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by

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Abstract. The solvable Bianchi I-VII groups which arise as homogeneity groups in cosmological models are analyzed in a uniform manner. The dual spaces (the equivalence classes of unitary irreducible representations) of these groups are computed explicitly. It is shown how parameterizations of the dual spaces can be chosen to obtain explicit Plancherel formulas. The Laplace operator Δ arising from an arbitrary left invariant Riemannian metric on the group is considered, and its spectrum and eigenfunctions are given explicitly in terms of that metric. The spectral Fourier transform is given by means of the eigenfunction expansion of Δ. The adjoint action of the group automorphisms on the dual spaces is considered. It is shown that Bianchi I-VII type cosmological spacetimes are well suited for mode decomposition. The example of the mode decomposed Klein-Gordon field on these spacetimes is demonstrated as an application.

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1. Introduction

In light of the apparent anisotropy of the cosmic background radiation map (CMB) obtained by space missions, the investigation of anisotropic cosmological models seems to gradually regain scientific interest almost forty years after its peak in seventies [1],[2]. These models are described by a three dimensional homogeneity group and a trivial or reduced isotropy group as compared with Friedman-Robertson-Walker (FRW) models. Of particular interest are different scenarios of the early epoch of the universe including also quantum effects. One theoretical paradigm that is adequate for describing such situations is quantum field theory in curved spacetimes (QFT in CST). This theory pertains mainly to mathematical physics and aims at a rigorous description of physical phenomena starting from an axiomatic basis. The way from the axioms to explicit results readily applicable at the observational level usually requires a rich supply of mathematical methods. In the context of (anisotropic) cosmological models, harmonic analysis, and related, Fourier analysis with respect to the isometry group of the spacetime under consideration are powerful tools for obtaining explicit results.

There is a prevalent tendency among different scientific disciplines to specialize to their own aims and perspectives and thus diverge from each other. This results in an isolation of scientific results inside a limited community without a wide access from outside. Perhaps one of such fields is the mathematical theory of harmonic analysis. The Levi decomposition effectively breaks apart the general harmonic analysis to those for solvable and semisimple groups separately. Both branches have been investigated in great generality. The Kirillov orbit theory for solvable groups, along with such developments as the Currey theory for exponential solvable Lie groups, has given methods for obtaining explicit results for arbitrary dimensions. On the other hand the theory developed by Harish-Chandra, Helgason and others provides a very deep insight into general semisimple homogeneous spaces and related structures. At the same time, abstract harmonic analysis for locally compact groups, and the Mackey machine based upon it, yielded remarkably in describing general principles and phenomena. However, when a theoretical cosmologist wants to perform a mode decomposition of some physical field on a homogeneous spacetime it is almost of no use to him to know that, for instance, a Borel measure exists, or that it can be computed up to equivalence for N dimensions. What a cosmologist really needs is an explicit description with formulas which can be used without expert knowledge in harmonic analysis, as it is available for the traditional Abelian group $\mathbb{R}^n$. But as far as we were able to see, results of that kind do not seem to be available in the literature; possibly because they are usually not in the focus of interest of mathematicians. It may be regarded as one of the duties of mathematical physics to provide bridges between the increasingly diverging interests of mathematics and physics. With this in mind we take to the task of giving an explicit description of harmonic analysis of a number of groups of cosmological interest.

As stated above, in quantum field theory on cosmological spacetimes one is often interested in Fourier analysis with respect to the symmetries of the underlying geometry.

In the framework of homogeneous cosmologies, $M = \mathcal{I} \times \Sigma$ where $\mathcal{I}$ is an open interval, and Σ is a smooth three dimensional manifold with a Riemannian metric h on it. The spacetime metric g on M takes the line element form

$$ds_g^2 = dt^2 - a_{ij}(t) ds_h^i ds_h^j$$

with a positive definite matrix $a_{ij}(t)$ depending smoothly on the "time coordinate" $t \in \mathcal{I}$. (Σ, h) is a homogeneous space with respect to a Lie group of isometries G. The variety of all such homogeneous spaces which arise in cosmology can be found in several works including [3],[4],[5]. We have analyzed the geometrical setup of general linear hyperbolic fields on cosmological spacetimes in [6]. In particular we have seen that in nearly all cases one deals with a semidirect product group $G = \Sigma \rtimes \mathcal{O}$ where $\mathcal{O}$ is either of $SO(3)$, $SO(2)$, $\{1\}$, and the role of Σ is played by Bianchi I-IX groups $Bi(N)$ $(N = I, II, ..., IX)$ and their quotients $Bi(N)/\Gamma$ by discrete normal subgroups Γ. Such structures are called *semidirect homogeneous spaces*, and a few important results have been obtained in [6] in this generality using abstract harmonic analysis. We moreover have seen how tightly harmonic and spectral analysis is related to mode decomposition. The next step towards physics will be to describe harmonic analysis of all those possible semidirect homogeneous spaces explicitly. The spaces of maximal symmetry with $\mathcal{O} = SO(3)$ are the FRW spaces, which are described by isometry groups $SO(4)$, $E(3)$ or $SO^+(1,3)$. The spaces with one rotational symmetry are described by $\mathcal{O} = SO(2)$ and are called LRS (locally rotationally symmetric) spaces. And finally the purely homogeneous spaces are given by trivial isotropy groups $\mathcal{O} = \{1\}$. The isometry groups of FRW spaces are classical groups and their harmonic analysis is also a classical subject. For purely homogeneous spaces (otherwise called Bianchi spaces) this is known partially. The Bianchi I group is the additive group $\mathbb{R}^3$ of which harmonic analysis is textbook standard. The Bianchi II group is the famous Heisenberg group of dimension $2+1$, which is well studied, and its harmonic analysis can be found in [7],[8],[9]. The Bianchi III group is the $ax+b$ group in 2+1 dimensions whose harmonic analysis is known as well [9]. The Bianchi VIII group is the universal covering group $\widetilde{SL}(2,\mathbb{R})$, which is a notorious non-linear group. Its harmonic analysis can be found in [10]. The last group, Bianchi IX, is simply $SU(2)$ which is again classical. Little is known about the Bianchi IV-VII groups beyond the structure of their Lie algebras which are semidirect products of Abelian algebras $\mathbb{R}^2$ and $\mathbb{R}$. In fact, although there are principally no obstacles on the way of their investigation, we were not able to locate any explicit description of their harmonic analysis in the literature; we also asked some prominent experts in the field, and none was able to point to such references. Even less is known about the semidirect products of Bianchi groups with $SO(2)$ describing LRS models. We stress again that there is no obstacle to applying the Mackey machine and perform all calculations, but it seems that this has not been done so far. Because the discrete subgroups Γ can be readily found from the group structure [11], once having control over harmonic analysis of the group G it is not hard to reduce it to the quotient G/Γ, but it again needs to be done somewhere. We have chosen to start with harmonic analysis of the solvable

Bianchi I-VII groups in a uniform manner deferring the remaining structures to the future. One could argue that this might be done straightforwardly by the exponential solvable methods of [12], but if one actually starts to do that (what we indeed did) one needs to construct an enormous amount of spaces, dual spaces and intersections, which are designed to handle arbitrary groups, and seem to be too bulky to be performed by hand. Therefore we preferred the original Mackey construction.

The harmonic analytical Fourier transform is only half the way to applications. In fact what is really useful in concrete computations is the spectral Fourier transform given by the eigenfunction expansion of a left invariant Laplace operator Δ acting on the sections in the subbundle of $\mathcal{T}$ over Σ. In [6] we have seen how the mode decomposition of linear fields can be performed knowing the eigenfunction expansion of Δ. We moreover have seen how in principle one can identify the harmonic analytical Fourier transform with the spectral Fourier transform of Δ to translate the general results on semidirect spaces into the language of concrete calculations. For this purpose one needs to find the spectral theory of Δ explicitly for all possible Σ and invariant Riemannian metrics on them. This is again something that can hardly be found in the literature, although spectral analysis in Riemannian spaces is huge and very well developed a subject in mathematics. In particular, one needs to know the spectrum and a complete system of eigenfunctions of Δ explicitly. In general, the eigenfunction problem of Δ is a vector valued elliptic partial differential equation on a manifold without boundary, which is difficult to compute even numerically. If this equation admits separation of variables so that the eigenfunctions are given by combinations of functions of one variable subject to ordinary differential equations then we can consider these eigenfunctions as given explicitly in terms of special functions. In this work we will give such an explicit description of the spectrum and eigenfunctions of Δ in terms of an arbitrary left invariant Riemannian metric on Σ for the line bundle over Bianchi I-VII groups. For arbitrary bundle dimension this is much more complicated. There is a bit of hope to obtain explicit solutions by transforming the original vector valued eigenfunction equation on the manifold to a scalar elliptic eigenfunction equation with constraints on the holonomy bundle of the linear connection associated with the given fiber metric. This is a non-trivial task to which we hope to return in the future.

We summarize the content of the current exposition as follows. First the Bianchi I-VII groups are realized as semidirect products of $\mathbb{R}^2$ and $\mathbb{R}$ and the main group properties are explicitly computed, such as the multiplication laws, exponential maps, modular functions and adjoint representations. Then the dual spaces of the groups are constructed, i.e., the equivalence classes of unitary irreducible representations. This is done by means of the Mackey machine. Next a look is given at the co-adjoint orbits of the groups in the sense of the Kirillov theory, and it is described explicitly how the cross sections can be chosen to parameterize the dual space. Afterwards an explicit Plancherel formula is given for all these groups. Thereafter we turn to spectral analysis. The spectra and the eigenfunctions of Δ are found explicitly in terms of the chosen arbitrary left invariant Riemannian metric. Then it is shown that these eigenfunctions are complete in L^2 and give rise to a conventional Fourier transform in sense of [6]. In the final part the applications in quantum field theory are discussed. First it is shown that Bianchi spacetimes are ideally adapted for the mode decomposition as given in [6]. Then this mode decomposition is demonstrated in the example of the Klein-Gordon field on Bianchi I-VII spacetimes. A number of interesting consequences are indicated including those for the quantum Klein-Gordon field, where results from [13] are used as well.

2. Semidirect structure of Bianchi I-VII groups

As a first step we will try to explicitly realize the solvable Bianchi II-VII groups (I is Abelian and will serve as a starting point in the analysis of others) as semidirect products of Abelian subgroups. A classification of solvable real Lie algebras with respect to such products can be inferred from [14].

Semidirect products of Lie algebras and Lie groups. We start by recalling some definitions. Let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras, and let $D(\mathfrak{a})$ be the Lie algebra of derivations on $\mathfrak{a}$. Let further $f : \mathfrak{b} \mapsto D(\mathfrak{a})$ be a Lie algebra homomorphism. The *semidirect product Lie algebra* $\mathfrak{a} \times_f \mathfrak{b}$ is the algebra modelled on $\mathfrak{a} \oplus \mathfrak{b}$ with the Lie bracket

$$[(a, b), (a', b')] = ([a, a'] + f(b)a' - f(b')a, [b, b']),\ (a, b), (a', b') \in \mathfrak{a} \oplus \mathfrak{b}.$$

Let, on the other hand, A and B be Lie groups, and $F : B \mapsto Aut(A)$ a Lie group homomorphism ($Aut(A)$ embedded into $GL(A)$). The *semidirect product* $A \times_F B$ of groups A and B is defined as the Lie group modelled on the product manifold $A \times B$ with the multiplication

$$(a, b)(a', b') = (aF(b)a', bb'),\ (a, b), (a', b') \in A \times B.$$

Following the notation of [15], denote by $F^\circ : B \mapsto Aut(\mathfrak{a})$ the map $B \ni b \mapsto d[F(b)] \in Aut(\mathfrak{a})$, where $\mathfrak{a}$ is the Lie algebra of A. Then the derivative of this map, $f = dF^\circ$, will be a Lie algebra homomorphism $f : \mathfrak{b} \mapsto D(\mathfrak{a})$ ($\mathfrak{b}$ the Lie algebra of B), and the Lie algebra of the direct product Lie group $A \times_F B$ is the direct product Lie algebra $\mathfrak{a} \times_f \mathfrak{b}$ [15].

Bianchi I-VII groups as semidirect products. With this in mind, let us start with realizing Bianchi I-VII algebras as semidirect product algebras $\mathfrak{g} = \mathbb{R}^2 \times_f \mathbb{R}$ with some Lie algebra homomorphism $f : \mathbb{R} \mapsto D(\mathbb{R}^2)$. This correspondence between Bianchi algebras and homomorphisms f can be obtained by combination of [16] and [14]. (Those uncomfortable with Russian may simply perform the semidirect product construction and check the commutation relations.) Namely, in each case $f(r) = r \cdot M$, $r \in \mathbb{R}$, in a suitable basis, where M is a 2×2 matrix. The matrix M for each algebra is given in Table 1 below.

The corresponding integral homomorphisms F° will be the exponentials $F^\circ(r) = e^{rM}$ (note that the exponential map on the group $\mathbb{R}$ is given by the identity map). If a

I	II	III	IV	V	VI	VIII
0	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}$ $-1 < q \leq 1$	$\begin{pmatrix} p & -1 \\ 1 & p \end{pmatrix}$ $p \geq 0$

Table 1. The matrices M for Bianchi I-VII groups

I	II	III	IV	V	VI	VIII
1	$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$	$\begin{pmatrix} e^r & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} e^r & 0 \\ re^r & e^r \end{pmatrix}$	$\begin{pmatrix} e^r & 0 \\ 0 & e^r \end{pmatrix}$	$\begin{pmatrix} e^r & 0 \\ 0 & e^{-qr} \end{pmatrix}$ $-1 < q \leq 1$	$\begin{pmatrix} e^{pr}\cos(r) & -e^{pr}\sin(r) \\ e^{pr}\sin(r) & e^{pr}\cos(r) \end{pmatrix}$ $p \geq 0$

Table 2. The matrices $F(r)$ for Bianchi I-VII groups

diffeomorphism is given locally by a linear coordinate map, $x_i' = A_i^j x_j$ with the matrix A, then its differential will be given by the same matrix A. Now that $F^\circ(r) = d[F(r)]$ and that $F(r)$ are linear automorphisms, it follows that $F(r) = e^{rM}$. Thus all Bianchi groups I-VII are given by semidirect products $G = \mathbb{R}^2 \times_F \mathbb{R}$, where for each class the group homomorphism $F : \mathbb{R} \mapsto Aut(\mathbb{R}^2)$ is given as in Table 2 above.

We appoint to use capital symbols X, Y, Z for Lie algebra coordinates and small symbols x, y, z for Lie group coordinates, but these may interfere in some calculations involving exponential maps. It follows that the group multiplication is

$$(x, y, z)(x', y', z') = ((x, y) + F(z)(x', y'), z + z'),\ (x, y, z), (x', y', z') \in G = \mathbb{R}^2 \times_F \mathbb{R}.$$

The exponential map. Finally we note that all 7 groups are exponential, and the exponential map is given as follows. Let $(X, Y, Z) \in \mathfrak{g} = \mathbb{R}^2 \times_f \mathbb{R}$ with $(X, Y) \in \mathbb{R}^2$ and $Z \in \mathbb{R}$. We use the Zassenhaus formula

$$\exp(A + B) = \exp(A)\exp(B)\exp(C_2)\exp(C_3)...,$$

where the coefficients C_m are homogeneous Lie algebra elements composed of nested commutators of order m. We will use the convenient method of obtaining C_m recursively as given in [17]. If we set $A = (X, Y, 0)$ and $B = (0, 0, Z)$, we obtain

$$[A, B] = -f(Z)A.$$

Now equating the homogeneous summands of any order of (4.7) and (4.8) of [17], we obtain recursion formulas for C_m which are bulky in general. However, trying an ansatz $C_m = \alpha_m(-f)^{m-1}(Z)A$, $\alpha_m \in \mathbb{R}$, and checking directly for $m = 2$, one can easily prove it inductively, and find

$$\alpha_m = \frac{1 - m}{m!}.$$

It remains to calculate

$$\exp(C_2)\exp(C_3)... = \exp\left(\sum_{m=1}^{\infty} \frac{1 - m}{m!}(-f)^{m-1}(Z)A\right).$$

If $f(Z)$ is invertible for all Z then we write

$$\frac{1-m}{m!}(-f)^{m-1}(Z) = (-f)^{-1}(Z)\frac{(-f)^m(Z)}{m!} - \frac{(-f)^{m-1}(Z)}{(m-1)!},$$

and obtain

$$D(Z) \doteq \sum_{m=1}^{\infty} \frac{1-m}{m!}(-f)^{m-1}(Z) = (-f)^{-1}(Z)\left(e^{-f(Z)}-1\right) - e^{-f(Z)} =$$
$$= f^{-1}(Z)\,(1-F(-Z)) - F(-Z). \tag{1}$$

It is only for Bianchi II and III that $f(Z)$ is degenerate, and for these two we can compute directly

$$D(Z) = \sum_{m=1}^{\infty} \frac{1-m}{m!}(-f)^{m-1}(Z) = \frac{1}{2}f(Z) \qquad \text{for Bianchi II}$$

and

$$D(Z) = \sum_{m=1}^{\infty} \frac{1-m}{m!}(-f)^{m-1}(Z) = (1-2e^{-1})f(Z) \qquad \text{for Bianchi III.}$$

Thus we arrive at

$$\exp((X,Y,0)+(0,0,Z)) = \exp((X,Y,0))\exp((0,0,Z))\exp(D(Z)(X,Y),0).$$

The exponential maps of $\mathbb{R}^2$ and $\mathbb{R}$ are the identity maps, therefore

$$(x,y,z) = \exp((X,Y,Z)) = (X,Y,Z)(D(Z)(X,Y),0) = ([1+F(Z)D(Z)](X,Y),Z),$$

where $F(Z)$ should be understood as $F(\exp(Z))$. The matrices $D(Z)$ appear somewhat bulky so we refrain from presenting them in a table.

The adjoint representations Ad **and** ad**.** Let $(g_x,g_y,g_z),(x,y,z) \in G$. Their conjugation $(x',y',z') = (g_x,g_y,g_z)(x,y,z)(g_x,g_y,g_z)^{-1}$ is given by

$$(x',y',z') = ((1-F(z))(g_x,g_y) + F(g_z)(x,y),Z).$$

The adjoint representation Ad is the differential of this map at the identity element $(x,y,z)=(0,0,0)$, and so it is given by the matrix field Ad_g,

$$Ad_g = \begin{pmatrix} F(g_z) & & -F'(0)(g_x,g_y)^\top \\ & & \\ 0 \quad 0 & & 1 \end{pmatrix}.$$

The adjoint representation of the Lie algebra is given by the matrix‡

$$ad_{(X,Y,Z)} = \begin{pmatrix} f(Z) & & -f'(0)(X,Y)^\top \\ & & \\ 0 \quad 0 & & 0 \end{pmatrix}.$$

‡ We use the general relation $ad_{\mathsf{X}}\mathsf{Y}=[\mathsf{X},\mathsf{Y}]$ for elements X, Y in a general Lie algebra.

The Haar measure and the modular function. The Haar measure on the Lie group is given by

$$d(exp(X,Y,Z)) = j(X,Y,Z)dXdYdZ,$$

where

$$j(X,Y,Z) = \mathfrak{h} \det \frac{1-e^{-ad_{(X,Y,Z)}}}{ad_{(X,Y,Z)}},\ (X,Y,Z) \in \mathfrak{g},$$

and $0 < \mathfrak{h} \in \mathbb{R}$ is an arbitrary constant. In group coordinates one can check that it is given by

$$dg = \mathfrak{h} \det F(-g_z) dg_z dg_y dg_z,\ (g_x, g_y, g_z) \in G.$$

The groups are all non-compact, so there is no preferred normalization for the constant $\mathfrak{h}$. Later it will be determined as related to the chosen left invariant Riemannian metric on G. The modular function $\Delta(g) = \det Ad_g^{-1}$ can be readily seen to be $\Delta(g) = \det F(-g_z)$.

This temporarily completes our task of analyzing the Bianchi I-VII groups as semidirect products. In the next section we will concentrate on their dual spaces.

3. The irreducible representations of Bianchi I-VII groups

In this section we will try to find the dual spaces of Bianchi I-VII groups using the Mackey procedure. Let us start with Bianchi I, which is simply the additive group $\mathbb{R}^3$. Its dual group $\hat{\mathbb{R}}^3$ is homeomorphic to itself, $\hat{\mathbb{R}}^3 = \mathbb{R}^3$, and the irreducible 1-dimensional representations are given by

$$\xi_{\vec{k}}(\vec{x}) = e^{i\{\vec{k},\vec{x}\}},\ \vec{x} \in \mathbb{R}^3,\ \vec{k} \in \hat{\mathbb{R}}^3 = \mathbb{R}^3,$$

where we appoint to denote by $\{\vec{a}, \vec{b}\}$ the usual Euclidean product of three-vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$. These scalar functions $\xi_{\vec{k}}$ can be viewed as unitary operator valued functions acting on the one complex dimensional Hilbert space $\mathbb{C}$.

The Mackey procedure for normal Abelian subgroups. We cite here the setup of the Mackey theory for groups with a normal Abelian subgroup as given in [9]. Let G be a locally compact group and N an Abelian normal subgroup. Then G acts on N by conjugation, and this induces an action of G on the dual group $\hat{N}$ defined by

$$g\nu(n) = \nu(g^{-1}ng),\ g \in G,\ \nu \in \hat{N},\ n \in N.$$

For each $\nu \in \hat{N}$, we denote by G_ν the *stabilizer* of ν,

$$G_\nu = \{g \in G\colon\ g\nu = \nu\},$$

which is a closed subgroup of G, and we denote by $\mathcal{O}_\nu$ the *orbit* of ν:

$$\mathcal{O}_\nu = \{g\nu\colon\ g \in G\}.$$

The action of G on $\hat{N}$ is said to be *regular* if some conditions are satisfied. To avoid presenting excessive information we only mention that if G is second countable (which is true for a Lie group), then the condition for a regular action is equivalent to the following: for each $\nu \in \hat{N}$, the natural map $gG_\nu \mapsto g\nu$ from G/G_ν to $\mathcal{O}_\nu$ is a homeomorphism. In our case $\hat{N}$ is a smooth manifold, and the group actions are all smooth, hence this map is not only a homeomorphism but even a diffeomorphism. The constructions become simpler under the assumption that G is a semidirect product of N and the factor group $H = G/N$. We define the *little group* H_ν of $\nu \in \hat{N}$ to be $H_\nu = G_\nu \cap H$. Now we cite a beautiful theorem which appears as **Theorem 6.42** in [9] and expresses the essence of the Mackey procedure. The functor **Ind** and the inducing construction are briefly introduced in the **Appendix A**.

Theorem 1 (Folland,[9]) *Suppose $G = N \ltimes H$, where N is Abelian and G acts regularly on $\hat{N}$. If $\nu \in \hat{N}$ and ρ is an irreducible representation of H_ν, then $\mathbf{Ind}_{G_\nu}^{G}(\nu\rho)$ is an irreducible representation of G, and every irreducible representation of G is equivalent to one of this form. Moreover, $\mathbf{Ind}_{G_\nu}^{G}(\nu\rho)$ and $\mathbf{Ind}_{G_\nu}^{G}(\nu'\rho')$ are equivalent if and only if ν and ν' belong to the same orbit, say $\nu' = g\nu$, and $h \mapsto \rho(h)$ and $h \mapsto \rho'(g^{-1}hg)$ are equivalent representations of H_ν.*

Application to the Bianchi groups. It is easy to see that Bianchi groups II-VII satisfy the assumptions of the theorem. In this case $N = \mathbb{R}^2$ and $H = \mathbb{R}$, the dual of N is $\hat{N} = \mathbb{R}^2$ and is given by

$$\hat{N} = \{e^{i\{\breve{k},\breve{x}\}}\colon\, \breve{x}, \breve{k} \in \mathbb{R}^2\},$$

where we overload the notation by brackets $\{\breve{a}, \breve{b}\}$ to denote the two dimensional Euclidean product of $\breve{a}, \breve{b} \in \mathbb{R}^2$. Let $\imath_N : \mathbb{R}^2 \mapsto G$ be the natural inclusion. The action of G on $\hat{N}$ is given by

$$g\xi_{\breve{k}}(\breve{x}) = \xi_{\breve{k}}(\imath_N^{-1}(g^{-1}\imath_N(\breve{x})g)).$$

All Bianchi solvable groups are homeomorphic to $\mathbb{R}^3$, and we may choose a global chart on them. In particular we choose one adapted to the semidirect structure $\mathbb{R}^2 \times_F \mathbb{R}$ presented in the previous section. Then the multiplication law in G is given by

$$(x, y, z)(x', y', z') = ((x, y) + F(z)(x', y'), z + z').$$

The unit $e \in G$ is given by $e = (0, 0, 0)$, and the inverse map by

$$(x, y, z)^{-1} = (-F^{-1}(z)(x, y), -z).$$

In particular, if $(\breve{x}, 0) = (x, y, 0) \in \imath_N(\mathbb{R}^2)$ and $(g_x, g_y, g_z) \in G$, then

$$(g_x, g_y, g_z)^{-1}(x, y, 0)(g_x, g_y, g_z) = (F^{-1}(g_z)(x, y), 0),$$

that is, the conjugation map $n \mapsto g^{-1}ng$ is given by $(x, y) \mapsto F^{-1}(g_z)(x, y)$. Thus the action of G on $\hat{N}$ is

$$g\xi_{\breve{k}}(\breve{x}) = \xi_{\breve{k}}(F^{-1}(g_z)\breve{x}) = e^{i\{\breve{k},F^{-1}(g_z)\breve{x}\}} = e^{i\{F^{\perp}(g_z)\breve{k},\breve{x}\}},$$

where $F^{\perp}(g_z)$ is the inverse transpose of the matrix $F(g_z)$. This means that this action can be described by

$$g\breve{k} = F^{\perp}(g_z)\breve{k},\ g \in G,\ \breve{k} \in \mathbb{R}^2.$$

Denote by $V^0 \subset \mathbb{R}^2$ the eigenspace of $M^\top$ corresponding to the eigenvalue 0 (the null space). Then it will be also the joint eigenspace of the matrices $F^{\perp}(g_z) = e^{-g_z M^\top}$ corresponding to the eigenvalue 1 simultaneously for all $g_z \in \mathbb{R}$. Let us write the stabilizer condition,

$$e^{-g_z M^\top}\breve{k} = \breve{k}.$$

Then the stabilizer $G_{\breve{k}}$ and the little group $H_{\breve{k}}$ will be

$$G_{\breve{k}} = \imath_N(\mathbb{R}^2) \cdot H_{\breve{k}}$$

and

$$H_{\breve{k}} = \begin{cases} \mathbb{R} & \text{if } \breve{k} \in V^0, \\ \{0\} & \text{else.} \end{cases}$$

Define the following space of irreducible representations of G:

$$\hat{J} = (V^0 \times \mathbb{R}) \cup (\mathbb{R}^2 \setminus V^0).$$

For each $\mu \in \hat{J}$ the corresponding irreducible representation is given by

$$T_\mu(g) = e^{i\{\breve{k},\breve{g}\}} e^{ik_3 g_3},\ \mu = (\breve{k}, k_3) = (k_1, k_2, k_3)$$

if $\mu \in V^0 \times \mathbb{R}$, and

$$T_\mu = T_{\breve{k}} = \mathbf{Ind}^G_{\mathbb{R}^2}(e^{i\{\breve{k},.\}}),\ \mu = \breve{k},$$

if $\mu \in \mathbb{R}^2 \setminus V^0$. The orbit $\mathcal{O}_{\breve{k}}$ is $\{\breve{k}\}$ if $\breve{k} \in V^0$ and $F^{\perp}(\mathbb{R})\breve{k}$ otherwise. As mentioned in the theorem, two representations $\mu, \mu' \in \hat{J}$ are equivalent if and only if $\breve{k}$ and $\breve{k}'$ are on the same orbit, $\breve{k} = F^{\perp}(z)\breve{k}'$, and the corresponding representations of $H_{\breve{k}}$ and $H_{\breve{k}'}$ are equivalent when intertwined with the action of z. The first condition can be satisfied non-trivially if $\breve{k}, \breve{k}' \in \mathbb{R}^2 \setminus V^0$, but then $H_{\breve{k}} = H_{\breve{k}'} = \{0\}$, and thus there exists only the trivial representation $\rho = 1$. Thus representations $\mu, \mu' \in \mathbb{R}^2 \setminus V^0$ are equivalent if and only if they are on the same orbit. On the other hand, let $\mu, \mu' \in V^0 \times \mathbb{R}$ such that $\breve{k} = \breve{k}'$, and the first condition is satisfied trivially. Then $G_{\breve{k}} = G$, and $G/G_{\breve{k}} = \{1\}$, so the action of 1 cannot intertwine inequivalent representations of $H_{\breve{k}}$. Thus $\mu \sim \mu'$ means $\mu = \mu'$. Therefore the dual space $\hat{G}$ of G will be

$$\hat{G} = (V^0 \times \mathbb{R}) \cup (\mathbb{R}^2 \setminus V^0)/F^{\perp}(\mathbb{R}).$$

The null spaces V^0. Finally let us find the eigenspaces V^0 for different Bianchi groups. By a calculation of eigenvectors and eigenvalues of M we obtain

$$V^0_I = \mathbb{R}^2,\ V^0_{II} = \mathbb{R} \oplus \{0\},\ V^0_{III} = \{0\} \oplus \mathbb{R},$$

$$V^0_{IV} = \{0\},\ V^0_V = \{0\},\ V_{VII} = \{0\},$$

and

$$V_{VI} = \begin{cases} \{0\} \oplus \mathbb{R} & \text{if } q = 0, \\ \{0\} & \text{else.} \end{cases}$$

Note that as always with solvable groups, the irreducible representations are either 1-dimensional or infinite dimensional.

To obtain explicit descriptions of the dual groups $\hat{G}$ for each Bianchi class we have to calculate the orbits $\mathcal{O}_{\breve{k}} = F^{\perp}(\mathbb{R})\breve{k}$ explicitly, which is done in the next section. Note that the entire construction could have been performed through the machinery of exponential solvable Lie groups developed in [18] and [12], where the problem is treated exhaustively. In particular, it was shown that (as adapted to our terminology) there exists a cross section $\tilde{K}$, an algebraic submanifold of $\mathbb{R}^2$ which crosses each *generic orbit* (i.e., an orbit of maximal dimension) exactly once, and thus parameterizes the infinite dimensional representations. Having explicitly calculated $\tilde{K}$ we find $\hat{G} = (V^0 \times \mathbb{R}) \cup \tilde{K}$. But the methods of [18] are extremely general and involve simple but lengthy algebraic calculations; this is why we have preferred the original topological Mackey constructions.

4. Co-adjoint orbits of Bianchi II-VII groups

The term co-adjoint orbits would probably suit better to the solvable Lie theoretical method of orbits as established by Kirillov and accomplished by Currey. At this point we deviate to present a little digression demonstrating the equivalence of that approach with that we have adopted.

The Kirillov approach. The Lie algebra $\mathfrak{g} = \mathbb{R}^2 \times_f \mathbb{R}$ of G is modelled on the vector space $\mathbb{R}^3$, and as such its dual space $\mathfrak{g}'$ is again isomorphic to $\mathbb{R}^3$. We will fix this isomorphism by choosing the basis in $\mathfrak{g}'$ dual to our adapted basis of $\mathfrak{g}$. With this identification the co-adjoint action of G on $\mathfrak{g}' = \mathbb{R}^3$ is given by the matrix field $Ad^*_g = Ad^{\perp}_g$,

$$Ad^*_g = \begin{pmatrix} & F^{\perp}(g_z) & & 0 \\ & & & 0 \\ & (g_x, g_y)F^{\perp}(g_z)M^{\top} & & 1 \end{pmatrix}.$$

For any $\mathfrak{l} = (X^*, Y^*, Z^*) \in \mathfrak{g}'$ its orbit $\mathcal{O}_{\mathfrak{l}}$ is given by

$$\mathcal{O}_{\mathfrak{l}} = (F^{\perp}(\mathbb{R})(X^*, Y^*), (\mathbb{R}, \mathbb{R})F^{\perp}(\mathbb{R})M^{\top}(X^*, Y^*) + Z^*),$$

and the space of orbits $\{\mathcal{O}_{\mathfrak{l}}\}$ with the quotient topology induced from $\mathfrak{g}'$ is homeomorphic to $\hat{G}$ with the Fell topology [9]. One can easily see that the orbits are of two types: those of (X^*, Y^*, Z^*) with $(X^*, Y^*) \in V^0$ or $(X^*, Y^*) \notin V^0$. The former are the so called *degenerate* orbits with dimension 0 (singletons), and the latter are the *generic* orbits with maximal dimension 3. This is exactly the same result we obtained above by Mackey machine.

The generic orbits and the cross sections. We will denote the range of a parameterized quantity $Q(p)$ of a parameter $p \in P$ by $Q(P)$. For instance, $F^{\perp}(\mathbb{R})$ will denote the range of the quantity $F^{\perp}(r)$ when r runs over $\mathbb{R}$. Here we will try to find the generic orbits $F^{\perp}(\mathbb{R})\breve{k}_0 \in \hat{G}$ mentioned in the previous section and corresponding cross-sections $\tilde{K} \in \mathbb{R}^2$. The latter will be algebraic manifolds composed of one or more connected components. In all cases V^0 is a subset of Lebesgue measure 0 in $\mathbb{R}^2$. By the definition of the cross section $\tilde{K}$, the submanifold $\mathbb{R}^2 \setminus V^0$ can be parameterized by a global chart $\breve{k} = \breve{k}(k,r)$, $(k,r) \in \mathfrak{K} \times \mathbb{R}$, such that $\breve{k}(k,r) = F^{\perp}(r)\breve{k}_0(k)$ and $\breve{k}_0(k) = \breve{k}(k,0) \in \tilde{K}$. Under this diffeomorphism the Lebesgue measure $d\breve{k}$ becomes $\rho(k,r)dkdr$, where $\rho(k,r) = |\det \partial(\breve{k})/\partial(k,r)|$.

Now let us proceed to the determination of the orbits and the cross sections case by case. Figure 1 in **Appendix B** illustrates them qualitatively.

II. We have

$$F^{\perp}(r)(k_x, k_y) = (k_x - rk_y, k_y),$$

hence the orbit through $\breve{k} \in \mathbb{R}^2 \setminus V^0$ is $F^{\perp}(\mathbb{R})(k_x, k_y) = (\mathbb{R}, k_y)$. The cross section can be chosen to be $\tilde{K} = \breve{k}_0(\mathfrak{K})$, $\mathfrak{K} = \mathbb{R} \setminus \{0\}$, $\breve{k}_0(k) = (0,k)$. Indeed, any orbit $(\mathbb{R}, k_y)$ meets $\tilde{K}$ exactly once at $\breve{k}_0(k_y)$. Then

$$\rho(k,r) = \left|\det\left(F^{\perp}(r)\frac{\partial\breve{k}_0(k)}{\partial k}, \frac{\partial F^{\perp}(r)}{\partial r}\breve{k}_0(k)\right)\right| = |k|.$$

III. In this case

$$F^{\perp}(r)(k_x, k_y) = (e^{-r}k_x, k_y),$$

and the orbit through $\breve{k} \in \mathbb{R}^2 \setminus V^0$ is $F^{\perp}(\mathbb{R})(k_x, k_y) = (sgn(k_x)\mathbb{R}_+, k_y)$. Let $\mathfrak{K} = \mathbb{R} \times \{-1, 1\}$, $k = (k_1, k_2)$. The cross section is the image $(-1, \mathbb{R}) \cup (1, \mathbb{R})$ of the map $\breve{k}_0(k) = (k_2, k_1)$. We find

$$\rho(k,r) = e^{-r}.$$

IV. For this group

$$F^{\perp}(r)(k_x, k_y) = (e^{-r}k_x - re^{-1}k_y, e^{-r}k_y),$$

and the orbits are complicated. We set $\mathfrak{K} = \mathbb{R}_{+0} \times \{-1, 1\}$, $k = (k_1, k_2)$ and $\breve{k}_0(k) = (k_2, k_2k_1)$. That this is a cross section can be checked immediately. The measure density ρ is

$$\rho(k,r) = e^{-2r}(1 + k_1).$$

V. Now

$$F^{\perp}(r)(k_x, k_y) = e^{-r}(k_x, k_y),$$

and the orbits are simply the incoming radial rays. Set $\mathfrak{K} = \mathbb{R}/2\pi\mathbb{Z}$ and $\breve{k}_0(k) = (\cos(k), \sin(k))$. It follows that

$$\rho(k,r) = e^{-2r}.$$

VI. For this group we consider only the case $q \neq 0$ as $q = 0$ is just the group III.

$$F^{\perp}(r)(k_x, k_y) = (e^{-r}k_x, e^{qr}k_y),$$

and the orbits are incoming polynomial curves if $q < 0$ and hyperbolic curves if $q > 0$. For $q < 0$ set $\mathfrak{K} = \mathbb{R}/2\pi\mathbb{Z}$ and $\breve{k}_0(k) = (\cos(k), \sin(k))$. Then

$$\rho(k, r) = e^{-(1-q)r}(\cos^2(k) - q\sin^2(k)).$$

For $q > 0$ set $\mathfrak{K} = \mathbb{R}_{+0} \times \{0, 1, 2, 3\}$, $k = (k_1, k_2)$ and

$$\breve{k}_0(k) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{k_2} \begin{pmatrix} 1 \\ k_1 \end{pmatrix}.$$

Thus

$$\rho(k, r) = q^{k_2 \bmod 2} e^{-(1-q)r}.$$

VII. The co-adjoint action in this group is given by

$$F^{\perp}(r)(k_x, k_y) = e^{-pr}(k_x \cos r - k_y \sin r, k_x \sin r + k_y \cos r),$$

and the orbits are incoming or outgoing spirals depending on whether $p < 0$ or $p > 0$. We take $\mathfrak{K} = (-e^{\pi p}; -1] \cup [1, e^{\pi p})$ and $\breve{k}_0(k) = (k, 0)$. Each orbit clearly intersects $\tilde{K}$ exactly once. Finally

$$\rho(k, r) = e^{-2pr}|k|.$$

Note that in all cases we have chosen $\tilde{K}$ such that it possesses an involution $\breve{k}_0(-k) = -\breve{k}_0(k)$, which will be useful in later constructions. Of course, these choices of cross sections are not unique, neither need they correspond to those suggested by Currey theory. In fact, one may make any other choice for convenience and calculate the corresponding measure density ρ precisely as we did.

5. The explicit Plancherel formula for Bianchi II-VII groups

We will obtain the Plancherel measure by extending the idea suggested in [9] for Heisenberg groups to all solvable Bianchi groups. Namely, we will exploit the Euclidean Parseval equality on the homeomorphic space $\mathbb{R}^3$.

Introductory material. Before going to the solvable groups II-VII let us recall the well-known form of the Plancherel formula for the Abelian group $\mathbb{R}^3$. The Fourier transform of a function $f \in C_0^\infty(\mathbb{R}^3)$ is defined by

$$\hat{f}(\vec{k}) = \int_{\mathbb{R}^3} d\vec{x} e^{-i\{\vec{k}, \vec{x}\}} f(\vec{x}),$$

and the Plancherel formula is

$$\int_{\mathbb{R}^3} d\vec{x} |f(\vec{x})|^2 = (2\pi)^3 \int_{\mathbb{R}^3} d\vec{k} |\hat{f}(\vec{k})|^2.$$

The Plancherel measure is simply $d\nu(\vec{k}) = (2\pi)^3 d\vec{k}$, proportional to the Lebesgue measure on $\mathbb{R}^3$.

We start by noting that, being an algebraic (matrix) group, G is necessarily type I (Theorem 7.8 or 7.10 [9]), and the normal subgroup N is unimodular and therefore in the kernel of the modular function Δ. It follows from **Theorem 7.6** in [9] that the Mackey Borel structure on $\hat{G}$ is standard, and thereby due to **Lemma 7.39** in [9] we have a measurable field of representations π_p on $p \in \hat{G}$, such that $\pi_p \in p$ (or equivalently, we have a measurable choice of representatives of each equivalence class $[\pi] \in \hat{G}$). Henceforth we will speak of a representation $\pi \in \hat{G}$ meaning the value of this field at a given point $[\pi] \in \hat{G}$. As can be inferred from [12] in the language of solvable Lie groups, only those irreducible representations corresponding to the generic orbits (i.e., orbits of maximal dimension) admit a non-zero Plancherel measure. Therefore only T_μ with $\mu \in \mathbb{R}^2 \setminus V^0$ (generic representations) will play a role in the Fourier transform. We proceed to their construction as $T_{\breve{k}} = \mathbf{Ind}_{\mathbb{R}^2}^G(e^{i\{\breve{k},.\}})$ following §6.1 in [9].

The Fourier transform at generic representations. For each $\breve{k} \in \mathbb{R}^2 \setminus V^0$ the representation Hilbert space $H_{\breve{k}}$ of $\nu_{\breve{k}} = e^{i\{\breve{k},.\}}$ is $H_{\breve{k}} = \mathbb{C}$. The homogeneous space $G/N = \mathbb{R}$ has a natural G-invariant measure, which is the Lebesgue measure dz. The representation space of T_μ is then the completion $L^2(\mathbb{R}, \mathbb{C})$ of the space of compactly supported continuous sections in the homogeneous Hermitian line bundle $\mathbb{R} \times \mathbb{C}$, and the action of G on it is given by

$$T_{\breve{k}}(g)f[z] = e^{-i\{\breve{k},(g^{-1}z)_N\}}f[(g^{-1}z)_H] = e^{i\{\breve{k},F(-z)\breve{g}\}}f[z-g_z],\ g = (\breve{g}, g_z) \in G,\ f \in C_0(\mathbb{R}, \mathbb{C}),$$

where for any $g \in G$ we write $g = g_N g_H$, $g_N \in N$, $g_H \in H$. For $f \in C_0(G)$ define the (harmonic analytical) Fourier transform by

$$\hat{f}(\pi) = \pi(f)D_\pi = \int_G f(g)\pi(g)D_\pi dg,$$

where the operator D_π is defined on $\phi \in C_0(\mathbb{R}, \mathbb{C})$ by

$$D_\pi\phi[z] = \Delta(z)^{+\frac{1}{2}}\phi[z] = (\det F(z))^{-\frac{1}{2}}\phi[z].$$

(Note that there is a misprint in the formula (7.49) of [9], and the sign "$-$" in the power of Δ should be replaced by "$+$". The author of [9] confirmed this in a private communication.) By **Theorem 7.50** in [9], the operator fields $\hat{f}(\pi)$ are measurable fields of Hilbert-Schmidt operators, and if we identify the space of Hilbert-Schmidt operators on H_π with the tensor product space $H_\pi \otimes H_\pi^*$, then the Fourier transform gives an isomorphism

$$L^2(G) \sim \int_{\hat{G}}^{\oplus} d\nu(\pi) H_\pi \otimes H_\pi^*.$$

To find the Plancherel measure $d\nu(\pi)$ we calculate the Fourier transforms $\hat{f}(T_{\breve{k}})$ directly. For $\phi \in C_0(\mathbb{R}, \mathbb{C})$ we have

$$\hat{f}(T_{\breve{k}})\phi[r] = \int_G f(g)T_{\breve{k}}(g)D_\pi\phi[r]dg =$$

$$= \mathfrak{h}\int_{\mathbb{R}^3} dg_x dg_y dg_z f(g_x, g_y, g_z) e^{i\{\breve{k}, F(-r)\breve{g}\}} (\det F(g_z))^{-1} (\det F(r-g_z))^{-\frac{1}{2}} \phi[r-g_z] =$$

by a substitution $g'_z = r - g_z$

$$= \mathfrak{h}\int_{\mathbb{R}^3} dg_x dg_y dg'_z f(g_x, g_y, r-g'_z) e^{i\{\breve{k}, F(-r)\breve{g}\}} (\det F(r-g'_z))^{-1} (\det F(g'_z))^{-\frac{1}{2}} \phi[g'_z] =$$

$$= \mathfrak{h}\int_{\mathbb{R}^3} dg_x dg_y dg'_z f(g_x, g_y, r-g'_z) e^{i\{F^{\perp}(r)\breve{k}, \breve{g}\}} (\det F(r-g'_z))^{-1} (\det F(g'_z))^{-\frac{1}{2}} \phi[g'_z].$$

Thus $\hat{f}(T_{\breve{k}})$ is an integral operator with a smooth kernel

$$\mathcal{K}^f_{\breve{k}}(r, g'_z) = \mathfrak{h}\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,r-g'_z)](F^{\perp}(r)\breve{k})(\det F(r-g'_z))^{-1}(\det F(g'_z))^{-\frac{1}{2}},$$

where

$$\bar{\mathfrak{F}}_{\mathbb{R}^2}[\psi(.,.)](\breve{k}) = \int_{\mathbb{R}^2} dxdy\psi(x,y)e^{i\{\breve{k},\breve{x}\}}.$$

The Hilbert-Schmidt norm $|||\hat{f}(T_{\breve{k}})|||$ is given by

$$|||\hat{f}(T_{\breve{k}})|||^2 = \mathfrak{h}^2 \int_{\mathbb{R}^2} drdg'_z |\mathcal{K}^f_{\breve{k}}(r, g'_z)|^2.$$

Coming back to the original variable $g_z = r - g'_z$, we have

$$|||\hat{f}(T_{\breve{k}})|||^2 = \mathfrak{h}^2 \int_{\mathbb{R}^2} drdg_z \left|\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,g_z)](F^{\perp}(r)\breve{k})\right|^2 (\det F(g_z))^{-2}(\det F(r-g_z))^{-1} =$$

$$= \mathfrak{h}^2 \int_{\mathbb{R}^2} drdg_z \left|\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,g_z)](F^{\perp}(r)\breve{k})\right|^2 (\det F(g_z))^{-1}(\det F(r))^{-1} =$$

(by Fubini's theorem)

$$= \mathfrak{h}^2 \int_{\mathbb{R}} dg_z (\det F(g_z))^{-1} \int_{\mathbb{R}} dr \left|\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,g_z)](F^{\perp}(r)\breve{k})\right|^2 (\det F(r))^{-1}.$$

The Plancherel formula. Now we refer to the previous section about the co-adjoint orbits, and note that in all cases $\rho(k,r) = \dot{\nu}(k)(\det F(r))^{-1}$ with some continuous non-negative function $\dot{\nu}(k)$ on $\tilde{K}$. We will shortly see that

$$d\nu(k) = \mathfrak{h}^{-1}\dot{\nu}(k)dk \tag{2}$$

is exactly the Plancherel measure desired. Indeed,

$$\int_{\tilde{K}} dk\mathfrak{h}^{-1}\dot{\nu}(k)|||\hat{f}(T_{\breve{k}_0(k)})|||^2 = \mathfrak{h}\int_{\tilde{K}} dk\dot{\nu}(k)\int_{\mathbb{R}} dg_z(\det F(g_z))^{-1}\times$$

$$\times \int_{\mathbb{R}} dr \left|\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,g_z)](F^{\perp}(r)\breve{k}_0(k))\right|^2 (\det F(r))^{-1} =$$

by another application of Fubini's theorem (see [19],chapter XIII),

$$= \mathfrak{h}\int_{\mathbb{R}} dg_z (\det F(g_z))^{-1} \int_{\tilde{K}} dk \int_{\mathbb{R}} dr \rho(k,r) \left|\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,g_z)](F^{\perp}(r)\breve{k}_0(k))\right|^2 =$$

by definition of $\rho(k,r)$,

$$= \mathfrak{h} \int_{\mathbb{R}} dg_z (\det F(g_z))^{-1} \int_{\mathbb{R}^2} d\breve{k} \left| \bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,g_z)](\breve{k}) \right|^2 =$$

by the Euclidean Parseval formula,

$$= \mathfrak{h} \int_{\mathbb{R}} dg_z (\det F(g_z))^{-1} \int_{\mathbb{R}^2} dg_x dg_y \left| f(g_x, g_y, g_z) \right|^2 = \int_G dg |f(g)|^2,$$

thus we arrive at an explicit Plancherel forumla,

$$\int_{\tilde{K}} d\nu(k) |||\hat{f}(T_{\breve{k}_0(k)})|||^2 = \int_G dg |f(g)|^2.$$

The Plancherel measures for groups II-VII are thus given by

$$\dot{\nu}_{II}(k) = |k|,\ \dot{\nu}_{III}(k) = 1,\ \dot{\nu}_{IV}(k) = 1 + k_1,\ \dot{\nu}_V(k) = 1,\ \dot{\nu}_{VI-}(k) = \cos^2(k) - q\sin^2(k)$$

$$\dot{\nu}_{VI+}(q) = q^{k_2 \bmod 2},\ \dot{\nu}_{VII}(k) = |k|.$$

Note that we could have chosen the cross section for VI, $q < 0$ in the same way as for VI, $q > 0$ to get a uniform Plancherel measure $\dot{\nu}_{VI} = \dot{\nu}_{VI_+}$ for all Bianchi VI groups, but we preferred the more conventional circle to the quartet of rays in Figure 1 when it was possible. This can be altered for any technical purposes when needed.

6. Scalar spectral analysis on Bianchi I-VII groups

Here the term scalar spectral analysis is understood as the spectral theory of the scalar Laplacian. Of course, there is no distinguished Laplacian on these groups. We will consider *any Laplacian* which arises as the metric operator with respect to any conserved metric on the group.

Let G be one of these groups, and let $\mathfrak{L}(G)$ be its Lie algebra generated by three right invariant vector fields ξ_1, ξ_2, ξ_3. Let further X_1, X_2, X_3 be a basis of left invariant vector fields on G, and $d\omega^1, d\omega^2, d\omega^3$ the dual basis. Any left invariant metric h_{ab} on G can be written as

$$h_{ab} = \sum_{i,j=1}^{3} \breve{h}_{ij} d\omega^i_a d\omega^j_b,$$

where $\breve{h}_{ij}$ is any symmetric positive definite 3×3 matrix, and the corresponding metric Laplacian will be

$$\mathbf{\Delta}_h = \sum_{i,j=1}^{3} \breve{h}^{ij} X_i X_j, \tag{3}$$

with $\breve{h}^{ij} = (\breve{h}_{ij})^{-1}$. To see this first note that

$$\sum_{i,j=1}^{3} \breve{h}^{ij} X_i X_j f = \sum_{i,j=1}^{3} \breve{h}^{ij} \sum_{l,m=1}^{3} [X^l_i X^m_j \partial_l \partial_m + X^l_i (\partial_l X^m_j) \partial_m] f.$$

On the other hand the connection Laplacian related to the Levi-Civita connection is given by

$$\mathbf{\Delta}_h = \sum_{i,j=1}^{3} \check{h}^{ij} \sum_{l,m=1}^{3} [X_i^l X_j^m \partial_l \partial_m - \sum_{k=1}^{3} X_i^l X_j^m \mathbf{\Gamma}_{lm}^k \partial_k],$$

where the $\mathbf{\Gamma}_{lm}^k$ are the Christoffel symbols. This together with the observation

$$\sum_{l,m=1}^{3} X_i^l X_j^m \mathbf{\Gamma}_{lm}^k = -\frac{1}{2}(X_i^l \partial_l X_j^m + X_j^l \partial_l X_i^m),$$

which follows from $\nabla_{X_i} X_j = \frac{1}{2}[X_i, X_j]$, gives (3). Our aim will be to find the eigenfunctions and the spectrum of $\mathbf{\Delta}_h$. If ξ_1 and ξ_2 commute and also commute with all X_i, then $\xi_1, \xi_2, \mathbf{\Delta}_h$ are a triple of commuting operators, and have common eigenfunctions. We will find those eigenfunctions and show that they are complete in the sense we desire. For the ease of notation let us denote

$$\check{h}^{2\times 2} = \check{h}^{ij}|_{i,j<3},$$

$$\check{h}^{\bullet 3} = \check{h}^{ij}|_{i<j=3}, \qquad \check{h}^{3\bullet} = \check{h}^{ij}|_{j<i=3}. \tag{4}$$

First let us describe the spectrum $\mathrm{Spec}(\mathbf{\Delta}_h)$ of the Laplacian $\mathbf{\Delta}_h$. We note that $\mathbf{\Delta}_h$ is a negative semidefinite operator, as $(\mathbf{\Delta}_h f, f)_{L^2(G)} = -(d_h f, d_h f)_{L^2(G)} \leq 0$, where d_h is the exterior derivative with respect to the metric h_{ab}. Thus $\mathbf{\Delta}_h$ is a semibounded and real symmetric operator on $L^2(G)$. There are several ways of extending $\mathbf{\Delta}_h$ to a self-adjoint operator on $L^2(G)$. A real symmetric operator has a self-adjoint extension by von Neumann's theorem [20]. A semibounded symmetric operator has a self-adjoint by Friedrich's extension theorem [20]. But we have something stronger. The Lie group G with its left invariant Riemannian metric h_{ab} is a complete Riemannian manifold [21]. Then following [22] $\mathbf{\Delta}_h$ is essentially self-adjoint on $C_0^\infty(G)$. Being a negative self-adjoint operator $\mathbf{\Delta}_h$ has a real non-positive spectrum, $\mathrm{Spec}(\mathbf{\Delta}_h) \subset (-\infty; 0]$. The semidirect structure of our groups satisfies the conditions of Lemma 5.6 of [21], and we have for the scalar curvature R_h the following formula,

$$R_h = -Tr[S^2] - (Tr[S])^2,$$

where we took into account that the normal Lie subgroup $\mathbb{R}^2$ with the induced metric is flat. The matrix S is given by

$$S = \frac{1}{2}(ad_{(0,0,1)}|_{\mathbb{R}^2} + ad_{(0,0,1)}|^*_{\mathbb{R}^2}) = \frac{1}{2}(f(1) + f(1)^*),$$

where the adjoint * is understood as

$$h(Af, g) = h(f, A^* g),\ \forall f, g \in \mathfrak{L}(G),\ A \in Aut(\mathfrak{L}(G)).$$

Here $h(f, g)$ for $f, g \in \mathfrak{L}(G)$ means the evaluation of the Riemannian metric h on the vector fields f,g. Thus all our groups endowed with any left invariant Riemannian metric are spaces of constant negative curvature equal to R_h, which is given explicitly in terms of the matrices $f(1) = M$ and $\check{h}^{2\times 2}$ with $S = \frac{1}{2}(M + (\check{h}^{2\times 2})^{-1} M \check{h}^{2\times 2})$. This in turn implies, following [23], that the essential spectrum of $\mathbf{\Delta}_h$ is precisely $\mathrm{EssSpec}(\mathbf{\Delta}_h) = (-\infty; R_h]$. Recall that the essential spectrum of a self adjoint operator consists of eigenvalues of infinite multiplicity (see [20]). For the group Bianchi I, all irreps are 1-dimensional, and as we will see later in the section, each eigenspace representation includes an infinite number of them, thus there is no discrete spectrum. For the remaining groups, we have seen in the previous section that no finite dimensional representation enters the Plancherel formula. On the other hand, in the next section we will see that the infinite dimensional eigenspaces exhaust $L^2(G)$, hence no finite dimensional eigenspace exists, i.e., the discrete spectrum is empty, and therefore $\mathrm{Spec}(\mathbf{\Delta}_h) = \mathrm{EssSpec}(\mathbf{\Delta}_h)$.

To find the generators ξ_i for Bianchi I-VII groups we differentiate the left translation map $\vec{x} \mapsto g\vec{x}$,

$$g(x, y, z) = ((g_x, g_y) + F(g_z)(x, y), g_z + z),$$

and obtain

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (x, & y)\dot{F}^\top(0) & 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$$

We see that $\xi_1 = \partial_x$ and $\xi_2 = \partial_y$ do indeed commute. To find the left invariant vectors X_i (which are the generators of right translations) we differentiate the right translation map $\vec{x} \mapsto \vec{x}g$,

$$(x, y, z)g = ((x, y) + F(z)(g_x, g_y), z + g_z),$$

and get

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} F^\top(z) & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}. \tag{5}$$

Thus ξ_1, ξ_2 do commute with all X_i. Now let $\zeta(\vec{x}) \in C^\infty(G)$ be a joint eigenfunction for $\{\xi_1, \xi_2, \mathbf{\Delta}_h\}$. Then it is necessarily of the form

$$\zeta(\vec{x}) = e^{i\{\check{k}_\mathbb{C}, \check{x}\}} P(z),$$

where $\check{k}_\mathbb{C} \in \mathbb{C}^2$, $\check{x} = (x, y)$, and satisfies

$$\mathbf{\Delta}_h \zeta(\vec{x}) = \lambda \zeta(\vec{x}),$$

for some $\lambda \in \mathbb{C}$. A matrix representation of equation (3) and a bit of manipulation yields the following equation

$$\check{h}^{33} \ddot{P}(z) + i(\check{k}_\mathbb{C}^\top F(z)[\check{h}^{\bullet 3} + (\check{h}^{3\bullet})^\top]) \dot{P}(z) -$$

$$-(\lambda + \check{k}_\mathbb{C}^\top F(z) \check{h}^{2\times 2} F^\top(z) \check{k}_\mathbb{C} - i\check{h}^{3\bullet} F^\top(z) M^\top \check{k}_\mathbb{C}) P(z) = 0,$$

where $\dot{F}^\top(z) = \partial_z e^{zM^\top} = F^\top(z)M^\top$ was used. This is a generalized time-dependent harmonic oscillator equation, which always have solutions, and those solutions comprise a two complex dimensional space. For given λ and $\breve{k}_\mathbb{C}$ let us choose two linearly independent solutions $P_{\lambda,k_\mathbb{C}}(z)$ and $Q_{\lambda,k_\mathbb{C}}(z)$ (the choice of initial data may be arbitrary).

First we consider the group Bianchi I. Here $M = 0$, $F(z) = 1$ and the equation becomes

$$\check{h}^{33}\ddot{P}(z) + i(\breve{k}_\mathbb{C}^\top[\check{h}^{\bullet 3} + (\check{h}^{3\bullet})^\top])\dot{P}(z) - (\lambda + \breve{k}_\mathbb{C}^\top \check{h}^{2\times 2}\breve{k}_\mathbb{C})P(z) = 0.$$

One can easily check that $P(z) = e^{ik_z \cdot z}$ is a solution if

$$\lambda = -\vec{k}_\mathbb{C}^\top \check{h}^{ij}\vec{k}_\mathbb{C},$$

where $\vec{k}_\mathbb{C} = (\breve{k}_\mathbb{C}, k_z)$. This is a consequence of the fact that for this group ξ_3 also commutes with all ξ_i and X_i, so that there exist joint eigenfunctions of the commuting operators $\xi_1, \xi_2, \xi_3, \mathbf{\Delta}_h$ of the form

$$\zeta(\vec{x}) = e^{i\{\vec{k}_\mathbb{C},\vec{x}\}},$$

corresponding to the eigenvalues

$$\lambda = -\vec{k}_\mathbb{C}^\top \check{h}^{ij}\vec{k}_\mathbb{C}.$$

In particular, when we restrict ourselves to the irreducibles $\vec{k}_\mathbb{C} = \vec{k} \in \mathbb{R}^3$, we obtain

$$\zeta_{\vec{k}}(\vec{x}) = e^{i\{\vec{k},\vec{x}\}},$$

and we observe immediately that each eigenspace corresponding to the eigenvalue λ includes infinitely many $\vec{k}$ which satisfy

$$\lambda = -\vec{k}^\top \check{h}^{ij}\vec{k}.$$

Of course, $e^{ik_z \cdot z}$ do not exhaust all solutions $P(z)$. But it turns out that the $\zeta_{\vec{k}}$ constructed in this way are already complete in $L^2(G)$. Indeed, that is the essence of the Euclidean Parseval equality. To be more precise, we need to take $d\nu(\vec{k}) = \frac{1}{\mathfrak{h}}d\vec{k}$ as the Plancherel measure for the Euclidean Plancherel formula to hold. Equivalently we can renormalize $\zeta_{\vec{k}}$ by taking

$$\zeta_{\vec{k}}(\vec{x}) = \frac{1}{\sqrt{\mathfrak{h}}}e^{i\{\vec{k},\vec{x}\}}$$

so that the Plancherel measure is independent of the metric. But this ease of construction is a peculiarity which the remaining groups Bianchi II-VII do not share, and we proceed to determine their eigenfunctions.

For the groups II-VII let us now restrict to $0 > \lambda \in \mathbb{R}$ and $\breve{k}_\mathbb{C} = F^\perp(r)k_0(-k) \in \mathbb{R}^2 \setminus V^0$, $k \in \mathfrak{K}$, $r \in \mathbb{R}$ (minus sign for convenience). The equation now becomes

$$\begin{aligned}&\check{h}^{33}\ddot{P}(z) + i(\breve{k}_0(-k)^\top F(z-r)[\check{h}^{\bullet 3} + (\check{h}^{3\bullet})^\top])\dot{P}(z) - \\ &(\lambda + \breve{k}_0(-k)^\top F(z-r)\check{h}^{2\times 2}F^\top(z-r)\breve{k}_0(-k) \\ &-i\check{h}^{3\bullet}F^\top(z-r)M^\top\breve{k}_0(-k))P(z) = 0, \end{aligned} \tag{6}$$

and the two independent solutions will be denoted by $P_{\lambda,k,r}(z)$ and $Q_{\lambda,k,r}(z)$. If we set $P_{\lambda,k,0}(z) = P_{\lambda,k}(z)$, $Q_{\lambda,k,0}(z) = Q_{\lambda,k}(z)$, then a variable substitution $z - r \mapsto z$ shows that we can choose $P_{\lambda,k,r}(z) = P_{\lambda,k}(z-r)$, $Q_{\lambda,k,r}(z) = Q_{\lambda,k}(z-r)$. Another point that can be noticed in equation (6) by taking the complex conjugate is that we can choose $P_{\lambda,-k}(z) = \bar{P}_{\lambda,k}(z)$, $Q_{\lambda,-k}(z) = \bar{Q}_{\lambda,k}(z)$. Finally we construct the eigenfunctions

$$\zeta_{k,\lambda,r,s}(\vec{x}) = (\det F(-r))e^{i\{F^{\perp}(r)\breve{k}_0(-k),\breve{x}\}}P_{\lambda,k,s}(z-r), \tag{7}$$

where to $s = 1$ (-1) corresponds $P_{\lambda,k,s} = P_{\lambda,k}$ $(Q_{\lambda,k})$. Note that each $\zeta_{k,\lambda,r,s}$ enters with its conjugate, $\bar{\zeta}_{k,\lambda,r,s} = \zeta_{-k,\lambda,r,s}$. As we will see in the next section, $P_{\lambda,k,s}$ are orthogonal with respect to the weight $\det F(-z)$, which shows that $\zeta_{k,\lambda,r,s}$ just defined are orthogonal with respect to the same weight. Again, instead of using the Plancherel measure (2) we can use $d\nu(k) = \dot{\nu}(k)dk$ and renormalize according to

$$\zeta_{k,\lambda,r,s}(\vec{x}) = \frac{1}{\sqrt{\mathfrak{h}}}(\det F(-r))e^{i\{F^{\perp}(r)\breve{k}_0(-k),\breve{x}\}}P_{\lambda,k,s}(z-r).$$

Note that by (5) the number $\mathfrak{h}$ is just $\sqrt{\det \breve{h}_{ij}}$.

7. Fourier transform on Bianchi II-VII groups

As a first step on the way of establishing the completeness of $\{\zeta_{k,\lambda,r,s}\}$ we prove a simple proposition. Consider the differential operator

$$D_{\breve{k}} = \breve{h}^{33}\frac{d^2}{dz^2} + i(\breve{k}^{\top}F(z)[\breve{h}^{\bullet 3} + (\breve{h}^{3\bullet})^{\top}])\frac{d}{dz} - (\breve{k}^{\top}F(z)\breve{h}^{2\times 2}F^{\top}(z)\breve{k} - i\breve{h}^{3\bullet}F^{\top}(z)M^{\top}\breve{k}),$$

$$\breve{k} \in \mathbb{R}^2 \setminus V^0,$$

which by definition satisfies

$$D_{\breve{k}}f(z) = e^{-i\{\breve{k},\breve{x}\}}\mathbf{\Delta}_h\left[e^{i\{\breve{k},\breve{x}\}}f(z)\right],\ f \in C_0^{\infty}(\mathbb{R}).$$

Proposition 1 *The operator $D_{\breve{k}}$ with domain $C_0^{\infty}(\mathbb{R})$ is symmetric in $L^2(\mathbb{R}, \det F(-z)dz)$, for any $\breve{k} \in \mathbb{R}^2 \setminus V^0$.*

Proof: Let us first write Green's identity for the operator $\mathbf{\Delta}_h$ on the infinite tube $D^1 \times \mathbb{R} \subset G$ where D^1 is the unit disk in the $\breve{x}$-plane,

$$\int_{D^1\times\mathbb{R}} d\vec{x}\left(e^{-i\{\breve{k},\breve{x}\}}\bar{g}(z)\mathbf{\Delta}_h\left[e^{i\{\breve{k},\breve{x}\}}f(z)\right] - \mathbf{\Delta}_h\left[e^{-i\{\breve{k},\breve{x}\}}\bar{g}(z)\right]e^{i\{\breve{k},\breve{x}\}}f(z)\right) =$$

$$= \int_{S^1\times\mathbb{R}} dzdl(\breve{x})\left(e^{-i\{\breve{k},\breve{x}\}}\bar{g}(z)(\breve{x},\frac{\partial}{\partial\breve{x}})\left[e^{i\{\breve{k},\breve{x}\}}f(z)\right] - (\breve{x},\frac{\partial}{\partial\breve{x}})\left[e^{-i\{\breve{k},\breve{x}\}}\bar{g}(z)\right]e^{i\{\breve{k},\breve{x}\}}f(z)\right) =$$

$$= \int_{S^1\times\mathbb{R}} dzdl(\breve{x})2i\bar{g}(z)f(z)(\breve{x},\breve{k}) = 0.$$

Next we note that

$$\int_{D^1\times\mathbb{R}} d\vec{x}\left(e^{-i\{\breve{k},\breve{x}\}}\bar{g}(z)\mathbf{\Delta}_h\left[e^{i\{\breve{k},\breve{x}\}}f(z)\right]-\mathbf{\Delta}_h\left[e^{-i\{\breve{k},\breve{x}\}}\bar{g}(z)\right]e^{i\{\breve{k},\breve{x}\}}f(z)\right)=$$

$$=\int_{D^1\times\mathbb{R}} dxdydz(\det F(-z))\left(\bar{g}(z)D_{\breve{k}}f(z)-\bar{D}_{\breve{k}}[\bar{g}(z)]f(z)\right)=$$

$$=\pi\int_{\mathbb{R}} dz(\det F(-z))\left(\bar{g}(z)D_{\breve{k}}f(z)-\bar{D}_{\breve{k}}[\bar{g}(z)]f(z)\right)=0,$$

which holds on the dense subset of all f,g in $C_0^\infty(\mathbb{R})$ inside $L^2(\mathbb{R},\det F(-z)dz)$, and symmetry is thus proven.$\square$

Now from the definition it is clear that $D_{\breve{k}}$ is a negative definite operator (because $\mathbf{\Delta}_h$ is such), and is hence upper semibounded, and has a self-adjoint extension in $L^2(\mathbb{R},\det F(-z)dz)$ by Friedrichs extension theorem [20]. In particular, for $\breve{k}=\breve{k}_0(-k)$, $k\in\mathfrak{K}$, the generalized eigenfunctions $\{P_{\lambda,k,s}\}_{\lambda\in Sp(\mathbf{\Delta}_h),s=\pm1}$ are complete and give rise to a Fourier transform $\mathfrak{F}_{\breve{k}_0(-k)}$ on $L^2(\mathbb{R},\det F(-z)dz)$ by means of an abstract eigenfunction expansion. $\mathfrak{F}_{\breve{k}_0(-k)}$ is given by

$$(\mathfrak{F}_{\breve{k}_0(-k)}f)(\lambda,s)=\int_{\mathbb{R}} dz(\det F(-z))\bar{P}_{\lambda,k,s}(z)f(z).$$

Define now the linear isomorphism $\mathfrak{V}:L^2(\mathbb{R})\to L^2(\mathbb{R},\det F(-z)dz)$ by

$$f(z)=[\mathfrak{V}\phi](z)=\phi(-z)(\det F(z))^{\frac{1}{2}}.$$

This induces a Fourier transform $\mathfrak{F}_k=\mathfrak{F}_{\breve{k}_0(-k)}\mathfrak{V}$ which acts as

$$(\mathfrak{F}_k\phi)(\lambda,s)=\int_{\mathbb{R}} dz(\det F(z))^{\frac{1}{2}}P_{\lambda,k,s}(-z)\phi(z)\doteq\tilde{\phi}(\lambda,k,s).$$

The inversion formula is given by

$$\phi(z)=(\det F(z))^{\frac{1}{2}}\sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda\tilde{\phi}(\lambda,k,s)\bar{P}_{\lambda,k,s}(-z).$$

Now we are in the position to show how $\zeta_{k,\lambda,r,s}$ are related to the irreducible representations $T_{\breve{k}_0(k)}$. Consider the following transformation on $f\in C_0^\infty(G)$,

$$\tilde{f}(k,\lambda,r,s)=\int_G dg\bar{\zeta}_{k,\lambda,r,s}(g)f(g),$$

with eigenfunctions $\zeta_{k,\lambda,r,s}$ defined in **Section 6**. We will see that $\tilde{f}(k,\lambda,r,s)$ are in some sense proportional to the matrix columns of the operators $\hat{f}(T_{\breve{k}_0(k)})$. First we see that

$$\tilde{f}(k,\lambda,r,s)=\mathfrak{h}(\det F(-r))\int_{\mathbb{R}^3} dxdydz(\det F(-z))\times$$

$$\times f(x,y,z)e^{i(F^{\perp}(r)\check{k}_0(k),\check{x})}\bar{P}_{\lambda,k,s}(-(r-z)) =$$

$$= \mathfrak{h}(\det F(-r))\int_{\mathbb{R}^3} dxdydz(\det F(-z))(\det F(r-z))^{-\frac{1}{2}}f(x,y,z)e^{i(F^{\perp}(r)\check{k}_0(k),\check{x})}\times$$

$$\times(\det F(r-z))^{\frac{1}{2}}\bar{P}_{\lambda,k,s}(-(r-z)).$$

Next we recognize that this is related to the extension of the operator $\hat{f}(T_{\check{k}_0(k)})$ from $L^2(\mathbb{R})$ to $C^{\infty}(\mathbb{R})$,

$$\tilde{f}(k,\lambda,r,s) = (\det F(-r))\hat{f}(T_{\check{k}_0(k)})\left[(\det F(r))^{\frac{1}{2}}\bar{P}_{\lambda,k,s}(-r)\right].$$

Integrating we obtain

$$\sum_{s=\pm 1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda \tilde{f}(k,\lambda,r,s)\tilde{\phi}(\lambda,k,s) = (\det F(-r))\hat{f}(T_{\check{k}_0(k)})\phi[r]. \tag{8}$$

Recall now the Fourier inversion formula as given in [12] (notation there is different, and we have adapted them to ours adopted from [9]),

$$f(1) = \int_{\mathfrak{K}} d\nu(k)Tr\left[D_{\pi}\hat{f}(T_{\check{k}_0(k)})\right]. \tag{9}$$

Formally a matrix element of $D_{\pi}\hat{f}(T_{\check{k}_0(k)})$ would be an expression

$$\left((\det F(z))^{\frac{1}{2}}\bar{P}_{\lambda',k,s'}(-z), D_{\pi}\hat{f}(T_{\check{k}_0(k)})(\det F(z))^{\frac{1}{2}}\bar{P}_{\lambda,k,s}(-z)\right)_{L^2(\mathbb{R})} =$$

$$= \int_{\mathbb{R}} dz(\det F(z))P_{\lambda',k,s'}(-z)\tilde{f}(k,\lambda,z,s),$$

which does not make sense in precise terms. However, the trace of such elements,

$$\sum_{s=\pm 1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda \int_{\mathbb{R}} dz(\det F(z))P_{\lambda,k,s}(-z)\tilde{f}(k,\lambda,z,s),$$

can be given an exact sense if we change the order of integration,

$$\int_{\mathbb{R}} dz \sum_{s=\pm 1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda(\det F(z))P_{\lambda,k,s}(-z)\tilde{f}(k,\lambda,z,s).$$

Indeed, let $\{p_n(z)\}$ be an orthonormal system in $L^2(\mathbb{R})$. Consider the Fourier transforms $\tilde{p}_n(\lambda,k,s)$, and consider the following bi-distribution in the Fourier space, $\sum_{n=1}^{\infty}\overline{\tilde{p}_n}(\lambda,k,s)\tilde{p}_n(\lambda',k,s')$. Let $\tilde{f},\tilde{g}$ be the Fourier transforms of arbitrary $f,g\in L^2(\mathbb{R})$. We have

$$\sum_{n=1}^{\infty}\sum_{s=\pm 1}\sum_{s'=\pm 1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda\int_{Sp(\mathbf{\Delta}_h)} d\lambda'\overline{\tilde{p}_n}(\lambda,k,s)\tilde{p}_n(\lambda',k,s')\tilde{f}(\lambda,k,s)\bar{\tilde{g}}(\lambda',k,s') =$$

$$= \sum_{n=1}^{\infty}\left(\sum_{s=\pm 1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda\overline{\tilde{p}_n}(\lambda,k,s)\tilde{f}(\lambda,k,s)\right)\left(\sum_{s'=\pm 1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda'\tilde{p}_n(\lambda',k,s')\bar{\tilde{g}}(\lambda',k,s')\right) =$$

$$= \sum_{n=1}^{\infty} (p_n, f)_{L^2(\mathbb{R})} (g, p_n)_{L^2(\mathbb{R})} = (g, f)_{L^2(\mathbb{R})} = \sum_{s=\pm 1} \int_{Sp(\mathbf{\Delta}_h)} d\lambda \tilde{f}(\lambda, k, s) \bar{\tilde{g}}(\lambda, k, s),$$

thus $\sum_{n=1}^{\infty} \overline{\widetilde{p_n}}(\lambda, k, s) \widetilde{p_n}(\lambda', k, s') = \delta(\lambda - \lambda') \delta^s_{s'}$. Now

$$\int_{\mathbb{R}} dz \sum_{s=\pm 1} \int_{Sp(\mathbf{\Delta}_h)} d\lambda (\det F(z)) P_{\lambda,k,s}(-z) \tilde{f}(k, \lambda, z, s) =$$

$$= \int_{\mathbb{R}} dz (\det F(z)) \sum_{n=1}^{\infty} \sum_{s,s'} \iint\limits_{Sp(\mathbf{\Delta}_h)^2} d\lambda d\lambda' \overline{\widetilde{p_n}}(\lambda, k, s) \widetilde{p_n}(\lambda', k, s') P_{\lambda,k,s}(-z) \tilde{f}(k, \lambda', z, s') =$$

$$= \int_{\mathbb{R}} dz (\det F(z)) \sum_{n=1}^{\infty} \overline{\left(\sum_{s=\pm 1} \int\limits_{Sp(\mathbf{\Delta}_h)} d\lambda \widetilde{p_n}(\lambda, k, s) \bar{P}_{\lambda,k,s}(-z) \right)} \times$$

$$\times \left(\sum_{s'=\pm 1} \int\limits_{Sp(\mathbf{\Delta}_h)} d\lambda' \tilde{f}(k, \lambda', z, s') \widetilde{p_n}(\lambda', k, s') \right) =$$

using (8),

$$= \int_{\mathbb{R}} dz (\det F(z)) \sum_{n=1}^{\infty} (\det F(z))^{-\frac{1}{2}} \overline{p_n}(z) (\det F(-z)) \hat{f}(T_{\check{k}_0(k)}) p_n(z) =$$

as both the sum and the integral converge in L^2,

$$= \sum_{n=1}^{\infty} \int_{\mathbb{R}} dz \overline{p_n}(z) (\det F(z))^{-\frac{1}{2}} \hat{f}(T_{\check{k}_0(k)}) p_n(z) = \sum_{n=1}^{\infty} \int_{\mathbb{R}} dz \overline{p_n}(z) D_\pi \hat{f}(T_{\check{k}_0(k)}) p_n(z) =$$

$$= \sum_{n=1}^{\infty} (p_n, D_\pi \hat{f}(T_{\check{k}_0(k)}) p_n)_{L^2(\mathbb{R})} = Tr \left[D_\pi \hat{f}(T_{\check{k}_0(k)}) \right].$$

Hence from (9) we have

$$f(1) = \int_{\mathfrak{K}} d\nu(k) \int_{\mathbb{R}} dz \sum_{s=\pm 1} \int_{Sp(\mathbf{\Delta}_h)} d\lambda (\det F(z)) P_{\lambda,k,s}(-z) \tilde{f}(k, \lambda, z, s).$$

To find an inversion formula at an arbitrary point $g \in G$ we apply this to the left translated function $[L_{g^{-1}} f](x) = f(gx)$,

$$f(g) = [L_{g^{-1}} f](1) = \int_{\mathfrak{K}} d\nu(k) \int_{\mathbb{R}} dz \sum_{s=\pm 1} \int_{Sp(\mathbf{\Delta}_h)} d\lambda (\det F(z)) P_{\lambda,k,s}(-z) [\widetilde{L_{g^{-1}} f}](\lambda, k, z, s).$$

But from the definition

$$[\widetilde{L_{g^{-1}} f}](\lambda, k, r, s) = \int_G dh \bar{\zeta}_{k,\lambda,r,s}(h) [L_{g^{-1}} f](h) = \int_G dh' \bar{\zeta}_{k,\lambda,r,s}(g^{-1} h') f(h').$$

From the definition of $\zeta_{k,\lambda,r,s}$ we find

$$\bar{\zeta}_{k,\lambda,r,s}(g^{-1}h') = e^{-i(F^{\perp}(r+g_z)\check{k}_0(k),\check{g})}(\det F(g_z))\bar{\zeta}_{\lambda,k,r+g_z,s}(h'),$$

thus

$$\int_G dh'\bar{\zeta}_{k,\lambda,r,s}(g^{-1}h')f(h') = e^{-i(F^{\perp}(r+g_z)\check{k}_0(k),\check{g})}(\det F(g_z))\tilde{f}(k,\lambda,r+g_z,s).$$

Therefore

$$f(g) = \int_{\mathfrak{K}} d\nu(k)\int_{\mathbb{R}} dz \sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda(\det F(z))P_{\lambda,k,s}(-z)\times$$
$$\times e^{-i(F^{\perp}(z+g_z)\check{k}_0(k),\check{g})}(\det F(g_z))\tilde{f}(k,\lambda,z+g_z,s) =$$

by substitution $r = z + g_z$

$$= \int_{\mathfrak{K}} d\nu(k)\int_{\mathbb{R}} dr \sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda(\det F(r))\tilde{f}(k,\lambda,r,s)\times$$
$$\times e^{-i(F^{\perp}(r)\check{k}_0(k),\check{g})}P_{\lambda,k,s}(g_z - r) =$$
$$= \int_{\mathfrak{K}} d\nu(k)\int_{\mathbb{R}} dr \sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda(\det F(r))\tilde{f}(k,\lambda,r,s)\zeta_{k,\lambda,r,s}(g),$$

which is our final inversion formula.

It remains to note that by denoting $\alpha = (k,\lambda,r,s)$ we have satisfied all conditions for the eigenfunction expansion $\bar{\zeta}_\alpha(f)$ to give a conventional Fourier transform in sense of [6].

8. Automorphism groups of Bianchi I-VII groups

In this section we consider the automorphism groups $\mathrm{Aut}(G)$ of Bianchi I-VII groups. After performing the calculations we discovered that these automorphisms have been obtained earlier in [24]. However we give here also the dual actions of these automorphisms on $\hat{G}$ which is new. This may become important when analyzing the transformation in the Fourier space induced by automorphisms. We start by noting that Bianchi I-VII groups are matrix groups, and their matrix realization can be given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto G(x,y,z) = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{F(z)}} & x \\ & & y \\ 0 & 0 & 1 \end{pmatrix}.$$

It can be easily seen that in this realization the group multiplication indeed corresponds to the matrix multiplication. The respective Lie algebra realization will be

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \mathfrak{g}(x,y,z) = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{zM}} & x \\ & & y \\ 0 & 0 & 0 \end{pmatrix},$$

which again can be checked to intertwine the matrix commutation with the Lie bracket. Moreover, we could have obtained immediately the exponential map by setting $\exp(x, y, z) = \exp(\mathfrak{g}(x, y, z))$ instead of referring to the Zassenhaus formula, but the latter is a more Lie theoretical approach. Now that all Bianchi groups are connected and simply connected by Theorem 1 of III.6.1 in [25] it follows that $\mathrm{Aut}(G) = \mathrm{Aut}(\mathfrak{g})$ in the sense of a topological group isomorphism (see also [15]). An algebra homomorphism of matrix algebras is necessarily linear in the matrix elements. It follows that any $\breve{\alpha} \in \mathrm{Aut}(\mathfrak{g})$ depends linearly on x, y, z, and is therefore given by an affine transformation in $\mathbb{R}^3$, which is actually a linear transformation because it preserves 0. Therefore we first determine $\mathrm{Aut}(\mathfrak{g})$. Let the linear map $\breve{\alpha} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \breve{\alpha}_{2\times 2} & \breve{\alpha}_{\bullet 3} \\ \breve{\alpha}_{3\bullet} & \breve{\alpha}_{33} \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix},$$

where we use notation similar to (4). Then $\breve{\alpha} \in \mathrm{Aut}(\mathfrak{g})$ if and only if $\breve{\alpha}[\vec{x}, \vec{y}] = [\breve{\alpha}\vec{x}, \breve{\alpha}\vec{y}]$, where $[,]$ is the Lie bracket. Expanding this condition we get the system of requirements

$$\begin{aligned} \breve{\alpha}_{2\times 2} M - \breve{\alpha}_{33} M \breve{\alpha}_{2\times 2} + M \breve{\alpha}_{\bullet 3} \breve{\alpha}_{3\bullet} &= 0, \\ \breve{\alpha}_{2\times 2} M \sigma \breve{\alpha}_{3\bullet}^{\top} &= 0, \\ \breve{\alpha}_{3\bullet} M &= 0, \end{aligned} \tag{10}$$

where σ is the unit antisymmetric matrix. The patterns of admissible matrices $\breve{\alpha}$ satisfying this system have to be computed for each group independently. For Bianchi I we have $M = 0$ and all three conditions are satisfied trivially. For Bianchi IV-VII the matrix M is invertible hence the third requirement means $\breve{\alpha}_{3\bullet} = 0$, so that the second becomes trivial, and the first reduces to $\breve{\alpha}_{2\times 2} M - \breve{\alpha}_{33} M \breve{\alpha}_{2\times 2} = 0$. The cases of groups Bianchi II and III are a bit more involved, but the calculations are straightforward. We present the results in the Table 8. Note that whenever $\breve{\alpha}_{3\bullet} = 0$ the invertibility of $\breve{\alpha}$ requires $\breve{\alpha}_{33} \neq 0$. As it can be seen from the table some algebras allow for reflective automorphisms and their automorphism groups consist of two components (this is what the union symbol $\bigcup$ in Table 8 refers to). Matrices of these pattern forms exhaust the groups $\mathrm{Aut}(\mathfrak{g})$. One can compare this pattern of automorphisms to those available in the literature, for instance, of the Heisenberg algebra in [26].

Now the corresponding group homomorphisms $\breve{A} \in \mathrm{Aut}(G)$ can be found by composing $\breve{\alpha} \in \mathrm{Aut}(\mathfrak{g})$ with the exponential map, $\breve{A} \exp((x, y, z)) = \exp(\breve{\alpha}(x, y, z))$. Recall that the exponential map is given by

$$\exp((x, y, z)) = ([1 + F(z)D(z)](x, y), z),$$

(where $D(z)$ is defined in (1))and because this map is bijective we know that the matrix $[1 + F(z)D(z)]$ is invertible for all z. The logarithmic map can be written as

$$\log((x, y, z)) = ([1 + F(z)D(z)]^{-1}(x, y), z),$$

I	II	III	IV	V	VI, $q \neq 1$
$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$	$\begin{pmatrix} a & 0 & c \\ d & a \cdot j - c \cdot g & f \\ g & 0 & j \end{pmatrix}$	$\begin{pmatrix} a & 0 & c \\ 0 & e & f \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & 0 & c \\ d & a & f \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & 0 & c \\ 0 & e & f \\ 0 & 0 & 1 \end{pmatrix}$

VI, $q = 1$	VII, $p \neq 0$	VII, $p = 0$
$\begin{pmatrix} a & 0 & c \\ 0 & e & f \\ 0 & 0 & 1 \end{pmatrix} \bigcup \begin{pmatrix} 0 & b & c \\ d & 0 & f \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} a & b & c \\ -b & a & f \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & b & c \\ -b & a & f \\ 0 & 0 & 1 \end{pmatrix} \bigcup \begin{pmatrix} a & b & c \\ b & -a & f \\ 0 & 0 & -1 \end{pmatrix}$

Table 3. Patterns of permissible matrices $\breve{\alpha}$ for Bianchi I-VII algebras

and the action of the group homomorphism $\breve{A}$ related to the algebra homomorphism $\breve{\alpha}$ becomes

$$\breve{A} \begin{pmatrix} \breve{x} \\ z \end{pmatrix} = \begin{pmatrix} [1 + F(z')D(z')] \, (\breve{\alpha}_{2\times 2}[1 + F(z)D(z)]^{-1}\breve{x} + \breve{\alpha}_{\bullet 3} z) \\ z' \end{pmatrix}$$

$$z' = \breve{\alpha}_{3\bullet}[1 + F(z)D(z)]^{-1}\breve{x} + \breve{\alpha}_{33} z,$$

for Bianchi II-VII groups and

$$\breve{A} \begin{pmatrix} \breve{x} \\ z \end{pmatrix} = \breve{\alpha} \begin{pmatrix} \breve{x} \\ z \end{pmatrix}$$

for the Bianchi I group. From $\breve{\alpha}_{3\bullet} M = 0$ it follows that $\breve{\alpha}_{3\bullet}[1 + F(z)D(z)]^{-1} = \breve{\alpha}_{3\bullet}$. Thus the formula for Bianchi II-VII simplifies to

$$\breve{A} \begin{pmatrix} \breve{x} \\ z \end{pmatrix} = \begin{pmatrix} [1 + F(\breve{\alpha}_{3\bullet}\breve{x} + \breve{\alpha}_{33}z)D(\breve{\alpha}_{3\bullet}\breve{x} + \breve{\alpha}_{33}z)] \, (\breve{\alpha}_{2\times 2}[1 + F(z)D(z)]^{-1}\breve{x} + \breve{\alpha}_{\bullet 3} z) \\ \breve{\alpha}_{3\bullet}\breve{x} + \breve{\alpha}_{33} z \end{pmatrix}.$$

One more step can be done in this generality. From formula (10) and $\breve{\alpha}_{3\bullet} M = 0$ it follows that

$$\breve{\alpha}_{2\times 2} M^m = (\breve{\alpha}_{33} M)^m \breve{\alpha}_{2\times 2}$$

for $m \geq 2$ and therefore for any sequence of complex numbers $\{f_m\}_{m=0}^{\infty}$

$$\breve{\alpha}_{2\times 2} \sum_{m=0}^{\infty} f_m M^m = \sum_{m=0}^{\infty} f_m (\breve{\alpha}_{33} M)^m \breve{\alpha}_{2\times 2} + f_1 M \breve{\alpha}_{\bullet 3} \breve{\alpha}_{3\bullet}$$

whenever the left hand side exists. This can be used to establish that

$$[1 + F(\breve{\alpha}_{3\bullet}\breve{x} + \breve{\alpha}_{33}z)D(\breve{\alpha}_{3\bullet}\breve{x} + \breve{\alpha}_{33}z)]\breve{\alpha}_{2\times 2} = \breve{\alpha}_{2\times 2}[1 + F(\frac{\breve{\alpha}_{3\bullet}\breve{x}}{\breve{\alpha}_{33}} + z)D(\frac{\breve{\alpha}_{3\bullet}\breve{x}}{\breve{\alpha}_{33}} + z)].$$

This far on the explicit form of the group automorphisms.

Now let us look at the dual spaces $\hat{G}$. If $\breve{A} \in \mathrm{Aut}(G)$ and $\pi \in \hat{G}$ then $\pi \circ \breve{A} = \pi'$ for some $\pi' \in \hat{G}$. Thus $\breve{A}$ induces a pullback map $\breve{A}^* : \hat{G} \to \hat{G}$. Because $\dim \pi = \dim \pi'$ it follows that $\breve{A}^*$ maps generic representations into generic representations and singletons into singletons. The representations $\pi \in \hat{G}$ are in a bijective correspondence with the derived representations $d\pi$ which are irreducible representations of the Lie algebra $\mathfrak{g}$. In a similar fashion, any $\breve{\alpha} \in \mathrm{Aut}(\mathfrak{g})$ induces a pullback map $\alpha^* : d\hat{G} \to d\hat{G}$ between derived representations. This pullback map is easier to study than that for the group representations. Consider first the Bianchi I group. The irreducibles are given by

$$T_{\vec{k}}(\vec{g}) = e^{i\{\vec{k},\vec{g}\})},$$

and the derived representations are

$$dT_{\vec{k}}(\vec{x}) = i\{\vec{k},\vec{x}\}.$$

An automorphism $\vec{x} = \breve{\alpha}\vec{q}$ induces the pullback map $\breve{\alpha}^*(\vec{k}) = \breve{\alpha}^\top \vec{k}$. Consider now the singletons of a Bianchi II-VII group. They are given for $\vec{k} \in V^0 \oplus \mathbb{R}$ by

$$T_{\vec{k}}(\vec{g}) = e^{i\{\vec{k},\vec{g}\}} = e^{i\{\breve{k},\breve{g}\}} e^{ik_3 g_z},$$

and the derived singletons are

$$dT_{\vec{k}}(\vec{x}) = i\{\vec{k},\vec{x}\},$$

and again, an automorphism $\vec{x} = \breve{\alpha}\vec{q}$ induces the pullback map $\breve{\alpha}^*(\vec{k}) = \breve{\alpha}^\top \vec{k}$. This in particular means that $\breve{k}' = \breve{\alpha}_{2\times 2}^\top \breve{k} + k_3 \breve{\alpha}_{3\bullet}^\top$, and if $\breve{k} \in V^0$ then

$$M^\top \breve{k}' = M^\top \breve{\alpha}_{2\times 2}^\top \breve{k} + k_3 M^\top \breve{\alpha}_{3\bullet}^\top = 0,$$

where (10) and $\breve{\alpha}_{3\bullet} M = 0$ were used. We explicitly observe that the automorphisms map singletons into singletons, as expected. Finally we turn to the generic representations. Let $T_{\breve{k}}$ be a generic representation of G. Then it acts on $L^2(\mathcal{R})$ by

$$T_{\breve{k}}(\vec{g}) f[w] = e^{i\{\breve{k},F(-w)\breve{g}\}} f[w - g_z],\ \vec{g} = (\breve{g}, g_z) \in G.$$

Its derived representation will be

$$dT_{\breve{k}}(\vec{x}) f[w] = i\{\breve{k}, F(-w)\breve{x}\} f[w] - z\partial_w f[w].$$

Under the automorphism $\vec{x} = \breve{\alpha}\vec{q}$ it will turn into

$$dT_{\breve{k}}(\vec{q}) f[w] = i\{\breve{k}, F(-w)[\breve{\alpha}_{2\times 2}\breve{q} + \breve{\alpha}_{\bullet 3} s]\} f[w] - [\breve{\alpha}_{3\bullet}\breve{q} + \breve{\alpha}_{33} s]\partial_w f[w].$$

For simplicity we will consider only the automorphisms with $\breve{\alpha}_{3\bullet} = 0$. Thus we omit only some automorphisms of the Heisenberg group, but this group is a central subject in harmonic analysis, and the missing results can be found in the literature. Define the isometric isomorphism $\mathfrak{T} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\mathfrak{T}(f)[w] = \frac{1}{\sqrt{\breve{\alpha}_{33}}} e^{i\{\breve{k}, \int_0^{-w} F(\breve{\alpha}_{33}\xi) d\xi \breve{\alpha}_{\bullet 3}\}} f(\breve{\alpha}_{33} w).$$

Consider the representation $dT_{\breve{k}'}$ with $\breve{k}' = \breve{\alpha}_{2\times 2}^{\top}\breve{k}$. Note that because $\breve{\alpha}_{3\bullet} = 0$ we have that $\breve{\alpha}_{2\times 2}^{\top}$ is invertible, and from (10) we assure that it maps $\breve{k} \notin V^0$ to $\breve{k}' \notin V^0$. Thus $dT_{\breve{k}'}$ is generic. Its action on the image $\mathfrak{T}(f)[w]$ is given by

$$dT_{\breve{k}'}(\vec{q})\mathfrak{T}(f)[w] = i\{\breve{k}', F(-w)\breve{q}\}\mathfrak{T}(f)[w] - s\partial_w \mathfrak{T}(f)[w] =$$

$$= \mathfrak{T}(i\{\breve{k}', F(-\frac{\bullet}{\breve{\alpha}_{33}})\breve{q}\}f)[w] + \mathfrak{T}(is\{\breve{k}, F(-\bullet)\breve{\alpha}_{\bullet 3}\}f)[w] - \mathfrak{T}(s\breve{\alpha}_{33}\partial f).$$

Recall that we have seen that from (10) and $\breve{\alpha}_{3\bullet} = 0$ it follows $\breve{\alpha}_{2\times 2}F(z) = F(\breve{\alpha}_{33}z)\breve{\alpha}_{2\times 2}$, hence

$$i\{\breve{k}', F(-\frac{\bullet}{\breve{\alpha}_{33}})\breve{q}\} = i\{\breve{\alpha}_{2\times 2}^{\top}\breve{k}, F(-\frac{\bullet}{\breve{\alpha}_{33}})\breve{q}\} = i\{\breve{k}, F(-\bullet)\breve{\alpha}_{2\times 2}\breve{q}\}.$$

We finally see that

$$dT_{\breve{k}'}(\vec{q})\mathfrak{T}(f)[w] = \mathfrak{T}\left([i\{\breve{k}, F(-\bullet)\breve{\alpha}_{2\times 2}\breve{q}\} + is\{\breve{k}, F(-\bullet)\breve{\alpha}_{\bullet 3}\}]f - s\breve{\alpha}_{33}\partial f\right) = \mathfrak{T}\left(dT_{\breve{k}}(\vec{q})f\right),$$

which means that $\mathfrak{T}$ intertwines the irreducible representations $dT_{\breve{k}} \circ \breve{\alpha}$ and $dT_{\breve{\alpha}_{2\times 2}^{\top}\breve{k}}$. Thus these two representations are unitarily equivalent, $\breve{\alpha}^*(\breve{k}) = \breve{\alpha}_{2\times 2}^{\top}\breve{k}$. If the cross sections are chosen explicitly (for instance, as we did) then it is a straightforward calculation to find the action of $\breve{\alpha}^*$ on $\tilde{K}$ and $\mathfrak{K}$. We omit these calculations here because, first, they depend on the preferred choice of the cross sections, and second, they involve transcendental functions (e.g., the solution of the equation $e^y + ay = x$) and are not transparent visually, and do not provide a better insight into the matter.

9. Separation of time variable in homogeneous universes

We want to see to which extent the technique of mode decomposition developed in [6] is applicable to hyperbolic fields on Bianchi type and FRW cosmological models. For this aim we have to check whether the conditions of **Proposition 2.3** are satisfied. Recall that the metric g of a homogeneous spacetime $M = \mathcal{I} \times \Sigma$, where $\mathcal{I}$ is an open interval and Σ is a Bianchi type homogeneous space, is given by

$$ds_g^2 = dt^2 - \sum_{\alpha,\beta=1}^{3} \breve{h}_{\alpha\beta}(t)d\omega^{\alpha}(\vec{x})d\omega^{\beta}(\vec{x}),$$

where $\breve{h}_{ij}(t)$ ($t \in \mathcal{I}$) is a smooth positive definite symmetric matrix function, and $d\omega^i$ are the left invariant 1-forms on Σ. Condition (i) of **Proposition 2.3** is automatically satisfied because $g_{00} = 1$. For condition (ii) note that

$$\sum_{i,j=1}^{3} g^{ij}(x)\frac{\partial g_{ij}}{\partial t}(x) = \sum_{i,j=1}^{3}\sum_{\alpha,\beta=1}^{3}\sum_{\gamma,\delta=1}^{3} \breve{h}^{\alpha\beta}(t)\dot{\breve{h}}_{\gamma\delta}(t)X_{\alpha}^{i}(\vec{x})X_{\beta}^{j}(\vec{x})(d\omega^{\gamma})_i(\vec{x})(d\omega^{\delta})_j(\vec{x}) =$$

$$= \sum_{\alpha,\beta=1}^{3}\sum_{\gamma,\delta=1}^{3} \breve{h}^{\alpha\beta}(t)\dot{\breve{h}}_{\gamma\delta}(t)\langle X_{\alpha}, d\omega^{\gamma}\rangle_h \langle X_{\beta}, d\omega^{\delta}\rangle_h =$$

$$= \sum_{\alpha,\beta=1}^{3} \sum_{\gamma,\delta=1}^{3} \check{h}^{\alpha\beta}(t)\dot{\check{h}}_{\gamma\delta}(t)\delta_{\alpha}^{\gamma}\delta_{\beta}^{\delta} = Tr[\check{h}^{-1}(t)\dot{\check{h}}(t)].$$

This shows that the condition (ii) is also satisfied. We see that the homogeneous spacetimes are an ideal playground for mode decomposition. Note that FRW spacetimes correspond to the choice $\check{h}_{\alpha\beta}(t) = a^2(t)\delta_{\alpha\beta}$.

If conditions (iii) and (iv) are also satisfied depends on the chosen connection ∇. For the scalar field (iii) is automatically satisfied with $\Gamma = 0$. In [6] we defined the field operator $D = \Box + m^2(x)$, and the instantaneous field operator $D_{\Sigma_t} = -\Delta + m^2(x)$. The eigenfunction equation of the latter operator was written as $D_{\Sigma_t}\zeta_\alpha = \lambda_t(\alpha)\zeta_\alpha$. Condition (iv) of **Proposition 2.3** in [6] can be obviously satisfied if $\check{h}_{\alpha\beta}(t) = a^2(t)\check{h}^0_{\alpha\beta}$ as in this case the time evolution amounts only to a rescaling of $\lambda(\alpha)$ in $D_{\Sigma_t}\zeta_\alpha = \lambda_t(\alpha)\zeta_\alpha$. Note that because D_{Σ_t} is G-invariant, the term m^2 is a function of t only. This is the situation where the dynamics of the universe consists of merely an isotropic rescaling. Thus, for instance, in case of FRW spacetimes condition (iv) is satisfied automatically.

But the condition (iv) can be also satisfied non-trivially with an anistropic rescaling and even some shears and rotations. This is clearly possible for Bianchi I group, because the eigenfunctions do not depend on the matrix $\check{h}$. For Bianchi II-VII groups one has to look at the equation (6) to see to which extent the solution $P(z)$ depends on the matrix $\check{h}$. Suppose $\check{h}$ and $\check{j}$ are two matrices for which there exist two common linearly independent solutions $P(z)$ and $Q(z)$. Because we have already seen that an isotropic rescaling is always possible, without loss of generality we assume $\check{h}^{33} = \check{j}^{33}$ (we again use the notations 4). Fix $\check{k} \in \mathbb{R}^2 \setminus V^0$ and $0 < \lambda \in \mathbb{R}$. Now the condition that the two equations have the same solution spaces can be cast into the following pair of equations,

$$\check{h}^{3\bullet}F^\top(z)\check{k} = \check{j}^{3\bullet}F^\top(z)\check{q},$$

$$\lambda + \check{k}^\top F(z)\check{h}^{2\times 2}F^\top(z)\check{k} = \lambda' + \check{q}^\top F(z)\check{j}^{2\times 2}F^\top(z)\check{q}$$

for some $\check{q} \in \mathbb{R}^2 \setminus V^0$, $0 < \lambda \in \mathbb{R}$ and for all $z \in \mathbb{R}$. That non-trivial possibilities exist is clear visually, but we will not go into details here. Once these conditions are satisfied at all $t \in \mathcal{I}$ for the 1-parameter family of matrices $\check{h}(t)$ describing the evolution of the spatial metric then condition (iv) is satisfied, and we have an explicit formula for the time dependent eigenvalue $\lambda_t(\alpha)$.

As electromagnetism is of primary importance for us, let us finally show that the assumption $\check{h}_{\alpha\beta}(t) = a^2(t)\check{h}^0_{\alpha\beta}$ is sufficient to satisfy condition (iii) for the 1-form field. Indeed, the 1-form field is given by the Levi-Civita connection, for which the connection forms are $(\Gamma_i)^a_b = -\mathbf{\Gamma}^a_{ib}$. Let us compute the symbol $\mathbf{\Gamma}^a_{0b}$. It is easy to see that $\mathbf{\Gamma}^0_{0b} = \mathbf{\Gamma}^a_{00} = 0$. For $a, b > 0$ we have

$$\mathbf{\Gamma}^a_{0b} = \frac{1}{2}\sum_{m=1}^{3} g^{am}\frac{\partial g_{mb}}{\partial t} = \sum_{\alpha,\beta,\gamma=1}^{3} \check{h}^{\alpha\beta}(t)\dot{\check{h}}_{\beta\gamma}(t)X^a_\alpha(\vec{x})d\omega^\gamma_b(\vec{x}).$$

If $\check{h}_{\alpha\beta}(t) = a^2(t)\check{h}^0_{\alpha\beta}$ then $\partial_t\check{h}_{\alpha\beta}(t) = 2H(t)\check{h}_{\alpha\beta}(t)$ with $H(t) = \frac{\dot{a}(t)}{a(t)}$ being the Hubble constant, and we get

$$\mathbf{\Gamma}^a_{0b} = 2H(t)\sum_{\alpha,\beta,\gamma=1}^{3}\check{h}^{\alpha\beta}(t)\check{h}_{\beta\gamma}(t)X^a_\alpha(\vec{x})d\omega^\gamma_b(\vec{x}) = 2H(t)\delta^a_b.$$

Thus $\Gamma_0 = -2H(t)0\oplus\mathbf{1}_3$ is not only a function of t, but also commutes with any matrix, hence (iii) is trivially satisfied.

10. The mode decomposition of the Klein-Gordon field

The investigation of classical and quantum fields in anisotropic cosmological models has been carried out by different authors since decades (see, e.g., [27],[28],[29],[30]). However these works mainly concentrate on Bianchi I models where the harmonic analysis and mode decomposition are obvious. With the background developed above we can extend this to all Bianch I-VII models.

In [6] we have described how the mode decomposition can be performed explicitly for an arbitrary vector valued field given the explicit spectral theory of the model spatial sections Σ. As we have already explicitly constructed the spectral theory of the line bundle over Bianchi I-VII spacetimes, we can apply the mode decomposition to the Klein-Gordon field and see what can be gained by this technique.

Let M be Bianchi I-VII type spacetime, i.e., a 4-dimensional smooth globally hyperbolic Lorentzian manifold with a smooth global time function chosen [31], and with the isometry group G which is one of the groups Bianchi I-VII, so that G acts simply transitively on the equal time hypersurfaces Σ_t (for all missing details see [6]). Or to put it in simpler words, let topologically $M = \mathbb{R}\times G$ where G is a Bianchi I-VII group, and let the metric be given by $ds^2 = dt^2 - h_{ij}(t,\vec{x})dx^i dx^j$, $\vec{x} = (x^1, x^2, x^3) \in \Sigma_t$, so that for any $t\in\mathbb{R}$ the Riemannian metric $h_{ij}(t,.)$ is left invariant under the action of the underlying Lie group G. In [6] we cosnidered the field operator $D = \Box + m^2$, where $\Box$ is the d'Alembert operator related to the Levi-Civita connection, and $m^2\in\mathbb{R}_+$ is a positive constant. The Klein-Gordon field is described by the equation

$$D\phi = (\Box + m^2)\phi = 0.$$

As we have seen in the previous section for each Bianchi type there are certain restrictions on the dynamics of the spatial metric $h(t)$ for the mode decomposition to be applicable. Recall that $h(t)$ is described by the positive definite matrix $\check{h}(t)$. For simplicity let us consider only an isotropic rescaling,

$$\check{h}(t) = a^2(t)\check{h}(0).$$

We further wrote in [6] $D = D_t + D_{\Sigma_t}$, where D_t is a differential operator in variable t, and D_{Σ_t} is the instantaneous field operator $D_{\Sigma_t} = -\Delta_t + m^2$. Its eigenfunctions satisfying $D_{\Sigma_t}\zeta_\alpha = \lambda_\alpha(t)\zeta_\alpha$ will be the eigenfunctions of the Laplace operator, $-\Delta_t\zeta_\alpha =$

$(\lambda_\alpha(t) - m^2)\zeta_\alpha$. Because the time dependent Fourier transform is normalized at time $t = 0$, we have $\lambda_\alpha(0) = -\lambda + m^2$ where $\alpha = (k, \lambda, r, s)$. Using $\check{h}^{ij}(t) = a^{-2}(t)\check{h}^{ij}(0)$ it follows that $\Delta_t = a^{-2}(t)\Delta_0$ and hence

$$\lambda_\alpha(t) = \frac{-\lambda}{a^2(t)} + m^2.$$

As we have demanded that the spacetime $M = \mathcal{I} \times \Sigma$ is globally hyperbolic, the Klein-Gordon field operator $D = \Box + m^2$ has unique advanced(+) and retarded(-) fundamental solutions $E^\pm : C_0^\infty(M) \mapsto C_0^\infty(M)$ with the properties

(i) $(\Box + m^2)E^\pm f = f = E^\pm(\Box + m^2)f$

(ii) $\mathrm{supp}\{E^\pm f\} \subset J^\pm(\mathrm{supp} f)$

for all $f \in C_0^\infty(M)$. Here, $J^\pm(N)$ denotes the causal future(+) and past(-) of a subset $N \subset M$. We refer to [32] for full discussion and further references. Then $E = E^+ - E^-$ is called the causal propagator of the Klein-Gordon operator $\Box + m^2$ on (M, g). Any $\phi = Ef$, $f \in C_0^\infty(M)$ is a solution of the homogeneous Klein-Gordon equation $(\Box + m^2)\phi = 0$, and the restriction of ϕ to any Cauchy surface is compactly supported. We define $Sol_0(M) = EC_0^\infty(M)$. We also write $E(f, h) = \langle f, Eh\rangle_{L^2(M)}$ where

$$\langle f, h\rangle_{L^2(M)} = \int_M f(x)h(x)dvol_g(x);$$

$dvol_g$ is the volume form on M induced by the metric g. Moreover, we set $\mathcal{K} = C_0^\infty(M)/\ker E$. Then (the real part of) $\mathcal{K}$ becomes a symplectic space with symplectic form

$$\sigma([f], [h]) = E(f, h), \qquad [f] = f + \ker E, \qquad [h] = h + \ker E.$$

The map $\mathcal{K} \mapsto Sol_0(M)$, $[f] \mapsto Ef$ is a symplectomorphism upon endowing $Sol_0(M)$ with the symplectic form

$$\sigma(\phi, \psi) = \int_{\mathcal{C}} (\phi n^a \nabla_a \psi - \psi n^a \nabla_a \phi)\, d\eta_{\mathcal{C}}$$

for any Cauchy surface $\mathcal{C}$ in M having future-pointing unit normal field n^a and metric-induced hypersurface measure $d\eta_{\mathcal{C}}$. Again, we refer to [32] for a complete discussion and full proofs.

Now by **Proposition 2.3** in [6] any $\phi \in Sol_0(M)$ can be written as

$$\phi(t, \vec{x}) = \int_{\tilde{\Sigma}} d\mu(\alpha) \left[a^\phi(\alpha)T_\alpha(t)\zeta_\alpha(\vec{x}) + b^\phi(\alpha)\bar{T}_\alpha(t)\zeta_\alpha(\vec{x})\right],$$

where $\alpha = (k, \lambda, r, s)$ and $d\mu(\alpha) = d\nu(k)d\lambda F(r)dr$. The modes T_α are to this point arbitrary μ-measurable solutions of the mode equation

$$\ddot{T}_\alpha(t) + F(t)\dot{T}_\alpha(t) + G_\alpha(t)T_\alpha(t) = 0$$

such that T_α and $\bar{T}_\alpha$ are linearly independent solutions. As found in [6] for the scalar field on an isotropically expanding universe,

$$F(t) = P(t) = 3H(t) = 3\frac{\dot{a}(t)}{a(t)},\, G_\alpha(t) = \lambda_\alpha(t),\, I(t) = a^3(t).$$

The propagator $E[f]$ is obviously a weak solution of the Klein-Gordon equation. If we want to mode decompose it we have to satisfy (2.19) of [6]. By **Proposition 2.6** of [6] we can do it if M is an analytic manifold. But the spatial metric h is analytic in the spatial variable x because it is expressed in left invariant fields of the Lie group G. Thus we only need to choose $a(t)$ a real analytic function. The spectrum of D_{Σ_t} is strictly uniform with the prescription $\omega(\alpha) = -\lambda$. Now if we restrict the initial data $T_\alpha(0) = p(\lambda)$ and $\dot{T}_\alpha(0) = q(\lambda)$ where $p, q \in \mathcal{A}(\mathbb{H}_0)$ and $p\bar{q} - \bar{p}q = i$ then **Proposition 2.9** is applicable. Suppose this is done, now from the **Section 2.6** of [6] we find that (for the line bundle obviously $s(\alpha) = 1$)

$$E[f](x) = i\int_{\tilde{\Sigma}} d\mu(\alpha)\left[\langle \bar{T}_\alpha\bar{\zeta}_\alpha, f\rangle_M T_\alpha(t)\zeta_\alpha(\vec{x}) - \langle T_\alpha\bar{\zeta}_\alpha, f\rangle_M \bar{T}_\alpha(t)\zeta_\alpha(\vec{x})\right].$$

Now we proceed to the quantization. The mode decomposition of arbitrary CCR quantum fields is discussed in [13] which provides a generalization of the works by [33], [34], [35]. Here we summarize some results applied to the quantized Klein-Gordon field. The latter is given by the field algebra $\mathcal{A}$ generated by the unit $\mathbf{1}$ and the elements $\phi(f)$ satisfying

(i) $\phi(af + h) = a\phi(f) + \phi(h)$,

(ii) $\phi(\bar{f}) = \phi^*$,

(iii) $[\phi(f), \phi(h)] = -i \cdot E(f, h)\mathbf{1}$,

(iv) $\phi((\Box + m^2)f) = 0, \qquad \forall f, h \in \mathcal{D}(M),\, a \in \mathbb{C}$.

A state ω of the field is a linear functional $\omega \in \mathcal{A}'$ such that $\omega(\mathbf{1}) = 1$ and $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$. The 2-point function ω_2 of a state ω is the bilinear form $\omega_2(f, h) = \omega(\phi(f)\phi(h))$. It follows that $\omega_2((\Box + m^2)f, h) = \omega_2(f, (\Box + m^2)h) = 0$, $\omega_2(\bar{f}, f) \geq 0$ and $\omega_2(f, h) - \omega_2(h, f) = -iE(f, h)$. Moreover,

$$\overline{\omega_2(\bar{f}, \bar{h})} = \overline{\omega(\phi(f)^*\phi(h)^*)} = \overline{\omega([\phi(h)\phi(f)]^*)} = \omega(\phi(h)\phi(f)) = \omega_2(h, f),$$

ω_2 is hermitian. A quasifree state ω is a state which is completely determined by its 2-point function ω_2 (for precise definitions see, e.g., [36],[37]). Being a weak bi-solution of the field equation ω_2 can be mode decomposed and that is done by **Proposition 4.1** of [13]. (For earlier results using mode decomposition of 2-point functions see [33],[38],[39].) If we denote for convenience

$$\tilde{f}^u(\alpha) = \langle T_\alpha\zeta_\alpha, f\rangle_M,\, \tilde{f}^v(\alpha) = \langle \bar{T}_\alpha\zeta_\alpha, f\rangle_M,$$

where

$$\langle f, h\rangle_M = \int_M dx f(x)h(x),$$

then a 2-point function can be written as

$$\omega_2(f,h) = a^\omega(\tilde{f}^u, \tilde{h}^u) + \overline{a^\omega(\tilde{\bar{f}}^u, \tilde{\bar{h}}^u)} + b^\omega(\tilde{f}^u, \tilde{h}^v) + \overline{b^\omega(\tilde{\bar{f}}^u, \tilde{\bar{h}}^v)} + \delta(\tilde{f}^u, \tilde{h}^v),$$

where a^ω and b^ω are bi-distributions satisfying certain symmetry and positivity conditions, and δ is the bi-distribution given by the integral kernel of the usual delta function. But it differs from the delta function because we identify kernels with bi-distribution using the pairing

$$a(\tilde{f}, \tilde{h}) = \int_{\tilde{\Sigma}} \int_{\tilde{\Sigma}} d\mu(\alpha) d\mu(\beta) a(\alpha, \beta) \tilde{f}(\alpha) \tilde{h}(-\beta).$$

This strange convention is chosen only for calculational purposes and can be transformed to the usual form if needed. In particular, the δ term in the formula for the 2-point function is

$$\delta(\tilde{f}^u, \tilde{h}^v) = \int_{\tilde{\Sigma}} d\mu(\alpha) \tilde{f}(\alpha) \tilde{h}(-\beta).$$

This term by itself represents a quasifree pure state, and the remaining part of the generic 2-point function is symmetric. This state depends on the choice of the modes T_α. Choosing different modes S_α we will find a rich supply of such pure states. Then we can transform these states back to our original T_α as follows. Let the old and new modes be related by $S_\alpha = \mu_\alpha T_\alpha + \nu_\alpha \bar{T}_\alpha$ with $|\mu_\alpha|^2 - |\nu_\alpha|^2 = 1$. The pure state given by the δ term in modes S_α is determined by the choice $a^\omega = b^\omega = 0$. As described in [13] in the original modes these components will become $a^\omega(\alpha, \beta) = \delta(\alpha, \beta) \mu_\alpha \bar{\nu}_\alpha$ and $b^\omega(\alpha, \beta) = \delta(\alpha, \beta) |\nu_\alpha|^2$. But such states do not exhaust all pure quasifree states. By **Corollary 4.1** of [13] any pure quasifree state is given by $a^\omega(\alpha, \beta) = -\delta(\alpha, \beta) \circ \tilde{S}^{v,u}$ and $b^\omega(\alpha, \beta) = -\delta(\alpha, \beta) \circ \tilde{S}^{v,v}$, where the linear maps $\tilde{S}^{v,u}, \tilde{S}^{v,v}$ are subject to certain conditions. In fact, the pure states given by δ terms are special in that they are homogeneous states, i.e., they are invariant under the isometry group G. More generally, by **Proposition 4.3** of [13] any homogeneous quasifree state is given by coefficients a^ω, b^ω which are of the form

$$a^\omega(\tilde{f}, \tilde{h}) = \mathfrak{a}^\omega \left(\int_{\mathbb{R}} dr \tilde{f}(-k, \lambda, r, s) \tilde{h}(k, \lambda', r, s') \right),$$

$$b^\omega(\tilde{f}, \tilde{h}) = \mathfrak{b}^\omega \left(\int_{\mathbb{R}} dr \tilde{f}(-k, \lambda, r, s) \tilde{h}(k, \lambda', r, s') \right),$$

with distributions $\mathfrak{a}^\omega(k, \lambda, \lambda', s, s')$ and $\mathfrak{b}^\omega(k, \lambda, \lambda', s, s')$.

Another important notion is the notion of Hadamard states. A quasifree state ω is said to be Hadamard if its 2-point function ω_2 satisfies the microlocal spectral condition (μSC) (see [40]). Hadamard states are believed to be the states of physical importance for several reasons discussed in the literature ([36],[41],[34],[42]). One way of checking whether a given ω_2 satisfies the μSC is to try to compute its wave front set directly [40]. The mode decomposition suggests another way of doing this. By **Proposition 4.6** of [13] there exists a wide variety of modes T_α such that the homogeneous pure states given by δ terms in these modes are Hadamard. Such modes can be computed easily. For some conceptually related approaches (however emphasizing somewhat different aspects), see also [33],[43],[44],[38],[39]. For convenience we switch to the variable

$$s(t) = \int_0^t \frac{d\tau}{a^3(\tau)}$$

as described in [13], then the mode equations become

$$\ddot{T}_\alpha(s) + \Lambda_\alpha(s) T_\alpha(s) = 0,$$

where

$$\Lambda_\alpha(s) = a^6(t(s))\left(\frac{\lambda}{a^2(t(s))} + m^2\right),$$

and $t(s)$ is the inverse function of $s(t)$. Now choose an arbitrary non-negative function $\eta \in C_0^\infty[-1,0]$ and let $\tilde{\eta}(s) = \int_0^s d\sigma \eta(\sigma)$. Denote

$$\Lambda'_\alpha(s) = (1 - \tilde{\eta}(s))\Lambda_\alpha(-1) + \tilde{\eta}(s)\Lambda_\alpha(s),$$

and let

$$\ddot{T}'_\alpha(s) + \Lambda'_\alpha(s) T'_\alpha(s) = 0.$$

Choose the initial data $T'_\alpha(-1) = p(\Lambda_\alpha(-1))$ and $T'_\alpha(0) = q(\Lambda_\alpha(-1))$ with corresponding functions p, q as before with the additional constraint that

$$p(\Lambda) - \frac{1}{\sqrt[4]{4\Lambda}} = \mathfrak{o}(\Lambda^{-\infty}),\ q(\Lambda) - i\sqrt[4]{\frac{\Lambda}{4}} = \mathfrak{o}(\Lambda^{-\infty}),$$

where $= \mathfrak{o}(\Lambda^{-\infty})$ means of rapid decay at $\Lambda \to +\infty$. Then the choice $T_\alpha(0) = T'_\alpha(0)$ and $\dot{T}_\alpha(0) = \dot{T}'_\alpha(0)$ yields modes T_α such that the homogeneous pure state given by the δ term is Hadamard. This is basically the essence of **Proposition 4.6**. Some relations of the rapid decay in the Fourier space with the Hadamard property can be found in [38]. It follows that if such modes T_α are chosen, then a generic quasifree state is Hadamard if and only if the remaining symmetric part of ω_2 given by coefficients a^ω and b^ω is smooth. A general criterion for smoothness of a distribution in terms of it Fourier transform is unfortunately not known in harmonic analysis. In principle **Proposition 4.4** of [6] could give at least sufficient conditions, but this is still a work in progress.

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Appendix A. Induced representations

Although the generalities of induced representations can be found in any standard textbook on group representations, for consistency we will very briefly give an overview of them in this appendix. We will follow the **Chapter 6** of [9] in our treatment.

The inducing procedure produces unitary representations of a locally compact topological group G from a unitary representation of its closed subgroup H. In this work we will be interested only in Lie groups, so that the majority of functional analytical issues are automatically settled. If ρ is a unitary representation of H, and ν the unitary representation of G induced from H (described below), we will write $\nu = \mathbf{Ind}_H^G \rho$. In particular the restriction of ν to H is unitarily equivalent to ρ, $\nu|_H \sim \rho$. Unless the homogeneous space G/H is a finite set, ν is infinite dimensional.

Denote $M = G/H$. We present the construction of the induced representation under assumptions which hold in cases of our interest. Namely, we suppose that there exists a G-invariant measure dx_M on M. Any $x \in G$ can be uniquely written as $x = x_M x_H$ with $x_M \in M$ and $x_H \in H$. If $\mathcal{H}_\rho$ is the representation Hilbert space of ρ, then the representation Hilbert space $\mathcal{H}_\nu$ of the induced representation is taken to be $\mathcal{H}_\nu = L^2(M, \mathcal{H}_\rho, dx_M)$, i.e., $\mathcal{H}_\rho$-valued dx_M-square integrable functions on M. The action of the representation ν on $\mathcal{H}_\nu$ is given by

$$\nu(x)f(y) = \rho((x^{-1}y)_H)^{-1} f((x^{-1}y)_M),\ \forall x \in G,\ y \in M,\ f \in \mathcal{H}_\nu.$$

That this is a natural construction can be seen by the following nice properties. If ρ and ρ' are unitarily equivalent unitary representations of H, then $\nu = \mathbf{Ind}_H^G \rho$ and $\nu' = \mathbf{Ind}_H^G \rho'$ are unitarily equivalent unitary representations of G. Moreover, it can be shown that if $\{\rho_i\}$ is a family of unitary representations of H then $\mathbf{Ind}_H^G \bigoplus \rho_i$ is unitarily equivalent to $\bigoplus \mathbf{Ind}_H^G \rho_i$.

Appendix B. An illustration of co-adjoint orbits

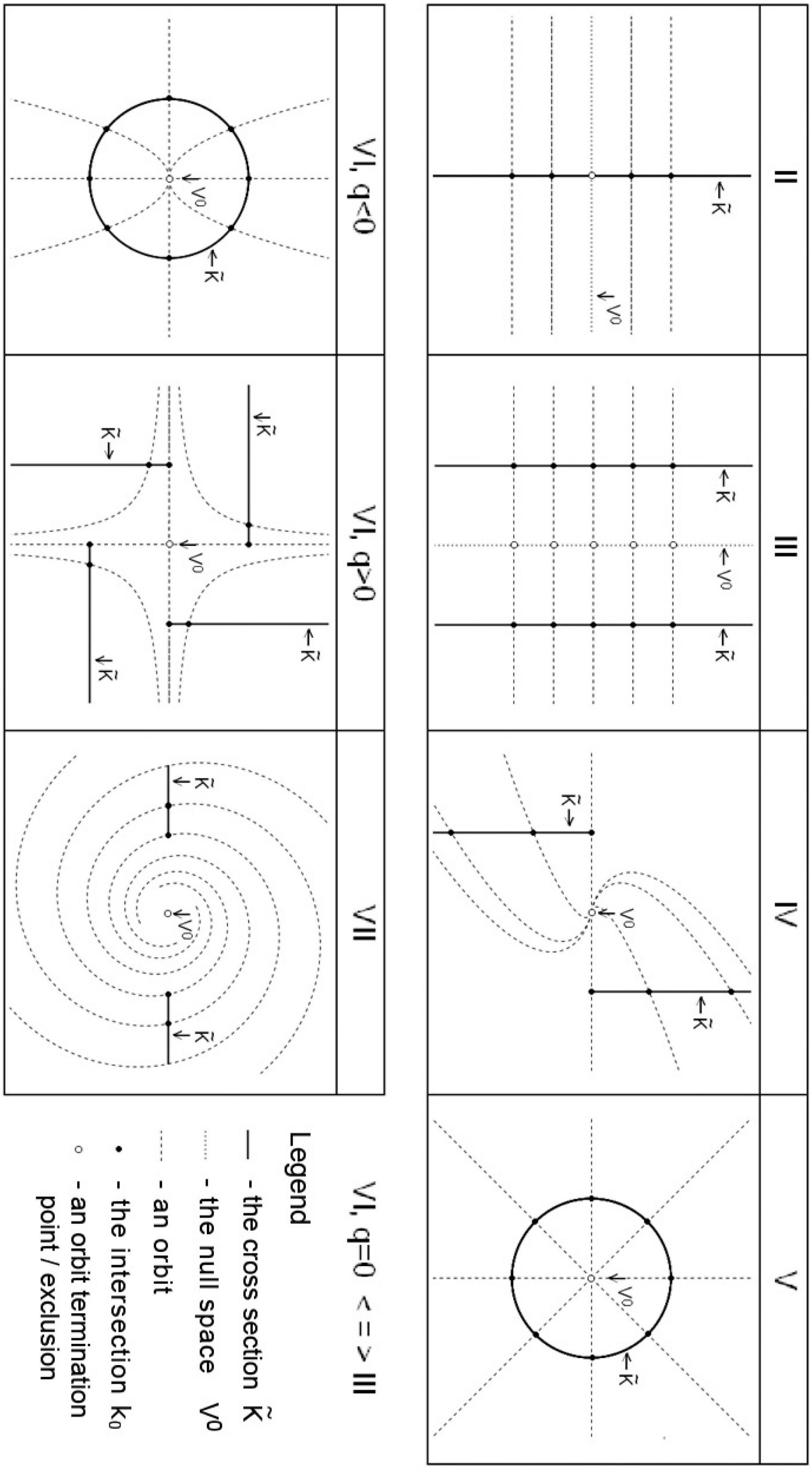

Figure 1. The co-adjoint orbit of Bianchi II-VII groups

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