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by

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Abstract

Let $\mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite system and ρ_{AB} a quantum state on $\mathcal{H}_A \otimes \mathcal{H}_B$, $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, $\rho_B = \mathrm{Tr}_A(\rho_{AB})$. Then each quantum operation Φ_B on quantum system $\mathcal{H}_B$ can induce a quantum ensemble $\{(p_\mu, \rho_{A,\mu})\}$ on quantum system $\mathcal{H}_A$. In this paper, we show that the Holevo quantity $\chi\{(p_\mu, \rho_{A,\mu})\}$ of the quantum ensemble $\{(p_\mu, \rho_{A,\mu})\}$ can be upper bounded by $\mathrm{S}(\rho_B)$. By using the result, we answer partly a conjecture of Fannes, de Melo, Roga and Życzkowski.

Keywords: Quantum state, Quantum operation, von Neumann entropy, Holevo quantity.

1 Introduction and preliminaries

Let $\mathcal{H}$ be a finite dimensional complex Hilbert space. A *quantum state* ρ on $\mathcal{H}$ is a positive semi-definite operator of trace one, in particular, for each unit vector $|\psi\rangle \in \mathcal{H}$, the

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operator $\rho = |\psi\rangle\langle\psi|$ is said to be a *pure state*. The set of all quantum states on $\mathcal{H}$ is denoted by $\mathrm{D}(\mathcal{H})$. For each quantum state $\rho \in \mathrm{D}(\mathcal{H})$, its von Neumann entropy is defined by $\mathrm{S}(\rho) = -\operatorname{Tr}(\rho \log_2 \rho)$. A *quantum operation* Φ on $\mathcal{H}$ is a trace-preserving completely positive linear mapping defined over the set $\mathrm{D}(\mathcal{H})$. It follows from ([**?**, Prop. 5.2 and Cor. 5.5]) that there exist linear operators $\{M_\mu\}_{\mu=1}^K$ on $\mathcal{H}$ such that $\sum_{\mu=1}^K M_\mu^\dagger M_\mu = \mathbb{1}$ and $\Phi = \sum_\mu \mathrm{Ad}_{M_\mu}$, that is, for each quantum state ρ, we have the Kraus representation

$$\Phi(\rho) = \sum_{\mu=1}^K M_\mu \rho M_\mu^\dagger.$$

Let $\mathcal{E} = \{(p_\mu, \rho_\mu)\}$ be a quantum ensemble on $\mathcal{H}$, that is, each $\rho_\mu \in \mathrm{D}(\mathcal{H})$, $p_\mu > 0$, and $\sum_\mu p_\mu = 1$. The *Holevo quantity* of the quantum ensemble $\{(p_\mu, \rho_\mu)\}$ is defined by the following expression:

$$\chi\{(p_\mu, \rho_\mu)\} = \mathrm{S}(\sum_\mu p_\mu \rho_\mu) - \sum_\mu p_\mu \mathrm{S}(\rho_\mu). \tag{1.1}$$

Let $\mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite system and ρ_{AB} a quantum state on $\mathcal{H}_A \otimes \mathcal{H}_B$, $\rho_A = \operatorname{Tr}_B(\rho_{AB})$, $\rho_B = \operatorname{Tr}_A(\rho_{AB})$, $\Phi_B = \sum_\mu \mathrm{Ad}_{M_{B,\mu}}$ be a quantum operation on quantum system $\mathcal{H}_B$. Then $\Phi = \sum_\mu \mathrm{Ad}_{\mathbb{1}_A \otimes M_{B,\mu}}$ is a quantum operation on the bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$.

Let

$$p_\mu = \operatorname{Tr}\left((\mathbb{1}_A \otimes M_{B,\mu})\, \rho_{AB} \left(\mathbb{1}_A \otimes M_{B,\mu}^\dagger\right)\right).$$

Then $p_\mu \geqslant 0$ and $\sum_\mu p_\mu = 1$. Without loss of generality, we assume that $p_\mu > 0$. Let

$$\begin{aligned} \rho_{A,\mu} &\equiv p_\mu^{-1} \operatorname{Tr}_B\left((\mathbb{1}_A \otimes M_{B,\mu})\, \rho_{AB} \left(\mathbb{1}_A \otimes M_{B,\mu}^\dagger\right)\right) \\ &= p_\mu^{-1} \operatorname{Tr}_B\left(\left(\mathbb{1}_A \otimes \sqrt{M_{B,\mu}^\dagger M_{B,\mu}}\right) \rho_{AB} \left(\mathbb{1}_A \otimes \sqrt{M_{B,\mu}^\dagger M_{B,\mu}}\right)\right). \end{aligned}$$

Then $\rho_{A,\mu}$ is a quantum state on $\mathcal{H}_A$. Thus, quantum operation Φ_B induced a quantum ensemble $\{(p_\mu, \rho_{A,\mu})\}$ on quantum system $\mathcal{H}_A$.

In this paper, the following result is obtained:

Theorem 1.1. $\chi\{(p_\mu, \rho_{A,\mu})\} \leqslant \min\{\mathrm{S}(\rho_A), \mathrm{S}(\rho_B)\}$.

By using this result, we answer partly a conjecture of Fannes, de Melo, Roga and Życzkowski.

2 The proof of Theorem ??

Clearly, $\chi\{(p_\mu, \rho_{A,\mu})\} \leqslant \mathrm{S}(\rho_A)$ is trivial by the definition of the Holevo information. It remains to prove $\chi\{(p_\mu, \rho_{A,\mu})\} \leqslant \mathrm{S}(\rho_B)$. The nontrivial part of the proof is divided into three parts as follows:

(i) If $\{|\psi_{B,\mu}\rangle\}_{\mu=1}^K$ is a standard orthonormal basis of $\mathcal{H}_B$ and $M_\mu = |\psi_{B,\mu}\rangle\langle\psi_{B,\mu}|$, then it follows, from Theorem 3.1 in [?] and its proof, that $\chi\{(p_\mu, \rho_{A,\mu})\} \leqslant \mathrm{S}(\rho_B)$.

(ii) If $M_{B,\mu} = P_{B,\mu}$, where $P_{B,\mu}$ is a projector on $\mathcal{H}_B$. Note that $\sum_\mu P_{B,\mu} = \mathbb{1}_B$, so there is a standard orthonormal basis $\{|u_{\mu,i}\rangle\}$ of $\mathcal{H}_B$ such that

$$P_{B,\mu} = \sum_i |u_{\mu,i}\rangle\langle u_{\mu,i}|$$

for each μ.

Denote $p_{\mu,i} = \langle u_{\mu,i} |\rho_B| u_{\mu,i}\rangle$, without loss of generality, we assume that $p_{\mu,i} > 0$, and denote

$$\rho_{A,\mu,i} = p_{\mu,i}^{-1} \langle u_{\mu,i} |\rho_{AB}| u_{\mu,i}\rangle .$$

Thus

$$\begin{aligned} p_\mu &= \mathrm{Tr}\left(\left(\mathbb{1}_A \otimes P_{B,\mu}\right)\rho_{AB}\left(\mathbb{1}_A \otimes P_{B,\mu}\right)\right) = \mathrm{Tr}\left(\left(\mathbb{1}_A \otimes P_{B,\mu}\right)\rho_{AB}\right) \\ &= \sum_i \mathrm{Tr}\left(\langle u_{\mu,i} |\rho_{AB}| u_{\mu,i}\rangle\right) = \sum_i \langle u_{\mu,i} |\rho_B| u_{\mu,i}\rangle = \sum_i p_{\mu,i} \end{aligned} \tag{2.1}$$

and

$$\left(\mathbb{1}_A \otimes P_{B,\mu}\right)\rho_{AB}\left(\mathbb{1}_A \otimes P_{B,\mu}\right) = \sum_{i,i'} \left\langle u_{\mu,i} |\rho_{AB}| u_{\mu,i'}\right\rangle \otimes |u_{\mu,i}\rangle\langle u_{\mu,i'}|. \tag{2.2}$$

It follows from Eq. (**??**) and Eq. (**??**) that

$$p_\mu \rho_{A,\mu} = \sum_i p_{\mu,i}\rho_{A,\mu,i}.$$

Therefore, by the concavity of von Neumann entropy, we have

$$p_\mu \mathrm{S}(\rho_{A,\mu}) \geqslant \sum_i p_{\mu,i}\mathrm{S}(\rho_{A,\mu,i}).$$

So,

$$\sum_\mu p_\mu \mathrm{S}(\rho_{A,\mu}) \geqslant \sum_\mu \sum_i p_{\mu,i}\mathrm{S}(\rho_{A,\mu,i}).$$

Thus, the desired inequality is obtained.

(iii) Now we prove the theorem generally. By the Naimark theorem [?], there exists a quantum system $\mathcal{H}_C$, a unit vector $|0_C\rangle \in \mathcal{H}_C$ and a projector $\{P_{BC,\mu}\}$ on the bipartite system $\mathcal{H}_B \otimes \mathcal{H}_C$ such that $\langle 0_C | P_{BC,\mu} | 0_C \rangle = M_{B,\mu}^\dagger M_{B,\mu}$. Thus,

$$\begin{aligned} p_\mu \rho_{A,\mu} &= \mathrm{Tr}_B \left(\left(\mathbb{1}_A \otimes \sqrt{M_{B,\mu}^\dagger M_{B,\mu}} \right) \rho_{AB} \left(\mathbb{1}_A \otimes \sqrt{M_{B,\mu}^\dagger M_{B,\mu}} \right) \right) \\ &= \mathrm{Tr}_{BC} \left(\left(\mathbb{1}_A \otimes P_{BC,\mu} \right) \left(\rho_{AB} \otimes |0\rangle\langle 0|_C \right) \left(\mathbb{1}_A \otimes P_{BC,\mu} \right) \right). \end{aligned}$$

So, the quantum ensemble $\{(p_\mu, \rho_{A,\mu})\}$ which is induced by the quantum operation Φ_B can be considered as one which is induced by the quantum operation $\Phi_{BC} = \sum_\mu \mathrm{Ad}_{P_{BC,\mu}}$ over $\mathcal{H}_B \otimes \mathcal{H}_C$. Thus, it follows from (ii) that

$$\chi\{(p_\mu, \rho_{A,\mu})\} \leqslant \mathrm{S}(\rho_B \otimes |0\rangle\langle 0|_C) = \mathrm{S}(\rho_B).$$

3 The conjecture of Fannes, de Melo, Roga and Życzkowski

Let $\mathcal{E}_N = \{(p_i, \rho_i)\}_{i=1}^N$ be a quantum ensemble on a finite dimensional quantum system $\mathcal{H}$, $\mathrm{F}_{ij} = \mathrm{F}(\rho_i, \rho_j) = \left(\mathrm{Tr}\left(\left|\sqrt{\rho}\sqrt{\sigma}\right|\right)\right)^2$ be the fidelity between ρ_i and ρ_j. The matrix

$$\mathrm{C}_{\sqrt{\mathrm{F}}}(\mathcal{E}_N) = \left[\sqrt{p_i p_j \mathrm{F}_{ij}}\right]_{ij}$$

is said to be a *correlation matrix* of the quantum ensemble $\mathcal{E}_N = \{(p_i, \rho_i)\}_{i=1}^N$.

For $N = 2$ or 3, the correlation matrix $\mathrm{C}_{\sqrt{\mathrm{F}}}(\mathcal{E}_N) = \left[\sqrt{p_i p_j \mathrm{F}_{ij}}\right]_{ij}$ is a legitimate state. However, if $N \geqslant 4$, then $\mathrm{C}_{\sqrt{\mathrm{F}}}(\mathcal{E}_N) = \left[\sqrt{p_i p_j \mathrm{F}_{ij}}\right]_{ij}$ fails to be a positive semi-definite matrix in general [?]. For $N = 2$, the correlation matrix

$$\mathrm{C}_{\sqrt{\mathrm{F}}}(\mathcal{E}_2) = \begin{bmatrix} p_1 & \sqrt{p_1 p_2 \mathrm{F}(\rho_1, \rho_2)} \\ \sqrt{p_1 p_2 \mathrm{F}(\rho_1, \rho_2)} & p_2 \end{bmatrix}$$

was shown to satisfy the following inequality [?]:

$$\chi(\mathcal{E}_2) \leqslant \mathrm{S}(\mathrm{C}_{\sqrt{\mathrm{F}}}(\mathcal{E}_2)).$$

Moreover, the upper bound $\mathrm{S}(\mathrm{C}_{\sqrt{\mathrm{F}}}(\mathcal{E}_2))$ is the tighter one in the above inequality.

Fannes, de Melo, Roga and Życzkowski conjectured that for $N = 3$, $\chi(\mathcal{E}_3) \leqslant \mathrm{S}(\mathrm{C}_{\sqrt{\mathrm{F}}}(\mathcal{E}_3))$ is also true [?].

In what follows, we apply Theorem **??** to answer partly the conjecture.

Lemma 3.1 ([?]). *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ be three finite dimensional complex Hilbert spaces. Then the block operator*

$$\begin{bmatrix} A & D & E \\ D^\dagger & B & F \\ E^\dagger & F^\dagger & C \end{bmatrix}$$

defined on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ is positive semi-definite if and only if the following statements are valid:

(i) $A \geqslant 0, B \geqslant 0, C \geqslant 0$;

(ii) there exist three contractive operators R_1, R_2 and R_3 such that $D = \sqrt{A}R_1\sqrt{B}, F = \sqrt{B}R_2\sqrt{C}$, and

$$E = \sqrt{A}R_1\mathrm{supp}(B)R_2\sqrt{C} + \sqrt{A - \sqrt{A}R_1\mathrm{supp}(B)R_1^\dagger\sqrt{A}}R_3\sqrt{C - \sqrt{C}R_2^\dagger\mathrm{supp}(B)R_2^\dagger\sqrt{C}},$$

where $\mathrm{supp}(B)$ *stands for the support projection of B.*

Lemma 3.2. *Let U, V and W be three unitary operators on finite dimensional complex Hilbert space $\mathcal{H}$ and $\mathbb{1}$ be the identity operator on $\mathcal{H}$. Then the operator*

$$\begin{bmatrix} \mathbb{1} & U & V \\ U^\dagger & \mathbb{1} & W \\ V^\dagger & W^\dagger & \mathbb{1} \end{bmatrix}$$

is positive semi-definite if and only if $V = UW$.

Proof. Taking $D = U, E = V, F = W$ and $A = B = C = \mathbb{1}$ in Lemma **??**, we have that $R_1 = U, R_2 = W, \mathrm{supp}(B) = \mathbb{1}$ and R_3 is a contractive operator. Moreover, $V = UW$. That is

$$\begin{bmatrix} \mathbb{1} & U & V \\ U^\dagger & \mathbb{1} & W \\ V^\dagger & W^\dagger & \mathbb{1} \end{bmatrix} \geqslant 0 \Longleftrightarrow V = UW.$$

□

Remark 3.3. The alternative proof of Lemma **??** may be given by Theorem 3.1 in [**?**].

Theorem 3.4. *Let $\mathcal{E}_3 = \{(p_1, \rho_1), (p_2, \rho_2), (p_3, \rho_3)\}$ be a quantum ensemble on the finite dimensional quantum system $\mathcal{H}$. It follows from the polar decomposition theorem that there exist three unitary operators V, U and W on $\mathcal{H}$ such that*

$$
\begin{aligned}
|\sqrt{\rho_2}\sqrt{\rho_1}| &= U\sqrt{\rho_2}\sqrt{\rho_1}, \\
|\sqrt{\rho_3}\sqrt{\rho_1}| &= V\sqrt{\rho_3}\sqrt{\rho_1}, \\
|\sqrt{\rho_3}\sqrt{\rho_2}| &= W\sqrt{\rho_3}\sqrt{\rho_2}.
\end{aligned}
$$

If $V = UW$, then

$$\chi(\mathcal{E}_3) \leqslant \mathrm{S}(\mathrm{C}_{\sqrt{\mathrm{F}}}(\mathcal{E}_3)).$$

Proof. By the conditions, it follows that

$$
\begin{aligned}
\mathrm{Tr}\left(\sqrt{\rho_1}U\sqrt{\rho_2}\right) &= \sqrt{\mathrm{F}_{12}}, \\
\mathrm{Tr}\left(\sqrt{\rho_1}V\sqrt{\rho_3}\right) &= \sqrt{\mathrm{F}_{13}}, \\
\mathrm{Tr}\left(\sqrt{\rho_2}W\sqrt{\rho_3}\right) &= \sqrt{\mathrm{F}_{23}}.
\end{aligned}
$$

Let $\mathcal{H}_A = \mathcal{H}$, $\mathcal{H}_B = \mathbb{C}^3$, and

$$\rho_{AB} = \begin{bmatrix} p_1\rho_1 & \sqrt{p_1p_2}\sqrt{\rho_1}U\sqrt{\rho_2} & \sqrt{p_1p_3}\sqrt{\rho_1}V\sqrt{\rho_3} \\ \sqrt{p_1p_2}\sqrt{\rho_2}U^\dagger\sqrt{\rho_1} & p_2\rho_2 & \sqrt{p_2p_3}\sqrt{\rho_2}W\sqrt{\rho_3} \\ \sqrt{p_1p_3}\sqrt{\rho_3}V^\dagger\sqrt{\rho_1} & \sqrt{p_2p_3}\sqrt{\rho_3}W^\dagger\sqrt{\rho_2} & p_3\rho_3 \end{bmatrix}.$$

Now, we only need to show that ρ_{AB} is a positive semi-definite operator on $\mathcal{H}_A \otimes \mathcal{H}_B$. Note that

$$\rho_{AB} = \begin{bmatrix} \sqrt{p_1\rho_1} & 0 & 0 \\ 0 & \sqrt{p_2\rho_2} & 0 \\ 0 & 0 & \sqrt{p_3\rho_3} \end{bmatrix} \begin{bmatrix} \mathbb{1} & U & V \\ U^\dagger & \mathbb{1} & W \\ V^\dagger & W^\dagger & \mathbb{1} \end{bmatrix} \begin{bmatrix} \sqrt{p_1\rho_1} & 0 & 0 \\ 0 & \sqrt{p_2\rho_2} & 0 \\ 0 & 0 & \sqrt{p_3\rho_3} \end{bmatrix},$$

and it follows from Lemma **??** that $\rho_{AB} \geqslant 0$ is equivalent to

$$\begin{bmatrix} \mathbb{1} & U & V \\ U^\dagger & \mathbb{1} & W \\ V^\dagger & W^\dagger & \mathbb{1} \end{bmatrix} \geqslant 0 \Longleftrightarrow V = UW.$$

Moreover, it is easy to show that

$$\rho_A = \mathrm{Tr}_B\left(\rho_{AB}\right) = \sum_{i=1}^{3} p_i\rho_i, \quad \rho_B = \mathrm{C}_{\sqrt{\mathrm{F}}}(\mathcal{E}_3).$$

Since $\dim(\mathcal{H}_B) = 3$, take a standard orthogonal basis $\{|\mu_B\rangle\}$ of $\mathcal{H}_B$ such that $p_\mu \rho_{A,\mu} = \langle \mu_B |\rho_{AB}| \mu_B\rangle$. By Theorem **??**, we have

$$\chi(\mathcal{E}_3) = \chi\{(p_\mu, \rho_{A,\mu})\} \leqslant \mathrm{S}(\rho_B) = \mathrm{S}(\mathrm{C}_{\sqrt{\mathrm{F}}}(\mathcal{E}_3)).$$

This completes the proof. □

Remark 3.5. In fact, Lemma **??** can be easily generalized to the case where 3-by-3 block matrix is replaced by K-by-K ($K \geqslant 3$) block matrix of unitary entries. The generalization is described as follows:

Assume that the following $K \times K$ block matrix of unitary entries is positive semi-definite:

$$\begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1K} \\ U_{21} & U_{22} & \cdots & U_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ U_{K1} & U_{K2} & \cdots & U_{KK} \end{bmatrix} \equiv P \geqslant 0.$$

Then these unitary operators satisfy the conditions:

- $U_{ii} = \mathbb{1}$ for each index i; $U_{ji} = U_{ij}^\dagger$ for all indices i, j.

Thus

$$P = \begin{bmatrix} \mathbb{1} & U_{12} & \cdots & U_{1K} \\ U_{12}^\dagger & \mathbb{1} & \cdots & U_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ U_{1K}^\dagger & U_{2K}^\dagger & \dots & \mathbb{1} \end{bmatrix}.$$

Furthermore, we have that P is of the following forms:

(a)

$$P = \begin{bmatrix} \mathbb{1} & U_1 & U_1U_2 & U_1U_2U_3 & \dots & \dots & U_1U_2\cdots U_{K-1} \\ U_1^\dagger & \mathbb{1} & U_2 & U_2U_3 & U_2U_3U_4 & \ddots & \vdots \\ U_2^\dagger U_1^\dagger & U_2^\dagger & \mathbb{1} & U_3 & U_3U_4 & \ddots & \vdots \\ U_3^\dagger U_2^\dagger U_1^\dagger & U_3^\dagger U_2^\dagger & U_3^\dagger & \mathbb{1} & \ddots & \ddots & U_{K-3}U_{K-2}U_{K-1} \\ \vdots & \ddots & \ddots & \dots & \ddots & \ddots & U_{K-2}U_{K-1} \\ \vdots & \ddots & \ddots & \dots & \ddots & \ddots & U_{K-1} \\ U_{K-1}^\dagger\cdots U_2^\dagger U_1^\dagger & \dots & \dots & U_{K-1}^\dagger U_{K-2}^\dagger U_{K-3}^\dagger & U_{K-1}^\dagger U_{K-2}^\dagger & U_{K-1}^\dagger & \mathbb{1} \end{bmatrix}$$

for a collection of unitary operators $\{U_i : i = 1, \dots, K-1\}$ on $\mathcal{H}$, or

(b)

$$P = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_K \end{bmatrix} \begin{bmatrix} V_1^\dagger & V_2^\dagger & \cdots & V_K^\dagger \end{bmatrix}$$

for a collection of unitary operators $\{V_i : i = 1, \ldots, K\}$ on $\mathcal{H}$.

The outline of the proof is the following. The fact that P is of the form (a) can be easily derived by applying repeatedly the Theorem 3.1 in **[?]** to a block matrix. Indeed, we first apply it to the new block matrix:

$$\begin{bmatrix} \mathbb{1} & X \\ X^\dagger & A \end{bmatrix} \geqslant 0,$$

where $X = [U_{12}, \ldots, U_{1K}]$ and

$$A = \begin{bmatrix} \mathbb{1} & U_{23} & \cdots & U_{2K} \\ U_{23}^\dagger & \mathbb{1} & \cdots & U_{3K} \\ \vdots & \vdots & \ddots & \vdots \\ U_{2K}^\dagger & U_{3K}^\dagger & \cdots & \mathbb{1} \end{bmatrix}.$$

Then apply it again to a similar block structure for A, and so on. Finally we obtain the form (a) of P. The forms (a) and (b) are equivalent via the following identification:

$$U_1 = V_1 V_2^\dagger, U_2 = V_2 V_3^\dagger, \ldots, U_{K-1} = V_{K-1} V_K^\dagger.$$

4 Concluding remarks

In this paper, we obtained an universal upper bound for the Holevo quantity which is induced by a quantum operation and proved that for a given quantum ensemble which consists of N quantum states on the same space, a so-called correlation matrix $C_{\sqrt{F}}(\mathcal{E}_N)$ can be constructed. Its von Neumann entropy is shown to be a upper bound of the Holevo quantity for $N = 3$ under some constraints. We also generalized Lemma **??** and obtained an interesting characterization of positivity of special operator matrix, which may shed new light on solving other related problems in quantum information theory.

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