Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Dimension reduction for compressible viscous fluids

by

Peter Bella, Eduard Feireisl, and Antonín Novotný

Preprint no.: 51 2013

Dimension reduction for compressible viscous fluids

Peter Bella   Eduard Feireisl *   Antonín Novotný

Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22, 04103 Leipzig, Germany

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

IMATH, EA 2134, Université du Sud Toulon-Var BP 20132, 83957 La Garde, France

Abstract

We consider the barotropic Navier-Stokes system describing the motion of a compressible viscous fluid confined to a cavity shaped as a thin rod $\Omega_\varepsilon = \varepsilon Q \times (0,1)$, $Q \subset R^2$. We show that the weak solutions in the $3D$ domain converge to (strong) solutions of the limit $1D$ Navier-Stokes system as $\varepsilon \to 0$.

Key words: Compressible Navier-Stokes system, dimension reduction, thin rod

1 Introduction

Although all fluid flows are in general three-dimensional, in many cases the specific shape of the physical domain enforces major changes in the density and velocity only in two directions or even only in one. A typical example is the fluid flow confined to a thin tube that can be effectively described by using only one spatial variable, while the influence of the cross section profile can be ignored. We consider a family of shrinking domains

$$\Omega_\varepsilon = Q_\varepsilon \times (0,1),\ Q_\varepsilon \subset R^2,\ Q_\varepsilon = \varepsilon Q,\ \varepsilon \to 0,$$

*Eduard Feireisl acknowledges the support of the GAČR (Czech Science Foundation) project P201-13-00522S in the framework of RVO: 67985840.

where $Q \subset R^2$ is a regular planar domain. We assume that the initial distribution of the fluid mass density $\varrho_{0,\varepsilon}$ and the velocity field $\mathbf{u}_{0,\varepsilon}$ are defined on Ω_ε, where the integral averages

$$\frac{1}{|Q_\varepsilon|}\int_{Q_\varepsilon} \varrho_{0,\varepsilon}(x_h, y)\ \mathrm{d}x_h,\ \frac{1}{|Q_\varepsilon|}\int_{Q_\varepsilon} \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}(x_h, y)\ \mathrm{d}x_h,\ x_h = (x_1, x_2),\ y = x_3,$$

converge weakly (with respect to the x_3-variable) to some limit, specifically,

$$\frac{1}{|Q_\varepsilon|}\int_{Q_\varepsilon} \varrho_{0,\varepsilon}(x_h, \cdot)\ \mathrm{d}x_h \to \varrho_0,\ \frac{1}{|Q_\varepsilon|}\int_{Q_\varepsilon} \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}(x_h, \cdot)\ \mathrm{d}x_h \to (\varrho\mathbf{u})_0 \text{ weakly in } L^1(0,1) \tag{1.1}$$

as $\varepsilon \to 0$. We suppose that the time evolution of $\varrho_\varepsilon = \varrho_\varepsilon(t, x)$ and $\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon(t, x)$ is governed by the standard *barotropic Navier-Stokes system*:

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \tag{1.2}$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_x p(\varrho_\varepsilon) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon), \tag{1.3}$$

where p is the pressure, and $\mathbb{S}$ is the viscous stress tensor given by *Newton's law*

$$\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) = \mu\left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I}\right) + \eta\,\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I}, \tag{1.4}$$

with the shear viscosity coefficient $\mu > 0$ and the bulk viscosity coefficient $\eta > 0$.

As the limit data depend only on the "vertical" variable y, a natural candidate for the limit problem is the $1D$ compressible Navier-Stokes system:

$$\partial_t \varrho + \partial_y(\varrho v) = 0, \tag{1.5}$$

$$\partial_t(\varrho v) + \partial_y(\varrho v^2) + \partial_y p(\varrho) = \nu \partial^2_{y,y} v,\ \nu = \frac{4}{3}\mu + \eta, \tag{1.6}$$

where $\varrho = \varrho(t, y)$, $v = v(t, y)$.

Our goal is to justify the above (formal) limit in the framework of weak solutions of the *primitive system* (1.2), (1.3), supplemented by the *complete slip boundary conditions*

$$\mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ [\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon)\cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \tag{1.7}$$

where the symbol $\mathbf{n}$ denotes the outer normal vector. We remark that the use of the slip instead of the more conventional no-slip condition $\mathbf{u}_\varepsilon|_{\partial\Omega_\varepsilon}$ is quite natural in the present context as the latter would completely stop the fluid motion in the asymptotic limit $\varepsilon \to 0$.

Although a rigorous justification of the limit passage from the $3D$-fluid motion to a linear one seems of obvious practical importance, there are only a few results available in the literature, at least in the context of *compressible* fluids. There are numerous studies of the *incompressible* fluid flows on thin domains, where the limit motion becomes planar and even regular, see Iftimie, Raugel and Sell [7], Raugel and Sell [17], [15], [16], and the references therein. Obviously, the $3D$ to $1D$ limit does not make too much sense in the incompressible setting. To the best of our knowledge, the first and so far the only attempt to address the $3D-1D$ limit in the compressible context is the paper of Vodák [19], containing several interesting ideas but quite unsatisfactory results.

Analysis of similar dimension reduction problems in the elasticity theory leans on variants of the celebrated Korn's inequality that provides estimates on the gradient of a vector function $\mathbf{v}$ in terms of its symmetric part, specifically,

$$\|\nabla_x \mathbf{v}\|_{L^2(\Omega_\varepsilon;R^{3\times 3})} \leq c(\varepsilon) \left\| \nabla_x \mathbf{v} + \nabla_x^t \mathbf{v} \right\|_{L^2(\Omega_\varepsilon;R^{3\times 3})}, \ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0. \tag{1.8}$$

Clearly, the validity of (1.8) requires the kernel of the linear operator $\mathbf{v} \mapsto \nabla_x \mathbf{v} + \nabla_x^t \mathbf{v}$, $\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon}$ to be empty, in particular, the "bottom" set Q must not be rotationally symmetric. Still, even if (1.8) holds for any fixed $\varepsilon > 0$, the constant $c(\varepsilon)$ blows up for $\varepsilon \to 0$ unless some necessary restrictions are imposed on the field $\mathbf{v}$, and this is true even if the set Q is not rotationally symmetric, cf. the interesting paper by Lewicka and Müller [9].

It is not difficult to see that the problems arising in the context of *compressible* fluids would need a stronger analogue of (1.8), namely

$$\|\nabla_x \mathbf{v}\|_{L^2(\Omega_\varepsilon;R^{3\times 3})} \leq c(\varepsilon) \left\| \nabla_x \mathbf{v} + \nabla_x^t \mathbf{v} - \frac{2}{3}\mathrm{div}_x \mathbf{v}\mathbb{I} \right\|_{L^2(\Omega_\varepsilon;R^{3\times 3})}, \ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \tag{1.9}$$

obviously related to the shear viscosity component of the viscous stress tensor (1.4), see Dain [2], Reshetnyak [18]. In view of the above mentioned difficulties related to the validity of (1.8) or (1.9), our approach relies on the structural stability of the family of solutions of the barotropic Navier-Stokes system encoded in the *relative entropy inequality* introduced in [4], [5]. This method is basically independent of the specific form of the viscous stress and of possible "dissipative" bounds for the Navier-Stokes system.

The paper is organized in the following manner. In Section 2 we summarize the necessary preliminary material including a proper definition of *finite energy weak solutions* to the Navier-Stokes system (1.2 - 1.4). We also introduce the relative entropy inequality and formulate our main result. In Section 3, we establish convergence towards the target system (1.5), (1.6).

2 Preliminaries, main result

We say that $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ is a *finite energy weak solution* to the barotropic Navier-Stokes (1.2 - 1.4), (1.7) in the space time cylinder $(0,T)\times\Omega_\varepsilon$ if the following holds:

- the functions $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ belong to the regularity class

$$\left\{ \begin{array}{c} \varrho_\varepsilon \in C_{\rm weak}([0,T]; L^\gamma(\Omega_\varepsilon)),\ \varrho_\varepsilon \geq 0 \text{ a.a. in } (0,T) \times \Omega_\varepsilon,\ \gamma > \frac{3}{2}, \\ \\ \mathbf{u}_\varepsilon \in L^2(0,T; W^{1,2}(\Omega_\varepsilon; R^3)),\ \mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ \varrho_\varepsilon \mathbf{u}_\varepsilon \in C_{\rm weak}([0,T]; L^{2\gamma/(\gamma+1)}(\Omega_\varepsilon)), \\ \\ p(\varrho_\varepsilon) \in L^\infty(0,T; L^1(\Omega_\varepsilon)); \end{array} \right\} \tag{2.1}$$

- the *continuity equation* (1.2) is replaced by the family of integral identities

$$\int_0^T \int_{\Omega_\varepsilon} \varrho_\varepsilon \Big(\partial_t \varphi + \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \Big) \ {\rm d}x \ {\rm d}t = - \int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} \varphi(0, \cdot) \ {\rm d}x \tag{2.2}$$

  for any test function $\varphi \in C^\infty_c([0,T) \times \overline{\Omega}_\varepsilon)$;

- the *momentum equation* (1.3) holds in the sense that

$$\int_0^T \int_{\Omega_\varepsilon} \Big(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \varphi + p(\varrho_\varepsilon) {\rm div}_x \varphi \Big) \ {\rm d}x \ {\rm d}t \tag{2.3}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \varphi \ {\rm d}x \ {\rm d}t - \int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \varphi(0, \cdot) \ {\rm d}x$$

  for any $\varphi \in C^\infty_c([0,T) \times \overline{\Omega}_\varepsilon; R^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0$;

- the *energy inequality*

$$\int_{\Omega_\varepsilon} \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + H(\varrho_\varepsilon) \right] (\tau, \cdot) \ {\rm d}x + \int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \ {\rm d}x \ {\rm d}t \tag{2.4}$$

$$\leq \int_{\Omega_\varepsilon} \left[\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + H(\varrho_{0,\varepsilon}) \right] \ {\rm d}x,\ \ H(\varrho) \equiv \varrho \int_1^\varrho \frac{p(z)}{z^2} \ {\rm d}z$$

  holds for a.a. $\tau \in (0,T)$.

The reader may consult the monograph [11] by P.-L.Lions and/or [3] for the mathematical theory of compressible viscous fluids in the framework of weak solutions. In particular, the weak solutions are known to exist globally in time for any finite energy initial data as soon as the pressure p satisfies

$$p \in C[0,\infty) \cap C^2(0,\infty),\ p'(\varrho) > 0 \text{ for all } \varrho > 0,\ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0,\ \gamma > \frac{3}{2}. \tag{2.5}$$

2.1 Relative entropy inequality

Motivated by [5] (see also Dafermos [1], Germain [6], Mellet, Vasseur [13]) we introduce the (scaled) relative entropy functional

$$\mathcal{E}_\varepsilon\left([\varrho,\mathbf{u}]\Big|[r,\mathbf{U}]\right) = \frac{1}{|Q_\varepsilon|}\int_0^1\int_{Q_\varepsilon}\left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r)\right]\ \mathrm{d}x_h\ \mathrm{d}x_3, \tag{2.6}$$

along with the *relative entropy inequality*

$$\mathcal{E}_\varepsilon\left([\varrho,\mathbf{u}]\Big|[r,\mathbf{U}]\right)(\tau) + \frac{1}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}\Big(\mathbb{S}(\nabla_x\mathbf{u}) - \mathbb{S}(\nabla_x\mathbf{U})\Big) : \Big(\nabla_x\mathbf{u} - \nabla_x\mathbf{U}\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{2.7}$$

$$\leq \mathcal{E}_\varepsilon\left([\varrho,\mathbf{u}]\Big|[r,\mathbf{U}]\right)(0) + \int_0^\tau \mathcal{R}_\varepsilon(\varrho,\mathbf{u},r,\mathbf{U})\ \mathrm{d}t,$$

with the remainder term

$$\mathcal{R}_\varepsilon\left(\varrho,\mathbf{u},r,\mathbf{U}\right) \equiv \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\varrho\Big(\partial_t\mathbf{U} + \mathbf{u}\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x \tag{2.8}$$

$$+\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\mathbb{S}(\nabla_x\mathbf{U}) : \nabla_x(\mathbf{U}-\mathbf{u})\ \mathrm{d}x$$

$$+\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left((r-\varrho)\partial_t H'(r) + \nabla_x H'(r)\cdot(r\mathbf{U} - \varrho\mathbf{u})\right)\ \mathrm{d}x - \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\mathrm{div}_x\mathbf{U}\Big(p(\varrho) - p(r)\Big)\ \mathrm{d}x.$$

Following the terminology of DiPerna and Lions [10] we say that $[\varrho_\varepsilon,\mathbf{u}_\varepsilon]$ is a *dissipative solution* to the barotropic Navier-Stokes system if $[\varrho_\varepsilon,\mathbf{u}_\varepsilon]$ enjoy the regularity specified in (2.1), and if the relative entropy inequality (2.8) is satisfies for $\varrho = \varrho_\varepsilon$, $\mathbf{u} = \mathbf{u}_\varepsilon$ and any pair of test functions

$$r \in C^\infty([0,T]\times\overline{\Omega}_\varepsilon),\ r > 0,\ \mathbf{U} \in C^\infty([0,T]\times\overline{\Omega}_\varepsilon; R^3),\ \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

We remark that the class of test functions can be extended to less regular $[r,\mathbf{U}]$ by means of density arguments. Now, the crucial observation is that *any* finite energy weak solution defined through (2.1 - 2.4) is a dissipative solution satisfying the relative entropy inequality, see [4, Section 3.2.1].

2.2 Solutions of the target system

Since the velocity fields $\mathbf{u}_\varepsilon$ satisfy the slip boundary conditions (1.7), it is natural to expect that the limit v will satisfy the no-slip boundary condition

$$v(\cdot,0) = v(\cdot,1) = 0, \tag{2.9}$$

together with the initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \varrho(0,\cdot)v(0,\cdot) = (\varrho u)_0^3 \text{ in } (0,1). \tag{2.10}$$

The initial-boundary value problem (2.9), (2.10) for the $1D$ Navier-Stokes system (1.5), (1.6) is nowadays well understood. The problem is well-posed, meaning admits global in time solutions, for any initial data satisfying

$$\varrho_0 \in W^{1,2}(0,1),\ \inf_{y\in(0,1)} \varrho_0(y) \geq \underline{\varrho} > 0,\ v \in W_0^{1,2}(0,1). \tag{2.11}$$

The corresponding (weak) solution $[\varrho, v]$ is unique in the class

$$\varrho \in L^\infty(0,T;W^{1,2}(0,1)),\ v \in L^\infty(0,T;W_0^{1,2}(0,1)) \cap L^2(0,T;W^{2,2}(0,1)), \tag{2.12}$$

and satisfies

$$0 < \underline{\varrho}(t) \leq \varrho(t,x) \leq \overline{\varrho}(t) < \infty,\ t \geq 0, \tag{2.13}$$

see Kazhikhov [8].

For future use, however, we will need more regular solutions. Supposing, in addition to (2.11), that

$$\left\{ \begin{array}{c} \varrho_0 \in C^{1+\beta}[0,1],\ v_0 = \frac{(\varrho u)_0^3}{\varrho_0} \in C^{2+\beta}[0,1],\ \beta > 0, \\ \\ \text{with the compatibility conditions } v_0|_{y=0,1} = \partial^2_{y,y} v_0|_{y=0,1} = \partial_y \varrho_0|_{0,1} = 0, \end{array} \right\} \tag{2.14}$$

one can show that the solution $[\varrho, v]$ is classical, see Kazhikhov [8].

2.3 Main result

We are ready to state our main result.

Theorem 2.1 *Let $Q \subset R^2$ be a Lipshitz domain. Suppose that the pressure $p = p(\varrho)$ satisfies the hypothesis (2.5), and that the viscous stress tensor $\mathbb{S}$ is given by (1.4), with the viscosity coefficients*

$$\mu > 0,\ \eta > 0. \tag{2.15}$$

Let

$$\frac{1}{|Q_\varepsilon|}\int_{Q_\varepsilon} \varrho_{0,\varepsilon}(x_h,\cdot)\ \mathrm{d}x_h \to \varrho_0,\ \frac{1}{|Q_\varepsilon|}\int_{Q_\varepsilon} \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}(x_h,\cdot)\ \mathrm{d}x_h \to (\varrho\mathbf{u})_0 \text{ weakly in } L^1(0,1), \tag{2.16}$$

where $\varrho_0 > \underline{\varrho}$, $(\varrho u)^3_0 = v_0$ *belong to the regularity class (2.14), and let*

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + H(\varrho_{0,\varepsilon})\right]\ \mathrm{d}x \to \int_0^1 \left[\frac{1}{2\varrho_0}|(\varrho u)^3_0|^2 + H(\varrho_0)\right]\ \mathrm{d}y \tag{2.17}$$

Finally, let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ *be a dissipative solution of the barotropic Navier-Stokes system (1.2 - 1.4) in* $(0,T)\times\Omega_\varepsilon$*, emanating from the initial data*

$$\varrho_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon},\ (\varrho_\varepsilon\mathbf{u}_\varepsilon)(0,\cdot) = \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}.$$

Then

$$\operatorname*{ess\ sup}_{t\in(0,T)}\frac{1}{|Q_\varepsilon|}\int_0^1\int_{Q_\varepsilon}|\varrho_\varepsilon(t,x_h,y) - \varrho(t,y)|^\gamma\ \mathrm{d}x_h\ \mathrm{d}y \to 0, \tag{2.18}$$

and

$$\operatorname*{ess\ sup}_{t\in(0,T)}\frac{1}{|Q_\varepsilon|}\int_0^1\int_{Q_\varepsilon}\left|\varrho_\varepsilon\mathbf{u}_\varepsilon(t,x_h,y) - [0,0,\varrho v](t,y)\right|^{2\gamma/(\gamma+1)}\ \mathrm{d}x_h\ \mathrm{d}y \to 0 \tag{2.19}$$

as $\varepsilon \to 0$*, where* $[\varrho, v]$ *is the unique solution of the* $1D$ *Navier-Stokes system (1.5), (1.6), with the initial data* $[\varrho_0, v_0]$ *and the no-slip boundary condition (2.9).*

Remark 2.1 *The hypothesis (2.17) is the same as in the paper by Vodák [19]. The meaning is that the averages of the initial energy of the barotropic Navier-Stokes system converge to the initial energy of the target system.*

Remark 2.2 *The hypothesis (2.15) is crucial in the analysis. In particular, we suppose that the* bulk *viscosity coefficient* η *is strictly positive. The physical relevance of such a stipulation was thoroughly discussed in a recent paper by Rajagopal [14].*

The rest of the paper is devoted to the proof of Theorem 2.1.

3 Convergence

Let $\underline{\varrho} > 0$, $\overline{\varrho} > 0$ be two positive constants such that

$$0 < \underline{\varrho} \leq \varrho(t,y) \leq \overline{\varrho} \text{ for all } t\in[0,T],\ y\in(0,1).$$

For each measurable function h, we define

$$h_{\mathrm{ess}}(t,x) = h\ 1_{\{\underline{\varrho}/2 < \varrho_\varepsilon(t,x) \leq 2\overline{\varrho}\}},\ h_{\mathrm{res}} = h - h_{\mathrm{ess}}.$$

From now on, we shall identify all functions depending only on the vertical variable y with functions defined on Ω_ε, extended to be constant in the x_h variable. Similarly, we shall write v for the vector-valued function $[0,0,v]$. In particular, we have

$$\mathcal{E}_\varepsilon\left([\varrho_\varepsilon,\mathbf{u}_\varepsilon]\Big|[\varrho,v]\right) \tag{3.1}$$

$$\geq \frac{c}{|Q_\varepsilon|}\left(\int_{\Omega_\varepsilon}|[\mathbf{u}_\varepsilon - v]_{\rm ess}|^2 + |\ [\varrho_\varepsilon-\varrho]_{\rm ess}|^2\right)\ {\rm d}x + \frac{c}{|Q_\varepsilon|}\left(\int_{\Omega_\varepsilon}\varrho_\varepsilon\,|[\mathbf{u}_\varepsilon - v]_{\rm res}|^2 + |\ [\varrho_\varepsilon-\varrho]_{\rm res}|^\gamma\right)\ {\rm d}x.$$

3.1 Application of the relative entropy inequality

The main idea of the proof of Theorem 2.1 is to take $\varrho=\varrho_\varepsilon$, $\mathbf{u}=\mathbf{u}_\varepsilon$, $r=\varrho$, $\mathbf{U}=[0,0,v]$ as test functions in the relative entropy inequality (2.7). To begin, observe that the hypotheses (2.16), (2.17) guarantee that

$$\mathcal{E}_\varepsilon\left([\varrho_{0,\varepsilon},\mathbf{u}_{0,\varepsilon}]\Big|[\varrho_0,v]\right)\to 0 \text{ as } \varepsilon\to 0, \tag{3.2}$$

and, by virtue of (2.15),

$$\int_{\Omega_\varepsilon}\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon)-\mathbb{S}(\nabla_x v):(\nabla_x\mathbf{u}_\varepsilon-\nabla_x v)\ {\rm d}x \geq c\int_{\Omega_\varepsilon}|\partial_y u^3_\varepsilon - \partial_y v|^2\ {\rm d}x,$$

with c independent of ε. Since

$$u^3_\varepsilon(\cdot,y)=v(\cdot,y)=0 \text{ for } y=0,1,$$

we may infer that

$$\int_{\Omega_\varepsilon}\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon)-\mathbb{S}(\nabla_x v):(\nabla_x\mathbf{u}_\varepsilon-\nabla_x v)\ {\rm d}x \geq c\int_{\Omega_\varepsilon}|u^3_\varepsilon - v|^2\ {\rm d}x. \tag{3.3}$$

Summing up the previous estimates, the relative entropy inequality takes the form

$$\mathcal{E}_\varepsilon\left([\varrho_\varepsilon,\mathbf{u}_\varepsilon]\Big|[\varrho,v]\right)(\tau) + \frac{c}{|Q_\varepsilon|}\int_0^\tau\int_{\Omega_\varepsilon}|u^3_\varepsilon - v|^2\ {\rm d}x\ {\rm d}t \tag{3.4}$$

$$\leq \delta_1(\varepsilon) + \int_0^\tau \mathcal{R}_\varepsilon(\varrho_\varepsilon,\mathbf{u}_\varepsilon,\varrho,v)\ {\rm d}t,\ \ \delta_1(\varepsilon)\to 0 \text{ as } \varepsilon\to 0.$$

Our ultimate goal is to show that all terms appearing in the remainder $\mathcal{R}_\varepsilon$ can be “absorbed” by the right-hand side of (3.4) by means of a Gronwall type argument.

Step 1:

We have

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \varrho_\varepsilon \Big(\partial_t v + u^3_\varepsilon \partial_y v\Big)(v - u^3_\varepsilon)\ \mathrm{d}x$$

$$= \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \varrho_\varepsilon \Big(\partial_t v + v \partial_y v\Big)(v - u^3_\varepsilon)\ \mathrm{d}x + \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \varrho_\varepsilon (u^3_\varepsilon - v)^2 \partial_y v\ \mathrm{d}x$$

$$= \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \frac{\varrho_\varepsilon}{\varrho} \Big(\nu \partial^2_{y,y} v - \partial_y p(\varrho)\Big)(v - u^3_\varepsilon)\ \mathrm{d}x + \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \varrho_\varepsilon (u^3_\varepsilon - v)^2 \partial_y v\ \mathrm{d}x.$$

Consequently, relation (3.4) takes the form

$$\mathcal{E}_\varepsilon \left([\varrho_\varepsilon, \mathbf{u}_\varepsilon] \Big| [\varrho, v] \right)(\tau) + \frac{c}{|Q_\varepsilon|}\int_0^\tau \int_{\Omega_\varepsilon} |u^3_\varepsilon - v|^2\ \mathrm{d}x\ \mathrm{d}t \tag{3.5}$$

$$\leq \delta_1(\varepsilon) + \int_0^\tau h_1(t) \mathcal{E}_\varepsilon \left([\varrho_\varepsilon, \mathbf{u}_\varepsilon] \Big| [\varrho, v] \right)(t)\ \mathrm{d}t + \int_0^\tau \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \frac{\nu}{\varrho} \partial^2_{y,y} v (\varrho_\varepsilon - \varrho)(v - u^3_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t$$

$$+ \int_0^\tau \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \left((\varrho - \varrho_\varepsilon) \frac{p'(\varrho)}{\varrho} (\partial_t \varrho + v \partial_y \varrho) \right)\ \mathrm{d}x\ \mathrm{d}t - \int_0^\tau \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \partial_y v \Big(p(\varrho_\varepsilon) - p(\varrho) \Big)\ \mathrm{d}x\ \mathrm{d}t,$$

where

$$h_1(t) = \| \partial_y v(t, \cdot) \|_{L^\infty(0,1)}.$$

Step 2:

Putting the last two integrals in (3.5) together and making use of the continuity equation (1.5) we obtain

$$\int_0^\tau \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \left((\varrho - \varrho_\varepsilon) \frac{p'(\varrho)}{\varrho} (\partial_t \varrho + v \partial_y \varrho) \right)\ \mathrm{d}x\ \mathrm{d}t - \int_0^\tau \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \partial_y v \Big(p(\varrho_\varepsilon) - p(\varrho) \Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$= - \int_0^\tau \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \partial_y v \Big(p(\varrho_\varepsilon) - p'(\varrho)(\varrho_\varepsilon - \varrho) - p(\varrho) \Big)\ \mathrm{d}x\ \mathrm{d}t,$$

where the resulting expression is controllable by means of the relative entropy. Thus (3.5) reduces to

$$\mathcal{E}_\varepsilon \left([\varrho_\varepsilon, \mathbf{u}_\varepsilon] \Big| [\varrho, v] \right)(\tau) + \frac{c}{|Q_\varepsilon|}\int_0^\tau \int_{\Omega_\varepsilon} |u^3_\varepsilon - v|^2\ \mathrm{d}x\ \mathrm{d}t \tag{3.6}$$

$$\leq \delta_1(\varepsilon) + \int_0^\tau \Big(h_1(t) + h_2(t) \Big) \mathcal{E}_\varepsilon \left([\varrho_\varepsilon, \mathbf{u}_\varepsilon] \Big| [\varrho, v] \right)(t)\ \mathrm{d}t + \int_0^\tau \frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon} \frac{\nu}{\varrho} \partial^2_{y,y} v (\varrho_\varepsilon - \varrho)(v - u^3_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t$$

where

$$h_2(t) \leq c \left\| \partial_y v(t, \cdot) \right\|_{L^\infty(0,1)}.$$

Step 3:

We write

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\frac{1}{\varrho}\partial^2_{y,y}v(\varrho_\varepsilon-\varrho)(v-u^3_\varepsilon)\ \mathrm{d}x$$

$$=\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left[\frac{1}{\varrho}\partial^2_{y,y}v\right]_{\rm ess}(\varrho_\varepsilon-\varrho)(v-u^3_\varepsilon)\ \mathrm{d}x+\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left[\frac{1}{\varrho}\partial^2_{y,y}v\right]_{\rm res}(\varrho_\varepsilon-\varrho)(v-u^3_\varepsilon)\ \mathrm{d}x,$$

where

$$\left|\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left[\frac{1}{\varrho}\partial^2_{y,y}v\right]_{\rm ess}(\varrho_\varepsilon-\varrho)(v-u^3_\varepsilon)\ \mathrm{d}x\right|\leq\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}h_3(t)\left(|\varrho_\varepsilon-\varrho|^2+\varrho_\varepsilon|v-\mathbf{u}_\varepsilon|^2\right)\ \mathrm{d}x,\tag{3.7}$$

$$h_3\leq c\left\|\frac{1}{\varrho}\partial^2_{y,y}v\right\|_{L^\infty(0,1)}.$$

Next, we write

$$\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}\left[\frac{1}{\varrho}\partial^2_{y,y}v\right]_{\rm res}(\varrho_\varepsilon-\varrho)(v-u^3_\varepsilon)\ \mathrm{d}x$$

$$=\frac{1}{|Q_\varepsilon|}\int_{\{\varrho_\varepsilon<\underline{\varrho}/2\}}\left(\frac{1}{\varrho}\partial^2_{y,y}v\right)(\varrho_\varepsilon-\varrho)(v-u^3_\varepsilon)\ \mathrm{d}x+\frac{1}{|Q_\varepsilon|}\int_{\{\varrho_\varepsilon>2\overline{\varrho}\}}\left(\frac{1}{\varrho}\partial^2_{y,y}v\right)(\varrho_\varepsilon-\varrho)(v-u^3_\varepsilon)\ \mathrm{d}x,$$

where

$$\left|\frac{1}{|Q_\varepsilon|}\int_{\{\varrho_\varepsilon<\underline{\varrho}/2\}}\left(\frac{1}{\varrho}\partial^2_{y,y}v\right)(\varrho_\varepsilon-\varrho)(v-u^3_\varepsilon)\ \mathrm{d}x\right|\tag{3.8}$$

$$\leq\frac{\delta}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}(v-u^3_\varepsilon)^2\ \mathrm{d}x+\frac{c_1(\delta)}{|Q_\varepsilon|}\int_{\{\varrho_\varepsilon<\underline{\varrho}/2\}}h_3^2|\varrho_\varepsilon-\varrho|^2\ \mathrm{d}x$$

$$\leq\frac{\delta}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}(v-u^3_\varepsilon)^2\ \mathrm{d}x+\frac{c_2(\delta)}{|Q_\varepsilon|}\int_{\{\varrho_\varepsilon<\underline{\varrho}/2\}}h_3^2|\varrho_\varepsilon-\varrho|^\gamma\ \mathrm{d}x\ \text{for any}\ \delta>0.$$

The parameter δ will be fixed so small that the integral is "absorbed" by its counterpart on the left hand side of (3.6).

Finally,

$$\frac{1}{|Q_\varepsilon|}\int_{\{\varrho_\varepsilon>2\overline{\varrho}\}}\left(\frac{1}{\varrho}\partial^2_{y,y}v\right)(\varrho_\varepsilon-\varrho)(v-u^3_\varepsilon)\ \mathrm{d}x\tag{3.9}$$

$$\leq\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}h_3\varrho_\varepsilon(v-u^3_\varepsilon)^2\ \mathrm{d}x+\frac{1}{|Q_\varepsilon|}\int_{\{\varrho_\varepsilon>2\overline{\varrho}\}}h_3\frac{(\varrho_\varepsilon-\varrho)^2}{\varrho_\varepsilon}\ \mathrm{d}x$$

$$\leq\frac{1}{|Q_\varepsilon|}\int_{\Omega_\varepsilon}h_3\varrho_\varepsilon(v-u^3_\varepsilon)^2\ \mathrm{d}x+\frac{c}{|Q_\varepsilon|}\int_{\{\varrho_\varepsilon>2\overline{\varrho}\}}h_3|\varrho_\varepsilon-\varrho|^\gamma\ \mathrm{d}x$$

as $\gamma > 3/2$.

Combining the relations (3.7 - 3.9), we obtain the desired conclusion

$$\mathcal{E}_\varepsilon \left([\varrho_\varepsilon, \mathbf{u}_\varepsilon] \Big| [\varrho, v] \right) (\tau) \tag{3.10}$$

$$\leq \delta_1(\varepsilon) + \int_0^\tau \Big(h_1(t) + h_2(t) + \nu h_3(t) \Big) \mathcal{E}_\varepsilon \left([\varrho_\varepsilon, \mathbf{u}_\varepsilon] \Big| [\varrho, v] \right) (t) \ \mathrm{d}t.$$

Thus a straightforward application of Gronwall's lemma completes the proof of Theorem 2.1.

4 Concluding remarks

The hypothesis (2.5) concerning the pressure p is necessary for the weak solutions of the $3D$ barotropic Navier-Stokes system to exist. Taking the existence of weak solutions for granted, we may replace (2.5) in Theorem 2.1 by the hypothesis

$$\left\{ \begin{array}{c} p \in C[0, \infty) \cap C^2(0, \infty), \\ \\ p'(\varrho) > 0, \ \text{for all } \varrho > 0, \ \int_1^\varrho \frac{p(z)}{z^2} \mathrm{d}z \geq c_1 \frac{p(\varrho)}{\varrho} \geq c_2 > 0 \text{ for all } \varrho > 2. \end{array} \right\} \tag{4.1}$$

Under these new assumptions all steps of the proof of Theorem 2.1 can be performed in the same spirit. We also note that the behavior of the pressure for large values of ϱ is irrelevant for the $1D$ problem as ϱ admits the bounds (2.13) *independent* of the specific form of p as soon as the pressure is non-negative, see Lovicar, Straškraba, and Valli [12].

References

[1] C.M. Dafermos. The second law of thermodynamics and stability. *Arch. Rational Mech. Anal.*, **70**:167–179, 1979.

[2] S. Dain. Generalized Korn's inequality and conformal Killing vectors. *Calc. Var. Partial Differential Equations*, **25**:535–540, 2006.

[3] E. Feireisl. *Dynamics of viscous compressible fluids.* Oxford University Press, Oxford, 2004.

[4] E. Feireisl, Bum Ja Jin, and A. Novotný. Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system. *J. Math. Fluid Mech.*, **14**:712–730, 2012.

[5] E. Feireisl, A. Novotný, and Y. Sun. Suitable weak solutions to the Navier–Stokes equations of compressible viscous fluids. *Indiana Univ. Math. J.*, **60**:611–631, 2011.

[6] P. Germain. Weak-strong uniqueness for the isentropic compressible Navier-Stokes system. *J. Math. Fluid Mech.*, **13**(1):137–146, 2011.

[7] D. Iftimie, G. Raugel, and G. R. Sell. Navier-Stokes equations in thin 3D domains with Navier boundary conditions. *Indiana Univ. Math. J.*, **56**(3):1083–1156, 2007.

[8] A. V. Kazhikhov. Correctness "in the large" of mixed boundary value problems for a model system of equations of a viscous gas. *Dinamika Splošn. Sredy*, (Vyp. 21 Tecenie Zidkost. so Svobod. Granicami):18–47, 188, 1975.

[9] M. Lewicka and S. Müller. The uniform Korn-Poincaré inequality in thin domains. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **28**(3):443–469, 2011.

[10] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.1, Incompressible models.* Oxford Science Publication, Oxford, 1996.

[11] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models.* Oxford Science Publication, Oxford, 1998.

[12] V. Lovicar, I. Straškraba, and A. Valli. On bounded solutions of one-dimensional compressible Navier-Stokes equations. *Rend. Sem. Mat. Univ. Padova*, **83**:81–95, 1990.

[13] A. Mellet and A. Vasseur. Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations. *SIAM J. Math. Anal.*, 39(4):1344–1365, 2007/08.

[14] K.R. Rajagopal. A new development and interpretation of the navier-stokes fluid which reveals why the stokes assumption is inapt. *International Journal of Non-Linear Mechanics*, **50**:141–151, 2013.

[15] G. Raugel and G. R. Sell. Navier-Stokes equations in thin 3D domains. III. Existence of a global attractor. In *Turbulence in fluid flows*, volume 55 of *IMA Vol. Math. Appl.*, pages 137–163. Springer, New York, 1993.

[16] G. Raugel and G. R. Sell. Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions. *J. Amer. Math. Soc.*, 6(3):503–568, 1993.

[17] G. Raugel and G. R. Sell. Navier-Stokes equations on thin 3D domains. II. Global regularity of spatially periodic solutions. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991)*, volume **299** of *Pitman Res. Notes Math. Ser.*, pages 205–247. Longman Sci. Tech., Harlow, 1994.

[18] Yu. G. Reshetnyak. *Stability theorems in geometry and analysis*, volume 304 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1994. Translated from the 1982 Russian original by N. S. Dairbekov and V. N. Dyatlov, and revised by the author, Translation edited and with a foreword by S. S. Kutateladze.

[19] R. Vodák. Asymptotic analysis of steady and nonsteady Navier-Stokes equations for barotropic compressible flow. *Acta Appl. Math.*, **110**(2):991–1009, 2010.