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A LOCAL ESTIMATE FOR SUPER-LIOUVILLE EQUATIONS ON CLOSED RIEMANN SURFACES

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Abstract. Continuing our work on the super-Liouville equations, a variational problem motivated by the supersymmetric extension of the Liouville functional in quantum field theory, we study the profile of blow-up solutions near the blow-up point and establish a local estimate for the bubbling sequences.

1. Introduction

The conformally invariant variational problems arising in geometry and theoretical physics often exhibit a very rich and subtle structure. Uncovering and utilizing this structure beyond the general phenomenon of limit cases of the Palais-Smale condition leads to some of the most difficult and most interesting problems of geometric analysis. Usually, however, this structure is very sensitive to the particular special form of the variational problems, and it disappears under any variations of it. One of the most studied examples in this context is the Liouville functional with its associated Euler-Lagrange equation, the Liouville equation. It has deep links with complex analysis, prescribed curvature problems on Riemann surfaces, and conformal field theory. Remarkably, in the context of string theory and conformal field theory, physicists ([Po1, Po2]) have discovered an extension of the Liouville functional with an even richer structure, the super-Liouville functional. Here, a scalar field as in the original Liouville problem is coupled with an anticommuting spinor field. The resulting Euler-Lagrange equations, the super-Liouville system, then exhibit supersymmetry between the two fields. The anticommuting character of the spinor field, however, leads outside the aforementioned context of geometric analysis and the geometry of Riemann surfaces. We have discovered, however, that there also exists a version of the super-Liouville system involving only ordinary, commuting fields, and we have started to study it using the tools of nonlinear analysis (see [JWZ], [JWZZ]). In mathematical terms, this is a variational problem on a closed Riemann surface (M, g) with a spin structure. The functional we consider couples the standard Liouville functional with a spinor term and is therefore also called the *super-Liouville functional* (see [JWZ]). In particular, it preserves the conformal invariance of the ordinary Liouville functional on Riemann surfaces.

It is then the challenge for nonlinear analysis to extend the detailed structural analysis of solutions of the Liouville equation to those of the super-Liouville system. The essential aspect here is the analysis of the blow-up behavior of sequences of solutions and the precise characterization of the possible blow-up limits. In the present paper, we continue our work [JWZ, JWZZ] in this direction, and we show

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that the elements of the blowing up sequence can be controlled by the rescaled blow-up limit within constants that are independent of the particular sequence. This will be our main result, Theorem 2.3 below. For the ordinary Liouville equation, this has been achieved in [BCLT] and [Ly]. In fact, for the scalar field, we can use the method of [BCLT] for our purposes, but handling the spinor field requires new estimates, of course.

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2. The super-Liouville functional

In this section, we introduce the super-Liouville functional, describe previous work about it, and then formulate our main result.

The super-Liouville functional is a conformally invariant functional that couples a real-valued function u and a spinor $\psi \in \Gamma(\Sigma M)$ on a closed Riemann surface M with conformal metric g and a fixed spin structure,

$$E(u,\psi) = \int_M \{\frac{1}{2}|\nabla u|^2 + K_g u + \langle (\not{D} + e^u)\psi, \psi\rangle - e^{2u}\} dv. \tag{1}$$

Here ΣM is the spinor bundle on M and K_g is the Gaussian curvature in M. The Dirac operator $\not{D}$ is defined by $\not{D}\psi := \sum_{\alpha=1}^{2} e_\alpha \cdot \nabla_{e_\alpha}\psi$, where $\{e_1, e_2\}$ is an orthonormal basis for TM, ∇ is the Levi-Civita connection on M with respect to g and $\cdot$ denotes Clifford multiplication for ΣM. Finally, $\langle \cdot, \cdot \rangle$ is the natural Hermitian metric on ΣM induced by g. Two useful formulas about Clifford multiplication between e_i and the spinors $\psi, \varphi \in \Sigma M$ are

$$e_i \cdot e_j \cdot \psi + e_j \cdot e_i \cdot \psi = -2\delta_{ij}\psi,$$

and

$$\langle \psi, \varphi \rangle = \langle e_i \cdot \varphi, e_i \cdot \psi \rangle.$$

For the geometric background of the theory of spinors and its calculus, we refer to [LM] and [Jo].

The Euler-Lagrange system for $E(u,\psi)$ is

$$\begin{cases} -\Delta u &= 2e^{2u} - e^u \langle \psi, \psi \rangle - K_g \\ \not{D}\psi &= -e^u \psi \end{cases} \quad \text{in } M. \tag{2}$$

where Δ is the Laplacian with respect to g. These equations are called the *super-Liouville equations*. Similarly to the classical Liouville equations ([Liou] [BM] [LSh]), the analytic foundations for solutions to (2) are established in [JWZ] and [JWZZ]. More precisely, we developed a blow-up theory for sequences of solutions to (2) via establishing the energy identity for blow-up solutions and calculating the blow-up values at the blow-up points. To summarize, we have

Theorem 2.1. *(Theorem 5.1,* [JWZ] *and Theorem 1.3,* [JWZZ]*) Let (M,g) be a closed Riemann surface with a fixed spin structure and let (u_n,ψ_n) be a sequence of smooth solutions of*

$$\begin{cases} -\Delta u_n &= 2e^{2u_n} - e^{u_n}\langle\psi_n,\psi_n\rangle - K_g, \\ \not{D}\psi_n &= -e^{u_n}\psi_n, \end{cases} \tag{3}$$

in M with the energy condition

$$\int_M e^{2u_n}dv < C \text{ and } \int_M |\psi_n|^4 dv < C \tag{4}$$

for some positive constant $C>0$.

Define the blow up set of (u_n,ψ_n) by:

$$\begin{aligned} \Sigma_1 &= \{x\in M,\ \text{there is a sequence } y_n\to x \text{ such that } u_n(y_n)\to+\infty\}, \\ \Sigma_2 &= \{x\in M,\ \text{there is a sequence } y_n\to x \text{ such that } |\psi_n(y_n)|\to+\infty\}. \end{aligned}$$

Then $\Sigma_2\subset\Sigma_1$ and (u_n,ψ_n) admits a subsequence, still denoted by (u_n,ψ_n), satisfying one of the following:

i) *u_n is bounded in $L^\infty(M)$.*
ii) *$u_n\to-\infty$ uniformly on M.*
iii) *Σ_1 is finite, nonempty,*

$$u_n\to-\infty \quad \text{uniformly on compact subsets of } M\backslash\Sigma_1,$$

and

$$2e^{2u_n} - e^{u_n}|\psi_n|^2 \rightharpoonup \sum_{x_i\in\Sigma_1}\alpha_i\delta_{x_i},$$

in the distribution sense and with $\alpha_i\geq 4\pi$.

The proof of Theorem 2.1 relies on understanding the behavior of the spinor part ψ_n in the neighborhood of the blow up point. This exhibits some similarities with the analysis for other conformally invariant variational problems, in particular two-dimensional harmonic maps. In fact, we have the following the energy identity for spinors, which tells us that the neck energy of ψ_n converges to zero:

Theorem 2.2. *(Thm 1.2,* [JWZZ]*) With the same notations and assumptions as in Theorem 2.1, suppose that $\Sigma_1=\{x_1,x_2,\cdots,x_l\}$. Then there are finitely many solutions of (2) on S^2: $(u^{i,k},\psi^{i,k})$, $i=1,2,\cdots,l; k=1,2,\cdots,L_i$, such that, after selection of a subsequence, ψ_n converges in C^∞_{loc} to ψ on $M\backslash\Sigma_1$ and the following energy identity holds:*

$$\lim_{n\to\infty}\int_M |\psi_n|^4 dv = \int_M |\psi|^4 dv + \sum_{i=1}^{l}\sum_{k=1}^{L_i}\int_{S^2}|\psi^{i,k}|^4 dv. \tag{5}$$

Further exploring the behavior of the bubbling solution (u_n,ψ_n), we calculated the blow-up values at blow-up points in Σ_1. Define the blow-up value $m(p)$ at $p\in\Sigma_1$ by

$$m(p) = \lim_{r\to0}\lim_{n\to\infty}\int_{B_r(p)}(2e^{2u_n} - e^{u_n}|\psi_n|^2)dv.$$

It is shown (see Lemma 5.1, [JWZZ]) that there exists $G \in W^{1,q}(M) \cap C^2_{loc}(M \backslash \Sigma_1)$ with $\int_M G = 0$ for $1 < q < 2$ such that

$$u_n - \frac{1}{|M|} \int_M u_n \to G \tag{6}$$

in $C^2_{loc}(M \backslash \Sigma_1)$ and weakly in $W^{1,q}(M)$. Moreover, in $\Sigma_1 = \{p_1, p_2, \cdots, p_l\}$, for $R > 0$ small such that $B_R(p_k) \cap \Sigma_1 = \{p_k\}$, $k = 1, 2, \cdots, l$, there holds

$$G = \frac{1}{2\pi} m(p_k) \log \frac{1}{|x - p_k|} + g(x)$$

for $x \in B_R(p_k) \backslash \{p_k\}$ with $g \in C^2(B_R(p_k))$. Then, by using a Pohozaev type identity for solutions (u_n, ψ_n) (see Proposition 2.7, [JWZZ]), we have shown (see Theorem 1.5, [JWZZ])

$$m(p) = \lim_{r \to 0} \lim_{n \to \infty} \int_{B_r(p)} (2e^{2u_n} - e^{u_n} |\psi_n|^2) dv = 4\pi. \tag{7}$$

Consequently, in iii) of Theorem 2.1, we have $\alpha_i = 4\pi$.

The purpose of the present paper is to study the profile of blow-up solutions near their blow-up points for the super-Liouville equation. This will extend the blow-up theory for the Liouville equation (see [BCLT], [Ly] and [JLW]).

To this end, we note that, by Theorem 1.1, ψ_n is uniformly bounded on compact subsets of $M \backslash \Sigma_1$ and due to (6), u_n has uniformly bounded oscillations on compact subsets of $M \backslash \Sigma_1$. Furthermore, to describe our result, by conformal invariance of the super-Liouville equations (2), it suffices to work with the Euclidean metric $g = dx_1^2 + dx_2^2$ around the point 0 on $B_2 = \{x \in \mathbb{R}^2 : |x|^2 \leq 2\}$, where 0 is the only blow up point of (u_n, ψ_n) in B_2. So. we consider the system of equations and inequalities

$$\begin{cases} -\Delta u_n = 2e^{2u_n} - e^{u_n} \langle \psi_n, \psi_n \rangle, & \text{in} \quad B_2 \\ \not{D} \psi_n = -e^{u_n} \psi_n, & \text{in} \quad B_2 \\ \max u_n - \min u_n \leq C, & \text{on} \quad \partial B_2 \\ \max |\psi_n| \leq C, & \text{on} \quad \partial B_2 \\ \int_{B_2} e^{2u_n} + |\psi_n|^4 dx \leq C, & \\ 2e^{2u_n} - e^{u_n} |\psi_n|^2 \rightharpoonup 4\pi\delta, \text{ in the sense of distribusions} & \text{in } B_2. \end{cases} \tag{8}$$

Here δ is the Dirac measure at the origin 0.

Assume that $\mu_n = u_n(x_n) = \max_{\bar{B}_2} u_n(x)$ and $\lambda_n = e^{-\mu_n}$. Then, $x_n \to 0$ and $\lambda_n \to 0$. Define the rescaled fields by

$$\begin{cases} \widetilde{u}_n(x) &= u_n(\lambda_n x + x_n) + \ln \lambda_n \\ \widetilde{\psi}_n(x) &= \lambda_n^{\frac{1}{2}} \psi_n(\lambda_n x + x_n) \end{cases} \tag{9}$$

for any $x \in B_{\frac{1}{\lambda_n}}(0)$. Then $(\widetilde{u}_n(x), \widetilde{\psi}_n(x))$ satisfies

$$\begin{cases} -\triangle \widetilde{u}_n(x) &= 2e^{2\widetilde{u}_n(x)} - e^{\widetilde{u}_n(x)} |\widetilde{\psi}_n(x)|^2 \\ \not{D} \widetilde{\psi}_n(x) &= -e^{\widetilde{u}_n(x)} \widetilde{\psi}_n(x) \end{cases}$$

in $B_{\frac{1}{\lambda_n}}(0)$ with the energy condition

$$\int_{B_{\frac{1}{\lambda_n}}(0)} e^{2\widetilde{u}_n(x)} + |\widetilde{\psi}_n(x)|^4 dv < C.$$

We know that $\widetilde{u}_n(0) = 0$ and $\widetilde{u}_n(x) \leq 0$. If we only consider the case that the bubble is a solution to the Super-Liouville equations, we may further assume that $0 \in \Sigma_1 \cap \Sigma_2$ (otherwise, if $0 \in \Sigma_1 \setminus \Sigma_2$, then the bubble is a solution to the Liouville equation). Then, as is discussed in [JWZZ], we obtain that $(\widetilde{u}_n, \widetilde{\psi}_n)$ admits a subsequence converging in $C^{1,\alpha}_{loc}(\mathbb{R}^2)$ to $(\widetilde{u}, \widetilde{\psi})$, which is an entire solution on $\mathbb{R}^2$ to the Super-Liouville equation (2), i.e.

$$\begin{cases} -\Delta \widetilde{u} &= 2e^{2\widetilde{u}} - e^{\widetilde{u}}\langle \widetilde{\psi}, \widetilde{\psi}\rangle, \\ \not{D}\widetilde{\psi} &= -e^{\widetilde{u}}\widetilde{\psi}, \end{cases} \qquad x \in \mathbb{R}^2 \tag{10}$$

with the energy condition

$$I(\widetilde{u}, \widetilde{\psi}) = \int_{\mathbb{R}^2} e^{2\widetilde{u}} + |\widetilde{\psi}|^4 dx < \infty. \tag{11}$$

By Proposition 6.3 in [JWZ], we know that

$$\int_{\mathbb{R}^2} 2e^{2\widetilde{u}} - e^{\widetilde{u}}|\widetilde{\psi}|^2 dx = 4\pi, \tag{12}$$

and $(\widetilde{u}, \widetilde{\psi})$ satisfies the following asymptotic behavior at infinity:

$$\widetilde{u}(x) = -2\ln|x| + C + O(|x|^{-1}) \qquad \text{for} \quad |x| \quad \text{near} \quad \infty, \tag{13}$$

$$\widetilde{\psi}(x) = -\frac{1}{2\pi}\frac{x}{|x|^2} \cdot \xi_0 + o(|x|^{-1}) \qquad \text{for} \quad |x| \quad \text{near} \quad \infty, \tag{14}$$

where $\cdot$ is the Clifford multiplication, $C \in \mathbb{R}$ is some constant and $\xi_0 = \int_{\mathbb{R}^2} e^{\widetilde{u}}\widetilde{\psi} dx$ is a constant spinor. Combining (7), (12) and Theorem 2.2, we conclude that the neck energy of u_n converges to 0. More presicely,

$$\lim_{R\to\infty}\lim_{r_0\to 0}\lim_{n\to\infty} \int_{R \leq |x| \leq r_0 \lambda_n^{-1}} \left(e^{2\widetilde{u}_n} + e^{\widetilde{u}_n}|\widetilde{\psi}_n|^2 + |\widetilde{\psi}_n|^4\right) dx = 0. \tag{15}$$

This completes the qualitative behavior for a sequence of bubbling solution to the super-Liouville equation derived in [JWZZ].

In this paper, we shall prove:

Theorem 2.3. *Suppose that (u_n, ψ_n) satisfies (8). Then there exist two constants $0 < r_0 < 2$ and $C > 0$ which both are independent of n, such that*

$$|u_n(x) - \mu_n - \widetilde{u}(\lambda_n^{-1}(x - x_n))| \leq C, \quad \textit{for } x \in B_{r_0} \tag{16}$$

$$|\lambda_n^{\frac{1}{2}}\psi_n(x) - \widetilde{\psi}(\lambda_n^{-1}(x - x_n)) \leq C, \quad \textit{for } x \in B_{r_0}. \tag{17}$$

When $\psi_n \equiv 0$, the problem we consider reduces to the usual Liouville equation problem, in which case the corresponding estimate as in (16) was established in [BCLT] and in [Ly] by using different methods. For our problem, we shall follow the approach developed in [BCLT]. The key point is to deal with the perturbation term, that is, to analyze the asymptotic behavior of ψ_n at the blow up point.

3. Proof of Theorem 2.3

To prove Theorem 2.3, we can follow the method of [BCLT] to deal with the function u. Therefore, we shall essentially have to deal with the spinor part ψ, for which we shall have to establish a decay estimate in a neighborhood of the blow-up point. So let us first state two useful lemmas, which will play an important role in the proof of our main result Theorem 2.3.

The first Lemma is a Pohozaev type identity for smooth solutions of (2).

Lemma 3.1. *(Proposition 2.7,* [JWZZ]*) Let (u,ψ) be a smooth solution to (2). Then for every geodesic ball $B_R \subseteq M$, we have*

$$\begin{aligned} & R\int_{\partial B_R} |\frac{\partial u}{\partial \nu}|^2 - \frac{1}{2}|\nabla u|^2 d\sigma \\ = & \int_{B_R} 2e^{2u} - e^u|\psi|^2 dv - R\int_{\partial B_R} e^{2u} d\sigma + \int_{B_R} K_g x\cdot \nabla u dv \\ & + \frac{1}{2}\int_{\partial B_R} \langle \frac{\partial \psi}{\partial \nu}, x\cdot\psi\rangle dv + \frac{1}{2}\int_{\partial B_R} \langle x\cdot\psi, \frac{\partial \psi}{\partial \nu}\rangle dv \end{aligned}$$

where ν is the outward normal vector to ∂B_R.

The second lemma is about the decay of the spinor part. Notice that equation (10) is invariant under conformal transformations. Let (v,ϕ) be the Kelvin transformation of $(\widetilde{u},\widetilde{\psi})$, i.e.

$$\begin{aligned} v(x) &= \widetilde{u}(\frac{x}{|x|^2}) - 2\ln|x| \\ \phi(x) &= |x|^{-1}\widetilde{\psi}(\frac{x}{|x|^2}). \end{aligned}$$

Then (v,ϕ) satisfies

$$\left\{ \begin{array}{rcll} -\Delta v &=& 2e^{2v} - e^v\left\langle \phi,\phi\right\rangle, & x\in\mathbb{R}^2\backslash\{0\}, \\ \not{D}\phi &=& -e^v\phi, & x\in\mathbb{R}^2\backslash\{0\}. \end{array} \right. \tag{18}$$

And, by a change of variables, we have

$$\begin{aligned} \int_{|x|\le r_0} e^{2v}dx &= \int_{|x|\ge \frac{1}{r_0}} e^{2\widetilde{u}}dx, \\ \int_{|x|\le r_0} |\phi|^4 dx &= \int_{|x|\ge \frac{1}{r_0}} |\widetilde{\psi}|^4 dx. \end{aligned}$$

Therefore, there is an $r_0>0$ small enough such that (v,ϕ) is a smooth solution to (18) on $B_{r_0}\backslash\{0\}$ with energy $\int_{|x|\le r_0} e^{2v}dx < \varepsilon_0 < \pi$ for any sufficiently small number $\varepsilon_0>0$ and $\int_{|x|\le r_0}|\phi|^4dx < C$. Since (18) are conformally invariant, in the sequel we may assume B_{r_0} to be the unit disk B_1. When the energy of v is sufficiently small, we have the following decay estimate of ϕ at the singularity $\{0\}$:

Lemma 3.2. *(Lemma 6.2,* [JWZ]*) There exists $0<\varepsilon_0<\pi$, such that if (v,ϕ) is a smooth solution to (18) on $B_1\backslash\{0\}$ with energy $\int_{|x|\le 1} e^{2v}dx < \varepsilon_0$ and $\int_{|x|\le 1}|\phi|^4 dx < C$, then for any $x\in B_{\frac{1}{2}}$ we have*

$$|\phi(x)||x|^{\frac{1}{2}} + |\nabla\phi(x)||x|^{\frac{3}{2}} \le C(\int_{B_{2|x|}} |\phi|^4 dx)^{\frac{1}{4}}.$$

Furthermore, if we assume that $e^{2v} = O(\frac{1}{|x|^{2-\varepsilon}})$, *then, for any* $x \in B_{\frac{1}{2}}$, *we have*

$$|\phi(x)||x|^{\frac{1}{2}} + |\nabla\phi(x)||x|^{\frac{3}{2}} \le C|x|^{\frac{1}{4C}}(\int_{B_1} |\phi|^4 dx)^{\frac{1}{4}},$$

for some positive constant C. *Here* ε *is any sufficiently small positive number.*

Proof of Theorem 1.3 We divide the proof into five steps.

Step 1. From the rescaled functions (9), it is easy to see that (16) and (17) are valid in $B_{\lambda_n R}(x_n)$ for any fixed large number $R > 0$ and for some constant $C > 0$ independent of n. Thus, we only need to prove that (16) and (17) are valid when $x \in B_{r_0} \setminus B_{\lambda_n R}(x_n)$ for some $r_0 > 0$. In the sequel, C will denote a universal positive constant independent of n, which may vary from line to line.

Step 2. It follows from the boundary condition in (8) that

$$0 \le u_n - \min_{\partial B_2} u_n \le C \qquad \text{on } \partial B_2.$$

Define w_n as the unique solution of the following Dirichlet problem

$$\begin{cases} -\triangle w_n &= 0, & \text{in } B_2 \\ w_n &= u_n - \min_{\partial B_2} u_n, & \text{on } \partial B_2 \end{cases}$$

By the maximum principle, w_n is uniformly bounded in B_2. Furthermore, the function $v_n = u_n - \min_{\partial B_2} u_n - w_n$ satisfies the Dirichlet problem

$$\begin{cases} -\triangle v_n = 2e^{2u_n} - e^{u_n}|\psi_n|^2, & \text{in } B_2 \\ v_n = 0, & \text{on } \partial B_2 \\ \int_{B_2} 2e^{2u_n} - e^{u_n}|\psi_n|^2 dx \le C. & \end{cases}$$

By Green's representation formula, we have

$$v_n(x) = \frac{1}{2\pi}\int_{B_2} \log\frac{1}{|x-y|}(2e^{2u_n(y)} - e^{u_n(y)}|\psi_n(y)|^2)dy + R_n(x)$$

where $R_n(x)$ is a uniformly bounded function in B_2. Therefore we have

$$\begin{aligned} & u_n(x) - \min_{\partial B_2} u_n \\ = \ & \frac{1}{2\pi}\int_{B_2} \log\frac{1}{|x-y|}(2e^{2u_n(y)} - e^{u_n(y)}|\psi_n(y)|^2)dy + O(1) \\ = \ & \frac{1}{2\pi}\int_{B_{r_0}(x_n)} \log\frac{1}{|x-y|}(2e^{2u_n(y)} - e^{u_n(y)}|\psi_n(y)|^2)dy + O(1). \end{aligned} \tag{19}$$

Here and in the sequel, $O(1)$ denotes a uniformly bounded term.

For the spinor ψ_n, we apply similar arguments. Since $\max_{\partial B_2}|\psi_n| \le C$, we define ϕ_n by

$$\begin{cases} \not{D}\phi_n &= 0, & \text{in } B_2 \\ \phi_n &= \psi_n, & \text{on } \partial B_2 \end{cases}$$

Recall that on $\mathbb{R}^2$ the Dirac operator $\not{D}$ is essentially the (doubled) Cauchy-Riemann operator (see [JWZ], P. 1108). By the maximum principle for holomorphic or anti-holomorphic functions, $|\phi_n|$ is uniformly bounded in B_2. We define $\varphi_n = \psi_n - \phi_n$.

Then φ_n satisfies

$$\begin{cases} \not{D}\varphi_n = -e^{u_n}\psi_n, & \text{in } B_2 \\ \varphi_n = 0, & \text{on } \partial B_2 \\ \int_{B_2} e^{u_n}\psi_n dx \le C. \end{cases}$$

By the Green function for the Dirac operator $\not{D}$ (see e.g. Section 2, [AHM]), we have

$$\varphi_n(x) = -\frac{1}{2\pi}\int_{B_2} \frac{x-y}{|x-y|^2}\cdot e^{u_n(y)}\psi_n(y)dy + \rho_n(x)$$

where $\rho_n(x)$ is a harmonic spinor which is uniformly bounded in B_2 and $\cdot$ is the Clifford multiplication. Therefore we have

$$\begin{aligned} \psi_n(x) &= -\frac{1}{2\pi}\int_{B_2} \frac{x-y}{|x-y|^2}\cdot e^{u_n(y)}\psi_n(y)dy + O(1) \\ &= -\frac{1}{2\pi}\int_{B_{r_0}(x_n)} \frac{x-y}{|x-y|^2}\cdot e^{u_n(y)}\psi_n(y)dy + O(1), \end{aligned} \tag{20}$$

Next we state the Green representation for the rescaled function $(\widetilde{u}_n(x), \widetilde{\psi}_n(x))$. Setting $x = x_0$ in (19), we get

$$\mu_n - \min_{\partial B_2} u_n = \frac{1}{2\pi}\int_{B_{r_0}(x_n)} \log\frac{1}{|x_n - y|}(2e^{2u_n(y)} - e^{u_n(y)}|\psi_n(y)|^2)dy + O(1),$$

and hence

$$\begin{aligned} \widetilde{u}_n(x) &= u_n(\lambda_n x + x_n) - \mu_n \\ &= \frac{1}{2\pi}\int_{B_{\frac{r_0}{\lambda_n}}} \log\frac{|y|}{|x-y|}(2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2)dy + O(1). \end{aligned} \tag{21}$$

Similarly, it follows from (20) that

$$\begin{aligned} \widetilde{\psi}_n(x) &= \lambda_n^{\frac{1}{2}}\psi_n(\lambda_n x + x_n) \\ &= -\frac{1}{2\pi}\int_{B_{\frac{r_0}{\lambda_n}}} \frac{x-y}{|x-y|^2}\cdot e^{\widetilde{u}_n(y)}\widetilde{\psi}_n(y)dy + O(1). \end{aligned} \tag{22}$$

Define the local mass by

$$M_n^1 = \int_{B_{r_0}(x_n)} 2e^{2u_n} - e^{u_n}|\psi_n|^2 dx, \quad \text{and} \quad M_n^2 = \int_{B_{r_0}(x_n)} e^{u_n}\psi_n dx.$$

Claim: For any $\delta > 0$ small, there exist $R = R_\delta > 1$ and $N = N_\delta \in \mathbb{N}$ such that when $|x| \ge 2R$ and $n > N$, there holds

$$\widetilde{u}_n(x) + \frac{M_n^1}{2\pi}\log|x| \le \delta\log|x| + O(1). \tag{23}$$

In fact, notice that $\lim_{n\to\infty} M_n^1 = 4\pi$. Therefore for any small $\delta > 0$, we can choose $R > 1$ large enough such that, for n large, there holds

$$\frac{1}{2\pi}\int_{|y|\le R} 2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2 dy \ge \frac{M_n^1}{2\pi} - \frac{\delta}{2}$$

Take $|x| > 2R$, n sufficiently large and set $\Omega_n = B_{r_0\lambda_n^{-1}} \setminus (B_{\frac{|x|}{2}} \cup B(x, \frac{|x|}{2}))$, rewrite $\widetilde{u}_n$ as

$$\begin{aligned}
\widetilde{u}_n(x) &= \frac{1}{2\pi}\int_{|y|\le R} \log\frac{|y|}{|x-y|}(2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2)dy \\
&\quad + \frac{1}{2\pi}\int_{R\le|y|\le\frac{|x|}{2}} \log\frac{|y|}{|x-y|}(2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2)dy \\
&\quad + \frac{1}{2\pi}\int_{B(x,\frac{|x|}{2})} \log\frac{|y|}{|x-y|}(2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2)dy \\
&\quad + \frac{1}{2\pi}\int_{\Omega_n} \log\frac{|y|}{|x-y|}(2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2)dy + O(1) \\
&= I_1 + I_2 + I_3 + I_4 + O(1).
\end{aligned}$$

Since $\int_{B_{\frac{r_0}{\lambda_n}}(0)} e^{2\widetilde{u}_n(x)} + |\widetilde{\psi}_n(x)|^4 dv < C$ and $\frac{|y|}{|x-y|} \le 1 + \frac{|x|}{|x-y|} \le 3$ for $y \in B_{\frac{|x|}{2}}$, we have

$$\begin{aligned}
I_2 &\le -\frac{1}{2\pi}\int_{R\le|y|\le\frac{|x|}{2}} \log\frac{|y|}{|x-y|} e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2 dy + O(1) \\
&= \frac{1}{2\pi}\int_{R\le|y|\le\frac{|x|}{2}} \log\frac{|x-y|}{|y|} e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2 dy + O(1) \\
&\le \frac{1}{2\pi}\int_{R\le|y|\le\frac{|x|}{2}} \log(1+|x|) e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2 dy + O(1) \\
&\le \frac{\delta}{8}\log|x| + O(1).
\end{aligned}$$

Here we have used (15).

For I_4, noticing that $\frac{1}{3} \le \frac{|y|}{|x-y|} \le 3$ for $y \in \Omega_n$, we have

$$I_4 \le O(1).$$

For I_3, let us recall that $\widetilde{u}_n \le 0$, and set $D_1 = B(x, \frac{|x|}{2}) \cap \{|x-y| < |x|^{-1}\}$ and $D_2 = B(x, \frac{|x|}{2}) \cap \{|x-y| \ge |x|^{-1}\}$. Noticing that $\frac{|x|}{2} \le |y| \le \frac{3}{2}|x|$ in $B(x, \frac{|x|}{2})$, we can also obtain

$$\begin{aligned}
I_3 &= \frac{1}{2\pi}\int_{D_1} \log\frac{1}{|x-y|} e^{2\widetilde{u}_n(y)} dy + \frac{1}{2\pi}\int_{D_2} \log\frac{1}{|x-y|} e^{2\widetilde{u}_n(y)} dy \\
&\quad + \frac{1}{2\pi}\int_{B(x,\frac{|x|}{2})} \log|y| e^{2\widetilde{u}_n(y)} dy + \frac{1}{2\pi}\int_{B(x,\frac{|x|}{2})} \log\frac{|x-y|}{|y|} e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2 dy \\
&\le C\int_{|x-y|\le|x|^{-1}} \log\frac{1}{|x-y|} dy + C\log|x|\int_{B(x,\frac{|x|}{2})} e^{2\widetilde{u}_n(y)} dy + O(1) \\
&\le \frac{\delta}{4}\log|x| + O(1).
\end{aligned}$$

For I_1, noticing that $\frac{1}{2} \leq \frac{|x-y|}{|x|} \leq \frac{3}{2}$ for $|y| \leq R$, $\widetilde{u}_n$ and $|\widetilde{\psi}_n|$ are uniformly bounded on $|y| \leq R$, we can estimate

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_{|y|\leq R} \log\frac{|y|}{|x-y|} 2e^{2\widetilde{u}_n(y)}dy - \frac{1}{2\pi}\int_{|y|\leq R} \log\frac{|y|}{|x-y|} e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2 dy \\ &\leq \frac{1}{2\pi}\log\frac{2R}{|x|}\int_{|y|\leq R} 2e^{2\widetilde{u}_n(y)}dy + \frac{1}{2\pi}\log|x|\int_{|y|\leq R} e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2 dy \\ &\quad -\frac{1}{2\pi}\int_{|y|\leq R}\log\frac{|x|}{|x-y|}e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2 dy - \frac{1}{2\pi}\int_{|y|\leq R}\log|y| e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2 dy \\ &\leq -\frac{1}{2\pi}\log|x|\int_{|y|\leq R}(2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2)dy + O(1) \\ &\leq -(\frac{M_n^1}{2\pi} - \frac{\delta}{2})\log|x| + O(1). \end{aligned}$$

Putting these estimates together, we get (23) and complete the proof of the claim.

By (23), we take some $0 < \delta_0 < 1$ such that

$$e^{\widetilde{u}_n(x)} \leq C|x|^{-\frac{M_n^1}{2\pi}+\delta_0},$$

for $|x| \geq 2R$. Recall that $M_n^1 = 4\pi + o(1)$ and $(\widetilde{u}_n, \widetilde{\psi}_n) \to (\widetilde{u}, \widetilde{\psi})$ in $C^{1,\alpha}_{loc}(\mathbb{R}^2)$. Applying the Kelvin transformation and Lemma 3.2, we see that there is some $0 < \delta_1 < 1$ such that for n large enough, the following asymptotic estimates for the spinor $\widetilde{\psi}_n$ hold:

$$|\widetilde{\psi}_n(x)| \leq C|x|^{-\frac{1}{2}-\delta_1}, \quad |\nabla\widetilde{\psi}_n(x)| \leq C|x|^{-\frac{3}{2}-\delta_1} \tag{24}$$

for $|x| \geq 2R$. Then, choosing $\delta < 2\delta_1$ and using (23) again, we have

$$e^{\widetilde{u}_n(x)} \leq C|x|^{-\frac{M_n^1}{2\pi}+\delta}. \tag{25}$$

for $|x| \geq 2R$. By using (24), (25) and some computations, we obtain

$$\int_{B_{\frac{r_0}{\lambda_n}}} |\log|y|(2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2)|dy \leq C, \tag{26}$$

and

$$\int_{B_{\frac{r_0}{\lambda_n}}} |y||(2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2)|dy \leq C. \tag{27}$$

Now we can follow similar arguments as in [BCLT] to obtain

$$|\widetilde{u}_n(x) + \frac{M_n^1}{2\pi}\log|x|| \leq C \tag{28}$$

$$|\nabla\widetilde{u}_n(x) + \frac{M_n^1}{2\pi}\frac{x}{|x|^2}| \leq \frac{C}{|x|^2} \tag{29}$$

for $\log\frac{1}{\lambda_n} \leq |x| \leq \frac{r_0}{\lambda_n}$. For the reader's convenience, we provide the proof of (28), (29).

In fact, by setting

$$\widetilde{M}_n^1(x) = \int_{|y|\leq \eta_0|x|}(2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2)dy$$

for any small $\eta_0 > 0$ (which can be fixed later) and taking $\delta > 0$ in (25) small enough, we have

$$\begin{aligned}
|\widetilde{M}_n^1(x) - M_n^1| &= |\int_{B_{\frac{r_0}{\lambda_n}} \setminus \{|y| \le \eta_0 |x|\}} (2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)} |\widetilde{\psi}_n(y)|^2) dy| \\
&\le \int_{|y| \ge \eta_0 \log \frac{1}{\lambda_n}} (2e^{2\widetilde{u}_n(y)} + e^{\widetilde{u}_n(y)} |\widetilde{\psi}_n(y)|^2) dy \\
&\le C \int_{|y| \ge \eta_0 \log \frac{1}{\lambda_n}} |y|^{-2(\frac{M_n^1}{2\pi} - \delta)} + |y|^{-(\frac{M_n^1}{2\pi} - \delta) - 1 - 2\delta_1} dy \\
&= O(1) (\log \frac{1}{\lambda_n})^{-1 - \frac{\delta_1}{2}} \qquad (30)
\end{aligned}$$

for $\log \frac{1}{\lambda_n} \le |x| \le \frac{r_0}{\lambda_n}$. On the other hand, by using Green's representation formula for $\widetilde{u}_n$ and estimate (26), we have

$$\begin{aligned}
\widetilde{u}_n(x) &= \frac{1}{2\pi} \int_{B_{\frac{r_0}{\lambda_n}} \setminus \{|y| \le \eta_0 |x|\}} \log \frac{|y|}{|x-y|} (2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)} |\widetilde{\psi}_n(y)|^2) dy \\
&\quad + \frac{1}{2\pi} \int_{|y| \le \eta_0 |x|} \log \frac{1}{|x-y|} (2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)} |\widetilde{\psi}_n(y)|^2) dy + O(1).
\end{aligned}$$

While, using (24) and (25) (taking $\delta > 0$ in (25) small enough), we can estimate

$$\begin{aligned}
& |\int_{|y| \ge \eta_0 |x|} \log \frac{|y|}{|x-y|} (2e^{2\widetilde{u}_n(y)} - e^{\widetilde{u}_n(y)} |\widetilde{\psi}_n(y)|^2) dy| \\
\le & \int_{|y| \ge \eta_0 |x|, |x-y| < 1} \log \frac{1}{|x-y|} (2e^{2\widetilde{u}_n(y)} + e^{\widetilde{u}_n(y)} |\widetilde{\psi}_n(y)|^2) dy \\
& + \int_{|y| \ge \eta_0 |x|, |x-y| \ge 1} \log |x-y| (2e^{2\widetilde{u}_n(y)} + e^{\widetilde{u}_n(y)} |\widetilde{\psi}_n(y)|^2) dy \\
& + \int_{|y| \ge \eta_0 |x|} \log |y| (2e^{2\widetilde{u}_n(y)} + e^{\widetilde{u}_n(y)} |\widetilde{\psi}_n(y)|^2) dy \\
\le & C \int_{|y| \ge \eta_0 \log \frac{1}{\lambda_n}} \log |y| (2e^{2\widetilde{u}_n(y)} + e^{\widetilde{u}_n(y)} |\widetilde{\psi}_n(y)|^2) dy \\
= & O(1) (\log \frac{1}{\lambda_n})^{-1 - \frac{\delta_1}{4}}
\end{aligned}$$

for $\log \frac{1}{\lambda_n} \le |x| \le \frac{r_0}{\lambda_n}$. Therefore, noticing that $(1 - \eta_0)|x| \le |x - y| \le (1 + \eta_0)|x|$ when $|y| \le \eta_0 |x|$, we get

$$\widetilde{u}_n(x) = -\frac{1}{2\pi} \widetilde{M}_n^1 \log |x| + O(1)$$

provided η_0 is small enough. Consequently, by (30) we get (28).

For the proof of (29), we use Green's representation formula for $\widetilde{u}_n(x)$ again to obtain

$$\begin{aligned}
&\nabla\widetilde{u}_n(x)+\frac{M_n^1}{2\pi}\frac{x}{|x|^2}\\
=\ &\frac{1}{2\pi}\int_{|y|\le\frac{r_0}{\lambda_n}}\{\frac{x}{|x|^2}-\frac{x-y}{|x-y|^2}\}(2e^{2\widetilde{u}_n(y)}-e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2)dy+O(1)\\
=\ &\frac{1}{2\pi}\int_{G_1}\{\frac{x}{|x|^2}-\frac{x-y}{|x-y|^2}\}(2e^{2\widetilde{u}_n(y)}-e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2)dy\\
&+\frac{1}{2\pi}\int_{G_2}\{\frac{x}{|x|^2}-\frac{x-y}{|x-y|^2}\}(2e^{2\widetilde{u}_n(y)}-e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2)dy+O(1)\\
=\ &H_1+H_2+O(1)
\end{aligned}$$

where $G_1=\{|y|\le\frac{r_0}{\lambda_n}\}\cap\{|x-y|\ge\frac{|x|}{2}\}$ and $G_2=\{|y|\le\frac{r_0}{\lambda_n}\}\cap\{|x-y|\le\frac{|x|}{2}\}$. Notice that, by the mean value theorem for any $|x|\ge 1$, there holds

$$|\frac{x}{|x|^2}-\frac{x-y}{|x-y|^2}|\le 4\frac{|y|}{|x|^2},\quad \text{in } G_1,$$

and

$$|\frac{x}{|x|^2}-\frac{x-y}{|x-y|^2}|\le \frac{2}{|x-y|},\quad \text{in } G_2.$$

Hence from (27) we have

$$H_1\le\frac{C}{|x|^2}.$$

On the other hand, by the decay estimates (24) and (25), we can take $\delta>0$ small enough and n large enough to get

$$|2e^{2\widetilde{u}_n(y)}-e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|^2|\le C|y|^{-3-\frac{\delta_1}{2}},\quad \text{for } |y|\ge 2R.$$

Then, for $|x|\ge 4R$ and n large enough, we have

$$H_2\le C|x|^{-3-\frac{\delta_1}{2}}\int_{G_2}\frac{1}{|x-y|}dy\le C|x|^{-2-\frac{\delta_1}{2}}.$$

Here we have used the fact that $\frac{|x|}{2}\le|y|\le\frac{3|x|}{2}$ for $y\in G_2$. Thus we get (29).

Scaling back from (28) and (29), we obtain

$$u_n(x)=\frac{M_n^1}{2\pi}\log\frac{1}{|x-x_n|}+(1-\frac{M_n^1}{2\pi})\log\frac{1}{\lambda_n}+O(1), \tag{31}$$

$$\nabla u_n(x)=-\frac{M_n^1}{2\pi}\frac{x-x_n}{|x-x_n|^2}+O(\frac{\lambda_n}{|x-x_n|^2}), \tag{32}$$

for $x\in B_{r_0}\setminus B_{\lambda_n\log\lambda_n^{-1}}(x_n)$.
For $\widetilde{\psi}_n(x)$, scaling back from (24), we also have

$$|\psi_n(x)|\le\frac{C\lambda_n^{\delta_1}}{|x-x_n|^{\frac{1}{2}+\delta_1}},\quad |\nabla\psi_n(x)|\le\frac{C\lambda_n^{\delta_1}}{|x-x_n|^{\frac{3}{2}+\delta_1}}, \tag{33}$$

for $x\in B_{r_0}\setminus B_{\lambda_n\log\lambda_n^{-1}}(x_n)$.

Step 3. We want to show that

$$M_n^1 = 4\pi + O(1)(\log \frac{1}{\lambda_n})^{-1}. \tag{34}$$

To this purpose, we apply the Pohozaev type identity (see Lemma 3.1) in the region $B_n := B_{\lambda_n \log \frac{1}{\lambda_n}}(x_n)$ to obtain

$$\begin{aligned} & \int_{\partial B_n} r(|\frac{\partial u_n}{\partial \nu}|^2 - \frac{1}{2}|\nabla u_n|^2) d\sigma \\ = & \int_{B_n} 2e^{2u_n} - e^{u_n}|\psi_n|^2 dv - \int_{\partial B_n} re^{2u_n} d\sigma \\ & + \frac{1}{2}\int_{\partial B_n} \langle \frac{\partial \psi_n}{\partial \nu}, x \cdot \psi_n \rangle d\sigma + \frac{1}{2}\int_{\partial B_n} \langle x \cdot \psi_n, \frac{\partial \psi_n}{\partial \nu} \rangle d\sigma, \end{aligned}$$

with $r = |x|$. Substituting (31) , (32) and (33) into both sides of the above identity, we have

$$\begin{aligned} & \int_{\partial B_n} r(|\frac{\partial u_n}{\partial \nu}|^2 - \frac{1}{2}|\nabla u_n|^2) d\sigma \\ = & \int_{\partial B_n} r(\frac{1}{2}(\frac{M_n^1}{2\pi})^2 \frac{1}{r^2} + O(\frac{\lambda_n}{r^3})) d\sigma \\ = & \frac{(M_n^1)^2}{4\pi} + O(1)(\log \frac{1}{\lambda_n})^{-1}. \end{aligned} \tag{35}$$

and

$$\begin{aligned} & \int_{B_n} 2e^{2u_n} - e^{u_n}|\psi_n|^2 dv \\ = & M_n^1 - \int_{B_{r_0} \setminus B_n} 2e^{2u_n} - e^{u_n}|\psi_n|^2 dv \\ = & M_n^1 + O(1)(\log \frac{1}{\lambda_n})^{-\frac{M_n^1}{2\pi}+1} \end{aligned} \tag{36}$$

and

$$\begin{aligned} & -\int_{\partial B_n} re^{2u_n} d\sigma + \frac{1}{2}\int_{\partial B_n} \langle \frac{\partial \psi_n}{\partial \nu}, x \cdot \psi_n \rangle d\sigma + \frac{1}{2}\int_{\partial B_n} \langle x \cdot \psi_n, \frac{\partial \psi_n}{\partial \nu} \rangle d\sigma \\ = & O(1)(\log \frac{1}{\lambda_n})^{-\frac{M_n^1}{\pi}+2} + o((\lambda_n \log \frac{1}{\lambda_n})^{\frac{1}{2}}) \\ = & O(1)(\log \frac{1}{\lambda_n})^{-1}. \end{aligned} \tag{37}$$

Putting (35),(36) and (37) together, we get (34).

Step 4. Now let us prove the local estimate (16). From step 1, it is sufficient to show

$$|\widetilde{u}_n(x) - \widetilde{u}(x)| \le C \tag{38}$$

for $R \le |x| \le \frac{r_0}{\lambda_n}$, where $R > 0$ is sufficiently large.
Notice that

$$|\widetilde{u}_n(x) - \widetilde{u}(x)| \le |\widetilde{u}_n(x) + 2\log|x|| + |\widetilde{u}(x) + 2\log|x||$$

and from the asymptotic behavior of entire solutions $\widetilde{u}$ (see (13)), we have

$$|\widetilde{u}(x)+2\log|x||\leq C$$

for $|x|\geq R$ and for R large enough. So, to prove (38), it is sufficient to prove

$$|\widetilde{u}_n(x)+2\log|x||\leq C$$

for $R\leq|x|\leq\frac{r_0}{\lambda_n}$.

For this purpose, by (28) and (34), we firstly have

$$|\widetilde{u}_n(x)+2\log|x||\leq|\widetilde{u}_n(x)+\frac{M_n^1}{2\pi}\log|x||+|\frac{M_n^1}{2\pi}\log|x|-2\log|x||\leq C$$

for $\log\lambda_n^{-1}\leq|x|\leq\frac{r_0}{\lambda_n}$.

Since $(\widetilde{u}_n,\widetilde{\psi}_n)$ converges to $(\widetilde{u},\widetilde{\psi})$ uniformly for $|x|\leq R$ for any large $R>0$, and $\widetilde{u}(x)$ satisfies $|\widetilde{u}(x)+2\log|x||\leq C$ for $|x|\geq R$, we have

$$|\widetilde{u}_n(x)+2\log|x||\leq|\widetilde{u}_n(x)-\widetilde{u}(x)|+|\widetilde{u}(x)+2\log|x||\leq 2C$$

for $|x|=R$ and n large enough. Now we define $w_\pm$ by

$$w_\pm(x)=-2\log|x|\pm(C_1-C_1|x|^{-\frac{1}{2}}).$$

Then it is clear that

$$\triangle w_\pm(x)=\mp\frac{1}{4}C_1|x|^{-\frac{5}{2}}$$

for $|x|\geq R$. Therefore, by using the decay estimates (24), (25), and suitably choosing $C_1>0$, we have

$$\begin{cases}-\triangle w_-(x)\leq-\triangle\widetilde{u}_n(x)\leq-\triangle w_+(x), & \text{for } R\leq|x|\leq\log\frac{1}{\lambda_n},\\ w_-(x)\leq\widetilde{u}_n(x)\leq w_+(x), & \text{on } |x|=R,|x|=\log\frac{1}{\lambda_n}\end{cases}\tag{39}$$

Hence, by the maximum principle, we conclude that

$$w_-(x)\leq\widetilde{u}_n(x)\leq w_+(x)$$

for $R\leq|x|\leq\log\frac{1}{\lambda_n}$. Thus we complete the local estimate (16) of u_n.

Step 5 Now we establish the local estimate (17) of ψ_n.
From step 1, it is sufficient to show

$$|\widetilde{\psi}_n(x)-\widetilde{\psi}(x)|\leq C\tag{40}$$

for $R\leq|x|\leq\frac{r_0}{\lambda_n}$, where R is sufficiently large.

At this point, by (14), we notice that

$$\widetilde{\psi}(x)=-\frac{1}{2\pi}\frac{x}{|x|^2}\cdot\xi_0+o(|x|^{-1})$$

for $|x| \geq R$, and

$$\begin{aligned}
|M_n^2 - \lambda_n^{\frac{1}{2}}\xi_0| &= |\int_{B_{r_0}\setminus B_{\lambda_n R}(x_n)} e^{u_n}\psi_n dx + \int_{B_{\lambda_n R}(x_n)} e^{u_n}\psi_n dx - \lambda_n^{\frac{1}{2}}\int_{\mathbb{R}^2}\widetilde{\psi}e^{\widetilde{u}}dx| \\
&\leq |\int_{B_{r_0}\setminus B_{\lambda_n R}(x_n)} e^{u_n}\psi_n dx| + |\lambda_n^{\frac{1}{2}}\int_{B_R} e^{\widetilde{u}_n}\widetilde{\psi}_n dx - \lambda_n^{\frac{1}{2}}\int_{\mathbb{R}^2}\widetilde{\psi}e^{\widetilde{u}}dx| \\
&\leq |\lambda_n^{\frac{1}{2}}\int_{B_{\frac{r_0}{\lambda_n}}\setminus B_R(x_n)} e^{\widetilde{u}_n}\widetilde{\psi}_n dx| + |\lambda_n^{\frac{1}{2}}\int_{B_R} e^{\widetilde{u}_n}\widetilde{\psi}_n dx - \lambda_n^{\frac{1}{2}}\int_{\mathbb{R}^2}\widetilde{\psi}e^{\widetilde{u}}dx| \\
&= o(\lambda_n^{\frac{1}{2}}).
\end{aligned}$$

Here we have used the decay estimates (24), (25) for $(\widetilde{u}_n, \widetilde{\psi}_n)$.

Recall that

$$x \cdot x \cdot \psi = -|x|^2\psi,$$

for any $x = x_1e_1 + x_2e_2 \in \mathbb{R}^2$ and any spinor ψ on $\mathbb{R}^2$, where $\{e_1, e_2\}$ is the standard orthonormal basis for $\mathbb{R}^2$ and $\cdot$ is the Clifford multiplication. Then, using Green's representation formula (22) for $\widetilde{\psi}_n(x)$, we calculate

$$\begin{aligned}
&|\widetilde{\psi}_n(x) + \frac{1}{2\pi\lambda_n^{\frac{1}{2}}}\frac{x}{|x|^2}\cdot M_n^2| \\
=\ & |\frac{x}{|x|^2}\cdot(x\cdot\widetilde{\psi}_n(x) - \frac{1}{2\pi\lambda_n^{\frac{1}{2}}}M_n^2)| \\
\leq\ & \frac{1}{2\pi|x|}|\int_{B_{r_0\lambda_n^{-1}}}(\frac{x\cdot(x-y)}{|x-y|^2}+1)\cdot e^{\widetilde{u}_n(y)}\widetilde{\psi}_n(y)dy| + O(1)|x|^{-1} \\
=\ & \frac{1}{2\pi|x|}|\int_{B_{r_0\lambda_n^{-1}}}\frac{y\cdot(x-y)}{|x-y|^2}\cdot e^{\widetilde{u}_n(y)}\widetilde{\psi}_n(y)dy| + O(1)|x|^{-1} \\
\leq\ & \frac{1}{2\pi|x|}\int_{B_{r_0\lambda_n^{-1}}}\frac{|y|}{|x-y|}\cdot e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|dy + O(1)|x|^{-1}
\end{aligned}$$

for $R \leq |x| \leq \frac{r_0}{\lambda_n}$.

By the decay estimates (24), (25) for $(\widetilde{u}_n, \widetilde{\psi}_n)$, there exists $0 < \delta_2 < 1$ such that

$$e^{\widetilde{u}_n(x)}|\widetilde{\psi}_n(x)| \leq c|x|^{-2-\delta_2}$$

for $|x| \geq R$ and for n large enough. Then, similarly to the derivation of the gradient estimates in [CK], we can estimate

$$\begin{aligned}
&\frac{1}{2\pi|x|}\int_{B_{r_0\lambda_n^{-1}}}\frac{|y|}{|x-y|}\cdot e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|dy \\
=\ & \frac{1}{2\pi|x|}\int_{|y|\leq\frac{|x|}{2}}\frac{|y|}{|x-y|}\cdot e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|dy \\
&+\frac{1}{2\pi|x|}\int_{\frac{|x|}{2}\leq|y|\leq 2|x|}\frac{|y|}{|x-y|}\cdot e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|dy \\
&+\frac{1}{2\pi|x|}\int_{2|x|\leq|y|\leq\frac{r_0}{\lambda_n}}\frac{|y|}{|x-y|}\cdot e^{\widetilde{u}_n(y)}|\widetilde{\psi}_n(y)|dy \\
\leq\ & C|x|^{-1}
\end{aligned}$$

for $R \le |x| \le \frac{r_0}{\lambda_n}$. Hence, there holds

$$|\widetilde{\psi}_n(x) + \frac{1}{2\pi\lambda_n^{\frac{1}{2}}}\frac{x}{|x|^2} \cdot M_n^2| \le C|x|^{-1}$$

for $R \le |x| \le \frac{r_0}{\lambda_n}$.

Put these estimates together, we have

$$\begin{aligned} & |\widetilde{\psi}_n(x) - \widetilde{\psi}(x)| \\ \le\ & |\widetilde{\psi}_n(x) + \frac{1}{2\pi}\frac{x}{\lambda_n^{\frac{1}{2}}|x|^2}M_n^2| + |\frac{1}{2\pi}\frac{x}{\lambda_n^{\frac{1}{2}}|x|^2}M_n^2 - \frac{1}{2\pi}\frac{x}{|x|^2}\xi_0| + |\widetilde{\psi}(x) + \frac{1}{2\pi}\frac{x}{|x|^2}\xi_0| \\ \le\ & C|x|^{-1} \\ \le\ & C \end{aligned}$$

for $R \le |x| \le \frac{r_0}{\lambda_n}$.

Thus we complete the proof of Theorem 2.3. □

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