Patterns from bifurcations: A symmetry analysis of networks with delayed coupling

by

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A Symmetry Analysis of Networks with Delayed Coupling 

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Abstract 

We study systems of coupled units in a general network configuration with a coupling delay. We show that the destabilizing bifurcations from an equilibrium are governed by the extreme eigenvalues of the coupling matrix of the network. Based on the equivariant degree method and its computational packages, we perform a symmetry classification of destabilizing bifurcations in bidirectional rings of coupled units, for bifurcating solutions either of steady-states or of oscillating states. We also introduce the concept of secondary dominating orbit types to capture bifurcating solutions of submaximal nature. 

Keywords: Symmetry, equivariant degree, bifurcation theory, network, delay, dynamical patterns. 

1 Introduction 

We consider \( n \) identical dynamical systems of form \( \dot{x} = f(x) \) coupled together in a general network configuration and possibly with a time delay \( \tau \geq 0 \): 

\[
\dot{x}_i(t) = f(x_i(t)) + \kappa g_i(x_1(t-\tau), x_2(t-\tau), \ldots, x_n(t-\tau)), \quad i = 1, 2, \ldots, n, \tag{1.1}
\]

where the scalar \( \kappa > 0 \) plays the role of coupling strength and \( g_i \) describes the interaction among the coupled systems. For simplicity of notations and a clear focus on the structural aspect of the system, we consider only scalar systems, i.e. \( x_i \in \mathbb{R} \). The functions \( f: \mathbb{R} \to \mathbb{R} \) and \( g_i: \mathbb{R}^n \to \mathbb{R} \) are assumed to be \( C^1 \) and \( g_i \) will be assumed equivariant when we consider symmetry. We also assume that \( f \) and the \( g_i \) vanish at the origin, hence (1.1) admits the zero solution. We will study the stability of the zero solution and its loss of stability through bifurcations, in terms of the time delay and network structure of the system.

The tool we are using for symmetry bifurcation analysis is the equivariant degree and the “Equivariant Degree Maple® Library Package” that performs exact computations of values of equivariant degrees. The idea of using equivariant degree theory for equivariant bifurcation problems has been explored in various texts, see [7, 6] and the references therein. In short, one associates to a given bifurcating equilibrium a bifurcation invariant in form of an equivariant degree. Based on the precise value of the bifurcation invariant, one derives a full topological classification of the bifurcating branches respecting their symmetry properties. The calculation task of bifurcation invariants is completely taken over by the “Equivariant Degree Maple® Library Package”. This equivariant degree approach together with assistance of the Maple® package has 

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been employed for example, in \[1, 4, 5, 2\]. In the monograph \[6\], one can find a complete exposition on the subject including the construction of the equivariant degree, its fundamental properties and many applications in equivariant nonlinear problems. The “Equivariant Degree Maple\textsuperscript{©} Library Package” was created by A. Biglands and W. Krawcewicz at the University of Alberta in 2006, supported by an NSERC summer research grant. The package is open source and is free to be downloaded, for example, at http://www.math.uni-hamburg.de/home/ruan/download.

There is also a newly developed GAP\textsuperscript{*}-based algorithm as well as a web application\textsuperscript{†} of it available for computation of equivariant degrees for dihedral groups. The results presented in this article are obtained however, using the “Equivariant Degree Maple\textsuperscript{©} Library Package”, which is more straightforward to clarify.

The exact value of bifurcation invariant is computed by calling showdegree[\Gamma] for a symmetry group \(\Gamma\). The command showdegree[\Gamma] takes several parameters as input, which are solely determined by the critical spectrum of the linearized operator. Corresponding to (1.1), the linearized system around zero is of form

\[
\dot{y}(t) = f'(0)y(t) + \kappa Cy(t - \tau), \quad y \in \mathbb{R}^n, \tag{1.2}
\]

where \(C = [c_{ij}] = [\partial g_i(0)/\partial x_j]\) is the coupling matrix of the network configuration of the system. In other words, the exact value of the bifurcation invariant associated to the zero solution of (1.1) depends only on the characteristic operator of (1.2).

In fact, all results that one retreat from the bifurcation invariant of (1.1), indeed, remain valid for any \(\Gamma\)-symmetric system whose linearization is of form (1.2). We give several examples of such systems.

The well-known neural network model

\[
\dot{x}_i(t) = -x_i(t) + g \left( \sum_{j=1}^{n} a_{ij} x_j(t - \tau) \right), \tag{1.3}
\]

where \(g\) is typically a sigmoidal function and \(a_{ij} \in \mathbb{R}\) are entries of the adjacency matrix \(A\) that describes the coupling among the neurons. Linearization about the zero solution has the form (1.2) with \(\kappa = g'(0)\) and \(C\) can be identified with the adjacency matrix \(A\).

A more general form can be used to model pulse-coupled systems

\[
\dot{x}_i(t) = f(x_i(t)) + h(x_i(t)) \cdot g \left( \sum_{j=1}^{n} a_{ij} x_j(t - \tau) \right), \tag{1.4}
\]

indicating that the influence of the network on the \(i\)th unit may be different depending on the state of the \(i\)th unit at that particular time instant. Although (1.4) is not of form (1.1), its linearization is given by (1.2) with \(\kappa = h(0)g'(0)\) and \(C = A\).

In other models that involve diffusive-type interactions, say of form

\[
\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{n} a_{ij} g(x_j(t - \tau) - x_i(t - \tau)), \tag{1.5}
\]

---

\*GAP ("Groups, Algorithms, Programming") is a system for computational discrete algebra. It provides a programming language and large data libraries of algebraic objects. The system is distributed freely at http://www.gap-system.org

\†See Dihedral Calculator from MuchLearning, http://dihedral.muchlearning.org
the linearized equation (1.2) arises with $C$ given by the negative of the Laplacian matrix, i.e. $C = -L = A - D$, where $D = \text{diag}\{k_1, \ldots, k_n\}$ is the diagonal matrix of vertex degrees $k_i = \sum_j a_{ij}$. If the delay originates only from the finite speed of information transmission from $j$ to $i$, then one has a slightly variant system

$$
\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^n a_{ij} g(x_j(t - \tau) - x_i(t)),
$$

(1.6)

whose linearization has the form (1.2) (with the identification $f(x_i) \rightarrow f(x_i) + g'(0)k_i x_i$) provided that all vertices have the same degree $k_i = k$, i.e., the network is regular.

In this paper we confine ourselves to bi-directional interactions and assume $C$ to be a symmetric matrix; thus $C$ has real eigenvalues. We take two quantities $\alpha := \tau f'(0)$ and $\beta := \tau k \xi$ as bifurcation parameters, where $\xi \in \sigma(C)$. As we shall see, bifurcations, either of steady states or of oscillating states, that destabilize the zero solution, are related only to the extreme eigenvalues of $C$. Consequently, networks with the same extreme eigenvalues of the coupling matrix will exhibit the same destabilizing bifurcation behavior.

For symmetrically coupled system, we consider systems that are coupled in a bidirectional ring configuration, i.e., they possess dihedral symmetries. We derive the input parameters for computing the bifurcation invariant. To illustrate how to interpret values of bifurcation invariant, we present bifurcation classification results for bidirectional rings of 12 coupled units in Example 6.1. The classification results we present are not restricted to specific ring configurations, and they are derived using extreme eigenvalues of the coupling matrix only. See Table 2 for steady-state bifurcations and Table 3-5 for Hopf bifurcations. For bidirectional rings of larger size, the method can be systematically applied. Computational packages for dihedral symmetry are currently available for $D_n$ up to $n = 200$. See Dihedral Calculator from Much-Learning http://dihedral.muchlearning.org. Other symmetry groups that are supported by computational packages are the quaternion group $Q_8$, the alternating groups $A_4$, $A_5$ and the symmetric group $S_4$, using the “Equivariant Degree Maple© Library Package”.

It should be mentioned that since the bifurcation invariant is a topological invariant, it remains invariant against all (admissible, equivariant) continuous deformations on the system. As a consequence, the classification result one obtains using the bifurcation invariant remains valid even if the modeling of the system varies within the framework of symmetry. In short, our results are robust against model variations.

2 Preliminaries

2.1 Groups and Group Representations

Throughout we consider groups that are either finite or of form $\Gamma \times S^1$, where $\Gamma$ is a finite group and $S^1$ is the group of complex numbers of unit length.

Let $G$ be a group and $H$ be a closed subgroup of $G$, written as $H \subset G$. Let $N(H) = \{g \in G : gHg^{-1} = H\}$ be the normalizer of $H$ and $W(H) = N(H)/H$ the Weyl group of $H$. The set of all closed subgroups of $G$ can be ordered by set inclusion. For subgroups $H, K \subset G$, we write $H \leq K$ if $H \subseteq K$; $H < K$ if $H \subsetneq K$. The symbol $(H)$ stands for the conjugacy class of the subgroup $H$ in $G$; that is $(H) = \{gHg^{-1} : g \in G\}$. The set of all conjugacy classes of closed subgroups of $G$ affords a partial order given by: $(H) \leq (K)$ if $H \subseteq gKg^{-1}$ for some $g \in G$; similarly, $(H) < (K)$ if $H \subsetneq gKg^{-1}$ for some $g \in G$. 
Example 2.1 (cf. [6]) Let $\Gamma = D_{12}$ be the dihedral group of order 24, which is represented as the group of 12 rotations: 1, $\eta$, $\eta^2$, ..., $\eta^{11}$ and 12 reflections: $\kappa$, $\kappa\eta$, $\kappa\eta^2$, ..., $\kappa\eta^{11}$ of the complex plane $\mathbb{C}$, where $\eta$ stands for the complex multiplication by $e^{i\pi}$ and $\kappa$ denotes the complex conjugation. There are two kinds of subgroups in $D_{12}$: cyclic and dihedral. The cyclic subgroups are $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_6, \mathcal{Z}_{12}$, where $\mathcal{Z}_k$ denotes the cyclic subgroup generated by $\eta^j$ with $l = \frac{12}{k}$.

The dihedral subgroups are

$$D_{k,j} = \{1, \eta^j, \eta^{2j}, \ldots, \eta^{(k-1)j}, \kappa\eta^j, \kappa\eta^{j+1}, \kappa\eta^{j+2}, \ldots, \kappa\eta^{j+(k-1)j}\}, \quad \text{for } 0 \leq j < l := \frac{12}{k},$$

where $k \in \{1, 2, 3, 4, 6, 12\}$. If $l$ is odd, then all subgroups $D_{k,j}$ for $0 \leq j < l$ are conjugate to $D_{k,0} := D_k$. If $l$ is even, then all subgroups $D_{k,j}$ with $j$ being odd are conjugate to $D_{k,0} = D_k$; all subgroups $D_{k,j}$ with $j$ being even are conjugate to $D_{k,1} := D_k$. Thus, up to conjugacy relation, we have the dihedral subgroups: $D_1, D_2, D_2, D_3, D_3, D_4, D_6, D_6, D_{12}$.

A real (resp. complex) representation of $G$ is a finite-dimensional real (resp. complex) vector space $X$ with a continuous map, or action, $\psi: G \times X \to X$ such that the map $\psi(g, \cdot): X \to X$ is linear, for every $g \in G$. Banach representations are similarly defined for Banach spaces with an action for which $\psi(g, \cdot)$ is bounded linear. We abbreviate $\psi(g, x)$ with $gx$.

A subset $\Omega \subset X$ is called invariant, if $gx \in \Omega$ whenever $x \in \Omega$ for all $g \in G$. A representation $X$ of $G$ is called irreducible, if $\{0\}$ and $X$ are the only invariant subspaces in $X$. An action is called free, if $gx = x$ for some $x \in X$ implies $g = e$ is the neutral element.

Example 2.2 (cf. [6]) The dihedral group $D_n$, for $n \in \mathbb{N}$ even, has the following real irreducible representations:

(i) The trivial representation $\mathcal{V}_0 \simeq \mathbb{R}$, where every element acts as the identity map.

(ii) For $1 \leq i \leq \frac{n}{2} - 1$, there is the representation $\mathcal{V}_i \simeq \mathbb{R}^2 \simeq \mathbb{C}$ given by the following actions:

$$\eta z = \eta^i \cdot z, \quad \kappa z = \bar{z},$$

where “$\cdot$” is the complex multiplication and “$-$” is the complex conjugation.

(iii) The representation $\mathcal{V}_\frac{n}{2} \simeq \mathbb{R}$ given by: $\eta x = x$ and $\kappa x = -x$.

(iv) The representation $\mathcal{V}_{\frac{n}{2} + 1} \simeq \mathbb{R}$ given by: $\eta x = -x$ and $\kappa x = x$.

(v) The representation $\mathcal{V}_{\frac{n}{2} + 2} \simeq \mathbb{R}$ given by: $\eta x = -x$ and $\kappa x = -x$.

It has the following complex irreducible representations:

(i) The trivial representation $\mathcal{U}_0 \simeq \mathbb{C}$, where every element acts as the identity map.

(ii) For $1 \leq j \leq \frac{n}{2} - 1$, there is the representation $\mathcal{U}_j \simeq \mathbb{C} \times \mathbb{C}$ given by the following actions:

$$\eta(z_1, z_2) = (\eta^j \cdot z_1, \eta^{-j} \cdot z_2), \quad \kappa(z_1, z_2) = (\bar{z}_2, \bar{z}_1),$$

where “$\cdot$” is the complex multiplication.

(iii) The representation $\mathcal{U}_{\frac{n}{2}} \simeq \mathbb{C}$ given by: $\eta z = z$ and $\kappa z = -z$.

(iv) The representation $\mathcal{U}_{\frac{n}{2} + 1} \simeq \mathbb{C}$ given by: $\eta z = -z$ and $\kappa z = z$. 

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(v) The representation $U_{2+2}^\mathbb{Z} \simeq \mathbb{C}$ given by: $\eta z = -z$ and $\kappa z = -z$.

For $n \in \mathbb{N}$ odd, the dihedral group $D_n$ has the above listed irreducible representations (i)-(iii), where $n$ is replaced with $(n+1)$.

Let $x \in X$. By the symmetry of $x$, we mean the isotropy subgroup of $x$ given by $\text{Iso}(x) := \{g \in G : gx = x\}$ with respect to the group action on $X$. The set $\text{Orb}(x) := \{gx : g \in G\}$ is called the orbit of $x$ and the symmetry of the orbit is defined by the orbit type of $x$ which is the conjugacy class of $\text{Iso}(x)$ of $\text{Iso}(x)$. Note that $\text{Iso}(gx) = g \text{Iso}(x)g^{-1}$ for $g \in G$, thus the symmetry of the orbit is independent of the choice of $x$ from the orbit.

Let $\Omega \subset X$ be a subset and $H \subset G$ be a closed subgroup. Define $\Omega_H = \{x \in X : \text{Iso}(x) = H\}$. It can be verified that the Weyl group $W(H)$ acts freely on $\Omega_H$. Denote the $H$-fixed point subspace in $\Omega$ by $\Omega^H = \{x \in X : gx = x, \forall g \in H\}$. Note that $\Omega_H \subset \Omega^H$. Moreover, $\Omega^H$ is the disjoint union of $\Omega_H$ for all $\hat{H} \supset H$.

**Example 2.3** Let $\Gamma = D_{12}$ and $X = \mathcal{V}_1$ be the real irreducible representation of $D_{12}$ given in Example 2.2. Then, orbit types that occur in $X$ are: $(D_{12})$, $(D_1)$, $(D_1)$ and $(Z_1)$ (cf. Example 2.1 for notations), with the corresponding fixed point subspaces:

$$X^{D_{12}} = \{(0,0)\}, \quad X^{D_1} = \{(x,0) : x \in \mathbb{R}\}, \quad X^{D_1} = \{re^{-\pi i r} : r \in \mathbb{R}\}, \quad X^{Z_1} = X.$$

Note that $X^{D_1}$ is the disjoint union of subsets $X_{D_1} = \{(x,0) : x \in \mathbb{R}, x \neq 0\}$ and $X_{D_{12}} = \{(0,0)\}$. On the subset $X_{D_1}$, the Weyl group $W(D_1) = D_2/D_1 \simeq \mathbb{Z}_2$ acts freely by the reflection. On the subset $X_{D_{12}}$, the Weyl group $W(D_{12}) = D_{12}/D_{12} \simeq \mathbb{Z}_4$ acts freely by the neutral element.

Finally, we remark that there is a natural way of “converting” a complex $\Gamma$-representation into a real $\Gamma \times S^1$-representation. Let $U$ be a complex $\Gamma$-representation. Define a $\Gamma \times S^1$-action on $U$ by

$$\gamma(z)u = z \cdot (\gamma u), \quad \text{for} \ (\gamma, z) \in \Gamma \times S^1, \ u \in U,$$

where $\cdot$ stands for the complex multiplication. The obtained representation is denoted by $\bar{U}$ and called the $\Gamma \times S^1$-representation induced from $U$. Note that $\bar{U}$ is irreducible as a real $\Gamma \times S^1$-representation if $U$ is irreducible as a complex $\Gamma$-representation.

### 2.2 Equivariant Maps and Equivariant Degree

Let $X, Y$ be two Banach representations of $G$. A continuous map $f : X \to Y$ is called equivariant, if $f(g_0x) = g_0f(x)$, for all $x \in X$ and $g \in G$, where $g_0$ and $a$ stand for the $G$-actions on $X$ and $Y$, respectively. A subset $\Omega \subset X$ is called invariant, if $g_0x \in \Omega$ whenever $x \in \Omega$ for all $g \in G$. In equivariant nonlinear analysis, one is interested in finding zeros of an equivariant map $f$ in an invariant domain $\Omega$. Note that by equivariance, the set of all zeros of $f$ in $\Omega$ is composed of disjoint group orbits, thus one speaks of zero orbits, instead of zeros, of $f$.

A map $f$ is called admissible on $\Omega$, if $f(x) \neq 0$ for all $x \in \partial \Omega$. A homotopy $h : [0, 1] \times X \to Y$ is called admissible, if $h(t, \cdot)$ is admissible for all $t \in [0, 1]$. An equivariant degree, intuitively speaking, is an algebraic count of zero orbits of an admissible $f$ in $\Omega$ with respect to orbit types, which remains unchanged against all admissible (equivariant) homotopies from $f$.

In the next two subsections, we review from [6] two types of equivariant degrees that will be used in Section 5 for bifurcation analysis. In both cases, the equivariant degree is first defined in finite-dimensional representations for continuous maps, and then extended to infinite-dimensional Banach representations for compact vector fields.
2.2.1 Equivariant Degree without Parameters

Let $G = \Gamma$ be a finite group acting on a finite-dimensional $\Gamma$-representation $X$. Let $\Phi$ be the set of all orbit types that appear in $X$. That is, every element of $\Phi$ is a conjugacy class of a finite subgroup of $\Gamma$. Consider a continuous equivariant map $f : X \to X$ on an open bounded invariant domain $\Omega \subset X$ such that $f$ is admissible on $\Omega$. Define an equivariant degree (without parameter) of $f$ in $\Omega$ by a finite sum of integer-indexed orbit types:

$$\Gamma \text{-Deg} (f, \Omega) = \sum_{(K) \in \Phi} n_K \cdot (K),$$

where $n_K \in \mathbb{Z}$ is an integer counting zero orbits of orbit type $(K)$. The precise definition of $n_K$ can be given by the following recurrence formula:

$$n_K = \frac{\deg (f|_{\Omega^K}, \Omega^K) - \sum_{(\tilde{K}) > (K)} n_{\tilde{K}} \cdot |W(\tilde{K})| \cdot n(K, \tilde{K})}{|W(K)|}. \quad (2.9)$$

We explain the notations used in (2.9) and their geometric meanings. Recall that $\Omega^K$ denotes the fixed point subspace of $K$ in $\Omega$. By restricting $f$ on $\Omega^K$, one obtains an (admissible) map $f|_{\Omega^K} : \Omega^K \to \Omega^K$. Using the classical Brouwer degree “deg”, the integer “$\deg (f|_{\Omega^K}, \Omega^K)$” counts the zeros of $f$ in $\Omega^K$. Since not every element in $\Omega^K$ has the precise isotropy $K$, one needs to subtract those zeros of larger isotropies. This is done by subtracting the summands in (2.9). Within each summand, $n_{\tilde{K}}$ is the integer counting zero orbits of orbit type $(\tilde{K})$. Since the Weyl group $W(\tilde{K})$ acts freely on $\Omega_{\tilde{K}}$, the integer $n_{\tilde{K}} \cdot |W(\tilde{K})|$ then counts the zeros of isotropy $\tilde{K}$. The number $n(K, \tilde{K})$ is defined as the number of distinct conjugate copies of $\tilde{K}$ that contain $K$, formally by

$$n(K, \tilde{K}) = \left| \{ g \in \Gamma : K \subseteq g\tilde{K}g^{-1} \} \right|. \quad N(\tilde{K})$$

Thus, the number $n_{\tilde{K}} \cdot |W(\tilde{K})| \cdot n(K, \tilde{K})$ counts the zeros of isotropy $K'$ for all $K'$ with $(K') = (\tilde{K})$. It follows that the expression of the numerator in (2.9) gives the count of zeros of $f$ having precise isotropy $K$. Again, since $W(K)$ acts freely on $\Omega_K$, we have then the total expression on the right hand side of (2.9) gives the count of zero orbits of $f$ having orbit type $(K)$.

**Example 2.4** Let $\Gamma = D_{12}$ and $X = V_1$ be the real irreducible representation of $D_{12}$ given in Example 2.2. Consider the antipodal map $f = -\text{Id} : X \to X$ on the unit disc $B \subset X$, which is $D_{12}$-equivariant and $B$-admissible. As mentioned in Example 2.3, orbit types that occur in $V_1$ are: $(D_{12})$, $(D_1)$, $(\bar{D}_1)$ and $(Z_1)$. Thus,

$$\Gamma \text{-Deg} (-\text{Id}, B) = n_{D_{12}} \cdot (D_{12}) + n_{D_1} \cdot (D_1) + n_{\bar{D}_1} \cdot (\bar{D}_1) + n_{Z_1} \cdot (Z_1).$$

We compute $n_{D_1}$ using (2.9). To do so, we first need to compute $n_{D_{12}}$:

$$n_{D_{12}} = \frac{\deg (-\text{Id}, B^{D_{12}})}{|W(D_{12})|} = \frac{1}{1} = 1,$$

where we used the fact $B^{D_{12}} = X^{D_{12}} \cap B = \{(0,0)\}$, $W(D_{12}) = Z_1$ from Example 2.3 and $\deg (-\text{Id}, \mathbb{R}^m) = (-1)^m$ for $m \in \{0\} \cup \mathbb{N}$. Thus, we have

$$n_{D_1} = \frac{\deg (-\text{Id}, B^{D_1}) - 1 \cdot 1 \cdot 1}{|W(D_1)|} = \frac{-1 - 1}{2} = -1,$$
where we used the fact \( n(D_1, D_{12}) = \frac{|D_{12}|}{D_{12}} = 1 \) and \( W(D_1) = \mathbb{Z}_2 \). Following (2.9) further, one shows that

\[
\Gamma\text{-Deg} (\text{Id}, B) = (D_{12}) - (D_1) - (\tilde{D}_1) + (\mathbb{Z}_1).
\]

\( \diamond \)

The definition of equivariant degree can be extended, in a standard way, to infinite-dimensional Banach representations for compact equivariant fields, namely, equivariant maps of form \( f = \text{Id} - F : D \subset X \to X \) that are admissible on a bounded domain \( D \) such that \( \overline{F(D)} \) is compact. It was shown in [3] that the equivariant degree defined by (2.8)-(2.9), as well as its infinite-dimensional extension, satisfies usual properties of a degree theory such as the existence property, which states that

\[
n_K \neq 0 \quad \Rightarrow \quad f^{-1}(0) \cap \Omega^K \neq \emptyset,
\]

which can be useful for predicting zero orbits of orbit type at least \((K)\).

### 2.2.2 Equivariant Degree with One Parameter

Let \( G = \Gamma \times S^1 \) be the product of a finite group \( \Gamma \) and the circle group \( S^1 \). There are two types of closed subgroups in \( G \): those subgroups that are of form \( K \times S^1 \) for some subgroups \( K \subset \Gamma \), or otherwise, they are the twisted subgroups of \( G \), defined as follows.

**Definition 2.5** A subgroup \( H \subset \Gamma \times S^1 \) is called a twisted 1-folded subgroup, if there exists a subgroup \( K \subset \Gamma \), an integer \( l \geq 0 \) and a group homomorphism \( \varphi : K \to S^1 \) such that

\[
H = K^{\varphi,l} := \{(\gamma, z) : \varphi(\gamma) = z^l\}.
\]

Conjugacy classes of twisted subgroups are called twisted orbit types.

\( \diamond \)

**Example 2.6** Let \( G = D_{12} \times S^1 \) be the product group of the dihedral \( D_{12} \) and the unit circle \( S^1 \subset \mathbb{C} \). We describe its twisted subgroups \( H = K^\phi \). Clearly, all subgroups of \( D_{12} \) are twisted subgroups with \( \phi \equiv 1 \in S^1 \). Besides that, there are twisted subgroups that are not contained in \( D_{12} \). These can be classified into two categories: those for which \( K = \mathbb{Z}_k \) and those for which \( K = D_k \) (cf. Example 2.1 for notations).

Let \( K = \mathbb{Z}_k \) for some \( k \in \{1, 2, 3, 4, 6, 12\} \) and \( \phi : K \to S^1 \) be given by \( \phi(\eta^l) = \eta^{jl} \) for some \( j \) with \( 1 \leq j < k \). Then,

\[
K^\phi = \{(1, 1), (\eta^l, \eta^{jl}), (\eta^{2l}, \eta^{2jl}), \ldots, (\eta^{(k-1)l}, \eta^{(k-1)jl})\} := \mathbb{Z}_k^{l_j}, \quad \text{for } 1 \leq j < k.
\]

Among these subgroups, \( \mathbb{Z}_k^{l_j} \) and \( \mathbb{Z}_k^{l_{k-j}} \) are conjugate to each other, for \( 1 \leq j < k \). Thus, for \( k \) even, up to conjugacy relation, we have the twisted subgroups \( \mathbb{Z}_k^{l_1}, \mathbb{Z}_k^{l_2}, \ldots, \mathbb{Z}_k^{l_{\frac{k}{2}}} := \mathbb{Z}_k^{l_1} \); for \( k \) odd, \( \mathbb{Z}_k^{l_1}, \mathbb{Z}_k^{l_2}, \ldots, \mathbb{Z}_k^{l_{\frac{k+1}{2}}} \). That is, we have \( \mathbb{Z}_2^{l_1}, \mathbb{Z}_3^{l_1}, \mathbb{Z}_4^{l_1}, \mathbb{Z}_6^{l_1}, \mathbb{Z}_6^{l_2}, \mathbb{Z}_6^{l_2}, \mathbb{Z}_6^{l_3}, \mathbb{Z}_6^{l_4}, \mathbb{Z}_6^{l_5}, \mathbb{Z}_6^{l_6} \).

Let \( K = D_k \) for some \( k \in \{1, 2, 3, 4, 6, 12\} \) and \( 0 \leq j < l = \frac{k}{2} \). Up to conjugacy, it is sufficient to consider \( K = D_k \) in case \( l \) is odd; and \( K = D_k \), \( K = \tilde{D}_k \) in case \( l \) is even (cf. Example 2.1). Let \( \phi : K \to S^1 \) be the group homomorphism such that \( \ker \phi = \mathbb{Z}_k \). Then,

\[
D_k^\phi = \{(1, 1), (\eta^l, 1), \ldots, (\eta^{(k-1)l}, 1), (\kappa, -1), (\kappa \eta^l, -1), \ldots, (\kappa \eta^{(k-1)l}, -1)\} := D_k^\tilde{\phi},
\]

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We consider the main equation (1.1), which describes coupled systems of identical cells.

In the case $k$ is even, there is a group homomorphism $\phi : K \to S^1$ for which $\ker \phi = D_\varphi$. Then,

$$D^\phi_k = \{(1,1),(\eta^l,1),\ldots,(\eta^{(k-1)}l,1),(\kappa,1),(\kappa \eta^l,1),\ldots,(\kappa \eta^{(k-1)}l,1)\} := D^d_k,$$

and

$$\tilde{D}^\phi_k = \{(1,1),(\eta^l,1),\ldots,(\eta^{(k-1)}l,1),(\kappa,1),(\kappa \eta^l,1),\ldots,(\kappa \eta^{(k-1)}l,1)\} := \tilde{D}^d_k,$$

if $l$ is even.

Also, there is a group homomorphism $\phi : K \to S^1$ for which $\ker \phi = \tilde{D}^\varphi_k$. Then,

$$D_k^\varphi = \{(1,1),(\eta^l,-1),\ldots,(\eta^{(k-1)}l,-1),(\kappa,1),(\kappa \eta^l,1),\ldots,(\kappa \eta^{(k-1)}l,1)\} := D_k^d,$$

and

$$\tilde{D}_k^\varphi = \{(1,1),(\eta^l,-1),\ldots,(\eta^{(k-1)}l,-1),(\kappa,1),(\kappa \eta^l,1),\ldots,(\kappa \eta^{(k-1)}l,1)\} := \tilde{D}_k^d,$$

if $l$ is even.

One shows that for $l$ even, $D_k^d$ and $D_k^d$ are conjugate; $\tilde{D}_k^d$ and $\tilde{D}_k^d$ are conjugate. Thus, in the case $k$ is even, up to conjugacy relation, we have the twisted subgroups $D_k^d$ and $D_k^d$ if $l$ is odd; $D_k^d$ and $D_k^d$ if $l$ is even. That is, for $D_{12}$, we have $D_1^d$, $\tilde{D}_1^d$, $D_2^d$, $\tilde{D}_2^d$, $D_4^d$, $\tilde{D}_4^d$, $D_6^d$, $\tilde{D}_6^d$, $\tilde{D}_{12}^d$, $\tilde{D}_{12}^d$.

Let $X$ be a finite-dimensional representation of $G$ and $\mathbb{R}$ be the one-dimensional parameter space on which $G$ acts trivially. Let $\Phi_1$ be the set of all twisted orbit types that appear in $\mathbb{R} \times X$. Consider a continuous equivariant map $f : \mathbb{R} \times X \to X$ on an open bounded invariant domain $\Omega \subset \mathbb{R} \times X$ such that $f$ is admissible on $\Omega$. Define an equivariant degree (with one parameter) of $f$ in $\Omega$ by a finite sum of integer-indexed twisted orbit types:

$$\Gamma \times S^1 \text{-Deg} (f, \Omega) = \sum_{(H) \in \Phi_1} n_H \cdot (H), \quad (2.10)$$

where $n_H \in \mathbb{Z}$ is an integer counting zero orbits of the twisted orbit type $(H)$. There is another recurrence formula, in resemblance of (2.9), that can be used to define the coefficients $n_H$'s. We omit the precise formula here and refer to [6]. It is sufficient to mention that this degree can be extended to infinite-dimensional Banach representations for compact equivariant fields. The resulting degree satisfies all classical properties of an equivariant degree theory, among which the existence property plays an important role for our purpose:

$$n_H \neq 0 \text{ in } (2.10) \Rightarrow f^{-1}(0) \cap \Omega^H \neq \emptyset.$$

### 3 Coupled Systems of Identical Cells

We consider the main equation (1.1), which describes $n$ identical dynamical systems of form $\dot{x} = f(x)$ with $f(0) = 0$ coupled together in a general network configuration including a possible time delay $\tau \geq 0$. For convenience, we repeat (1.1) here:

$$\dot{x}_i(t) = f(x_i(t)) + \kappa g_i(x_1(t-\tau), x_2(t-\tau), \ldots, x_n(t-\tau)), \quad i = 1, 2, \ldots, n,$$
where $x_i \in \mathbb{R}$, $\kappa > 0$ is the coupling strength, $f : \mathbb{R} \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ are assumed to be $C^1$. Furthermore, the $g_i$ are assumed to vanish at the origin. Thus, (1.1) admits the zero solution, for which we study the stability and bifurcations in terms of the time delay and symmetry of the system.

By linearizing (1.1) at zero, we obtain (1.2) which is

$$\dot{y}(t) = f'(0)y(t) + \kappa Cy(t - \tau)$$

where $y = (y_1, \ldots, y_n)$ and $C = [c_{ij}] = [\partial g_i(0)/\partial x_j]$. The network structure is encoded in the coupling matrix $C$. The component $c_{ij} \in \mathbb{R}$ describes how strongly the $j$-th cell influences the $i$-th cell. The influence is enhancing if $c_{ij} > 0$ or inhibiting if $c_{ij} < 0$.

As mentioned earlier in the Introduction, (1.1) and (1.2) represent a broad class of coupled systems, for which our bifurcation analysis applies. Based on the linearized system (1.2), we obtain bifurcation existence results that are valid for systems whose linearization has the form (1.2), such as (1.3), (1.4), (1.5) and (1.6). For simplicity, we assume the coupling matrix $C$ to be symmetric so that it has only real eigenvalues. All the time delay are assumed to be identical. These assumptions allow us to carry out a stability and bifurcation analysis of manageable size.

4 Stability Analysis and the Bifurcation Diagram

For $\tau > 0$, the time in the linearized equation (1.2) can be scaled $t \to t/\tau$ to yield

$$\dot{y}(t) = \tau f'(0)y(t) + \tau \kappa Cy(t - 1)$$

(4.11)

Thus, the characteristic operator for (4.11) is

$$\Delta(\lambda) = (\lambda - \tau f'(0))I_n - \tau \kappa e^{-\lambda}C : \mathbb{C}^n \to \mathbb{C}^n$$

(4.12)

and the corresponding characteristic equation is

$$\det \Delta(\lambda) = \prod_{\xi \in \sigma(C)} (\lambda - \tau f'(0) - \tau \kappa e^{-\lambda}\xi) = 0.$$  

(4.13)

Since $C$ is assumed to be a symmetric matrix, we have $\sigma(C) \subset \mathbb{R}$. Let $\xi \in \sigma(C)$ and consider the corresponding factor in (4.13). If $\lambda = u + iv$ is a characteristic root, then separating real and imaginary parts leads to

$$\left\{ \begin{array}{l}
u - \alpha - \beta e^{-u} \cos v = 0 \\
v + \beta e^{-u} \sin v = 0, \end{array} \right.$$  

(4.14)

where $\alpha = \tau f'(0)$ and $\beta = \tau \kappa \xi$. For purely imaginary roots, we have $u = 0$, giving

$$\left\{ \begin{array}{l} -\alpha - \beta \cos v = 0 \\
v + \beta \sin v = 0. \end{array} \right.$$  

(4.15)

For $v = 0$ the solution is the line $L_1$ defined by $\beta = -\alpha$, which corresponds to parameter values for which $\lambda = 0$ is a characteristic root. Over the intervals $v \in (k\pi, (k + 1)\pi)$, $k \in \mathbb{Z}$, the solution can be expressed in the parametric form $(\alpha(v), \beta(v)) = (v/\tan(v), -v/\sin(v))$, which gives parametric curves for which there exists a pair of purely imaginary characteristic roots of the form $\lambda = \pm iv$. These bifurcation curves are depicted in Figure 1. Knowing that the zero
solution is stable for $\beta = 0$ and $\alpha < 0$, and because characteristic roots can cross the imaginary axis only for parameter values belonging to the bifurcation curves, one can then move vertically in the parameter plane, increasing the number of roots with positive real parts appropriately each time a bifurcation curve is crossed. Implicit differentiation on bifurcation curves shows that the characteristic roots on the imaginary axis move to the right as $|\beta|$ increases, yielding the picture shown in Figure 1.

Figure 1: Bifurcation diagram of the characteristic equation. The curves indicate the parameter values for which the characteristic equation has a root on the imaginary axis. The curves separate the $\alpha$-$\beta$ parameter plane into regions in which the number of characteristic roots with positive real parts is a constant, the value of which is indicated in the figure. Hence “0” indicates the region where the origin is stable, which is bounded from above by the straight line $L_1$ and from below by the curve $C_2$.

The region of stability is indicated in Figure 1 by the label “0”. It is bounded from above by the straight line $L_1$ and from below by the curve $C_2$. The latter is given by the parametric branch $(\alpha, \beta) = (v/\tan(v), -v/\sin(v)), \; v \in (0, \pi)$, and meets the line $L_1$ at the point $(1, -1)$. This is of course for one particular spatial mode corresponding to the eigenvalue $\xi$. One can then repeat the same argument for all eigenmodes $\xi \in \sigma(C)$. If a parameter pair $(\alpha, \beta)$ escapes the stable region by crossing the line $L_1$, a bifurcation of steady states may occur. If it crosses the curve $C_2$, then a bifurcation of oscillating states can take place. The codimension of these bifurcations is related to the multiplicity of the eigenvalue $\xi$ given by the critical value of $\beta = \tau\kappa\xi$.

4.1 Effect of Network Structure

Suppose we start with stable systems ($f'(0) < 0$) without coupling, so we are initially on the negative $\alpha$-axis. As we increase the coupling $\kappa$ stability may be lost via a stationary or an oscillatory bifurcation through the first eigenmode $\xi$ to hit $L_1$ or $C_2$. The important observation is that this first bifurcation depends only on the extremal eigenvalues $\xi$ of the coupling matrix $C$. Hence, the number of relevant parameters is greatly reduced and one needs to check only the
two extremal eigenvalues of the coupling matrix regardless of the network size. Thus one can classify networks by defining equivalence classes according to the extreme eigenvalues: networks having the same smallest and largest eigenvalues will have identical stability properties with regard to the class.

It is possible to give more precise statements. For diffusively coupled systems such as (1.5) or (1.6), the coupling matrix $C$ is given by the negative of the Laplacian matrix; therefore, all its eigenvalues are non-positive, the largest one always being zero. In fact, for connected networks, all eigenvalues of $C$ are strictly negative, except for a single zero eigenvalue. In this case, it is the smallest eigenvalue of $C$ (i.e., the largest Laplacian eigenvalue) that determines the first bifurcation. As far as the network structure is concerned, this is the only relevant quantity.

For systems of the form (1.3) or (1.4), $C$ is given by the adjacency matrix $A$, which can have both negative and positive eigenvalues. Thus both $\xi_{\min}$ and $\xi_{\max}$ should be considered for the first bifurcation.

### 4.2 Effect of Delays

For $\tau = 0$, the characteristic equation for (1.2) is

$$\prod_{\xi \in \sigma(C)} (\lambda - f'(0) - \kappa \xi) = 0. \quad (4.16)$$

from which the characteristic roots can be directly read off as $\lambda = f'(0) + \kappa \xi$, $\xi \in \sigma(C)$. The roots are real for real network eigenvalues $\xi$: hence the only critical root is $\lambda = 0$, which occurs when $f'(0) = -\kappa \xi$. The corresponding critical curve is a straight line on the parameter plane of $f'(0)$ versus $\kappa \xi$, which can be identified with the line $L1$ of Figure 1. Thus, one has stability below this line and one real positive characteristic root above, for a given spatial mode corresponding to $\xi$. In particular, Hopf bifurcations are not possible.

Hence, stationary bifurcations given by $L1$ of Figure 1 are independent of the delay, whereas the remaining set of curves of oscillatory bifurcation are a result of delay. In the following sections we will consider both stationary and oscillatory bifurcations in our symmetry analysis. The former type will be relevant for both delayed and undelayed systems, whereas the latter will be a feature of delayed systems only.

### 5 Symmetry Aspect and Equivariant Bifurcations

By a symmetry of a dynamical system, we mean a group of elements acting on the phase space that keep the system invariant.

Let $S_n$ be the group of all permutations of $n$ symbols. For $\varepsilon \in S_n$ and consider its natural action on $\mathbb{R}^n$ by $(x_1, \ldots, x_n) \mapsto (x_{\varepsilon(1)}, \ldots, x_{\varepsilon(n)})$. Consider a subgroup $\Gamma \subset S_n$.

**Lemma 5.1** Let $\kappa \neq 0$. Then $\Gamma$ is a symmetry of systems of form (1.1) if and only if

$$g_{\varepsilon(i)}(x_1, x_2, \ldots, x_n) = g_i(x_{\varepsilon(1)}, x_{\varepsilon(2)}, \ldots, x_{\varepsilon(n)}), \quad (5.17)$$

for all $\varepsilon \in \Gamma$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

**Proof** Let $\varepsilon \in \Gamma$ and apply its action on (1.1). We obtain

$$\dot{x}_{\varepsilon(i)}(t) = f(x_{\varepsilon(i)}(t)) + \kappa g_i(x_{\varepsilon(1)}(t - \tau), x_{\varepsilon(2)}(t - \tau), \ldots, x_{\varepsilon(n)}(t - \tau)). \quad (5.18)$$
Comparing with (1.1), we see that (5.18) is the same system as (1.1) if and only if
\[ \kappa g_i(x_{\Sigma(1)}(t-\tau), x_{\Sigma(2)}(t-\tau), \ldots, x_{\Sigma(n)}(t-\tau)) = \kappa g_{\Sigma(i)}(x_1(t-\tau), x_2(t-\tau), \ldots, x_n(t-\tau)) \].

This leads to (5.17), since \( \kappa \neq 0 \).

\[ \square \]

**Remark 5.2** Note that a necessary condition for (5.17) to hold is
\[ c_{ij} = c_{\Sigma(i)\Sigma(j)}, \quad \forall \Sigma \in \Gamma, \tag{5.19} \]
is satisfied for the coupling matrix \( C \) in the linearization (1.2). For systems (1.3), (1.4), (1.5) and (1.6), however, it is sufficient, since (5.17) reduces to \( a_{ij} = a_{\Sigma(i)\Sigma(j)} \) for all \( \Sigma \in \Gamma \).

\[ \diamond \]

### 5.1 Bifurcation Analysis Using showdegree[\Gamma]

In what follows, \( \Gamma \subset S_n \) stands for the group of symmetries of the system (??), which is determined by (5.17). In case the system takes special form of (1.3), (1.4), (1.5) or (1.6), the symmetry is determined directly by the coupling matrix \( C \) in (??) (cf. Remark 5.2). We are interested in studying the bifurcations that destabilize the equilibrium \( x = 0 \).

The tool we are using for bifurcation analysis is the equivariant degree and the “Equivariant Degree Maple© Library Package” that performs exact computations of values of equivariant degrees. This package is free to be downloaded at \texttt{http://www.math.uni-hamburg.de/home/ruan/download}.

We provide details of using the Maple© package for our computations. In fact, all computations of exact values of the associated bifurcation invariants are done by calling

\[ \text{showdegree}[\Gamma](n_0, n_1, \ldots, n_r, n_{m_0}, n_{m_1}, \ldots, n_{m_s}), \quad \text{for } n_i, m_j \in \mathbb{Z}, \tag{5.20} \]

where \( \Gamma \) is a finite group describing the permutational symmetry of the coupled system, \( n_i \)'s and \( m_j \)'s are integers to be determined by the critical spectrum of the linearized system at the equilibrium.

The number \( r \) and \( s \) in (5.20) are predetermined by \( \Gamma \). They are the number of all distinct (nontrivial) irreducible representations of \( \Gamma \) over reals and over complex numbers, respectively. In what follows, we use \( V_0, V_1, \ldots, V_r \) for the distinct real irreducible representations and \( U_0, U_1, \ldots, U_s \) for the complex ones, where \( V_0 \) and \( U_0 \) are reserved for the trivial representations.

### 5.2 Steady-State Bifurcations

Assume that \((\alpha, \beta)\) crosses L1 through \((\alpha_0, \beta_0)\) from the shaded region in Figure 1. Then,

\[ \alpha_0 = -\beta_0 = \tau \kappa \xi_0, \tag{5.21} \]

for an eigenvalue \( \xi_0 \in \sigma(C) \). For \( \tau, \kappa > 0 \), \( \xi_0 \) is the maximal eigenvalue of \( C \). Let \( E(\xi_0) \) be the generalized eigenspace of \( \xi_0 \). Given the \( \Gamma \)-action on \( \mathbb{R}^n \), we decompose \( \mathbb{R}^n \) into pieces of \( V_i \)'s:

\[ \mathbb{R}^n = V_0 \times V_1 \times \cdots \times V_r, \]

where every \( V_i \)

\[ V_i = V_i \times \cdots \times V_i \quad \text{n_i times} \tag{5.22} \]
is a product of $n_i$ copies of $V_i$ for some integer $n_i \in \mathbb{N} \cup \{0\}$. Also, since $E(\xi_o)$ is a $\Gamma$-invariant subspace of $\mathbb{R}^n$, we can decompose $E(\xi_o)$ as: \[ E(\xi_o) = E_0 \times E_1 \times \cdots \times E_r, \] where every $E_i$ is given by \[ E_i = V_i \times \cdots \times V_i \] \[ \text{ ei times } \] (5.23) is a product of $e_i$ copies of $V_i$ for some integer $e_i \in \mathbb{N} \cup \{0\}$. Using (5.22)-(5.23), define \[ u_i := n_i - e_i, \] (5.24) for $i = 0, 1, \ldots, r$. Then, the bifurcation invariant around $(\alpha_o, \beta_o)$ is given by \[ \omega_0 := \text{showdegree}[\Gamma](n_0, \ldots, n_r, 1, 0, \ldots, 0) - \text{showdegree}[\Gamma](u_0, \ldots, u_r, 1, 0, \ldots, 0). \] (5.25) Running the Maple© package, we obtain the value of $\omega_0$ which is of form \[ c_1(K_1) + c_2(K_2) + \cdots + c_p(K_p), \] for integers $c_i \in \mathbb{Z}$ and conjugacy classes $(K_i)$ of subgroups $K_i$ in $\Gamma$.

**Theorem 5.3** Let $(\alpha_o, \beta_o)$ be such that $\alpha_o = -\beta_o$ and $\xi_o \in \sigma(C)$ be given by (5.21). If $\omega_0$ given by (5.25) is of form \[ \omega_0 = c_1(K_1) + c_2(K_2) + \cdots + c_p(K_p), \] for some $c_i \neq 0$, then there exists a bifurcating branch of steady states of symmetry at least $(K_i)$.

**Proof** The formula (5.25) of the bifurcation invariant was established in [8] (cf. Theorem 8.5.2). We provide a different and more straightforward proof in the appendix (cf. Appendix A). The rest of the statement follows from the existence property of equivariant degree. \[ \square \]

**Corollary 5.4** Under the hypotheses of Theorem 5.3, if moreover, the subgroup $K_i$ satisfies \[ \xi_o \notin \sigma(C|_{\text{Fix}(H)}), \quad \forall H \supseteq K_i, \] (5.26) then there exists a bifurcating branch of steady states of symmetry precisely $(K_i)$.

**Proof** By Theorem 5.3, there exists a bifurcating branch of steady states of symmetry at least $(K_i)$. Let $(H)$ be the symmetry of this branch of solutions. Then, $(H) \supseteq (K_i)$. Up to the group conjugacy, we have $H \supseteq K_i$. We need to show $H = K_i$. Assume to the contrary that $H \supsetneq K_i$. Then, by (5.26), we have that when restricted to $\text{Fix}(H)$, the characteristic operator $\Delta(0)|_{\text{Fix}(H)} : \text{Fix}(H) \to \text{Fix}(H)$ is an isomorphism, for $(\alpha, \beta)$ in a neighborhood of $(\alpha_o, \beta_o)$. By the theorem of implicit functions, there can be no additional solution in neighborhood of the trivial solution $x = 0 \in \text{Fix}(H)$, which is a contradiction. \[ \square \]
5.3 Hopf Bifurcations

Assume that \((\alpha, \beta)\) crosses \(C_2\) through \((\alpha_0, \beta_0)\) from the shaded region in Figure 1. Since \(C_2\) bounds the region from below and \(\tau, \kappa > 0\), the first parameter pair that crosses \(C_2\) must be related to the minimal eigenvalue \(\xi_{\min}\) of \(C\).

Let \(\xi_0 \in \sigma(C)\) be the corresponding eigenvalue, i.e.

\[
\beta_0 = \tau \kappa \xi_0. \tag{5.27}
\]

That is, \(\xi_0 = \xi_{\min}\) becomes critical. Consider the complexification \(\mathbb{C}^n = \mathbb{C} \otimes \mathbb{R}^n\) of the phase space \(\mathbb{R}^n\) and extend the \(\Gamma\)-action on \(\mathbb{C}^n\) by defining

\[
\gamma(z \otimes x) = z \otimes (\gamma x), \quad \text{for} \quad \gamma \in \Gamma, \ x \in \mathbb{R}^n. \tag{5.28}
\]

The (generalized) eigenspace \(E(\xi_0)\) remains \(\Gamma\)-invariant as a complex subspace of \(\mathbb{C}^n\). Thus, we decompose \(E(\xi_0)\) into irreducible representations \(U_0, U_1, \ldots, U_s\) as:

\[
E(\xi_0) = F_0 \times F_1 \times \cdots \times F_s,
\]

where every \(F_j\) is given by

\[
F_j = U_j \times \cdots \times U_j \quad \text{Times} \ m_j
\]

is a product of \(m_j\) copies of \(U_j\) for some integer \(m_j \in \mathbb{N} \cup \{0\}\). Then, the bifurcation invariant around \((\alpha_0, \beta_0)\) for Hopf bifurcation is given by

\[
\omega_1 := \text{showdegree}_{\Gamma}(0, 0, \ldots, 0, -m_0, -m_1, \ldots, -m_s). \tag{5.30}
\]

Running the Maple\(^\circledast\) package, we obtain the value of \(\omega_1\) being of form

\[
c_1(H_1) + c_2(H_2) + \cdots + c_q(H_q),
\]

for integer coefficients \(c_j \in \mathbb{Z}\) and conjugacy classes \((H_j)\) of subgroups \(H_j \subset \Gamma \times S^1\).

**Theorem 5.5** Let \((\alpha_0, \beta_0)\) be such that \((\alpha_0, \beta_0) \in C_2\) in Figure 1 and \(\omega_1\) be given by (5.30). If

\[
\omega_1 = c_1(H_1) + c_2(H_2) + \cdots + c_q(H_q),
\]

contains a non-zero coefficient \(c_j \neq 0\) for some \((H_j)\), then there exists a bifurcating branch of oscillating states of symmetry at least \((H_j)\).

**Proof** Using equivariant degree theory, the bifurcation invariant is computed by (cf. [6])

\[
\omega_1 = \text{showdegree}_{\Gamma}(k_0, k_1, \ldots, k_r, t_0, t_1, \ldots, t_s),
\]

where \(k_i\)'s are related to the positive spectrum of the right hand side of (4.11) in the constant function space, and the \(t_j\)'s are the crossing numbers which are equal to either \(m_j\) or \(-m_j\), depending on the direction of the crossing of the critical characteristic roots.

Consider (4.11) in the constant function space. Then,

\[
(\tau f'(0)\text{Id} + \tau \kappa C)x = 0, \quad x \in \mathbb{R}^n.
\]
The positive spectrum $\sigma_+$ of the linear operator $(\tau f'(0)\text{Id} + \tau kC)$ is

$$\sigma_+ = \{ \tau f'(0) + \tau k\xi : \tau f'(0) + \tau k\xi > 0, \quad \xi \in \sigma(C) \} = \{ \alpha + \beta(\xi) : \alpha + \beta(\xi) > 0, \quad \xi \in \sigma(C) \},$$

which is an empty set, since the curve $C_2$ lies in the area $\alpha + \beta < 0$. Since the integer $k_i$ is the total number of copies of $V_i$ in $E(\mu)$ for $\mu \in \sigma_+$, we have $k_i = 0$ for all $i = 0, 1, \ldots, r$.

The crossing numbers are positive if the critical characteristic roots cross from the right to the left of the complex plane; and negative otherwise. As $(a, \beta)$ crosses $C_2$ at $(\alpha_o, \beta_o)$ from the shaded region in Figure 1, the count of characteristic roots with positive real part increases by 2, thus all nonzero $t_j$’s are negative and equal to $-m_j$.

Theorem 5.5 gives an existing result of bifurcating branches together with their least symmetry. To sharpen to the precise symmetry, one can work with orbit types that satisfy certain maximal condition. Here, we recall the concept of dominating orbit types from [6] and introduce a new complementing definition of secondary dominating orbit types.

**Definition 5.6** Let $\{U_1, U_2, \ldots, U_m\}$ be the set of irreducible $\Gamma$-representations that occur in $C^n$, where $C^n$ is the complexification of the phase space $\mathbb{R}^n$ of the system (1.1). Let $\tilde{U}_j$ be the $\Gamma \times S^1$-representation induced from $U_j$, for $j = 1, 2, \ldots, m$ (cf. (2.7)). Collect maximal orbit types from $U_j$, for $j = 1, 2, \ldots, m$. Denote this collection by $\mathcal{M}$. An orbit type $(H) \in \mathcal{M}$ is called dominating, if $(H)$ is maximal in $\mathcal{M}$. A non-dominating orbit type $(L) \in \mathcal{M}$ is called secondary dominating, if all orbit types $(H) \in \mathcal{M}$ satisfying $(L) < (H)$ are dominating.

**Proposition 5.7** Let $(\alpha_o, \beta_o)$ be such that $(\alpha_o, \beta_o) \in C_2$ in Figure 1 and $\xi_o$ be the corresponding eigenvalue of $C$ given by (5.27). Assume that $\omega_1$ defined by (5.30) contains $(H)$ with a nonzero coefficient. Then, the following holds:

(i) If $(H)$ is a dominating orbit type, then there exists a bifurcating branch of oscillating states of symmetry precisely equal to $(H)$.

(ii) If $(H)$ is a secondary dominating orbit type and for every dominating orbit type $(\tilde{H})$ with $(H) < (\tilde{H})$, there exists a $C$-invariant subspace $S \subset \mathbb{R}^n$ such that

(a) $S$ contains every state of symmetry $\tilde{H}$; and

(b) $\xi_o \notin \sigma(C|_S)$,

then there exists a bifurcating branch of oscillating states of symmetry precisely being $(H)$.

**Proof** The statement (i) follows from [6]. (ii) follows from the theorem of implicit functions, in the same spirit as Corollary 5.4. More precisely, let $(H)$ be a secondary dominating orbit type with a no-zero coefficient in $\omega_1$. By Theorem 5.5, there exists a bifurcating branch of oscillating states of symmetry at least $(H)$. Let $(\tilde{H})$ be the precise symmetry of this branch and suppose that $(H) < (\tilde{H})$. By definition of secondary dominating orbit types, the only orbit types that are strictly larger than $(H)$ are dominating orbit types. Thus, $(\tilde{H})$ is dominating and so there exists a $C$-invariant subspace $S$ in $\mathbb{R}^n$ satisfying (a)-(b). Note that this subspace $S$ is also flow invariant for the system (1.1), thus one can consider the restricted flow on $S$. The bifurcating branch of oscillating states, by condition (a), is contained in $S$. However, by condition (b) and the Implicit Function Theorem, there can be no bifurcation taking place in $S$. This leads to a contradiction.

□
6  Bidirectional Ring Configuration

In this section, we study the bifurcations of the system (1.1) with a bidirectional ring configuration. That is, we assume \( q_i \)'s satisfy (5.17) for \( I = D_n \). If the system takes form of (1.3), (1.4), (1.5) or (1.6), this assumption can be weakened to (5.19). In either case, the coupling matrix \( C \) in (1.2) satisfies (5.19), which in case of dihedral configuration implies that \( C \) is a circulant matrix\(^1\) with \( c_{1j} = c_{1,(n+2-j)} \) for \( 1 \leq j \leq n \). In particular, \( C \) is a symmetric matrix.

It is known that a circulant matrix with first row entries \( c_1, c_2, \ldots, c_n \) has the following eigenvalues

\[
\xi_j = c_0 + c_1 \omega_j + c_2 \omega_j^2 + \cdots + c_n \omega_j^{n-1}, \quad j = 0, 1, 2, \ldots, n-1,
\]

with their eigenvectors \( v_j = (1, \omega_j, \omega_j^2, \ldots, \omega_j^{n-1})^T \), where \( \omega_j = e^{2\pi j/n} \)'s are the \( n \)-th roots of unity. Moreover, if the circulant matrix is \( D_n \)-symmetric, then \( \xi_j = \xi_{n-j} \) for \( 0 < j, k < n \), which is essentially induced by the \( D_n \)-symmetry. In fact, we have

\[
\begin{align*}
E(\xi_0) &= V_0, \\
E(\xi_j) &= E(\xi_{n-j}) = V_j & \text{for } 0 < j < \frac{n}{2} \\
E(\xi_{\frac{n}{2}}) &= V_{\frac{n}{2}+2}, & \text{if } n \text{ is even}
\end{align*}
\]

(cf. Example 2.2 for notations \( V_j \)). An eigenvalue \( \xi \in \sigma(C) \) is called simple, if \( E(\xi) \) is irreducible. To a critical eigenvalue \( \xi_o \), we associate an index set

\[
I = \{ i : \xi_i = \xi_o \}
\]

(in case \( n \) is even and \( \xi_{\frac{n}{2}} = \xi_o \), we put \( \frac{n}{2} + 2 \) into \( I \) instead of \( \frac{n}{2} \)), which collects all indices of irreducible representations that have to do with the critical eigenvalue \( \xi_o \).

6.1 Steady-State Bifurcations for Bidirectional Rings

Recall that \( D_n \) acts on the phase space \( \mathbb{R}^n \) by

\[
\begin{align*}
\eta(x_1, x_2, \ldots, x_n) &= (x_n, x_1, x_2, \ldots, x_{n-1}) \\
\kappa(x_1, x_2, \ldots, x_n) &= (x_n, x_{n-1}, \ldots, x_1)
\end{align*}
\]

for \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \). Using characters of representations, \( \mathbb{R}^n \) can be decomposed into irreducible representations of \( D_n \). In case of even \( n \), we have

\[
\mathbb{R}^n = V_0 \times V_1 \times V_2 \times \cdots \times V_{\frac{n}{2}-1} \times V_{\frac{n}{2}+2}
\]

and in case of odd \( n \), we have

\[
\mathbb{R}^n = V_0 \times V_1 \times V_2 \times \cdots \times V_{\frac{n-1}{2}},
\]

(cf. Example 2.2 for notations \( V_j \)). It follows that the non-zero \( n_i \)'s in (5.25) are (cf. (5.22))

\[
\begin{align*}
n_0 &= n_1 = n_2 = \cdots = n_{\frac{n}{2}-1} = n_{\frac{n}{2}+2} = 1, & \text{if } n \text{ is even}, \\
n_0 &= n_1 = n_2 = \cdots = n_{\frac{n-1}{2}} = 1, & \text{if } n \text{ is odd}.
\end{align*}
\]

\(^1\)Recall that an \( n \times n \)-matrix is called circulant, if every row is the right shift of the previous row (mod \( n \)). A circulant matrix \( C = (c_{ij}) \) is also denoted by \( \text{circ}[c_{11}, c_{12}, \ldots, c_{1n}] \) using the entries of its first row.
The integers $u_i$’s in (5.25) are determined by the critical eigenvalue $\xi_o$ and the corresponding $I$ (cf. (6.32)). Based on (6.31) and the definition (5.24) of $u_i$, we have the non-zero $u_i$’s are
\[
\begin{cases}
  u_i = 1, & \text{for } i \in \{0, 1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2} + 2\} \setminus I, \text{ if } n \text{ is even}, \\
  u_i = 1, & \text{for } i \in \{0, 1, 2, \ldots, \frac{n-1}{2}\} \setminus I, \text{ if } n \text{ is odd}.
\end{cases}
\]

(6.38)

Thus, the bifurcation invariant $\omega_0$ can be computed using (5.25), accompanied by (6.37)-(6.38).

**Example 6.1** (Simple critical eigenvalues for bidirectional rings) Let $C$ be a coupling matrix satisfying (5.19) for $\Gamma = D_n$. Then, $C$ is determined by $\left(\frac{n}{2} + 1\right)$ resp. $\left(\frac{n-1}{2}\right)$ different entries if $n$ is even resp. odd. These entries decide which eigenvalue is maximal. Let $\xi_o \in \sigma(C)$ be the maximal eigenvalue. Assume that $\xi_o$ is simple, i.e. $E(\xi_o)$ is irreducible. Then, the index set $I$ is a singleton and there are only possibly $\frac{n}{2}$ or $\frac{n-1}{2}$ different values of $\omega_0$, depending on if $n$ is even or odd. As an example, for $n = 12$, we have
\[
\omega_0 = \left(\begin{array}{c}
-2(D_{12}) + 2(D_6) + 4(D_4) - 2(D_3) - 2(D_2) - 2(D_2) + 2(Z_4) + 2(Z_2), \\
(D_1) - (D_1), \\
-(D_2) + (D_2) + 2(D_1) - 2(D_1), \\
-(D_3) + (D_3), \\
2(D_4) - 2(D_2) - (Z_4) + (Z_2) - 2(D_1) + 2(D_1), \\
-(D_1) + (D_1), \\
(D_6) - 2(D_3) + (Z_4),
\end{array}\right)
\]

(6.39)

These values combined with fixed point subspaces of subgroups of $D_{12}$ (cf. Table 1) lead to the classification result summarized in Table 2. To illustrate, in case $\xi_o = \xi_1$, we have two orbit types

| $K$ | Fix $(K)$ | $\sigma(C|_{\text{Fix}(K)})$ |
|-----|-----------|-----------------------------|
| $D_{12}$ | \{ $x_1 = x_2 = \cdots = x_{12}$ \} | $\xi_0$ |
| $D_6$ | \{ $x_1 = x_2 = \cdots = x_{12}$ \} | $\xi_0$ |
| $D_8$ | \{ $x_1 = x_3 = \cdots = x_{11}, x_2 = x_4 = \cdots = x_{12}$ \} | $\xi_0, \xi_6$ |
| $Z_6$ | \{ $x_1 = x_3 = \cdots = x_{11}, x_2 = x_4 = \cdots = x_{12}$ \} | $\xi_0, \xi_6$ |
| $D_4$ | \{ $x_1 = x_3 = x_4 = x_6 = x_7 = x_9 = x_{10} = x_{12}, x_2 = x_5 = x_8 = x_{11}$ \} | $\xi_0, \xi_4, \xi_4$ |
| $Z_4$ | \{ $x_1 = x_4 = x_7 = x_{10}, x_2 = x_5 = x_8 = x_{11}, x_3 = x_6 = x_9 = x_{12}$ \} | $\xi_0, \xi_4, \xi_4$ |
| $D_4$ | \{ $x_1 = x_4 = x_5 = x_8 = x_9 = x_{12}, x_2 = x_3 = x_6 = x_7 = x_{10} = x_{11}$ \} | $\xi_0, \xi_4, \xi_4$ |
| $D_3$ | \{ $x_1 = x_3 = x_5 = x_7 = x_9 = x_{11}, x_2 = x_4 = x_6 = x_{10} = x_{12}$ \} | $\xi_0, \xi_3, \xi_6$ |
| $Z_3$ | \{ $x_1 = x_3 = x_5 = x_7 = x_9 = x_{11}, x_2 = x_4 = x_6 = x_{10}, x_3 = x_6 = x_{12}$ \} | $\xi_0, \xi_3, \xi_3, \xi_6$ |
| $D_2$ | \{ $x_1 = x_6 = x_7 = x_{12}, x_2 = x_5 = x_8 = x_{11}, x_3 = x_4 = x_9 = x_{10}$ \} | $\xi_0, \xi_2, \xi_4$ |
| $Z_4$ | \{ $x_1 = x_5 = x_7 = x_{11}, x_2 = x_4 = x_8 = x_{10}, x_3 = x_9, x_6 = x_{12}$ \} | $\xi_0, \xi_2, \xi_4, \xi_6$ |
| $D_1$ | \{ $x_1 = x_2, x_2 = x_3, x_3 = x_9, x_4 = x_{10}, x_5 = x_{11}, x_6 = x_{12}$ \} | $\xi_0, \xi_2, \xi_4, \xi_4, \xi_4, \xi_6$ |
| $D_1$ | \{ $x_1 = x_{12}, x_2 = x_{11}, x_3 = x_{10}, x_4 = x_9, x_5 = x_8, x_6 = x_7$ \} | $\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$ |
| $Z_4$ | $\mathbb{R}^{12}$ | $\xi_0, \xi_1, \xi_1, \xi_2, \xi_3, \xi_4, \xi_4, \xi_5, \xi_5, \xi_5, \xi_6$ |

Table 1: Fixed point subspaces of $K \subset D_{12}$ and eigenvalues of the coupling matrix $C|_{\text{Fix}(K)} : \text{Fix}(K) \to \text{Fix}(K)$ (up to conjugacy classes of subgroups).
### Table 2: Summary of distinct forms of steady states bifurcating from the equilibrium $x = 0$ of the system (1.1) for $n = 12.$

<table>
<thead>
<tr>
<th>Critical Eigenvalue</th>
<th>Symmetry</th>
<th>Form of Bifurcating Steady-States (for distinct $a, b, c, d, e, f, g \in \mathbb{R}$)</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_0$</td>
<td>$D_{12}$</td>
<td>$(a, a, a, a, a, a, a, a, a, a, a, a)$</td>
<td></td>
</tr>
<tr>
<td>$\xi_1$ or $\xi_5$</td>
<td>$D_1$</td>
<td>$(a, b, c, d, e, f, f, e, d, c, b, a)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$D_1 \eta^{-1}$</td>
<td>$(a, a, b, c, d, e, f, e, d, c, b)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta^2 D_1 \eta^{-2}$</td>
<td>$(b, a, a, b, c, d, e, f, e, d, c)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta^3 D_1 \eta^{-3}$</td>
<td>$(c, b, a, a, b, c, d, e, f, e, d)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta^4 D_1 \eta^{-4}$</td>
<td>$(d, c, b, a, a, b, c, d, e, f, e)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta^5 D_1 \eta^{-5}$</td>
<td>$(e, d, c, b, a, a, b, c, d, e, f)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\tilde{D}_1$</td>
<td>$(a, b, c, d, e, f, e, d, c, b, a, g)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta \tilde{D}_1 \eta^{-1}$</td>
<td>$(g, a, b, c, d, e, f, e, d, c, b, a)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta^2 \tilde{D}_1 \eta^{-2}$</td>
<td>$(a, g, a, b, c, d, e, f, e, d, c, b)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta^3 \tilde{D}_1 \eta^{-3}$</td>
<td>$(b, a, g, a, b, c, d, e, f, e, d, c)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta^4 \tilde{D}_1 \eta^{-4}$</td>
<td>$(c, b, a, g, a, b, c, d, e, f, e, d)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta^5 \tilde{D}_1 \eta^{-5}$</td>
<td>$(d, c, b, a, g, a, b, c, d, e, f, e)$</td>
<td></td>
</tr>
<tr>
<td>$\xi_2$</td>
<td>$D_2$</td>
<td>$(a, b, c, c, b, a, a, b, c, c, b, a)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$D_2 \eta^{-1}$</td>
<td>$(a, a, b, c, c, b, a, a, b, c, c, b)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta^2 D_2 \eta^{-2}$</td>
<td>$(b, a, a, b, c, c, b, a, a, b, c, c)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta D_2 \eta^{-1}$</td>
<td>$(g, a, b, c, c, b, a, a, b, c, c, b)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta^2 D_2 \eta^{-2}$</td>
<td>$(a, g, a, b, c, c, b, a, a, b, c, c)$</td>
<td></td>
</tr>
<tr>
<td>$\xi_3$</td>
<td>$D_3$</td>
<td>$(a, b, b, a, a, b, b, a, b, a, b, a)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta D_3 \eta^{-1}$</td>
<td>$(a, a, b, b, a, a, b, b, a, b, a, b)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\tilde{D}_3$</td>
<td>$(c, a, b, c, a, b, c, a, b, c, a)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta \tilde{D}_3 \eta^{-1}$</td>
<td>$(g, a, b, c, c, b, a, a, b, c, c, b)$</td>
<td></td>
</tr>
<tr>
<td>$\xi_4$</td>
<td>$D_4$</td>
<td>$(a, b, a, a, b, a, a, b, a, a, b, a)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$D_4 \eta^{-1}$</td>
<td>$(a, a, b, a, a, b, a, a, b, a, a, b)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta D_4 \eta^{-1}$</td>
<td>$(a, a, b, a, a, b, a, a, b, a, a, b)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\eta^2 D_4 \eta^{-2}$</td>
<td>$(b, a, a, b, a, a, b, a, a, b, a, a)$</td>
<td></td>
</tr>
<tr>
<td>$\xi_6$</td>
<td>$\tilde{D}_6$</td>
<td>$(a, b, b, b, a, b, a, b, b, a, b, b)$</td>
<td></td>
</tr>
</tbody>
</table>
$k = 0, 1, \ldots, 5$, we derive the form of the solution for each of these isotropies. The same can be applied to $(\tilde{D}_1)$.

Note that the all possible values of $\omega_0$ do not depend on the entries of $C$ directly, but rather their choice of the maximal eigenvalue. For example, if every cell is connected only with its 2 nearest neighbors, then $\xi_o = \xi_0$ if the coupling is enhancing; and $\xi_o = \xi_6$ if it is inhibiting. That is, this configuration does not allow $\xi_o$ to be $\xi_i$ for $i \in \{1, 2, 3, 4, 5\}$. However, if every cell is connected with its 4 nearest neighbors, then every eigenvalue can be maximal for some choices of the two coupling strength $d_1, d_2$. See Figure 2 for their precise relation.

![Figure 2: The maximal eigenvalue of the coupling matrix $C$, if every cell is connected with its 4 nearest neighbors, in relation to $d_1, d_2$.](image)

Besides those values listed in (6.39), $\omega_0$ can take other values if $\xi_o$ is non-simple. For example, the coupling configuration with 4 nearest neighbors allows double critical eigenvalues as shown in Figure 2, when the relation between $d_1, d_2$ follows one of the lines there. In this case, one can work out the index set $I$ and compute $\omega_0$ individually. The same result using Theorem 5.3 and Corollary 5.4 applies.

### 6.2 Hopf Bifurcations for Bidirectional Rings

The complexification of $E(\xi_j)$ for $\xi_j \in \sigma(C)$ satisfies

$$
\begin{align*}
E'(\xi_0) &= U_0, \\
E'(\xi_j) &= E'(\xi_{n-j}) = U_j \quad \text{for } 0 < j < \frac{n}{2} \\
E'(\xi_{\frac{n}{2}}) &= U_{(\frac{n}{2}+2)}, \quad \text{if } n \text{ is even}
\end{align*}
$$

(cf. Example 2.2 for notations $U_j$). It follows that the non-zero integers $m_j$’s in (5.30) are

$$
m_j = 1, \quad \text{for } j \in I.
$$
where $I$ is given by (6.32). The bifurcation invariant $\omega_1$ can then be computed using (5.30) together with (6.41).

**Example 6.2** (Simple critical eigenvalues for bidirectional rings) Following Example 6.1, we take $C$ that satisfies (5.19) with $\Gamma = D_n$. The $(\frac{n}{2} + 1)$ resp. $(\frac{n-1}{2})$ different entries of $C$ decide which eigenvalue is minimal. Let $\xi_o \in \sigma(C)$ be the minimal eigenvalue. Assume that $\xi_o$ is simple. Then, the index set $I$ is a singleton and there are only possibly $\frac{n}{2}$ or $\frac{n-1}{2}$ different values of $\omega_0$, depending on if $n$ is even or odd. Again for $n = 12$, we have

$$\omega_1 = \begin{cases} 
-(D_{12}), & \text{if } \xi_o = \xi_0 \\
-(Z_{12}^i) - (D_{12}^i) - (D_{12}^j) + (Z_{12}^j), & \text{if } \xi_o = \xi_1 \\
-(Z_{12}^i) - (D_{12}^i) - (D_{12}^j) + (Z_{12}^j), & \text{if } \xi_o = \xi_2 \\
-(Z_{12}^i) - (D_{12}^i) - (D_{12}^j) + (Z_{12}^j), & \text{if } \xi_o = \xi_3 \\
-(Z_{12}^i) - (D_{12}^i) - (D_{12}^j) + (Z_{12}^j), & \text{if } \xi_o = \xi_4 \\
-(Z_{12}^i) - (D_{12}^i) - (D_{12}^j) + (Z_{12}^j), & \text{if } \xi_o = \xi_5 \\
-(D_{12}^i), & \text{if } \xi_o = \xi_6 
\end{cases}$$

To find dominating and secondary dominating orbit types, consider the maximal orbit types in $U_i$'s. They are $(D_{12})$ in $U_0$: $(Z_{12}^i)$, $(D_{12}^i)$, $(D_{12}^j)$ in $U_4$: $(Z_{12}^i)$, $(D_{12}^i)$, $(D_{12}^j)$ in $U_5$: $(D_{12}^i)$ in $U_6$: $(D_{12}^i)$, $(D_{12}^j)$, $(D_{12}^j)$ and the secondary dominating orbit types: $(D_{12}^i)$, $(D_{12}^j)$, $(D_{12}^j)$. The values of $\omega_1$ together with the dominating and secondary dominating orbit types lead to the classification result summarized in Table 3-5 using Proposition 5.7.

\[\diamond\]

### A The Proof of Theorem 5.3

**Theorem 5.3** Let $(\alpha_o, \beta_o)$ be such that $\alpha_o = -\beta_o$ and $\xi_o \in \sigma(C)$ be given by (5.21). If $\omega_0$ given by (5.25) is of form

$$\omega_0 = c_1(K_1) + c_2(K_2) + \cdots + c_p(K_p),$$

for some $c_i \neq 0$, then there exists a bifurcating branch of steady states of symmetry at least $(K_1)$.

**Proof** The parameter pair $(\alpha, \beta)$ escapes the shaded region in Figure 1 by crossing over $L_1$ through $(\alpha_o, \beta_o)$ (cf. Figure 3).

Let $c : [\lambda_-, \lambda_+] \subset \mathbb{R} \to \mathbb{R}^2$ be a parametrization of the crossing curve such that $c(\lambda_-) = (\alpha_-, \beta_-)$, $c(\lambda_o) = (\alpha_o, \beta_o)$ and $c(\lambda_+) = (\alpha_+, \beta_+)$. Then, the initial bifurcation problem becomes a bifurcation problem around $\lambda_o$. More precisely, we have a $\Gamma$-equivariant map $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that $F(\lambda, 0) = 0$ for all $\lambda \in [\lambda_-, \lambda_+]$. The spectrum of $D_xF(\lambda, 0)$ belongs to $C_-$ (the left half of the complex plane) for all $\lambda \in [\lambda_-, \lambda_o]$ and as $\lambda$ crosses $\lambda_o$, the spectrum of $D_xF(\lambda_o, 0)$ intersects with $i\mathbb{R}$ at $0$.

Without loss of generality, let $\lambda_\pm = \pm 4$ and $\lambda_o = 0$. Define a box around the bifurcation point $(0, 0) \in \mathbb{R} \times \mathbb{R}^n$ by (cf. Figure 4)

$$\Omega_1 := \{(\lambda, x) : |\lambda| < 4, \ ||x|| < \rho\},$$

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<table>
<thead>
<tr>
<th>Critical Eigenvalue</th>
<th>Symmetry</th>
<th>Form of Oscillating-States (for some period $T$)</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_0$</td>
<td>$D_{12}$</td>
<td>$(x(t), x(t), x(t), \ldots, x(t))$</td>
<td><img src="image1.png" alt="Figure" /></td>
</tr>
<tr>
<td>$\xi_1$</td>
<td>$Z_{12}^{\xi_1}$</td>
<td>$(x_1(t), x_1(t + \frac{T}{12}), x_1(t + \frac{2T}{12}) \ldots, x_1(t + \frac{11T}{12}))$</td>
<td><img src="image2.png" alt="Figure" /></td>
</tr>
<tr>
<td>$\kappa Z_{12}^{\xi_1}$</td>
<td>$(x_1(t), x_1(t + \frac{11T}{12}), x_1(t + \frac{10T}{12}) \ldots, x_1(t + \frac{T}{12}))$</td>
<td><img src="image3.png" alt="Figure" /></td>
<td></td>
</tr>
<tr>
<td>$\xi_1$ or $\xi_5$</td>
<td>$\eta D_2^\eta$</td>
<td>$(x_1(t), x_2(t), x_3(t), x_3(t + \frac{T}{2}), x_2(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_2(t + \frac{T}{2}), x_3(t), x_2(t), x_1(t))$</td>
<td><img src="image4.png" alt="Figure" /></td>
</tr>
<tr>
<td></td>
<td>$\eta D_2^\eta$</td>
<td>$(x_1(t), x_1(t), x_2(t), x_3(t), x_3(t + \frac{T}{2}), x_2(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_3(t), x_2(t), x_1(t))$</td>
<td><img src="image5.png" alt="Figure" /></td>
</tr>
<tr>
<td></td>
<td>$\eta^2 D_2^\eta$</td>
<td>$(x_2(t), x_1(t), x_1(t), x_2(t), x_3(t), x_3(t + \frac{T}{2}), x_2(t + \frac{T}{2}), x_3(t + \frac{T}{2}), x_3(t), x_2(t))$</td>
<td><img src="image6.png" alt="Figure" /></td>
</tr>
<tr>
<td></td>
<td>$\eta^3 D_2^\eta$</td>
<td>$(x_3(t), x_2(t), x_1(t), x_1(t), x_2(t), x_3(t), x_3(t + \frac{T}{2}), x_2(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_3(t))$</td>
<td><img src="image7.png" alt="Figure" /></td>
</tr>
<tr>
<td></td>
<td>$\eta^4 D_2^\eta$</td>
<td>$(x_3(t + \frac{T}{2}), x_3(t), x_2(t), x_1(t), x_1(t), x_2(t), x_3(t), x_3(t + \frac{T}{2}), x_2(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_3(t), x_2(t), x_1(t))$</td>
<td><img src="image8.png" alt="Figure" /></td>
</tr>
<tr>
<td></td>
<td>$\eta^5 D_2^\eta$</td>
<td>$(x_2(t + \frac{T}{2}), x_3(t + \frac{T}{2}), x_3(t), x_2(t), x_1(t), x_1(t), x_2(t), x_3(t), x_2(t), x_1(t))$</td>
<td><img src="image9.png" alt="Figure" /></td>
</tr>
</tbody>
</table>

Table 3: Summary of distinct forms of oscillating states bifurcating from the equilibrium $x = 0$ of the system (1.1), where cells are coupled to their nearest and next nearest neighbors (Part I).

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<table>
<thead>
<tr>
<th>Critical Eigenvalue</th>
<th>Symmetry</th>
<th>Form of Oscillating-States (for some period $T$)</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_2$</td>
<td>$Z_{12}^{\ell_2}$</td>
<td>$(x_1(t), x_1(t + \frac{T}{6}), x_1(t + \frac{2T}{6}), \ldots, x_1(t + \frac{5T}{6})),$ $x_1(t), x_1(t + \frac{T}{6}), x_1(t + \frac{2T}{6}), \ldots, x_1(t + \frac{5T}{6}))$</td>
<td><img src="image1.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$\xi_2$</td>
<td>$\kappa Z_{12}^{\ell_2} \kappa^{-1}$</td>
<td>$(x_1(t), x_1(t + \frac{2T}{6}), x_1(t + \frac{4T}{6}), \ldots, x_1(t + \frac{5T}{6}));$ $x_1(t), x_1(t + \frac{2T}{6}), x_1(t + \frac{4T}{6}), \ldots, x_1(t + \frac{5T}{6}))$</td>
<td><img src="image2.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$\xi_3$</td>
<td>$D_4^d$</td>
<td>$(x_1(t), x_2(t), x_1(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_2(t), x_1(t), x_2(t), x_1(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_2(t), x_1(t))$</td>
<td><img src="image3.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$\xi_3$</td>
<td>$\eta D_4^d \eta^{-1}$</td>
<td>$(x_1(t), x_1(t), x_2(t), x_1(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_2(t), x_1(t), x_2(t), x_1(t))$</td>
<td><img src="image4.png" alt="Diagram" /></td>
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<tr>
<td>$\xi_3$</td>
<td>$\eta^2 D_4^d \eta^{-2}$</td>
<td>$(x_2(t), x_1(t), x_1(t), x_2(t), x_1(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_2(t), x_1(t), x_2(t), x_1(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_2(t), x_1(t))$</td>
<td><img src="image5.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$\xi_3$</td>
<td>$Z_{12}^{\ell_3}$</td>
<td>$(x_1(t), x_1(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_1(t + \frac{3T}{4}),)$ $x_1(t), x_1(t + \frac{T}{4}), \ldots, x_1(t + \frac{3T}{4}))$</td>
<td><img src="image6.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$\xi_3$</td>
<td>$\kappa Z_{12}^{\ell_3} \kappa^{-1}$</td>
<td>$(x_1(t), x_1(t + \frac{3T}{4}), x_1(t + \frac{T}{2}), x_1(t + \frac{T}{2}),)$ $x_1(t), x_1(t + \frac{3T}{4}), \ldots, x_1(t + \frac{T}{2}))$</td>
<td><img src="image7.png" alt="Diagram" /></td>
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<tr>
<td>$\xi_3$</td>
<td>$D_6^d$</td>
<td>$(x_1(t), x_1(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_1(t), x_1(t), x_1(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_1(t), x_1(t))$</td>
<td><img src="image8.png" alt="Diagram" /></td>
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<td>$\xi_3$</td>
<td>$\eta D_6^d \eta^{-1}$</td>
<td>$(x_1(t), x_1(t), x_1(t + \frac{T}{2}), x_1(t + \frac{T}{2}), x_1(t), x_1(t), x_1(t + \frac{T}{2}), x_1(t), x_1(t))$</td>
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<td>$\xi_3$</td>
<td>$D_6^d$</td>
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<tr>
<td>$\xi_3$</td>
<td>$\eta D_6^d \eta^{-1}$</td>
<td>$(x_2(t + \frac{T}{2}), x_1(t), x_2(t), x_1(t), x_2(t + \frac{T}{2}), x_1(t), x_2(t), x_1(t), x_2(t), x_1(t), x_2(t), x_1(t), x_2(t), x_1(t))$</td>
<td><img src="image11.png" alt="Diagram" /></td>
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</table>

Table 4: Summary of distinct forms of oscillating states bifurcating from the equilibrium $x = 0$ of the system (1.1), where cells are coupled to their nearest and next nearest neighbors (Part II).
<table>
<thead>
<tr>
<th>Critical Eigenvalue</th>
<th>Symmetry</th>
<th>Form of Oscillating-States (for some period $T$)</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_4$</td>
<td>$\mathbb{Z}_{12}^{14}$</td>
<td>$(x_1(t), x_1(t + \frac{T}{3}), x_1(t + \frac{2T}{3}), x_1(t), x_1(t + \frac{T}{3}), \ldots, x_1(t + \frac{2T}{3}))$</td>
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<td>$\kappa\mathbb{Z}_{12}^{14}\kappa^{-1}$</td>
<td>$(x_1(t), x_1(t + \frac{2T}{3}), x_1(t + \frac{T}{3}), x_1(t), x_1(t + \frac{2T}{3}), \ldots, x_1(t + \frac{T}{3}))$</td>
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<tr>
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<td>$D_3^x$</td>
<td>$(x_1(t), x_2(t), x_1(t + \frac{T}{3}), x_1(t), x_2(t), x_1(t + \frac{T}{3}), x_1(t), x_2(t), x_1(t), x_2(t), x_1(t + \frac{T}{3}), x_1(t), x_2(t), x_1(t), x_2(t))$</td>
<td><img src="image3" alt="Diagram" /></td>
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<tr>
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<td>$\eta D_3^x \eta^{-1}$</td>
<td>$(x_1(t + \frac{T}{3}), x_1(t), x_2(t), x_1(t + \frac{T}{3}), x_1(t), x_2(t), x_1(t + \frac{T}{3}), x_1(t), x_2(t), x_1(t), x_2(t), x_1(t + \frac{T}{3}), x_1(t), x_2(t))$</td>
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<td>$\eta^2 D_3^x \eta^{-2}$</td>
<td>$(x_2(t), x_1(t + \frac{T}{3}), x_1(t), x_2(t), x_1(t + \frac{T}{3}), x_1(t), x_2(t), x_1(t + \frac{T}{3}), x_1(t), x_2(t), x_1(t), x_2(t), x_1(t + \frac{T}{3}), x_1(t), x_2(t))$</td>
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<td>$(x_1(t), x_2(t), x_1(t), x_2(t), x_1(t), x_2(t), x_1(t), x_2(t), x_1(t), x_2(t), x_1(t))$</td>
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<td>$\eta D_4 \eta^{-1}$</td>
<td>$(x_1(t), x_1(t), x_2(t), x_1(t), x_1(t), x_2(t), x_1(t), x_1(t), x_2(t), x_1(t), x_2(t))$</td>
<td><img src="image7" alt="Diagram" /></td>
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<td>$\eta^2 D_4 \eta^{-2}$</td>
<td>$(x_2(t), x_1(t), x_1(t), x_2(t), x_1(t), x_1(t), x_2(t), x_1(t), x_1(t), x_2(t), x_1(t))$</td>
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<tr>
<td>$\xi_5$</td>
<td>$\mathbb{Z}_{12}^{15}$</td>
<td>$(x_1(t), x_1(t + \frac{5T}{12}), x_1(t + \frac{10T}{12}), x_1(t + \frac{15T}{12}), x_1(t), x_1(t + \frac{T}{12}), x_1(t + \frac{6T}{12}), x_1(t + \frac{11T}{12}), x_1(t + \frac{4T}{12}), x_1(t + \frac{9T}{12}), \ldots, x_1(t + \frac{T}{12}), x_1(t + \frac{3T}{12})$</td>
<td><img src="image9" alt="Diagram" /></td>
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<td></td>
<td>$\kappa\mathbb{Z}_{12}^{15}\kappa^{-1}$</td>
<td>$(x_1(t), x_1(t + \frac{7T}{12}), x_1(t + \frac{T}{12}), x_1(t + \frac{6T}{12}), x_1(t), x_1(t + \frac{T}{12}), x_1(t + \frac{4T}{12}), x_1(t + \frac{9T}{12}), x_1(t + \frac{2T}{12}), x_1(t + \frac{7T}{12}), x_1(t + \frac{1T}{12}), x_1(t + \frac{5T}{12}), x_1(t + \frac{3T}{12})$</td>
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<tr>
<td>$\xi_6$</td>
<td>$D_4^{12}$</td>
<td>$(x_1(t), x_1(t + \frac{T}{3}), x_1(t), x_1(t + \frac{T}{3}), \ldots, x_1(t + \frac{T}{3}))$</td>
<td><img src="image11" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Table 5: Summary of distinct forms of oscillating states bifurcating from the equilibrium $x = 0$ of the system (1.1), where cells are coupled to their nearest and next nearest neighbors (Part III).
Figure 3: The crossing of $(\alpha, \beta)$ through $(\alpha_o, \beta_o) \in L_1$.

Figure 4: An isolating box $\Omega_1$ around the bifurcating point $\lambda = \lambda_o$, where the red line is the equilibrium, the blue curves are potential bifurcating solutions and the plus signs “+” are the signs of auxiliary function $\zeta_1$.

where $\rho > 0$ is such that $F(\pm 4, \cdot)$ is a homeomorphism on $\{x \in \mathbb{R}^n : \|x\| < \rho\}$. Without loss of generality, let $\rho = 2$. Define $\mathcal{F}_1 : \overline{\Omega}_1 \to \mathbb{R} \times \mathbb{R}^n$ by

$$\mathcal{F}_1(\lambda, x) := (|\lambda|(|x| - 2) + \|x\| - 1, F(\lambda, x)) := (\zeta_1(\lambda, x), F(\lambda, x)).$$

Note that $\zeta_1 > 0$ for $\|x\| = 2$ and $\zeta_1 < 0$ for $\|x\| = 0$. Functions with such property are called auxiliary functions on $\Omega_1$. Thus, by construction, zeros of $\mathcal{F}_1$ in $\Omega_1$ are contained properly in $\Omega_1$, i.e. $\mathcal{F}_1^{-1}(0) \cap \overline{\Omega}_1 \subset \Omega_1$, and if $\mathcal{F}_1(\lambda, x) = 0$, then $x \neq 0$. In other words, zeros of $\mathcal{F}_1$ correspond precisely those non-trivial zeros of $F$ in $\Omega_1$. The bifurcation invariant $\omega_0$ is defined by

$$\omega_0 = \Gamma\text{-Deg} (\mathcal{F}_1, \Omega_1).$$

To compute $\omega_0$, we perform several homotopies on $\mathcal{F}_1$. Define $\mathcal{F}_2 : \overline{\Omega}_1 \to \mathbb{R} \times \mathbb{R}^n$ by

$$\mathcal{F}_2(\lambda, x) := (|\lambda|(|x| - 1) + \|x\| + 1, F(\lambda, x)) := (\zeta_2(\lambda, x), F(\lambda, x)).$$

Since $\zeta_2 > 0$ for $\|x\| = 2$, we have $\mathcal{F}_1$ and $\mathcal{F}_2$ are homotopic on $\Omega_1$ by a linear homotopy. Also, $\zeta_2 > 0$ for $|\lambda| \leq \frac{1}{2}$. Thus, zeros of $\mathcal{F}_2$ in $\Omega_1$ are contained in the following subset of $\Omega_1$

$$\Omega_2 := \{(\lambda, x) : \frac{1}{2} < \lambda < 4, \|x\| < 2\}.$$ 

By excision property, we have $\Gamma\text{-Deg} (\mathcal{F}_1, \Omega_1) = \Gamma\text{-Deg} (\mathcal{F}_2, \Omega_2)$. 

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Moreover, $\mathcal{F}_2$ is homotopic to $\mathcal{F}_3 : \Omega_2 \to \mathbb{R} \times \mathbb{R}^n$ defined by

$$\mathcal{F}_3(\lambda, x) := (\zeta_2(\lambda, x), D_x F(\lambda, 0)).$$

Decompose $\mathbb{R}^n$ into the sum of the critical eigenspace and the eigenspaces of the rest (all negative) eigenvalues of $D_x F(\lambda_0, 0)$, say $\mathbb{R}^n = R_0 \times R_1$. Then, for $x = (x_1, x_2) \in R_0 \times R_1$, the linear map $D_x F(\lambda, 0)(x_1, x_2)$ is homotopic to $(\lambda x_1, -x_2)$. Thus, $\mathcal{F}_3$ is homotopic to $\mathcal{F}_4 : \Omega_2 \to \mathbb{R} \times \mathbb{R}^n$ defined by

$$\mathcal{F}_4(\lambda, x) := (\zeta_2(\lambda, x), (\lambda x_1, -x_2)),$$

for $x = (x_1, x_2) \in R_0 \times R_1$. Note that $\mathcal{F}_4(\lambda, x) = 0$ only if $x = 0$. Substituting $x = 0$ into $\zeta_2(\lambda, x)$, we have $\zeta_2(\lambda, 0) = 0$ if and only if $\lambda = \pm 1$. That is,

$$\mathcal{F}_4^{-1}(0) \cap \Omega_2 = \{(-1, 0), (1, 0)\}.$$

It follows that

$$\Gamma\text{-Deg} (\mathcal{F}_2, \Omega_2) = \Gamma\text{-Deg} (\mathcal{F}_4, \Omega_2) = \Gamma\text{-Deg} (\mathcal{F}_4, \mathcal{N}_{-1}) + \Gamma\text{-Deg} (\mathcal{F}_4, \mathcal{N}_1),$$

where $\mathcal{N}_{-1}$ resp. $\mathcal{N}_1$ is a small neighborhood of $(-1, 0)$ resp. $(1, 0)$. On $\mathcal{N}_{-1}$, we have $\mathcal{F}_4$ is homotopic to $(1 + \lambda, -x_1, -x_2)$. By suspension, we obtain

$$\Gamma\text{-Deg} (\mathcal{F}_4, \mathcal{N}_{-1}) = \Gamma\text{-Deg} (-\text{Id}, B_1(\mathbb{R}^n)),$$

where $B_1(\cdot)$ denotes the unit ball. On the other hand, $\mathcal{F}_4$ is homotopic to $(1 - \lambda, x_1, -x_2)$ on $\mathcal{N}_1$, so by multiplication, we have

$$\Gamma\text{-Deg} (\mathcal{F}_4, \mathcal{N}_1) = -\Gamma\text{-Deg} (-\text{Id}, B_1(R_1)).$$

Therefore,

$$\omega_0 = \Gamma\text{-Deg} (-\text{Id}, B_1(\mathbb{R}^n)) - \Gamma\text{-Deg} (-\text{Id}, B_1(R_1)).$$

Using $\text{showdegree}[\Gamma]$, it is expressed as

$$\omega_0 = \text{showdegree}[\Gamma](n_0, n_1, \ldots, n_r, 1, 0, \ldots, 0) - \text{showdegree}[\Gamma](u_0, u_1, \ldots, u_r, 1, 0, \ldots, 0),$$

where $n_i$'s and $u_i$'s are defined by (5.22)-(5.24).

The statement then follows from the existence property of degree.

\[\square\]

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**References**


