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On the spectrum of the normalized Laplacian for  
signed graphs: Interlacing, contraction, and  
replication

by

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# On the Spectrum of the Normalized Laplacian for Signed Graphs: Interlacing, Contraction, and Replication

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## Abstract

We consider the normalized Laplacian matrix for signed graphs and derive interlacing results for its spectrum. In particular, we investigate the effects of several basic graph operations, such as edge removal and addition and vertex contraction, on the Laplacian eigenvalues. We also study vertex replication, whereby a vertex in the graph is duplicated together with its neighboring relations. This operation causes the generation of a Laplacian eigenvalue equal to one. We further generalize to the replication of motifs, i.e. certain small subgraphs, and show that the resulting signed graph has an eigenvalue 1 whenever the motif itself has eigenvalue 1.

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*Keywords:* Signed graph, interlacing, normalized Laplacian, contraction, replication

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## 1. Introduction

Let  $G = (V, E)$  be a simple graph with vertex set  $V = V(G) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E = E(G) = \{e_1, e_2, \dots, e_m\}$ . The *Laplacian matrix* of the graph  $G$  is the  $n \times n$  matrix  $L(G) = D(G) - A(G)$ , where  $D(G) = \text{diag}\{d_{v_1}, d_{v_2}, \dots, d_{v_n}\}$  is the diagonal matrix of vertex degrees and  $A(G) = [a_{ij}]$  is the  $(0, 1)$ -adjacency matrix of the graph  $G$ , that is,  $a_{ij} = 1$  if  $v_i$  and  $v_j$  are adjacent, and  $a_{ij} = 0$  otherwise. The *normalized Laplacian* is the matrix  $\bar{L}(G)$  is given by [5]

$$\bar{L}(u, v) = \begin{cases} 1, & \text{if } u = v \text{ and } d_v \neq 0, \\ -1/\sqrt{d_u d_v} & \text{if } u \text{ and } v \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

We can write  $\bar{L} = D^{-1/2} L D^{-1/2}$  with the convention that  $D^{-1/2}(v, v) = 0$  whenever  $d_v = 0$ . The normalized Laplacian arises in several applications such as random walks and spreading problems on networks. The eigenvalue spectrum of the Laplacian determines the convergence rates of such dynamics; furthermore, it gives useful information about the graph structure.

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Signed graphs were introduced by Harary [7] in connection with the study of theory of social balance. A signed graph  $\Gamma = (G, \sigma)$  consists of a simple graph  $G = (V, E)$  and a mapping  $\sigma : E \rightarrow \{+, -\}$ , called the edge labeling. In this context  $G$  is called the underlying graph of  $\Gamma$ . We may write  $V(\Gamma)$  for the vertex set and  $E(\Gamma)$  for the edge set if necessary. The degree of  $v_i \in V(\Gamma)$  is defined as  $d_{v_i} = d_{v_i}^+ + d_{v_i}^-$  where  $d_{v_i}^+$  is number of positive edges and  $d_{v_i}^-$  is number of negative edges incident on  $v_i$ . Thus, a signed graph  $\Gamma = (G, \sigma)$  and its underlying graph  $G$  have the same degree sequence, and the degree matrix of  $\Gamma$  is  $D(\Gamma) = D(G)$ . The signed adjacency matrix of  $\Gamma$  is  $A(\Gamma) = [a_{ij}^\sigma]$ , where  $a_{ij}^\sigma = \sigma(v_i, v_j)a_{ij}$ . The *Laplacian matrix* of  $\Gamma$ , denoted by  $L(\Gamma)$  or  $L(G, \sigma)$ , is defined by  $D(\Gamma) - A(\Gamma)$ . Clearly,  $L(G) = L(G, +)$  and  $L(G, -) = D(G) + A(G)$ , where “+” and “-” denote all-positive and all-negative edge labelings, respectively. The Laplacian  $L(\Gamma)$  is a symmetric matrix whose row sum vector is  $2(d_{v_1}^-, d_{v_2}^-, \dots, d_{v_n}^-)^\top$ . The *normalized Laplacian* of  $\Gamma$ , denoted by  $\bar{L}(\Gamma)$  or  $\bar{L}(G, \sigma)$ , is the matrix whose components are given by

$$\bar{L}(u, v) = \begin{cases} 1 & \text{if } u = v \text{ and } d_v \neq 0, \\ -\sigma(u, v)/\sqrt{d_u d_v} & \text{if } uv \in E(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

We can write  $\bar{L} = D^{-1/2}L(\Gamma)D^{-1/2} = I - D^{-1/2}A(\Gamma)D^{-1/2}$ , with the convention  $D^{-1/2}(v, v) = 0$  whenever  $d_v = 0$ .

Eigenvalue interlacing provides a useful tool for obtaining regularity and comparison results regarding the graph structure and various graph matrices. Much research has been done in this area concerning the adjacency and Laplacian matrices of unsigned graphs [6, 9, 13, 4, 14]. In contrast, there exist considerably fewer results on the spectra of signed graphs. Among the relevant works, we mention Hou et al. [11, 12], who studied the spectrum of the Laplacian  $L(\Gamma)$  for signed graphs and obtained some bounds for the largest and smallest Laplacian eigenvalues of unbalanced signed graphs. The notion of the normalized Laplacian for signed graphs were introduced in Li et al. [15].

In this paper we consider the normalized Laplacian for signed graphs and derive interlacing results for its spectrum. We start with some basic results on the Laplacian spectrum for signed graphs in Section 2. In Section 3 we present eigenvalue interlacing results for several graph operations, including edge removal and addition (Section 3.2) and vertex contraction (Section 3.3). We further generalize to successive contractions of a vertex using the concepts of dominating sets and private neighborhoods (Section 3.4). In Section 4 we study the replication operation. Vertex replication refers to duplicating a vertex together with its neighboring relations (Section 4.1). This operation causes the generation of a Laplacian eigenvalue equal to 1 (Section 4.2). We conclude the paper by extending the replication operation from single vertices to entire *motifs*, i.e. certain small connected subgraphs, and showing that the resulting signed graph has an eigenvalue 1 whenever the motif itself has eigenvalue 1 (Section 4.3).

## 2. Eigenvalues of the normalized Laplacian

Without loss of generality, the graphs considered in this paper can be assumed to have no isolated vertices, because, by the definition of the Laplacian, an isolated vertex simply contributes a zero eigenvalue to the spectrum. The normalized Laplacian  $\bar{L}$  can

thus be viewed as an operator on the space of functions  $f : V(\Gamma) \rightarrow \mathbb{R}$  satisfying

$$\bar{L}f(u) = f(u) - \sum_{v, v \sim u} \sigma(u, v) \frac{f(v)}{\sqrt{d_u d_v}}. \quad (1)$$

As a symmetric operator, the basic properties of its spectrum can be obtained through Rayleigh quotients. To this end, first notice that, for  $f = (f_1, \dots, f_n)$  and the usual Laplacian  $L(\Gamma) = D(\Gamma) - A(\Gamma)$  for signed graphs, we have

$$\begin{aligned} f^\top L(\Gamma) f &= \sum_i f_i^2 d_i - 2 \sum_{i \sim j} \sigma(i, j) f_i f_j \\ &= \sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2, \end{aligned}$$

where  $\sum_{i \sim j}$  denotes a sum over all unordered pairs  $\{i, j\}$  for which  $v_i$  and  $v_j$  are adjacent. If  $g = D^{1/2} f$ , then

$$\begin{aligned} \frac{g^\top \bar{L}(\Gamma) g}{g^\top g} &= \frac{(D^{1/2} f)^\top \bar{L}(\Gamma) (D^{1/2} f)}{(D^{1/2} f)^\top (D^{1/2} f)} = \frac{f^\top L(\Gamma) f}{f^\top D f} \\ &= \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2}{\sum_i f_i^2 d_i}. \end{aligned} \quad (2)$$

The right hand side of (2) is obviously nonnegative; moreover, it is bounded from above by 2 since

$$\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 \leq \sum_{i \sim j} 2(f_i^2 + f_j^2) = 2 \sum_i f_i^2 d_i.$$

As  $D^{1/2}$  is a vector space isomorphism, it follows by the Courant-Fisher theorem (see Theorem 6 below for a statement) that the eigenvalues of  $\bar{L}$  belong to the interval  $[0, 2]$  (see [15]).

The switching operation introduced by Seidel [2] plays an important role in discussions of signed graphs. Let  $\Gamma = (G, \sigma)$  be a signed graph and  $\theta : V \rightarrow \{+, -\}$  be a sign function on its vertex set. *Switching*  $\Gamma$  by  $\theta$  means forming a new signed graph  $\Gamma^\theta = (G, \sigma^\theta)$  whose underlying graph is the same as  $G$ , but whose sign function is defined on an edge  $e = v_i v_j$  by  $\sigma^\theta(e) = \theta(v_i) \sigma(e) \theta(v_j)$ . Two signed graphs  $\Gamma_1 = (G, \sigma_1)$  and  $\Gamma_2 = (G, \sigma_2)$  with the same underlying graph are said to be *switching equivalent*, written  $\Gamma_1 \sim \Gamma_2$ , if there exists a switching function  $\theta$  such that  $\Gamma_2 = \Gamma_1^\theta$ . Switching leaves many signed-graph characteristics invariant, including the set of positive cycles. A *signature matrix* is a diagonal matrix  $S = \text{diag}\{s_1, s_2, \dots, s_n\}$  with diagonal entries  $s_i = \pm 1$ . Two square matrices  $M_1, M_2$  of order  $n$  are said to be *signature similar* if there exists a signature matrix  $S$  such that  $M_2 = S M_1 S$ . Clearly, two signature-similar matrices have identical eigenvalues, since  $S^{-1} = S$ . By the following lemma, switching equivalence of graphs can be described in terms of signature similarity of matrices. A direct consequence is that the switching operation does not alter the Laplacian eigenvalues.

**Lemma 1.** [12] *Let  $\Gamma_1 = (G, \sigma_1)$  and  $\Gamma_2 = (G, \sigma_2)$  be signed graphs on the same underlying graph. Then  $\Gamma_1 \sim \Gamma_2$  if and only if  $L(\Gamma_1)$  and  $L(\Gamma_2)$  are signature similar.*

The same conclusion holds also for the normalized Laplacian. As stated in the next lemma, the normalized Laplacian  $\bar{L}$  of switching-equivalent graphs have identical eigenvalues and the corresponding eigenfunctions are directly related by the switching function.

**Lemma 2.** *Let  $\Gamma = (G, \sigma)$  be a signed graph on  $n$  vertices and suppose  $f = (f(1), \dots, f(n))$  is an eigenfunction of  $\bar{L}(\Gamma)$  corresponding to the eigenvalue  $\lambda$ . Let  $\Gamma^\theta = (G, \sigma^\theta)$  be a switching-equivalent graph obtained from  $\Gamma$  through the switching function  $\theta$ . Then the function  $f^\theta := (\theta(1)f(1), \dots, \theta(n)f(n))$  is an eigenfunction of  $\bar{L}(\Gamma^\theta)$  corresponding to the eigenvalue  $\lambda$ .*

*Proof.* From (1),

$$(\bar{L}(\Gamma)f)(u) = f(u) - \sum_{v, v \sim u} \sigma(u, v) \frac{f(v)}{\sqrt{d_u d_v}} = \lambda f(u).$$

Multiplying through by  $\theta(u)$  and noting that  $\theta^2(v) = 1$ , we obtain

$$\theta(u)f(u) - \sum_{v, v \sim u} \theta(u)\sigma(u, v)\theta(v) \frac{\theta(v)f(v)}{\sqrt{d_u d_v}} = \lambda \theta(u)f(u),$$

that is,

$$f^\theta(u) - \sum_{v, v \sim u} \sigma^\theta(u, v) \frac{f^\theta(v)}{\sqrt{d_u d_v}} = \lambda f^\theta(u).$$

showing that  $f^\theta$  is an eigenfunction of  $\bar{L}(\Gamma^\theta)$  corresponding to the eigenvalue  $\lambda$ .  $\square$

A particular case of interest is signed graphs that are switching-equivalent to unsigned graphs. This is related to the concept of *balanced*, introduced as follows. If  $C$  is a cycle of a signed graph  $\Gamma$ , the *sign* of  $C$  is defined by  $\text{sign}(C) = \prod_{e \in C} \sigma(e)$ . A cycle whose sign is  $+$  (resp.,  $-$ ) is said to be *positive* (resp., *negative*). A signed graph is said to be *balanced* if all its cycles are positive. It can be shown that a signed graph  $\Gamma$  is balanced if and only if  $\Gamma = (G, \sigma) \sim (G, +)$  [11, Theorem 2.5]. Thus, for balanced graphs,  $\bar{L}(\Gamma)$  and  $\bar{L}(G)$  are signature similar and have identical spectrum. According to the matrix-tree theorem for signed graphs [3, 16], 0 is an eigenvalue of  $L(\Gamma)$  or  $\bar{L}(\Gamma)$  if and only if  $\Gamma$  is balanced [12]. Furthermore, it is well-known for unsigned graphs that 0 is a *simple* eigenvalue of  $L(G)$  or  $\bar{L}(G)$  if and only if  $G$  is connected. Combining, we obtain the following lemma.

**Lemma 3.** *For a connected, balanced signed graph  $\Gamma = (G, \sigma)$ , zero is a simple eigenvalue of both  $L(\Gamma)$  or  $\bar{L}(\Gamma)$ .*

Finally, for bipartite graphs, the following result carries over from unsigned graphs.

**Lemma 4.** *Let  $\Gamma = (G, \sigma)$  be a bipartite signed graph. If  $\lambda$  is an eigenvalue of  $\bar{L}(\Gamma)$ , then  $2 - \lambda$  is also an eigenvalue of  $\bar{L}(\Gamma)$ .*

*Proof.* Let  $\Gamma$  be a bipartite graph, with the two partitions  $V_1, V_2$ . Let  $f$  be an eigenfunction of  $\bar{L}(\Gamma)$  corresponding to the eigenvalue  $\lambda$ ; thus by (1),

$$(1 - \lambda)f(i) = \sum_{j \sim i} \frac{\sigma(i, j)f(j)}{\sqrt{d_i d_j}}. \quad (3)$$

Define the function  $g$  by

$$g(i) = \begin{cases} f(i) & \text{if } i \in V_1, \\ -f(i) & \text{if } i \in V_2. \end{cases}$$

Then from (3),

$$(1 - \lambda)g(i) = - \sum_{j \sim i} \frac{\sigma(i, j)g(j)}{\sqrt{d_i d_j}},$$

since all neighborhoods are across the partitions. It follows that

$$g(i) - \sum_{j \sim i} \frac{\sigma(i, j)g(j)}{\sqrt{d_i d_j}} = (2 - \lambda)g(i),$$

that is,  $g$  is an eigenfunction of  $\bar{L}$  corresponding to the eigenvalue  $2 - \lambda$ .  $\square$

In the following sections, we obtain further information on the Laplacian eigenvalues for some basic graph operations.

### 3. Edge removal and vertex contraction

#### 3.1. Main tools for eigenvalue interlacing

We briefly recall some useful facts from matrix analysis. The following result is one of the basic tools in eigenvalue interlacing (see e. g., [10]).

**Theorem 5.** (*Cauchy's interlacing theorem*) *Let  $A$  be a real  $n \times n$  symmetric matrix and  $B$  be an  $(n - 1) \times (n - 1)$  principal submatrix of  $A$ . If*

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \quad \text{and} \quad \theta_1 \leq \theta_2 \leq \dots \leq \theta_{n-1}$$

*denote the eigenvalues of  $A$  and  $B$ , respectively, then*

$$\lambda_i \leq \theta_i \leq \lambda_{i+1} \quad \text{for } i = 1, 2, \dots, n - 1.$$

The edge version of the interlacing property for the Laplacian  $L(G)$  is given in [9, 13]. Chen et al. [4] studied Cauchy interlacing-type properties of the normalized Laplacian  $\bar{L}(G)$  by using the Courant-Fischer Theorem [10].

**Theorem 6.** (*Courant-Fischer*) *If  $M$  is an  $n \times n$  real symmetric matrix with eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n,$$

*then*

$$\lambda_1 = \min \left\{ \frac{\langle Mf, f \rangle}{\langle f, f \rangle} : 0 \neq f \in \mathbb{R}^n \right\}$$

*and*

$$\lambda_n = \max \left\{ \frac{\langle Mf, f \rangle}{\langle f, f \rangle} : 0 \neq f \in \mathbb{R}^n \right\}.$$

*Moreover, the  $k$ -th smallest eigenvalue  $\lambda_k$  is given by*

$$\lambda_k = \min_{S_{n-k-1}} \max_{\substack{f \perp S_{n-k-1} \\ f \neq 0}} \frac{\langle Mf, f \rangle}{\langle f, f \rangle} = \max_{S_k} \min_{\substack{f \perp S_k \\ f \neq 0}} \frac{\langle Mf, f \rangle}{\langle f, f \rangle}$$

*where  $S_t$  denotes a  $t$ -dimensional subspace of  $\mathbb{R}^n$  and  $f \perp S_t$  indicates that  $f \perp g$  for all  $g \in S_t$ .*

We will also make use of the following lemma.

**Lemma 7.** [4] Suppose that for real  $a, b$ , and  $\gamma$ ,

$$a^2 - 2\gamma^2 \geq 0, \quad b^2 - \gamma^2 > 0, \quad \text{and} \quad \frac{a^2}{b^2} \leq 2. \quad (4)$$

Then

$$\frac{a^2 - 2\gamma^2}{b^2 - \gamma^2} \leq \frac{a^2}{b^2}. \quad (5)$$

### 3.2. Edge removal and addition

Eigenvalue interlacing results with respect to the removal of an edge are well known for unsigned graphs. Our next result gives an extension to the normalized Laplacian of signed graphs. It obviously pertains also to edge addition by reversing the roles of graphs.

**Theorem 8.** Let  $\Gamma$  be a signed graph without isolated vertices and let  $\Gamma - e$  be the signed graph obtained from  $\Gamma$  by removing an edge  $e$ . If

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \quad \text{and} \quad 0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_n$$

are the eigenvalues of  $\bar{L}(\Gamma)$  and  $\bar{L}(\Gamma - e)$ , respectively, then

$$\lambda_{i-1} \leq \theta_i \leq \lambda_{i+1} \quad (6)$$

for each  $i = 1, 2, \dots, n$ , with the convention that  $\lambda_0 = 0$  and  $\lambda_{n+1} = 2$ .

*Proof.* Consider the Rayleigh quotient given in (2), with  $g = D^{1/2}f$ . Since  $D^{1/2}$  is an invertible matrix, its action on a  $t$ -dimensional subspace yields again a  $t$ -dimensional subspace. Thus, the Courant-Fisher theorem for the  $k$ th smallest eigenvalue  $\lambda_k$  of  $\bar{L}(\Gamma)$  can be expressed as

$$\begin{aligned} \lambda_k &= \min_{S_{n-k-1}} \max_{g \perp S_{n-k-1}} \frac{g^\top \bar{L}(\Gamma)g}{g^\top g} \\ &= \min_{S_{n-k-1}} \max_{\substack{D^{1/2}f \perp S_{n-k-1} \\ D^{1/2}f \neq 0}} \frac{f^\top L(\Gamma)f}{f^\top Df} \\ &= \min_{S'_{n-k-1}} \max_{\substack{f \perp S'_{n-k-1} \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j)f_j)^2}{\sum_i f_i^2 d_i}. \end{aligned} \quad (7)$$

Similarly,

$$\lambda_k = \max_{S'_k} \min_{\substack{f \perp S'_k \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j)f_j)^2}{\sum_i f_i^2 d_i}. \quad (8)$$



Suppose now the edge  $e = v_1v_2 \in E$  is removed from  $\Gamma$  and assume that  $\sigma(v_1, v_2) = -$ . The degrees of  $v_1$  and  $v_2$  decrease by one after removing the edge  $v_1v_2 \in E$ . So, the denominator above changes to

$$\sum_j f_j^2 d_j \rightarrow \sum_j f_j^2 d_j - f_1^2 - f_2^2,$$

and, due to changing neighborhood relations, the numerator becomes

$$\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 \rightarrow \sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 - (f_1 + f_2)^2,$$

where the summations on the right are taken in the original graph. Thus,

$$\begin{aligned} \theta_k &= \max_{S'_k} \min_{\substack{f \perp S'_k \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 - (f_1 + f_2)^2}{\sum_j f_j^2 d_j - f_1^2 - f_2^2} \\ &\leq \max_{S'_k} \min_{\substack{f \perp S'_k \\ f_1 = f_2, f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 - (f_1 + f_2)^2}{\sum_j f_j^2 d_j - f_1^2 - f_2^2} \\ &= \max_{S'_k} \min_{\substack{f \perp S'_k \\ f \neq 0} \text{ and } f \perp e_1 - e_2} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 - 4f_1^2}{\sum_j f_j^2 d_j - 2f_1^2} \end{aligned} \quad (9)$$

where the vectors  $e_1$  and  $e_2$  denote the standard basis vectors. We will use Lemma 7 with  $\gamma^2 = 2f_1^2$ ,  $a^2 = \sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2$ , and  $b^2 = \sum_j f_j^2 d_j$  and recall from Section 2 that the eigenvalues are bounded from above by 2. Thus, continuing from (9),

$$\begin{aligned} \theta_k &\leq \max_{S'_k} \min_{\substack{f \perp S'_k, \text{ and } f \perp e_1 - e_2 \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2}{\sum_j f_j^2 d_j} \\ &\leq \max_{S'_{k+1}} \min_{\substack{f \perp S'_{k+1} \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2}{\sum_j f_j^2 d_j} = \lambda_{k+1}. \end{aligned} \quad (10)$$

Similarly, using the min-max form of the Courant-Fisher theorem,

$$\begin{aligned}
\theta_k &= \min_{S'_{n-k-1}} \max_{\substack{f \perp S'_{n-k-1} \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 - (f_1 + f_2)^2}{\sum_j f_j^2 d_j - f_1^2 - f_2^2} \\
&\geq \min_{S'_{n-k-1}} \max_{\substack{f \perp S'_{n-k-1} \\ f \neq 0 \\ f_1 = -f_2}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 - (f_1 + f_2)^2}{\sum_j f_j^2 d_j - f_1^2 - f_2^2} \\
&\geq \min_{S'_{n-k-1}} \max_{\substack{f \perp S'_{n-k-1} \text{ and } f \perp e_1 + e_2 \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2}{\sum_j f_j^2 d_j - 2f_1^2} \\
&\geq \min_{S'_{n-k}} \max_{\substack{f \perp S'_{n-k} \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2}{\sum_j f_j^2 d_j} = \lambda_{k-1} \tag{11}
\end{aligned}$$

Combining (10) and (11) proves (6). The case when  $\sigma(v_1, v_2) = +$  is proved similarly.  $\square$

**Remark 9.** It is known that the eigenvalues of the Laplacian  $L$  of unsigned graphs decrease or stay the same when an edge is removed [13]. Theorem 8 shows that for the normalized Laplacian  $\bar{L}$  of signed graphs, the eigenvalues may in fact increase, in which case the theorem gives an upper bound to the increase. As an example, consider the signed graph shown in Figure 1 and the two graphs obtained from it by removing an edge. Comparing their Laplacian spectra shows that eigenvalues may increase or decrease when an edge is removed.

### 3.3. Vertex contraction

Let  $G$  be a graph and let  $v \in V(G)$ . The *open neighborhood* of  $v \in V(G)$  is the set

$$N(v) = \{u \in V : uv \in E\}$$

and the *closed neighborhood* of  $v$  is  $N[v] = N(v) \cup \{v\}$ . For any two vertices  $u$  and  $v$  of  $G$ , we use  $G/\{u, v\}$  to denote the graph obtained from  $G$  by *contracting*  $u$  and  $v$  to a single vertex, i. e., by deleting the vertices  $u$  and  $v$  and adding a new vertex  $(uv)$  such that the neighborhood of  $(uv)$  is the union of neighborhoods of  $u$  and  $v$ . When  $u$  and  $v$  are adjacent,  $G/\{u, v\}$  is the graph obtained from  $G$  by contracting the edge  $uv$ .

The contraction operation can similarly be defined for a signed graph by preserving the signs provided that the signs on the edges can be assigned consistently, i.e., when the edges from  $u$  and  $v$  to any common neighbor have the same signs. We say that  $\Gamma/\{u, v\}$  is an *allowable contraction* for a signed graph  $\Gamma$  if  $\sigma(x, u) = \sigma(x, v)$  for all  $x \in N(u) \cap N(v)$ . Hence, an allowable contraction is a signed graph  $\Gamma/\{u, v\}$  obtained from  $\Gamma$  by deleting the vertices  $u$  and  $v$  and adding a new vertex  $(uv)$  such that the neighborhood of  $(uv)$  is the union of neighborhoods of  $u$  and  $v$ , and  $\sigma(x, (uv)) = \sigma(x, u) = \sigma(x, v)$  for all  $x \in N(u) \cup N(v)$ . In particular,  $\Gamma/\{u, v\}$  is an allowable contraction whenever  $N(u) \cap N[v] = \emptyset$ . The next interlacing result pertains to this case.

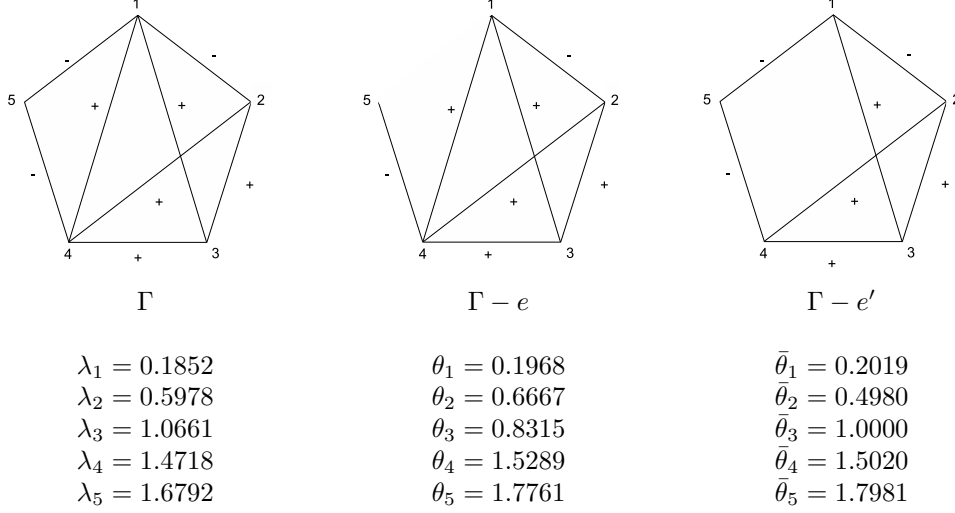


Figure 1: A signed graph  $\Gamma$  and two graphs,  $\Gamma - e$  and  $\Gamma - e'$ , obtained from it by removing an edge, together with the eigenvalue spectra of their normalized Laplacians  $\bar{L}$ . It can be seen that removing an edge may increase (e.g.  $\theta_2$ ) or decrease (e.g.  $\theta_2$ ) the eigenvalues.

**Theorem 10.** *Let  $\Gamma$  be a signed graph and let  $u$  and  $v$  be two vertices of  $\Gamma$  such that  $N(u) \cap N[v] = \emptyset$ . Let*

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \quad \text{and} \quad \theta_1 \leq \theta_2 \leq \dots \leq \theta_{n-1}$$

*denote the eigenvalues of  $\bar{L}(\Gamma)$  and  $\bar{L}(\Gamma/\{u, v\})$  respectively. Then*

$$\lambda_{i-1} \leq \theta_i \leq \lambda_{i+1},$$

*with the convention that  $\lambda_0 = 0$  and  $\lambda_{n+1} = 2$ .*

*Proof.* We denote  $u = v_1$  and  $v = v_2$ . Let  $J$  be an index set such that  $j \in J$  if and only if  $v_j \in N(v_1)$ . Since  $N(v_1) \cap N[v_2] = \emptyset$ ,  $\Gamma/\{v_1, v_2\}$  can be seen as the operation of removing edges  $v_1v_j$  and simultaneously adding edges  $v_2v_j$ , while preserving signs of them. Thus, using the Courant-Fisher theorem (7) and arguing as in the proof of Theorem 8, the eigenvalues  $\theta_k$  of  $\bar{L}(\Gamma/\{u, v\})$  can be expressed as

$$\theta_k = \min_{S'_{n-k-1}} \max_{\substack{f \perp S'_{n-k-1} \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j)f_j)^2 + \sum_{j \in J} (f_2 - \sigma(v_2, v_j)f_j)^2 - (f_1 - \sigma(v_1, v_j)f_j)^2}{\sum_j f_j^2 d_j - d_1 f_1^2 + d_1 f_2^2}$$

Here the summations are understood to be taken in the original graph  $\Gamma$ . Hence, similar to the case of unsigned graphs [4], taking  $f_1 = f_2$  yields the lower bound  $\lambda_{k-1}$ . The upper bound  $\lambda_{k+1}$  follows analogously from the max min statement (8) of Courant-Fisher theorem. Therefore,  $\lambda_{i-1} \leq \theta_i \leq \lambda_{i+1}$ , as required.  $\square$

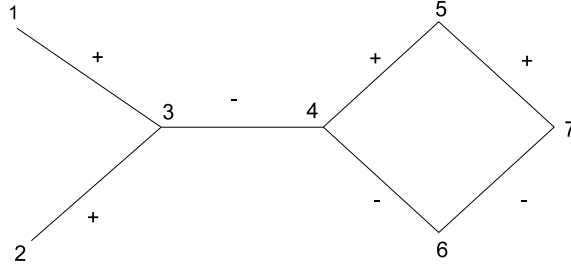


Figure 2: Example of a signed graph  $\Gamma$  and a dominating set  $S = \{3, 7\}$ . The vertices 1, 2 and 4 are  $S$ -private neighbors of vertex 3, and the vertices 5 and 6 are  $S$ -private neighbors of vertex 7. Note that  $N(3) \cap N(7) = \emptyset$ .

### 3.4. Dominating sets and successive contractions

Following the contraction of two vertices  $v_1, v_2$ , we can consider the contraction of the new vertex  $(v_1 v_2)$  with another vertex  $v_3$ , obtaining a new graph  $(G/\{v_1, v_2\})/\{(v_1 v_2), v_3\}$ . To reduce notational burden, we denote the ensuing graph simply by  $G/\{v_1, v_2, v_3\}$ . By generalization, we can consider  $k-1$  successive contractions yielding a graph  $G/\{v_1, v_2, \dots, v_k\}$ . Before giving the interlacing result, we study the problem of when such a sequence of contractions is well-defined for signed graphs.

A set  $S \subseteq V$  of vertices in graph  $G = (V, E)$  is called a *dominating set* if for every vertex  $v \in V - S$ , there exists a vertex  $u \in S$  such that  $v$  is adjacent to  $u$ . The *domination number* of  $G$ , denoted by  $\gamma(G)$ , is the minimum cardinality of dominating sets in  $G$ . A dominating set of  $G$  which has cardinality  $\gamma(G)$  is called a  $\gamma(G)$ -*set*. If  $S$  is a dominating set, a vertex  $w \in V$  is called an  $S$ -*private neighbor* of  $v \in S$  if  $N[w] \cap S = \{v\}$ . The  $S$ -*private neighborhood* of  $v \in S$ , denoted  $\text{pn}[v, S]$ , is the set of all  $S$ -private neighbors of  $v$ . The *open  $S$ -private neighborhood* is defined analogously by the condition  $N(w) \cap S = \{v\}$ . For a survey of the subject of domination in graphs, the reader is referred to [8]. If every vertex in  $V$  is an  $S$ -private neighbor of some  $v_i \in S$  then  $\bigcap_{i=1}^{\gamma(G)} N(v_i) = \emptyset$  (see Figure 2 for an illustration). Furthermore, in this case the successive contractions  $\Gamma/\{v_1, v_2, v_3, \dots, v_k\}$  is a well-defined operation.

**Theorem 11.** *Let  $\Gamma = (G, \sigma)$  be a signed graph,  $\gamma(G) = k$ , and  $S = \{v_1, \dots, v_k\}$  be a  $\gamma(G)$ -set, and suppose the successive contractions  $\Gamma/\{v_1, v_2, v_3, \dots, v_k\}$  are well-defined. Let*

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \quad \text{and} \quad \theta_1^{(k-1)} \leq \theta_2^{(k-1)} \leq \dots \leq \theta_{n-(k-1)}^{(k-1)}$$

*be the eigenvalues of  $\bar{L}(\Gamma)$  and  $\bar{L}(\Gamma/\{v_1, v_2, v_3, \dots, v_k\})$ , respectively. Then*

$$\lambda_{i-k+1} \leq \theta_i^{(k-1)} \leq \lambda_{i+k-1}, \tag{12}$$

*with the convention  $\lambda_i = 0$  if  $i \leq 0$  and  $\lambda_i = 2$  if  $i > n$ .*

*Proof.* By Theorem 10, the eigenvalues of  $\bar{L}(\Gamma)$  and  $\bar{L}(\Gamma/\{v_1, v_2\})$  satisfy

$$\lambda_{i-1} \leq \theta_i \leq \lambda_{i+1}.$$

Similarly, after contracting  $(v_1 v_2)$  and  $v_3$  the eigenvalues of  $\bar{L}(\Gamma/\{v_1, v_2\})$  and  $\bar{L}(\Gamma/\{v_1, v_2, v_3\})$  satisfy

$$\lambda_{i-2} \leq \theta_{i-1} \leq \theta_i^{(2)} \leq \theta_{i+1} \leq \lambda_{i+2}.$$

Repeating the argument yields (12).  $\square$

## 4. Vertex and motif replication

### 4.1. Vertex replication

Let  $\Gamma = (G, \sigma)$  be a signed graph on  $n$  vertices and  $v \in V(\Gamma)$ . We consider the operation of *replicating*  $v$ , that is, adding a new vertex  $v'$  and connecting it to all neighbors of  $v$ , preserving the signs between them. In other words,  $x \in N(v')$  iff  $x \in N(v)$ , and  $\sigma(v', x) = \sigma(v, x) \forall x \in N(v')$ . We denote the resulting graph by  $\Gamma^v$ . The next theorem pertains to the Laplacian eigenvalues of  $\Gamma^v$ .

**Theorem 12.** *Let  $\Gamma$  be a signed graph on  $n$  vertices and  $\Gamma^v$  be the graph obtained from  $\Gamma$  by replicating a vertex. Let*

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{n+1} \quad \text{and} \quad \theta_1 \leq \theta_2 \leq \dots \leq \theta_n$$

*denote the eigenvalues of  $\bar{L}(\Gamma^v)$  and  $\bar{L}(\Gamma)$ , respectively. Then,*

$$\theta_i \leq \lambda_{i+1} \leq \theta_{i+2},$$

*with the convention that  $\theta_{n+1} = \theta_{n+2} = 2$ .*

*Proof.* Let  $v'$  denote the copy of  $v \in V(\Gamma)$  and  $J$  be an index set such that  $j \in J$  if and only if  $v_j \in N(v)$ . Since  $\sigma(x, v) = \sigma(x, v')$  for all  $x \in N(v) \cap N(v')$ , contraction of the vertices  $v, v'$  in the new graph  $\Gamma^v$  is an allowable operation; and indeed  $\Gamma^v/\{v, v'\} = \Gamma$ . Suppose now  $v'$  is removed from  $\Gamma^v$ . Thus, using the Courant-Fisher theorem (7) and arguing as in the proof of Theorem 8, the  $k$ th smallest eigenvalue  $\theta_k$  of  $\bar{L}(\Gamma)$  can be expressed as

$$\theta_k = \min_{S'_{n-k-1}} \max_{\substack{f \perp S'_{n-k-1} \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 - \sum_{j \in J} (f_{v'} - \sigma(v', v_j) f_j)^2}{\sum_j f_j^2 d_j - d_{v'} f_{v'}^2 - \sum_{j \in J} f_j^2}$$

Here the summation over the edges as well as the sum in the denominator are evaluated in the graph  $\Gamma^v$ . Define the function  $f'$  as  $f'_{v'} = 1$  and  $f'_j = -\sigma(v', v_j)/d_{v'}$  for  $j \in J$ , and

0 elsewhere. Then,

$$\begin{aligned}
\theta_k &= \min_{S'_{n-k-1}} \max_{\substack{f \perp S'_{n-k-1} \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 - \sum_{j \in J} (f_{v'} - \sigma(v', v_j) f_j)^2}{\sum_j f_j^2 d_j - d_{v'} f_{v'}^2 - \sum_{j \in J} f_j^2} \\
&\geq \min_{S'_{n-k-1}} \max_{\substack{f \perp S'_{n-k-1} \\ f \neq 0 \\ f_j = f_{v'} / \sigma(v', v_j), \forall j \in J}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 - \sum_{j \in J} (f_{v'} - \sigma(v', v_j) f_j)^2}{\sum_j f_j^2 d_j - d_{v'} f_{v'}^2 - \sum_{j \in J} f_j^2} \\
&\geq \min_{S'_{n-k-1}} \max_{\substack{f \perp S'_{n-k-1} \text{ and } f \perp f' \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2}{\sum_j f_j^2 d_j - 2d_{v'} f_{v'}^2} \\
&\geq \min_{S'_{n-k}} \max_{\substack{f \perp S'_{n-k} \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2}{\sum_j f_j^2 d_j} = \lambda_{k-1}.
\end{aligned}$$

Similarly, define  $f''$  by  $f''_{v'} = 1$  and  $f''_j = \sigma(v', v_j)/d_{v'}$  for all  $j \in J$ , and 0 elsewhere. Then, using the max-min form of Courant-Fisher theorem,

$$\begin{aligned}
\theta_k &= \max_{S'_k} \min_{\substack{f \perp S'_k \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 - \sum_{j \in J} (f_{v'} - \sigma(v', v_j) f_j)^2}{\sum_j f_j^2 d_j - d_{v'} f_{v'}^2 - \sum_{j \in J} f_j^2} \\
&\leq \max_{S'_k} \min_{\substack{f \perp S'_k \\ f \neq 0 \\ f_j = -f_{v'} / \sigma(v', v_j), \forall j \in J}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 - \sum_{j \in J} (f_{v'} - \sigma(v', v_j) f_j)^2}{\sum_j f_j^2 d_j - d_{v'} f_{v'}^2 - \sum_{j \in J} f_j^2} \\
&\leq \max_{S'_k} \min_{\substack{f \perp S'_k \text{ and } f \perp f'' \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2 - 4d_{v'} f_{v'}^2}{\sum_j f_j^2 d_j - 2d_{v'} f_{v'}^2} \tag{13}
\end{aligned}$$

We will use Lemma 7 with  $\gamma^2 = 2d_{v'} f_{v'}^2$ ,  $a^2 = \sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2$ , and  $b^2 = \sum_j f_j^2 d_j$  and recall from Section 2 that the eigenvalues are bounded from above by 2. Indeed, it is easy to check that the conditions (4) of Lemma 7 are satisfied; so we can use its conclusion (5) to continue from (13) as

$$\begin{aligned}
\theta_k &\leq \max_{S'_k} \min_{\substack{f \perp S'_k \text{ and } f \perp f'' \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2}{\sum_j f_j^2 d_j} \\
&\leq \max_{S'_{k+1}} \min_{\substack{f \perp S'_{k+1} \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - \sigma(i, j) f_j)^2}{\sum_j f_j^2 d_j} = \lambda_{k+1},
\end{aligned}$$

which implies

$$\lambda_{i-1} \leq \theta_i \leq \lambda_{i+1} \leq \theta_{i+2} \leq \lambda_{i+3},$$

thus proving the theorem.  $\square$

#### 4.2. Eigenvalue 1

From the eigenvalue equation (3) for the normalized Laplacian of signed graphs, it is seen that 1 is an eigenvalue of  $\bar{L}$  iff

$$\sum_{j, j \sim i} \sigma(i, j) \frac{f(j)}{\sqrt{d_j}} = 0 \quad \forall i \quad (14)$$

for some nonzero function  $f : V \rightarrow \mathbb{R}$ . It is easy to show that vertex replication generates an eigenvalue 1. Indeed, suppose  $\Gamma^v$  is the signed graph obtained from  $\Gamma$  by replicating the vertex  $v$ , and denote the new vertex by  $v'$ . Then the localized eigenfunction  $f$  defined by

$$f(i) = \begin{cases} 1 & \text{if } i = v \\ -1 & \text{if } i = v' \\ 0 & \text{otherwise} \end{cases}$$

satisfies (14) and so corresponds to an eigenvalue equal to 1. By extension, a sequence of  $m$  vertex replications will increase the multiplicity of eigenvalue 1 by  $m$ .

#### 4.3. Motif replication

More generally, one can consider the replication of subgraphs consisting of several vertices, or the so-called *motifs*. For unsigned graphs, motif replication and its effect on eigenvalue 1 for the normalized Laplacian were studied in [1]. We extend these results to signed graphs.

Let  $\Sigma$  be a connected subgraph of a signed graph  $\Gamma$ , that is,  $V(\Sigma) \subset V(\Gamma)$ , and for  $u, v \in V(\Sigma)$  one has  $(u, v) \in E(\Sigma)$  iff  $(u, v) \in E(\Gamma)$ , with the sign  $\sigma(u, v)$  inherited from  $\Gamma$ . Such a small connected subgraph is sometimes referred to as a motif. By *motif replication* we refer to the enlarged graph  $\Gamma^\Sigma$  that contains a replica of the subgraph  $\Sigma$  with all its connections. More precisely, if  $\Sigma$  is a motif consisting of the vertices  $v_1, \dots, v_m$ , and  $\Sigma'$  denotes its copy with vertices  $v'_1, \dots, v'_m$ , then for all  $v'_i, v'_j \in V(\Sigma')$  we have  $(v'_i, v'_j) \in E(\Sigma')$  iff  $(v_i, v_j) \in E(\Sigma)$ , and  $(v'_i, u) \in E(\Gamma^\Sigma)$  whenever  $u \in N(v_i) \setminus V(\Sigma)$ . Moreover, the signs of the edges are preserved in the replication, i.e.,  $\sigma(v'_i, u) = \sigma(v_i, u)$  for all edges  $(v'_i, u) \in E(\Gamma^\Sigma)$ .

Let  $\Sigma$  be a motif of  $\Gamma$ , and suppose that 1 is an eigenvalue of  $\bar{L}(\Sigma)$  with eigenfunction  $f$ . The function  $f$  can be extended to a function  $\bar{f}$  over the whole graph  $\Gamma$  by defining it to be zero on  $V(\Gamma) \setminus V(\Sigma)$ ; however,  $\bar{f}$  need not be an eigenfunction of  $\bar{L}(\Gamma)$  and 1 need not be an eigenvalue of  $\bar{L}(\Gamma)$ . Nevertheless, if the motif  $\Sigma$  is replicated, then the new graph  $\Gamma^\Sigma$  also has an eigenvalue 1, as we prove next.

**Theorem 13.** *Let  $\Gamma$  be a signed graph,  $\Sigma$  be a motif of  $\Gamma$ , and  $\Gamma^\Sigma$  be obtained from  $\Gamma$  by replicating  $\Sigma$ . If  $\bar{L}(\Sigma)$  possesses an eigenvalue 1, then  $\bar{L}(\Gamma^\Sigma)$  also possesses the eigenvalue 1, with a localized eigenfunction that is nonzero only on  $\Sigma$  and its copy  $\Sigma'$ .*

*Proof.* As above, we use the notation  $v_\alpha$  and  $v'_\alpha$ ,  $\alpha = 1, \dots, m$ , to denote the vertices of  $\Sigma$  and  $\Sigma'$ , respectively. Let  $f$  be an eigenfunction of  $\bar{L}(\Sigma)$  corresponding to the eigenvalue 1. Then by (14),

$$\sum_{j: (i,j) \in E(\Sigma)} \sigma(i,j) \frac{f(j)}{\sqrt{d_j^\Sigma}} = 0, \quad \forall i \in V(\Sigma),$$

where  $d_j^\Sigma$  denotes the degree of vertex  $j$  in  $\Sigma$ . Define the function  $g$  on  $V(\Gamma^\Sigma)$  by

$$g(i) = \begin{cases} f(v_\alpha) \frac{\sqrt{d_{v_\alpha}}}{\sqrt{d_{v_\alpha}^\Sigma}} & \text{if } i = v_\alpha \in V(\Sigma), \\ -f(v_\alpha) \frac{\sqrt{d_{v_\alpha}}}{\sqrt{d_{v_\alpha}^\Sigma}} & \text{if } i = v'_\alpha \in V(\Sigma'), \\ 0 & \text{otherwise,} \end{cases}$$

where  $d$  denotes the degree in the graph  $\Gamma^\Sigma$ . It is easily checked that

$$\sum_{j: (i,j) \in E(\Gamma^\Sigma)} \sigma(i,j) \frac{g(j)}{\sqrt{d_j}} = 0, \quad \forall i \in V(\Gamma^\Sigma).$$

Hence, by (14), 1 is an eigenvalue of  $\bar{L}(\Gamma^\Sigma)$ .

If  $i \notin V(\Sigma)$  or  $V(\Sigma')$  then

$$\begin{aligned} \sum_{j, j \sim i} \sigma(i,j) \frac{g(j)}{\sqrt{d_i d_j}} &= \sum_{j \in V(\Sigma)} \sigma(i,j) \frac{g(j)}{\sqrt{d_j}} + \sum_{j \in V(\Sigma')} \sigma(i,j) \frac{g(j)}{\sqrt{d_j}} + \sum_{\substack{j \notin V(\Sigma) \\ j \notin V(\Sigma')}} \sigma(i,j) \frac{f(j)}{\sqrt{d_j}} \\ &= \sum_{j \in V(\Sigma)} \sigma(i,j) \frac{f(j)}{\sqrt{d_j}} - \sum_{j \in V(\Sigma')} \sigma(i,j) \frac{f(j)}{\sqrt{d_j}} + 0 \\ &= 0 \end{aligned}$$

Thus, for all  $i \in V(\Gamma^\Sigma)$ ,  $\sum_{j, j \sim i} \sigma(i,j) \frac{g(j)}{\sqrt{d_j}} = 0$ . Hence, 1 is an eigenvalue of  $\bar{L}(\Gamma^\Sigma)$  with  $g$  the corresponding eigenvector.  $\square$

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