A)$ on the whole domain $\mathcal{D}(A)$ guarantees the injectivity on the subspace $\text{Ker}(M) \cap \mathcal{D}(A)$, condition 2 implies that (10) holds with $\mathcal{V} = \text{Ker}(M)$. Now we apply Theorem 7 to obtain the invariance of $\text{Ker}(M)$ under the semigroup $\{T(t)\}_{t \geq 0}$ generated by A . Lumpability then follows by Theorem 8. \square

Observation 5. As a special case of condition 1 in Theorem 9, we could consider the case

$$\text{Ker}(M) \subset \mathcal{D}(A) \text{ and } A(\text{Ker}(M)) \subset \text{Ker}(M). \quad (13)$$

If (13) holds, then the condition on the resolvent set given in Theorem 7 is automatically verified. This can be seen by noting that the restricted operator

$$A|_{\text{Ker}(M)} : \text{Ker}(M) \rightarrow \text{Ker}(M)$$

is bounded by the closed graph theorem. Since the spectrum of a bounded operator is compact in the complex plane, one can obviously find an element λ satisfying (10). However, condition (13) is usually too strong and is generally not satisfied, as we will see in the examples below.

Observation 6 (Observability with unbounded operators). Let A be the unbounded generator of a strongly continuous semigroup with growth bound ω . It is can be shown [15] that that the system

$$\begin{cases} \dot{v}(t) = Av(t) \\ y(t) = Mv(t) \end{cases}$$

is observable if and only if, for any element $\mu \in \rho(A)$ such that $\operatorname{Re}(\mu) > \omega$, the following system is observable:

$$\begin{cases} \dot{v}(t) = \mathcal{R}(\mu, A)v(t), \\ y(t) = Mv(t), \end{cases}$$

where the resolvent operator $\mathcal{R}(\mu, A) := (\mu I - A)^{-1}$ is indeed bounded. Hence the condition for observability is reduced to

$$\bigcap_{k=0}^{\infty} \operatorname{Ker}(M\mathcal{R}(\mu, A)^k) = 0. \quad (14)$$

If the system is lumpable by M then $\operatorname{Ker}(M)$ is invariant under the semigroup, so it is invariant under the resolvent operators for $\operatorname{Re}(\mu) > \omega$ [17]. Since $\operatorname{Ker}(M) \neq 0$, this implies that (14) is not satisfied and the system is non-observable. Hence the observation stated in [3] for bounded operators holds also in the unbounded case.

Example 3 (Quotient semigroup). Let \mathcal{C} be a closed subspace that is invariant under a semigroup $\{T(t)\}_{t \geq 0}$ (or, equivalently, satisfying condition (10)). As in the bounded case, the quotient projection

$$\pi : X \rightarrow \frac{X}{\mathcal{C}}, \quad x \mapsto [x]$$

yields a lumping on the system associated with the generator A . The semigroup induced on the quotient space is

$$\widehat{T}(t)[x] := [T(t)x], \quad t \geq 0, x \in X,$$

generated by $\widehat{A}[x] := [Ax]$. For example, on $X = L^1((-\infty, 1])$ the differentiation operator $Af = f'$ defined on the domain

$$\mathcal{D}(A) := \{f \in X \text{ absolutely continuous such that } f' \in X, f(1) = 0\}$$

generates the semigroup of left translations $T(t)f(s) = f(s+t)$. Consider now the closed $T(t)$ -invariant subspace

$$\mathcal{C} := \{f \in X, f(s) = 0 \ \forall s \in [0, 1]\}.$$

The quotient $\frac{X}{\mathcal{C}}$ can be identified with the Banach space $L^1([0, 1])$. In fact all functions in a given equivalence class coincide on the interval $[0, 1]$, so that we can identify a class $[f]$ with the function $g \in L^1([0, 1])$ such that g is equal to the restriction of f on $[0, 1]$. The lumped dynamics on the upper level is now defined on a space of functions with compact support. The new generator on the quotient is again the derivative operator $\widehat{A}g := g'$, defined on the domain

$$\mathcal{D}(\widehat{A}) := \{g \in L^1([0, 1]) \text{ absolutely continuous, } g' \in L^1([0, 1]), g(1) = 0\},$$

(see [1]), which generates the nilpotent semigroup

$$\widehat{T}(t)g(s) := \begin{cases} g(s+t) & s+t \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4. Consider again the space $X := C_0(\mathbb{R})$, and let $h : \mathbb{R} \rightarrow \mathbb{C}$ be a continuous function. Define the multiplicative operator

$$Af(x) = h(x)f(x), \quad \mathcal{D}(A) = \{f \in X : hf \in X\},$$

(which is bounded iff h is a bounded function). One can show that A generates a strongly continuous semigroup if and only if $\sup_{x \in \mathbb{R}} \operatorname{Re}(h(x)) < \infty$, and in this case the semigroup is given by

$$T(t)f(x) = e^{th(x)}f(x), \quad \forall t \geq 0.$$

If h is nonzero, then for any positive integer k we can select k points $\{x_1, \dots, x_k\}$ on the real line such that at which h does not vanish. Define the linear bounded operator $M : C_0(\mathbb{R}) \rightarrow \mathbb{C}^k$ by $Mf = (f(x_1), \dots, f(x_k))^\top$, which simply evaluates a given function on the k points. We can write

$$\begin{aligned} MAf &= M(hf) = (h(x_1)f(x_1), \dots, h(x_k)f(x_k))^\top \\ &= \operatorname{Diag}(h(x_1), \dots, h(x_k)) \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_k) \end{pmatrix} := \widehat{A}Mf, \end{aligned}$$

where “Diag” denotes a diagonal matrix. Thus M yields a lumping on the system associated with A . Note that the kernel of M is invariant under A , but not fully contained in $\mathcal{D}(A)$; hence (13) is not satisfied. On the other hand, observe the resolvent condition given by (10) is satisfied. This can be easily seen considering that the resolvent set of A is the complementary set of

$$\sigma(A) := \{\lambda \in \mathbb{C} : h(x) = \lambda \text{ for some } x \in \mathbb{R}\}.$$

Taking $\lambda \in \rho(A)$, the operator $\lambda I - A$ is surjective from $\mathcal{D}(A) \cap \operatorname{Ker}(M)$ to $\operatorname{Ker}(M)$ if and only if for every $g \in \operatorname{Ker}(M)$ the function

$$f(x) := \frac{g(x)}{\lambda - h(x)}$$

belongs to $\mathcal{D}(A) \cap \operatorname{Ker}(M)$. This is indeed verified because:

1. since $\lambda \in \rho(A)$, $\frac{h(x)}{\lambda - h(x)}$ is always bounded, so that $h(x)f(x)$ tends to zero at infinity;
2. since g vanishes at the points x_i , and the previous property holds, f also vanishes on this set of points. Hence, we can take as λ in condition (10) every element in $\rho(A)$ that is greater than ω .

Example 5 (Delay differential equations). Let $X = C([-1, 0], \mathbb{R}^n)$ be the Banach space of continuous vector-valued functions on the compact interval $[-1, 0]$, and let $L : X \rightarrow \mathbb{R}^n$ be linear and continuous. A linear *delay differential equation* (DDE) is an equation of the form $\dot{u}(t) = Lu_t$, where $u_t \in X$ is the function defined by

$$u_t(s) := u(t + s), \quad s \in [-1, 0].$$

The unbounded linear operator A defined by

$$Af = f'; \quad \mathcal{D}(A) = \{f \in C^1([-1, 0], \mathbb{R}^n) : f'(0) = Lf\}$$

generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ that gives the solutions of the DDE. In other words, the unique solution $u(t)$ of the Cauchy problem

$$\begin{cases} \dot{u}(t) = Lu_t, & t \geq 0 \\ u(t) = f(t), & t \in [-1, 0] \end{cases}$$

with initial condition $f \in X$, satisfies

$$u_t(s) = T(t)f(s), \quad s \in [-1, 0], t \geq 0.$$

For definiteness, we choose $Lf(s) = f(-1)$ and consider the following equation with a single delay

$$\begin{cases} \dot{u}(t) = u(t-1), & t \geq 0 \\ u(t) = f(t), & t \in [-1, 0]. \end{cases}$$

As the lumping operator we take $M : X \rightarrow Y := C([-1, 0], \mathbb{R})$ defined by

$$M(f)(s) = a_1 f_1(s) + \cdots + a_n f_n(s)$$

for some non-zero real numbers a_i . Clearly M is bounded and surjective. It is also easy to verify that $\text{Ker}(M) \cap \mathcal{D}(A)$ is invariant under A . (Note that $\text{Ker}(M)$ is not fully contained in the domain of A , so the condition (13) does not hold). Furthermore, $(\lambda I - A)$ is surjective from $\text{Ker}(M) \cap \mathcal{D}(A)$ to $\text{Ker}(M)$ for every $\lambda \in \rho(A)$. In fact for any $g \in \text{Ker}(M)$ there exists $f \in \text{Ker}(M) \cap \mathcal{D}(A)$ such that $(\lambda I - A)f = g$ if and only if the following hold:

1. $f(x) = (c_0 + \int_0^x g(s)e^{\lambda s} ds) e^{\lambda x}, \quad c_0 \in \mathbb{R}^n,$
2. $Mc_0 = 0,$
3. $(\lambda I - e^{-\lambda}I)c_0 = -g(0) - e^{-\lambda} \int_{-1}^0 g(s)e^{\lambda s} ds.$

Since for $\lambda \in \rho(A)$ the matrix $(\lambda I - e^{-\lambda}I)$ is injective and surjective, we can always find a vector c_0 that satisfies condition 3 and belongs to $\text{Ker}(M)$. By inserting c_0 in the expression of f given by condition 1, it is easy to see that condition (10) is satisfied. Thus M yields a lumping of the system. The operator on the upper level Y is the derivative operator

$$\hat{A}g(s) := g'(s), \quad \mathcal{D}(\hat{A}) = \{g \in C^1([-1, 0], \mathbb{R}) : g'(0) = g(-1)\}.$$

It generates the strongly continuous semigroup of the solution operators associated with the linear DDE

$$\begin{cases} \dot{v}(t) = v(t-1), & t \geq 0 \\ v(t) = g(t), & t \in [-1, 0] \end{cases}$$

defined on a space of scalar-valued functions.

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