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Regularity of area minimizing currents I:
gradient L^p estimates

by

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REGULARITY OF AREA MINIMIZING CURRENTS I: GRADIENT L^p ESTIMATES

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ABSTRACT. In a series of papers, including the present one, we give a new, shorter proof of Almgren's partial regularity theorem for area minimizing currents in a Riemannian manifold, with a slight improvement on the regularity assumption for the latter. This note establishes a new a priori estimate on the excess measure of an area minimizing current, together with several statements concerning approximations with Lipschitz multiple valued graphs. Our new a priori estimate is an higher integrability type result, which has a counterpart in the theory of Dir-minimizing multiple valued functions and plays a key role in estimating the accuracy of the Lipschitz approximations.

0. FOREWORD: A NEW PROOF OF ALMGREN'S PARTIAL REGULARITY

In the present work we continue the investigations started in [13, 14], which together with the forthcoming papers [16, 17] lead to a proof of the following theorem.

Theorem 0.1. *Let $\Sigma \subset \mathbb{R}^{m+n}$ be a C^{3,ε_0} submanifold for some $\varepsilon_0 > 0$ and T an m -dimensional area minimizing integral current in Σ . Then, there is a closed set $\text{Sing}(T)$ of Hausdorff dimension at most $m - 2$ such that T is a C^{3,ε_0} embedded submanifold in $\Sigma \setminus (\text{spt}(\partial T) \cup \text{Sing}(T))$.*

Theorem 0.1 was first proved by Almgren in his monumental work [3], assuming slightly better regularity on Σ , namely $\Sigma \in C^5$. The improvement itself is therefore not so significant, but our proof, besides being much shorter, introduces new ideas and establishes several new results, which we hope will provide useful tools for further investigations in the area. Indeed, although we still follow Almgren's program and use many of his groundbreaking discoveries, the main steps are achieved in a more efficient way thanks to new estimates and techniques. A striking example is the construction of the so-called center manifold, which is by far the most intricate part of Almgren's work and the least explored, in spite of its importance: in this respect, our construction in [16] is considerably simpler and shorter than [3, Chapter 4], and establishes better results.

Some of our improvements are more transparent, although not substantially simpler, when $\Sigma = \mathbb{R}^{m+n}$ and in a book in preparation [11] we will provide a complete and self-contained account of Theorem 0.1 under such assumption. Moreover, building on our understanding of the various issues involved to the analysis of higher codimension singularities, we plan to tackle Chang's improvement [7], which shows that $\text{Sing}(T)$ consists of isolated points when $m = 2$. His arguments rely on a center manifold construction which does not match exactly the statements of [3] and it is not fully justified, but only briefly

sketched in the appendix of [7]. In [18], instead, we give a detailed, simple construction for such center manifold and a complete proof of this refined regularity result.

An alternative route to Chang’s result for J -holomorphic currents in symplectic manifolds has been given recently in [26, 27]. The interest in the regularity theory for this class of area minimizing 2-dimensional currents has been generated by the seminal paper of Taubes [31] on the equivalence between Gromov and Seiberg-Witten invariants, where it plays an important role. Moreover, the papers [26, 27] have stimulated a lot of activity in the area, cf., for example, [6, 23, 24, 25]. In [6] Bellettini and Rivière proved that, when T is a special Lagrangian cone in \mathbb{R}^6 , $\text{Sing}(T)$ consists of finitely many half-lines meeting at the origin. This is, to our knowledge, the only result of its type not covered by the Almgren-Chang works. We believe that the Bellettini-Rivière regularity theorem can be extended to general 3-dimensional area minimizing cones in any space dimension, combining the techniques developed in [13]–[18]. Most of the proofs in [6, 23, 24, 25, 26, 27, 31] take advantage of two specific assumptions, the underlying almost complex structure and the 2-dimensionality of the objects of study. Nonetheless these works have had a profound influence on our research.

0.1. A blow-up proof: a very brief overview. In the rest of this foreword we will give a rough outline of the proof of Theorem 0.1, highlighting the contents of this note and the way it merges with its companion papers [16, 17], while comparing them to [3]. Our discussion will be based on a well-known class of examples for which the statement of Theorem 0.1 is optimal, namely singular holomorphic curve of \mathbb{C}^2 . As it was first observed by Federer (cf. [20, 5.4.19]), the integral currents induced by holomorphic subvarieties of \mathbb{C}^n (with their natural orientation) are area minimizing.

We denote by $D_Q(T)$ the set of points in $\text{spt}(T) \setminus \text{spt}(\partial T)$ where the density of a current T equals the natural number $Q \geq 1$. One first pioneering contribution by Almgren is an elementary, but very clever, generalization of Federer’s reduction argument, which has been widely used in several contexts (see [28, Theorem 35.3] and [32]). This argument implies that, if T is area minimizing, then $\text{spt}(T) \setminus (\cup_Q D_Q(T) \cup \text{spt}(\partial T))$ has Hausdorff dimension at most $m - 3$. Thus, to prove Theorem 0.1 it suffices to show that the Hausdorff dimension of $\text{Sing}_Q(T) := \text{Sing}(T) \cap D_Q(T)$ is at most $m - 2$. Since the “classical” regularity theory ensures that T is a $C^{1,\alpha}$ submanifold in the neighborhood of any point $x \in D_1(T)$, it is natural to argue by induction on Q .

Let us therefore consider the case $Q = 2$ and a point $x \in D_2(T)$. By the monotonicity formula, in some neighborhood U of x , $\|T\|$ -almost all points have density 1 or 2. If the point of density 1 are a set of $\|T\|$ -measure zero, by the classical regularity theory x is a regular point for T . So any $x \in \text{Sing}_2(T)$ must be surrounded by many points of density 1, as it is, for instance, for the complex curve $\{z^2 = w^3\} \subset \mathbb{C}^2$ at $x = 0$. On the other hand, in such example 0 is an isolated singularity, whereas, if T were to contradict Theorem 0.1, by standard measure theoretic arguments there would be a point $x \in \text{Sing}_2(T)$ surrounded by many points of density 2. From now on we argue by contradiction and assume that this happens for some area minimizing T at the point $0 \in D_2(T)$. Moreover, by known facts in geometric measure theory, we can reduce the contradiction to the case that, for a suitable

sequence of radii $r_k \downarrow 0$, the homothetic rescalings of the current T by a factor $1/r_k$ (from now on denoted by T_k) converge to a double copy of an m -dimensional plane, while at the same time $D_2(T_k)$ remains rather large.

It was first recognized by De Giorgi that the convergence of T_k to a *single copy* of a flat plane implies that $\text{spt}(T_k)$ can be well approximated by the graph of Lipschitz functions which are “almost harmonic”. However, the example $\{z^2 = w^3\} \subset \mathbb{C}^2$ shows that this is not always the case if the limiting plane has higher multiplicity. Motivated by this fact, Almgren undertook in [3] the strikingly ambitious program of giving a rather complete existence and regularity theory for *multiple valued* functions minimizing a suitable generalization of the Dirichlet energy, called Dir-minimizers. The crowning achievement of this theory is that, except for a closed set of codimension at most 2, Dir-minimizers can be locally decomposed in classical (i.e. single-valued) non-intersecting harmonic sheets (possibly counted with multiplicity). Such “linear theory” is developed in [3, Chapter 2] and revisited in our paper [13]. Moreover, it is complemented by several technical statements linking the multiple valued graphs to the integral currents, a task which is accomplished in [3, Chapter 1] by Almgren and in [14] by us (we refer to the introduction to our previous two papers [13, 14] for more details).

The guiding idea in the contradiction argument is to approximate the currents T_k with Lipschitz *2-valued* functions and, after a suitable renormalization of their Dirichlet energy, show that they converge to a Dir-minimizer. If such limit inherits a large singular set from the currents T_k , then it contradicts the linear regularity theory. Obviously, this strategy requires suitable approximations of area minimizing currents with multiple valued graphs, accomplished by Almgren in [3, Chapter 3] and by us in the present paper. If one follows our approach, the convergence of these approximations to a Dir-minimizer can be concluded in a rather direct way. However, we cannot expect that such limit inherits the singular set of the current. For example, given the complex curve $\{(z, w) : (z - w^2)^2 = w^5\} \subset \mathbb{C}^2$, any reasonable approximations of homothetic rescalings of this algebraic variety in a neighborhood of the origin converge to a double copy of the classical holomorphic graph $\{(w, w^2) : w \in \mathbb{C}\}$, which has lost the singularity at the origin.

In order to perform the blow-up argument, we then need to “modulate lower order regularities out”. This is accomplished by the construction of a center manifold (see [3, Chapter 4] and [16]): such an object is a regular $C^{3,\alpha}$ submanifold which is very close to the average of the sheets of the current at any scale where the latter is “very collapsed”. The final blow-up argument is then carried over a new sequence of 2-valued approximations of T_k , performed on the normal bundles of the center manifolds (see [3, Chapter 5] and [17]). By a delicate unique continuation principle, based on a new monotonicity formula discovered by Almgren, a suitable normalization of the latter approximations does converge to a Dir-minimizer which would be forced to have a large singular set, reaching the desired contradiction. This final step builds upon very delicate computations, which thus require a lot of accuracy in the construction of the center manifold, that in turn needs very good estimates on the approximation results of this note. Thus, unlike the two works [13, 14], which can be considered separately, the papers [15, 16, 17] are intimately interconnected.

0.2. Our contribution; or, what is new. In their overall structure, our five papers match bijectively the five chapters of [3]. Moreover, it is clear that the ultimate reason for the success of the program is the very same prodigious and celebrated discovery of Almgren: the monotonicity of the frequency function and its astonishing robustness, which enters twice in the plan: at the very beginning, in the linear regularity theory, and at the end, in the convergence of the final approximations (cf. [13, 17]).

So, what is new in our proof? Aside from finer details, which are explained in the introductions to each of our papers, there are some new contributions which come at a higher level. Our investigations started with the idea that the machinery developed in metric analysis and metric geometry in the last 30 years could reduce the complexity of several arguments in Almgren’s program. This is, indeed, the case at many levels in the two papers [13, 14] and in this note. Approaching vast parts of Almgren’s theory with these tools, we not only get shorter and more transparent proofs, but often also achieve stronger analytic estimates, which give a better starting point for the PDE parts of the program. Moreover, as it often happens when “abstract nonsense” simplifies preexisting mathematical theories, such machinery provides also a better insight to the material of [3], as it highlights the important points in the proofs therein.

However, this alone would not explain the shortness of our papers compared to [3, Chapters 3,4,5]. The other important reason is that we also derive some fundamental, new “hard” estimates. A primary example is the present paper, where the main a priori estimate is a new higher integrability result, which comes from a Gehring-type argument and is inspired by a simple remark in the linear theory (the higher integrability of gradients of Dir-minimizers) which to our knowledge is not observed in Almgren’s monograph. Similar instances are present in the papers [16, 17], where some new quantities and guiding principles are introduced (for instance, the “modified frequency” function in [17] and the “splitting-before-tilting” principle in [16], inspired by [25]), which probably lead to the improvement on the regularity assumptions of the ambient manifold Σ . In all these cases we provide more efficient tools compared to [3] and invoke more PDE theory at several levels, drawing connections with fairly classical concepts from other areas of analysis (such as maximal functions, Lipschitz truncations, elliptic systems, Sobolev capacity). Unfortunately we do not understand Almgren’s arguments at a sufficiently deep level to draw a fine parallel between our papers [16, 17] and the last two chapters of his book, where the intricacy of the arguments in [3] is almost prohibitive. It remains the fact that our papers are much more accessible, and we hope that in the near future our work will be used to penetrate further in the richness and beauty of Almgren’s monograph and to go beyond Theorem 0.1.

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1. INTRODUCTION

1.1. A priori gradient L^p estimate. In order to state the main results, we start specifying some assumptions, which will hold throughout the paper. For the notation concerning submanifolds $\Sigma \subset \mathbb{R}^{m+n}$ we refer to [14, Section 1]. $\mathbf{C}_r(x)$ will always denote a cylinder of the form $B_r(x) \times \mathbb{R}^n \subset \mathbb{R}^{m+n}$ and the point x will be omitted when it is the origin. Let e_i be the unit vectors in the standard basis, π_0 the (oriented) plane $\mathbb{R}^m \times \{0\}$ and $\vec{\pi}_0$ the m -vector $e_1 \wedge \dots \wedge e_m$ orienting it. We denote by \mathbf{p} and \mathbf{p}^\perp the orthogonal projections onto, respectively, π_0 and its orthogonal complement π_0^\perp . In some cases we need orthogonal projections onto other planes π and their orthogonal complements π^\perp , for which we use the notation \mathbf{p}_π and \mathbf{p}_π^\perp .

Assumption 1.1. $\Sigma \subset \mathbb{R}^{m+n}$ is a C^2 submanifold of dimension $m + \bar{n} = m + n - l$, which is the graph of an entire function $\Psi : \mathbb{R}^{m+\bar{n}} \rightarrow \mathbb{R}^l$ and satisfies the bounds

$$\|\nabla \Psi\|_0 \leq c_0 \quad \text{and} \quad \mathbf{A} := \|A_\Sigma\|_0 \leq c_0, \quad (1.1)$$

where c_0 is a positive (small) dimensional constant. T is an integral current of dimension m supported in Σ which, for some open cylinder $\mathbf{C}_{4r}(x)$ (with $r \leq 1$) and some positive integer Q , satisfies

$$\mathbf{p}_\# T = Q \llbracket B_{4r}(x) \rrbracket \quad \text{and} \quad \partial T \llcorner \mathbf{C}_{4r}(x) = 0. \quad (1.2)$$

Definition 1.2 (Excess measure). For a current T as in Assumption 1.1 we define the cylindrical excess $\mathbf{E}(T, \mathbf{C}_{4r}(x))$, the excess measure \mathbf{e}_T and its density \mathbf{d}_T :

$$\begin{aligned} \mathbf{E}(T, \mathbf{C}_r(x)) &:= \frac{\|T\|(\mathbf{C}_r(x))}{\omega_m r^m} - Q, \\ \mathbf{e}_T(A) &:= \|T\|(A \times \mathbb{R}^n) - Q |A| \quad \text{for every Borel } A \subset B_r(x), \\ \mathbf{d}_T(y) &:= \limsup_{s \rightarrow 0} \frac{\mathbf{e}_T(B_s(y))}{\omega_m s^m} = \limsup_{s \rightarrow 0} \mathbf{E}(T, \mathbf{C}_s(y)), \end{aligned}$$

where ω_m is the measure of the m -dimensional unit ball (the subscripts T will be omitted if clear from the context).

The function \mathbf{d} is naturally an L^1 function. However, we can show the following higher integrability estimate when T is, in addition, area minimizing. We call it a gradient L^p

estimate because we will show that \mathbf{d} coincides with the gradient of an appropriate Lipschitz function on a large region.

Theorem 1.3 (Gradient L^p estimate). *There exist constants $p_1 > 1$ and $C, \varepsilon_{10} > 0$ (depending on m, n, \bar{n}, Q) with the following property. Let T be as in Assumption 1.1 in the cylinder \mathbf{C}_4 . If T is area minimizing and $E = \mathbf{E}(T, \mathbf{C}_4) < \varepsilon_{10}$, then*

$$\int_{\{\mathbf{d} \leq 1\} \cap B_2} \mathbf{d}^{p_1} \leq C E^{p_1-1} (E + \mathbf{A}^2). \quad (1.3)$$

In the case $Q = 1$ or $\bar{n} = 1$, it follows from the classical regularity theory (essentially due to De Giorgi, cf. [8]) that T is a $C^{1,\alpha}$ submanifold in \mathbf{C}_2 . However, when $\min\{Q, \bar{n}\} \geq 2$, T is not necessarily regular and Theorem 1.3 gives in fact an *a priori* regularity estimate: in this case (1.3) cannot be improved (except for optimizing the constants p_1, C and ε_{10}). Indeed, for $Q = m = 2$, $\Sigma = \mathbb{R}^4$ and $p_1 = 2$, (1.3) is false no matter how large ε_{10}^{-1} and C are chosen (cf. [9, Section 6.2]).

In order to achieve Theorem 1.3 we develop the following tools:

- (a) a general scheme to approximate integer rectifiable currents with multiple valued functions, relying heavily on the “metric analysis” of [13] and on a modified “Jerrard–Soner” BV estimate for the slicing of currents (cf. Proposition 2.2);
- (b) a simple and robust harmonic approximation of area minimizing currents with multiple valued functions (cf. Theorem 4.2);
- (c) the higher integrability of the gradient of Dir-minimizing multiple valued functions (cf. Theorem 5.1 – see also [29] for a different proof and related results).

In turn, Theorem 1.3 will be combined with (a) to achieve a very accurate approximation result for area minimizing current, stated in Theorem 1.4. This theorem and some corollaries of our analysis play a fundamental role in the papers [16, 17] and, as explained in the Foreword, have a counterpart in [3, Chapter 3]. However, our derivation of Theorem 1.4 differs substantially from Almgren’s and when we use some of his ideas, as it is for the existence of the almost projection $\boldsymbol{\rho}^*$ of Section 7, we give independent arguments for the main steps of the proof.

1.2. Strong approximation of area minimizing currents. We adopt the notation of our previous papers [13, 14], denoting by \mathbf{G}_f the integer rectifiable current naturally associated to the graph of a Lipschitz Q -valued function $f : \mathbb{R}^m \supset A \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$. Moreover, we will use the notation $\text{osc}(f)$ for the quantity $\inf_p \sup_x \mathcal{G}(f(x), Q \llbracket p \rrbracket)$.

Theorem 1.4 (Almgren’s strong approximation). *There exist constants $C, \gamma_1, \varepsilon_1 > 0$ (depending on m, n, \bar{n}, Q) with the following property. Assume that T is area minimizing, satisfies Assumption 1.1 in the cylinder $\mathbf{C}_{4r}(x)$ and $E = \mathbf{E}(T, \mathbf{C}_{4r}(x)) < \varepsilon_1$. Then, there is a map $f : B_r(x) \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$, with $\text{spt}(f(x)) \subset \Sigma$ for every x , and a closed set $K \subset B_r(x)$*

such that

$$\text{Lip}(f) \leq C E^{\gamma_1}, \quad (1.4)$$

$$\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n) \quad \text{and} \quad |B_r(x) \setminus K| \leq C E^{\gamma_1} (E + r^2 \mathbf{A}^2) r^m, \quad (1.5)$$

$$\left| \|T\|(\mathbf{C}_r(x)) - Q \omega_m r^m - \frac{1}{2} \int_{B_r(x)} |Df|^2 \right| \leq C E^{\gamma_1} (E + r^2 \mathbf{A}^2) r^m. \quad (1.6)$$

If in addition $\mathbf{h}(T, \mathbf{C}_{4r}(x)) := \sup\{|\mathbf{p}^\perp(x) - \mathbf{p}^\perp(y)| : x, y \in \text{spt}(T) \cap \mathbf{C}_{4r}(x)\} \leq r$, then

$$\text{osc}(f) \leq C \mathbf{h}(T, \mathbf{C}_{4r}(x), \pi_0) + C(E^{1/2} + r \mathbf{A}) r. \quad (1.7)$$

The gain of a small power E^{γ_1} in the three estimates (1.4)-(1.6) plays a crucial role in the papers [16, 17]. When $Q = 1$ and $\Sigma = \mathbb{R}^{m+1}$, this approximation theorem was first proved by De Giorgi in [8]. In the generality above it appears in the big regularity paper for the first time (cf. [3, Sections 3.28-3.30]). Its proof is an elementary consequence of Theorem 6.1 and the Lipschitz approximation algorithm mentioned above. In turn Theorem 6.1 will be derived from Theorem 1.3 using a suitable competitor argument. In the case $Q = 1$, the competitor is the convolution of (a first) Lipschitz approximation with a smooth kernel, a classical argument which in fact appears already in De Giorgi's seminal paper [8], although in a slightly different form (cf. [12, Appendix]).

Here we need a similar approach in the framework of multiple valued functions. However, since $\mathcal{A}_Q(\mathbb{R}^n)$ is highly nonlinear, it is not possible to regularize directly by convolution. We exploit at this point a key idea of Almgren, embedding $\mathcal{A}_Q(\mathbb{R}^n)$ in a euclidean space and using some suitable "almost projections" ρ_δ^* . Our proof of the existence of these almost projections is however different from the one given by Almgren in [3, Theorem 1.3] and, indeed, gives better bounds in terms of the relevant parameters (see Proposition 6.2).

1.3. Harmonic approximation. A second theorem which in [16, 17] will play a key role is the following harmonic approximation (already mentioned in (b) above).

Theorem 1.5 (Harmonic approximation). *Let γ_1 be the constant of Theorem 1.4. Then, for every $\bar{\eta}, \bar{\delta} > 0$, there is a positive constant $\bar{\varepsilon}_1$ with the following property. Assume that T is as in Theorem 1.4, $E := \mathbf{E}(T, \mathbf{C}_{4r}(x)) < \bar{\varepsilon}_1$ and $r \mathbf{A} \leq E^{1/4+\bar{\delta}}$. If f is the map in Theorem 1.4, then there exists a Dir-minimizing function $w : B_r \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ such that*

$$r^{-2} \int_{B_r(x)} \mathcal{G}(f, w)^2 + \int_{B_r} (|Df| - |Dw|)^2 + \int_{B_r} |D(\boldsymbol{\eta} \circ f) - D(\boldsymbol{\eta} \circ w)|^2 \leq \bar{\eta} E r^m. \quad (1.8)$$

This theorem is the multi-valued analog of De Giorgi's harmonic approximation (cf. [8]). We prove it via a compactness argument which, although very close in spirit to De Giorgi's original one, is to our knowledge new (even when $n = \bar{n} = 1$). Indeed, it uses neither the monotonicity formula nor a regularization by convolution of the Lipschitz approximation, and we expect it to be useful in different contexts.

1.4. Persistence of Q -points. A major ingredient in [17] is the persistence of points of maximal multiplicity in the approximation of Theorem 1.4, when interpreted in a suitable “limiting sense”. If the current T has a point of density Q , f must satisfy the following integral bound (even though f might have no values of multiplicity Q).

Theorem 1.6 (Persistence of Q -points). *For every $\hat{\delta}, C^* > 0$, there is $\bar{s} \in]0, \frac{1}{2}[$ such that, for every $s < \bar{s}$, there exists $\hat{\varepsilon}(s, \hat{\delta}) > 0$ with the following property. If T is as in Theorem 1.4, $E := \mathbf{E}(T, \mathbf{C}_{4r}(x)) < \hat{\varepsilon}$, $r^2 \mathbf{A}^2 \leq C^* E$ and $\Theta(T, (p, q)) = Q$ at some $(p, q) \in \mathbf{C}_{r/2}(x)$, then the approximation f of Theorem 1.4 satisfies*

$$\int_{B_{sr}(p)} \mathcal{G}(f, Q \llbracket \eta \circ f \rrbracket)^2 \leq \hat{\delta} s^m r^{2+m} E. \quad (1.9)$$

1.5. A remark on notation. Concerning multiple valued functions we will follow the notation of [13, 14], except for a subtle point. We denote by $\boldsymbol{\xi}$ the map in [13, Corollary 2.2], which there was denoted by $\boldsymbol{\xi}_{BW}$, since the symbol $\boldsymbol{\xi}$ was in fact used for the “precursor map” of [13, Theorem 2.1]. So, here $\boldsymbol{\xi} : \mathcal{A}_Q(\mathbb{R}^n) \rightarrow \mathbb{R}^{N(Q,n)}$ is an injective function satisfying the following three properties:

- (i) $\text{Lip}(\boldsymbol{\xi}) \leq 1$;
- (ii) $\text{Lip}(\boldsymbol{\xi}^{-1}|_{\mathcal{Q}}) \leq C(n, Q)$, where $\mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q)$;
- (iii) $|Df| = |D(\boldsymbol{\xi} \circ f)|$ almost everywhere for every $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$.

This “improved” $\boldsymbol{\xi}$ was suggested by Brian White and appears for the first time in [7]. The conclusion (iii) above is actually not explicitly stated in [13], but it follows easily: indeed [13, Corollary 2.2] implies the identity $|Df| = |D(\boldsymbol{\xi} \circ f)|$ at every point of differentiability of a Lipschitz map and, hence, almost everywhere. The case of a general $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ can then be concluded from [13, Proposition 2.5].

We will use the notation C and c for generic positive dimensional constants, which may possibly change from line to line: we will always understand that these constants depends only on the dimensional parameters m, \bar{n}, n, Q, c_0 of Assumption 1.1.

2. LIPSCHITZ APPROXIMATION

To begin with, we develop a robust algorithm to approximate currents T as in Assumption 1.1 with graphs of multiple valued functions. Following the work of Ambrosio and Kirchheim [5], we view the slice map $x \mapsto \langle T, \mathbf{p}, x \rangle$ as a function taking values in the space $\mathbf{I}_0(\mathbb{R}^n)$ of 0-dimensional integral currents. A key estimate of Jerrard and Sonner (cf. [5, 22]) implies that this map has bounded variation in the metric sense introduced by Ambrosio in [4]. On the other hand, following [13], Q -valued functions can be viewed as Sobolev maps taking values into (a subset of) $\mathbf{I}_0(\mathbb{R}^n)$. Thus, finding Lipschitz multiple valued approximations of T can be seen as a particular case of the more general task of finding Lipschitz approximations of BV maps with a fairly general target space.

Definition 2.1 (Maximal function of the excess measure). Given a current T as in Assumption 1.1 we introduce the “non-centered” maximal function of \mathbf{e}_T :

$$\mathbf{me}_T(y) := \sup_{y \in B_s(w) \subset B_{4r}(x)} \frac{\mathbf{e}_T(B_s(w))}{\omega_m s^m} = \sup_{y \in B_s(w) \subset B_{4r}(x)} \mathbf{E}(T, \mathbf{C}_s(w)).$$

We can now state the main result of the section, which provides the first Lipschitz approximation for rectifiable currents.

Proposition 2.2 (Lipschitz approximation). *There exists a constant $C > 0$ with the following property. Let T and Ψ be as in Assumption 1.1 in the cylinder $\mathbf{C}_{4s}(x)$. Set $E = \mathbf{E}(T, \mathbf{C}_{4s}(x))$, let $0 < \delta_{11} < 1$ be such that*

$$r_0 := 16 \sqrt[m]{\frac{E}{\delta_{11}}} < 1,$$

and define $K := \{\mathbf{me}_T < \delta_{11}\} \cap B_{3s}(x)$. Then, there is $u \in \text{Lip}(B_{3s}(x), \mathcal{A}_Q(\mathbb{R}^n))$ such that $\text{spt}(u(y)) \subset \Sigma$ for every $y \in B_{3s}(x)$ and

$$\text{Lip}(u) \leq C \delta_{11}^{1/2}, \quad \text{osc}(u) \leq C \mathbf{h}(T, \mathbf{C}_{4s}(x), \pi_0) + Cs \|\nabla \Psi\|_0,$$

$$\mathbf{G}_u \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n),$$

$$|B_r(x) \setminus K| \leq \frac{10^m}{\delta_{11}} \mathbf{e}_T(\{\mathbf{me}_T > 2^{-m} \delta_{11}\} \cap B_{r+r_0s}(x)) \quad \forall r \leq 3s. \quad (2.1)$$

The proof of the proposition is based on a BV estimate which differs from the ones of [5, 22]. Note that we do not assume that T is area minimizing. Indeed, even the assumption (1.2) could be relaxed, but we do not pursue this issue here.

2.1. The modified Jerrard–Soner estimate. Recall that each element $S \in \mathbf{I}_0(\mathbb{R}^{m+n})$ is simply a finite sum of Dirac delta, $S = \sum_{i=1}^h w_i \delta_{z_i}$, where $h \in \mathbb{N}$, $w_i \in \{-1, 1\}$ and the z_i 's are (not necessarily distinct) points in \mathbb{R}^{m+n} . Let T be a current as in Assumption 1.1 in the cylinder \mathbf{C}_4 . The slicing map $x \mapsto \langle T, \mathbf{p}, x \rangle$ takes values in $\mathbf{I}_0(\mathbb{R}^{m+n})$ and is characterized by (cf. [28, Section 28]):

$$\int_{B_4} \langle T, \mathbf{p}, x \rangle (\varphi) dx = T(\varphi dx) \quad \text{for every } \varphi \in C_c^\infty(\mathbf{C}_4). \quad (2.2)$$

Moreover $\text{spt}(\langle T, \mathbf{p}, x \rangle) \subseteq \mathbf{p}^{-1}(\{x\})$ and therefore $\langle T, \mathbf{p}, x \rangle = \sum_i w_i \delta_{(x, y_i)}$. The assumption (1.2) guarantees that $\sum_i w_i = Q$ for almost every x . In order to state our BV estimate, we consider the push-forwards of $\langle T, \mathbf{p}, x \rangle$ into the vertical directions:

$$T_x := \mathbf{p}_\#^\perp(\langle T, \mathbf{p}, x \rangle) \in \mathbf{I}_0(\mathbb{R}^n). \quad (2.3)$$

It follows from (2.2) that the currents T_x are characterized through the identity:

$$\int_{B_4} T_x(\psi) \varphi(x) dx = T(\varphi \psi dx) \quad \text{for every } \varphi \in C_c^\infty(B_4), \psi \in C_c^\infty(\mathbb{R}^n). \quad (2.4)$$

Proposition 2.3 (BV estimate). *Assume T satisfies Assumption 1.1 in \mathbf{C}_4 . For every $\psi \in C_c^\infty(\mathbb{R}^n)$, set $\Phi_\psi(x) := T_x(\psi)$. If $\|D\psi\|_\infty \leq 1$, then $\Phi_\psi \in BV(B_4)$ and satisfies*

$$(|D\Phi_\psi|(A))^2 \leq 2m^2 \mathbf{e}_T(A) \|T\|(A \times \mathbb{R}^n) \quad \text{for every Borel set } A \subseteq B_4. \quad (2.5)$$

Note that in the usual Jerrard-Soner estimate the RHS of (2.5) would be $(\|T\|(A \times \mathbb{R}^n))^2$.

Proof. It is enough to prove (2.5) for every open set $A \subseteq B_4$. To this aim, recall that:

$$|D\Phi_\psi|(A) = \sup \left\{ \int_A \Phi_\psi(x) \operatorname{div} \varphi(x) dx : \varphi \in C_c^\infty(A, \mathbb{R}^m), \|\varphi\|_\infty \leq 1 \right\}. \quad (2.6)$$

For any smooth vector field φ , it holds that $(\operatorname{div} \varphi(x)) dx = d\Xi$, where

$$\Xi = \sum_j \varphi_j d\hat{x}^j \quad \text{and} \quad d\hat{x}^j = (-1)^{j-1} dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^m.$$

From (2.4) and the assumption $\partial T \lrcorner \mathbf{C}_4 = 0$ in (1.2), we conclude that

$$\begin{aligned} \int_A \Phi_\psi(x) \operatorname{div} \varphi(x) dx &= \int_{B_4} T_x(\psi) \operatorname{div} \varphi(x) dx = T(\psi \operatorname{div} \varphi dx) \\ &= T(\psi d\Xi) = T(d(\psi \Xi)) - T(d\psi \wedge \Xi) = -T(d\psi \wedge \Xi). \end{aligned} \quad (2.7)$$

Observe that the m -form $d\psi \wedge \Xi$ has no dx component, since

$$d\psi \wedge \Xi = \sum_{j=1}^m \sum_{i=1}^n (-1)^{j-1} \frac{\partial \psi}{\partial y^i}(y) \varphi_j(x) dy^i \wedge d\hat{x}^j. \quad (2.8)$$

Write $\vec{T} = \langle \vec{T}, \vec{\pi}_0 \rangle \vec{\pi}_0 + \vec{S}$. Then,

$$(T(d\psi \wedge \Xi))^2 = \left(\int \langle \vec{S}, d\psi \wedge \Xi \rangle d\|T\| \right)^2 \leq \|d\psi \wedge \Xi\|_\infty^2 \|T\|(A \times \mathbb{R}^n) \int_{A \times \mathbb{R}^n} |\vec{S}|^2 d\|T\|,$$

($|\cdot|$ denotes the norms on Λ_m and Λ^m induced by the natural inner products $\langle \cdot, \cdot \rangle$). Since $|\vec{S}|^2 = 1 - \langle \vec{T}, \vec{\pi}_0 \rangle^2 \leq 2 - 2\langle \vec{T}, \vec{\pi}_0 \rangle$, we have

$$\int_{A \times \mathbb{R}^n} |\vec{S}|^2 d\|T\| \leq 2 \int_{A \times \mathbb{R}^n} \left(1 - \langle \vec{T}, \vec{\pi}_0 \rangle \right) d\|T\| = 2 \mathbf{e}_T(A).$$

Moreover, by (2.8), $\|d\psi \wedge \Xi\|_\infty \leq m \|D\psi\|_\infty \|\varphi\|_\infty \leq m$. Summarizing, we get

$$\int_A \Phi_\psi(x) \operatorname{div} \varphi(x) dx \leq (2m^2 \mathbf{e}_T(A) \|T\|(A \times \mathbb{R}^n))^{1/2}. \quad (2.9)$$

Taking the supremum in (2.9) over φ 's with $\|\varphi\|_\infty \leq 1$, we conclude (2.5) through (2.6). \square

2.2. Proof of Proposition 2.2. Since the statement is invariant under translations and dilations, without loss of generality we assume $x = 0$ and $s = 1$. Consider the slices $T_x := \mathbf{p}_\#^\perp \langle T, \mathbf{p}, x \rangle \in \mathbf{I}_0(\mathbb{R}^n)$ and recall that $\|T\|(A \times \mathbb{R}^n) \geq \int_A \mathbf{M}(T_x) dx$ for every open set A (cf. [28, Lemma 28.5]). Therefore,

$$\mathbf{M}(T_x) \leq \lim_{r \rightarrow 0} \frac{\|T\|(\mathbf{C}_r(x))}{\omega_m r^m} \leq \mathbf{me}_T(x) + Q \quad \text{for almost every } x.$$

Since $\delta_{11} < 1$, we infer $\mathbf{M}(T_x) < Q + 1$ for a.e. $x \in K$. There are, then, Q functions $g_i : K \rightarrow \mathbb{R}^n$ such that $T_x = \sum_{i=1}^Q \delta_{g_i(x)}$ for a.e. $x \in K$. Define $g : K \mapsto \mathcal{A}_Q(\mathbb{R}^n)$ as $g := \sum_i \llbracket g_i \rrbracket$ and fix $\psi \in C_c^\infty(\mathbb{R}^n)$. Proposition 2.3 gives

$$(|D\Phi_\psi|(B_r(y)))^2 \leq 2m (\mathbf{e}_T(B_r(y))) \|T\|(\mathbf{C}_r(y)) = 2m (\mathbf{e}_T(B_r(y))) (Q|B_r(y)| + \mathbf{e}_T(B_r(y))).$$

Hence, if we define the maximal function

$$\mathbf{m}|D\Phi_\psi|(x) := \sup_{x \in B_r(y) \subset B_{4r}} \frac{|D\Phi_\psi|(B_r(y))}{|B_r(y)|},$$

we conclude that

$$(\mathbf{m}|D\Phi_\psi|(x))^2 \leq 2m \mathbf{me}_T(x)^2 + 2m Q \mathbf{me}_T(x) \leq C\delta_{11} \quad \text{for every } x \in K.$$

Therefore, the theory of BV functions gives a dimensional constant C such that

$$|\Phi_\psi(x) - \Phi_\psi(y)| \leq C \delta_{11}^{1/2} |x - y| \quad \forall x, y \in K \text{ Lebesgue points of } \Phi_\psi, \quad (2.10)$$

(see for instance [19, Section 6.6.2]: although in that reference the authors use the *centered* maximal function, the proof works obviously also in our context). Consider next the Wasserstein distance of exponent 1:

$$W_1(S_1, S_2) := \sup \{ \langle S_1 - S_2, \psi \rangle : \psi \in C^1(\mathbb{R}^n), \|D\psi\|_\infty \leq 1 \}. \quad (2.11)$$

Obviously, when $S_1 = \sum_i \llbracket S_{1i} \rrbracket, S_2 = \sum_i \llbracket S_{2i} \rrbracket \in \mathcal{A}_Q(\mathbb{R}^n)$, the supremum in (2.11) can be taken over a suitable countable subset of $\psi \in C_c^\infty(\mathbb{R}^n)$, chosen independently of the S_i 's. Moreover, it follows easily from the definition in (2.11) that

$$W_1(S_1, S_2) = \inf_{\sigma \in \mathcal{P}_Q} \sum_i |S_{1i} - S_{2\sigma(i)}| \geq \inf_{\sigma \in \mathcal{P}_Q} \left(\sum_i |S_{1i} - S_{2\sigma(i)}|^2 \right)^{1/2} = \mathcal{G}(S_1, S_2).$$

So $\mathcal{G}(g(x), g(y)) \leq C \delta_{11}^{1/2} |x - y|$ for a.e. $x, y \in K$.

Next, write $g(x) = \sum_i \llbracket (h_i(x), \Psi(x, h_i(x))) \rrbracket$. Obviously $x \mapsto h(x) := \sum_i \llbracket h_i(x) \rrbracket \in \mathcal{A}_Q(\mathbb{R}^{\bar{n}})$ is a Lipschitz map. Recalling [13, Theorem 1.7], we can extend it to a map $\bar{u} \in \text{Lip}(B_3, \mathcal{A}_Q(\mathbb{R}^{\bar{n}}))$ satisfying $\text{Lip}(\bar{u}) \leq C \delta_{11}^{1/2}$ and $\text{osc}(\bar{u}) \leq C \text{osc}(h)$. Set finally $u(x) = \sum_i \llbracket (\bar{u}_i(x), \Psi(x, \bar{u}_i(x))) \rrbracket$. The estimates claimed on u follow easily.

The identity $\mathbf{G}_u \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n)$ is a consequence of $u(x) = T_x$ for a.e. $x \in K$. Indeed, recall that both T and \mathbf{G}_u are rectifiable and observe that $\langle \vec{T}, \vec{\pi}_0 \rangle \neq 0$ $\|T\|$ -a.e. on $K \times \mathbb{R}^n$, because $\mathbf{me}_T < \infty$ on K . Similarly, $\langle \vec{\mathbf{G}}_u, \vec{\pi}_0 \rangle \neq 0$ $\|\mathbf{G}_u\|$ -a.e. on $K \times \mathbb{R}^n$, by [14, Proposition 1.4]. Thus, $(\mathbf{G}_u - T) \llcorner K \times \mathbb{R}^n = 0$ if and only if $(\mathbf{G}_u - T) \llcorner dx \mathbf{1}_{K \times \mathbb{R}^n} = 0$. The

latter identity follows from the slicing formula and the property $\langle T, \mathbf{p}, x \rangle = \langle \mathbf{G}_u, \mathbf{p}, x \rangle = \sum_i \delta_{(x, u_i(x))}$, valid for a.e. $x \in K$.

Finally, for each $x \in B_r \setminus K$ choose a ball $x \in B^x = B_{r(x)}(y(x)) \subset B_4$ such that $\mathbf{e}_T(B^x) \geq 2^{-m} \delta_{11} \omega_m r(x)^m$. By the $5r$ -Covering theorem, we choose balls $\hat{B}^i = B_{5r(x_i)}(y(x_i))$ which cover $B_r \setminus K$ and such that the balls B^{x_i} are pairwise disjoint. We then conclude

$$|B_r \setminus K| \leq 10^m \delta_{11}^{-1} \mathbf{e}_T \left(\bigcup_i B^{x_i} \right). \quad (2.12)$$

Fix $y \in B^{x_i}$. Since $B^{x_i} \subset B_4$, we have $2^{-m} \delta_{11} \omega_m r(x_i)^m \leq \mathbf{e}_T(B^{x_i}) \leq \mathbf{e}_T(B_4) = 4^m \omega_m E$, which implies $2r(x_i) \leq r_0 < 1$. Thus, $y \in B_{r+r_0} \subset B_4$. By definition of \mathbf{me}_T we obviously have $\mathbf{me}_T(y) \geq 2^{-m} \delta_{11}$. So $\cup_i B^{x_i} \subset B_{r+r_0} \cap \{\mathbf{me}_T > 2^{-m} \delta_{11}\}$ and (2.12) implies (2.1).

3. PATCHING MULTIPLE VALUED GRAPHS

In this section we prove some complementary results to the theory of multiple valued functions as exposed in [13, 14]. In particular, we show here a concentration compactness principle for Q -valued functions, and give an algorithm to construct suitable competitors for the Dirichlet energy, which will be also used in [17]. We first introduce some terminology.

Definition 3.1 (Translating sheets). Let $\Omega \subset \mathbb{R}^m$ be a bounded open set. A sequence of maps $\{h_l\}_{l \in \mathbb{N}} \subset W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is called a sequence of *translating sheets* if there are:

- (a) integers $J \geq 1$ and $Q_1, \dots, Q_J \geq 1$ satisfying $\sum_{j=1}^J Q_j = Q$,
- (b) vectors $y_l^j \in \mathbb{R}^n$ (for $j \in \{1, \dots, J\}$ and $l \in \mathbb{N}$) with

$$\lim_l |y_l^j - y_l^i| = +\infty \quad \forall i \neq j, \quad (3.1)$$

- (c) and maps $\zeta^j \in W^{1,2}(\Omega, \mathcal{A}_{Q_j})$ for $j \in \{1, \dots, J\}$,

such that $h_l = \sum_{j=1}^J \llbracket \tau_{y_l^j} \circ \zeta^j \rrbracket$ (see [13, Section 3.3.3] for the notation).

Remark 3.2. Assume that h_l, Q_j, y_l^j and ζ^j satisfy all the requirements of Definition 3.1 except for (3.1). Up to subsequences and relabellings, assume that $y_l^1 - y_l^2$ converges to a vector $2\bar{y}$. We can replace

- the integers Q_1 and Q_2 with $Q' = Q_1 + Q_2$;
- the vectors y_l^1 and y_l^2 with $y_l' = (y_l^1 + y_l^2)/2$;
- the maps ζ^1 and ζ^2 with $\zeta' := \llbracket \tau_{\bar{y}} \circ \zeta^1 \rrbracket + \llbracket \tau_{-\bar{y}} \circ \zeta^2 \rrbracket$.

The new collections $Q', Q_3, \dots, Q_J, y_l', y_l^3, \dots, y_l^J$ and $\zeta', \zeta^3, \dots, \zeta^J$, and the function $h_l' := \llbracket \zeta' \rrbracket + \sum_{j=3}^J \llbracket \zeta^j \rrbracket$, satisfy again all the requirements of Definition 3.1 except, possibly, for (3.1). Moreover, $\|\mathcal{G}(h_l', h_l)\|_{L^2} \rightarrow 0$ and $|Dh_l'| = |Dh_l|$. Obviously, we can iterate this procedure only a finite number of times, obtaining a subsequence of translating sheets \hat{h}_l asymptotic to h_l in the L^2 distance with $|D\hat{h}_l| = |Dh_l|$.

3.1. Concentration compactness. Translating sheets give a useful device to recover a suitable “compactness statement” for sequences of maps with equi-bounded energy.

Proposition 3.3 (Concentration compactness). *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set and $(g_l)_{l \in \mathbb{N}} \subset W^{1,2}(\Omega, \mathcal{A}_Q)$ a sequence of functions with $\sup_l \int_\Omega |Dg_l|^2 < \infty$. Then, there exist a subsequence (not relabeled) and a sequence of translating sheets h_l such that $\|\mathcal{G}(g_l, h_l)\|_{L^2} \rightarrow 0$ and the following inequalities hold for every open $\Omega' \subset \Omega$ and any sequence of measurable sets J_l with $|J_l| \rightarrow 0$:*

$$\liminf_{l \rightarrow +\infty} \left(\int_{\Omega' \setminus J_l} |Dg_l|^2 - \int_{\Omega'} |Dh_l|^2 \right) \geq 0 \quad (3.2)$$

$$\limsup_{l \rightarrow +\infty} \int_\Omega (|Dg_l| - |Dh_l|)^2 \leq \limsup_l \int_\Omega (|Dg_l|^2 - |Dh_l|^2). \quad (3.3)$$

Proof. We start proving, by induction, the existence of traveling sheets $\{h_l\}$ (and a subsequence) with $\|\mathcal{G}(h_l, g_l)\|_{L^2} \rightarrow 0$ and satisfying the following additional property. If J, Q_j, y_l^j and ζ^j are as in Definition 3.1, then there are Q_j valued functions w_l^j such that, after setting $f_l = \sum_j \llbracket w_l^j \rrbracket$, we have

$$\|\mathcal{G}(\hat{f}_l, g_l)\|_{L^2} + |\{g_l \neq \hat{f}_l\}| \rightarrow 0, \quad \|\mathcal{G}(\tau_{-y_l^j} \circ w_l^j, \zeta^j)\|_{L^2} \rightarrow 0 \quad \text{and} \quad |D\hat{f}_l| \leq |Dg_l|. \quad (3.4)$$

If $Q = 1$ the claim is an easy corollary of the Poincaré inequality and the compact embedding $W^{1,2} \hookrightarrow L^2$. Assuming that the claim holds for any $Q^* < Q$, we prove it for Q . By the generalized Poincaré inequality [13, Proposition 2.12], there exist points $\bar{g}_l \in \mathcal{A}_Q(\mathbb{R}^n)$ and a real number M such that

$$\int_\Omega \mathcal{G}(g_l, \bar{g}_l)^2 \leq C \int_\Omega |Dg_l|^2 \leq M < \infty \quad \forall l \in \mathbb{N}.$$

Recall the separation $s(T)$ and the diameter $d(T)$ of a point $T = \sum_i \llbracket P_i \rrbracket$ introduced in [13, Definition 3.4]: $s(T) := \min\{|P_i - P_j| : P_i \neq P_j\}$ and $d(T) := \max\{|P_i - P_j|\}$. We distinguish between two cases.

Case 1: $\liminf_l d(\bar{g}_l) < \infty$. After passing to a subsequence, we find $y_l \in \mathbb{R}^n$ such that the functions $\tau_{y_l} \circ g_l$ are equi-bounded in the $W^{1,2}$ -metric. By the Sobolev embedding [13, Proposition 2.11], there exists a Q -valued map $\zeta \in W^{1,2}$ such that $\tau_{y_l} \circ g_l \rightarrow \zeta$ in $L^2(\Omega)$.

Case 2: $\lim_l d(\bar{g}_l) = +\infty$. By [13, Lemma 3.8] there are points $S_l \in \mathcal{A}_Q$ such that

$$s(S_l) \geq \beta d(\bar{g}_l) \quad \text{and} \quad \mathcal{G}(S_l, \bar{g}_l) \leq s(S_l)/32,$$

where β is a dimensional constant. Write $S_l = \sum_{i=1}^J k_i \llbracket P_l^i \rrbracket$, with $P_l^i \neq P_l^j$ for $i \neq j$. Both J and k_i may depend on l but they have a finite range: therefore, after extracting a subsequence, we can assume that they do not depend on l . Set next $r_l = \frac{s(S_l)}{16}$ and let ϑ_l be the retraction of $\mathcal{A}_Q(\mathbb{R}^n)$ into $\overline{B_{r_l}(S_l)}$ provided by [13, Lemma 3.7]. Clearly, the functions

$f_l = \vartheta_l \circ g_l$ satisfy $|Df_l| \leq |Dg_l|$ and there are k_i -valued functions z_l^i such that

$$f_l = \sum_{i=1}^J \llbracket z_l^i \rrbracket, \quad \text{with} \quad \|\mathcal{G}(z_l^i, k_i \llbracket P_l^i \rrbracket)\|_\infty \leq r_l.$$

Since $k_i < Q$, we apply the inductive hypothesis to each sequence $(z_l^i)_l$ and, using Remark 3.2 reach a subsequence (not relabeled) of f_l , a sequence of translating sheets h_l and corresponding functions \hat{f}_l which satisfy (3.4) with f_l replacing g_l .

We next claim that (3.4) holds even for g_l , i.e. that $\lim_l (\|\mathcal{G}(f_l, h_l)\|_{L^2} + |\{f_l \neq g_l\}|) = 0$. To this aim, recall first that

$$\{g_l \neq f_l\} = \{\mathcal{G}(g_l, S_l) > r_l\} \subseteq \{\mathcal{G}(g_l, \bar{g}_l) > r_l/2\}.$$

Thus,

$$|\{g_l \neq f_l\}| \leq |\{\mathcal{G}(g_l, \bar{g}_l) > r_l/2\}| \leq \frac{C}{r_l^2} \int_{\{\mathcal{G}(g_l, \bar{g}_l) > \frac{r_l}{2}\}} \mathcal{G}(g_l, \bar{g}_l)^2 \leq \frac{CM}{(d(\bar{g}_l))^2}. \quad (3.5)$$

Next, since $\vartheta_l(\bar{g}_l) = \bar{g}_l$ and $\text{Lip}(\vartheta_l) = 1$, we have $\mathcal{G}(f_l, \bar{g}_l) \leq \mathcal{G}(g_l, \bar{g}_l)$. Therefore, by the Sobolev embedding and the Poincaré inequality, for any $p \in]2, 2^*[$, we infer

$$\begin{aligned} \int_{B_2} \mathcal{G}(f_l, g_l)^2 &= \int_{\{g_l \neq h_l\}} \mathcal{G}(f_l, g_l)^2 \leq 2 \int_{\{f_l \neq g_l\}} \mathcal{G}(f_l, \bar{g}_l)^2 + 2 \int_{\{f_l \neq g_l\}} \mathcal{G}(\bar{g}_l, g_l)^2 \\ &\leq 4 \int_{\{f_l \neq g_l\}} \mathcal{G}(\bar{g}_l, g_l)^2 \leq C \|\mathcal{G}(g_l, \bar{g}_l)\|_{L^p}^2 |\{f_l \neq g_l\}|^{1-\frac{2}{p}} \stackrel{(3.5)}{\leq} \frac{CM^{1-2/p}}{d(\bar{g}_l)^{2-4/p}} \int_{B_2} |Dg_l|^2. \end{aligned}$$

Since $d(\bar{g}_l)$ diverges, this shows $\|\mathcal{G}(f_l, g_l)\|_{L^2} \rightarrow 0$.

We now show that (3.2) and (3.3) are consequences of (3.4). For each j we consider the corresponding embedding $\boldsymbol{\xi}_j : \mathcal{A}_{Q_j}(\mathbb{R}^n) \rightarrow \mathbb{R}^{N(Q_j, n)}$ and, by a slight abuse of notation, we drop the j subscript. Then, we conclude that $\boldsymbol{\xi} \circ \tau_{-y_l^j} \circ w_l^j \rightarrow \boldsymbol{\xi} \circ \zeta^j$ in L^2 and $\|D(\boldsymbol{\xi} \circ \tau_{-y_l^j} \circ w_l^j)\|_{L^2}$ is a bounded sequence, from which

$$D(\boldsymbol{\xi} \circ \tau_{-y_l^j} \circ w_l^j) \rightharpoonup D(\boldsymbol{\xi} \circ \zeta^j) \quad \text{in } L^2(\Omega). \quad (3.6)$$

Since by hypothesis $\mathbf{1}_{\Omega' \setminus J_l} \rightarrow \mathbf{1}_{\Omega'}$ in $L^2(\Omega)$, it follows from (3.6) that

$$D(\boldsymbol{\xi} \circ \tau_{-y_l^j} \circ w_l^j) \mathbf{1}_{\Omega' \setminus J_l} \rightharpoonup D(\boldsymbol{\xi} \circ \zeta^j) \mathbf{1}_{\Omega'},$$

and, hence,

$$\text{Dir}(\zeta^j, \Omega') = \int_{\Omega'} |D(\boldsymbol{\xi} \circ \zeta^j)|^2 \leq \liminf_l \int_{\Omega' \setminus J_l} |D(\boldsymbol{\xi} \circ \tau_{-y_l^j} \circ w_l^j)|^2 = \liminf_l \int_{\Omega' \setminus J_l} |Dw_l^j|^2.$$

Summing over j , we obtain (3.2). As for (3.3), set $J_l := \{g_l \neq f_l\}$. Thus,

$$\begin{aligned}
& \int_{\Omega \setminus J_l} (|Dg_l| - |Dh|)^2 = \sum_j \int_{\Omega \setminus J_l} (|Dw_l^j| - |D\zeta^j|)^2 \\
& = \sum_j \int_{\Omega \setminus J_l} (|D(\boldsymbol{\xi} \circ \tau_{-y_l^j} \circ w_l^j)| - |D(\boldsymbol{\xi} \circ \zeta^j)|)^2 \leq \sum_j \int_{\Omega \setminus J_l} |D(\boldsymbol{\xi} \circ \tau_{-y_l^j} \circ w_l^j) - D(\boldsymbol{\xi} \circ \zeta^j)|^2 \\
& = \sum_j \int_{\Omega \setminus J_l} \left(|D(\boldsymbol{\xi} \circ \tau_{-y_l^j} \circ w_l^j)|^2 + |D(\boldsymbol{\xi} \circ \zeta^j)|^2 - 2D(\boldsymbol{\xi} \circ \tau_{-y_l^j} \circ w_l^j) \cdot D(\boldsymbol{\xi} \circ \zeta^j) \right). \quad (3.7)
\end{aligned}$$

Therefore, by (3.6) (and taking into account that $|J_l| \rightarrow 0$) one gets

$$\begin{aligned}
& \limsup_{l \rightarrow +\infty} \int_{\Omega \setminus J_l} (|Dg_l| - |Dh|)^2 \\
& \leq \lim_{l \rightarrow +\infty} \sum_j \int_{\Omega \setminus J_l} \left(|D(\boldsymbol{\xi} \circ \tau_{-y_l^j} \circ w_l^j)|^2 + |D(\boldsymbol{\xi} \circ \zeta^j)|^2 - 2D(\boldsymbol{\xi} \circ \tau_{-y_l^j} \circ w_l^j) \cdot D(\boldsymbol{\xi} \circ \zeta^j) \right) \\
& = \limsup_{l \rightarrow +\infty} \int_{\Omega \setminus J_l} \sum_j |D(\boldsymbol{\xi} \circ \tau_{-y_l^j} \circ w_l^j)|^2 - \int_{\Omega} \sum_j |D(\boldsymbol{\xi} \circ \zeta^j)|^2 \\
& = \limsup_{l \rightarrow +\infty} \int_{\Omega \setminus J_l} |Dg_l|^2 - \int_{\Omega} |Dh|^2. \quad (3.8)
\end{aligned}$$

On the other hand, since $|J_l| \rightarrow 0$ we conclude

$$\limsup_{j \rightarrow \infty} \int_{J_l} (|Dg_l| - |Dh|)^2 = \limsup_{j \rightarrow \infty} \int_{J_l} |Dg_l|^2.$$

Observe that, after passing to a subsequence, we can actually assume that all limsups are in fact limits. Summing (3.7) and (3.8) we then conclude (3.3). \square

3.2. Dirichlet competitors. We consider next a standard procedure to construct competitors for the Dirichlet energy of a sequence of functions with equi-bounded energy.

Proposition 3.4 (Construction of a competitor). *Consider two radii $0 \leq r_0 < r_1 < 4$ and maps $f_l, h_l \in W^{1,2}(B_{r_1}, \mathcal{A}_Q(\mathbb{R}^n))$ such that $\{h_l\}_l$ is a sequence of translating sheets,*

$$\sup_l \text{Dir}(f_l, B_{r_1}) < +\infty \quad \text{and} \quad \|\mathcal{G}(f_l, h_l)\|_{L^2(B_{r_1} \setminus B_{r_0})} \rightarrow 0.$$

For every $\eta > 0$, there exist $r \in]r_0, r_1[$, a subsequence of $\{f_l\}_l$ (not relabeled) and functions $g_l \in W^{1,2}(B_{r_1}, \mathcal{A}_Q(\mathbb{R}^n))$ such that $g_l|_{B_{r_1} \setminus B_r} = f_l|_{B_{r_1} \setminus B_r}$ and $\text{Dir}(g_l, B_{r_1}) \leq \text{Dir}(h_l, B_{r_1}) + \eta$. In addition, there is a dimensional constant C and a constant $C^(\eta)$ (depending also on the two sequences, but not on l) such that*

$$\text{Lip}(g_l) \leq C^*(\eta) (\text{Lip}(f_l) + 1), \quad (3.9)$$

$$\|\mathcal{G}(g_l, h_l)\|_{L^2(B_r)} \leq C \text{Dir}(f_l, B_r) + C \text{Dir}(g_l, B_r), \quad (3.10)$$

$$\|\boldsymbol{\eta} \circ g_l\|_{L^1(B_{r_1})} \leq C \|\boldsymbol{\eta} \circ f_l\|_{L^1(B_{r_1})} + C \|\boldsymbol{\eta} \circ h_l\|_{L^1(B_{r_1})}. \quad (3.11)$$

In order to prove the proposition, we need to recall the following two lemmas, which are slight variants of [13, Proposition 2.5] and [13, Lemma 2.15].

Lemma 3.5 (Lipschitz approximation). *Let $f \in W^{1,2}(B_r, \mathcal{A}_Q)$. Then, for every $\varepsilon > 0$, there exists $f_\varepsilon \in \text{Lip}(B_r, \mathcal{A}_Q)$ such that*

$$\int_{B_r} \mathcal{G}(f, f_\varepsilon)^2 + \int_{B_r} (|Df| - |Df_\varepsilon|)^2 + \int_{B_r} (|D(\boldsymbol{\eta} \circ f)| - |D(\boldsymbol{\eta} \circ f_\varepsilon)|)^2 \leq \varepsilon. \quad (3.12)$$

If $f|_{\partial B_r} \in W^{1,2}(\partial B_r, \mathcal{A}_Q)$, then f_ε can be chosen to satisfy also

$$\int_{\partial B_r} \mathcal{G}(f, f_\varepsilon)^2 + \int_{\partial B_r} (|Df| - |Df_\varepsilon|)^2 \leq \varepsilon. \quad (3.13)$$

Proof. By an obvious scaling argument we can assume $r = 1$. We start noticing that (3.12) is a corollary of [13, Proposition 2.5]. On the other hand, if $f|_{\partial B_1} \in W^{1,2}(\partial B_1)$, we extend the map to B_2 by setting $f(x) = f(\frac{x}{|x|})$ if $|x| \geq 1$. We then can apply [13, Proposition 2.5] to find a sequence of Lipschitz maps f_k such that $f_k \rightarrow f$ strongly in $W^{1,2}(B_2)$. Given $\delta > 0$, define the maps $f^\delta(x) = f((1 + \delta)x)$ and $f_k^\delta(x) = f_k((1 + \delta)x)$. Obviously, $f_k^\delta \rightarrow f^\delta$ strongly in $W^{1,2}(B_1)$ and $f^\delta \rightarrow f$ strongly in $W^{1,2}(B_1)$ as $\delta \downarrow 0$. By a standard Fubini argument, for each j we can find a $\delta_j < \frac{1}{j}$ and a subsequence $\{f_{k,j}\}_k$ such that $f_{k,j}|_{\partial B_{1+\delta_j}} \rightarrow f|_{\partial B_{1+\delta_j}}$ (i.e. $f_{k,j}^{\delta_j}|_{\partial B_1} \rightarrow f^{\delta_j}|_{\partial B_1} = f|_{\partial B_1}$) strongly in $W^{1,2}(\partial B_{1+\delta_j})$ as $k \uparrow \infty$. By standard diagonal argument we can arrange the subsequences so that $\{f_{k,j}\} \supset \{f_{k,j+1}\}$. Thus, a suitable diagonal sequence $\bar{f}_j := f_{k(j),j}^{\delta_j}$ has the property that $\bar{f}_j \rightarrow f$ in $W^{1,2}(B_1)$ and $\bar{f}_j|_{\partial B_1} \rightarrow f|_{\partial B_1}$ in $W^{1,2}(\partial B_1)$. \square

Lemma 3.6 (Interpolation). *There exists a constant $C_0 = C_0(m, n, Q) > 0$ with the following property. Assume $r \in]1, 3[$, $f \in W^{1,2}(B_r, \mathcal{A}_Q)$ and $g \in W^{1,2}(\partial B_r, \mathcal{A}_Q)$ are given maps such that $f|_{\partial B_r} \in W^{1,2}(\partial B_r, \mathcal{A}_Q)$. Then, for every $\varepsilon \in]0, r[$ there exists a function $h \in W^{1,2}(B_r, \mathcal{A}_Q)$ such that $h|_{\partial B_r} = g$ and*

$$\int_{B_r} |Dh|^2 \leq \int_{B_r} |Df|^2 + \varepsilon \int_{\partial B_r} (|D_\tau f|^2 + |D_\tau g|^2) + \frac{C_0}{\varepsilon} \int_{\partial B_r} \mathcal{G}(f, g)^2, \quad (3.14)$$

$$\text{Lip}(h) \leq C_0 \left\{ \text{Lip}(f) + \text{Lip}(g) + \varepsilon^{-1} \sup_{\partial B_r} \mathcal{G}(f, g) \right\}, \quad (3.15)$$

$$\int_{B_r} |\boldsymbol{\eta} \circ h| \leq C_0 \int_{\partial B_r} |\boldsymbol{\eta} \circ g| + C_0 \int_{B_r} |\boldsymbol{\eta} \circ f|, \quad (3.16)$$

(here D_τ denotes the tangential derivative).

Proof. The first conclusion is an obvious corollary of [13, Lemma 2.15]. It is then straightforward to see that the map constructed in the proof of [13, Lemma 2.15] satisfies also (3.15). As for the final claim, let $\bar{g} := \sum [g_i - \boldsymbol{\eta} \circ g]$, $\bar{f} := \sum [f_i - \boldsymbol{\eta} \circ f]$ and consider the interpolation map \bar{h} between \bar{f} and \bar{g} given by [13, Lemma 2.15]. Set $\hat{h} = \sum_i [\bar{h}_i - \boldsymbol{\eta} \circ \bar{h}]$ and observe that $\text{Lip}(\hat{h}) \leq \text{Lip}(\bar{h})$ and $\text{Dir}(\hat{h}) \leq \text{Dir}(\bar{h})$. We apply again [13, Lemma 2.15]

in the case $Q = 1$ to $\boldsymbol{\eta} \circ f$ and $\boldsymbol{\eta} \circ g$, and get the interpolation u . It is then easy to check that the map $h := \sum_i [\hat{h}_i + u]$ has all the desired properties. \square

Proof of Proposition 3.4. Set for simplicity $A_l := \|\mathcal{G}(f_l, h_l)\|_{L^2(B_{r_1} \setminus B_{r_0})}$ and $B_l := \|\boldsymbol{\eta} \circ f_l\|_{L^1(B_{r_1})}$. If $A_l \equiv 0$, then there is nothing to prove. Assuming that for a subsequence (not relabeled) $B_l > 0$, we consider the function

$$\psi_l(r) := \int_{\partial B_r} (|Df_l|^2 + |Dh_l|^2) + A_l^{-2} \int_{\partial B_r} \mathcal{G}(f_l, h_l)^2 + B_l^{-1} \int_{\partial B_r} |\boldsymbol{\eta} \circ f_l|. \quad (3.17)$$

By assumption $\liminf_l \int_{r_0}^{r_1} \psi_l(r) dr < \infty$. So, by Fatou's Lemma, there is $r \in]r_0, r_1[$ and a subsequence, not relabeled, such that $\lim_l \psi_l(r) < \infty$. Thus, for some $M > 0$ we have

$$\int_{\partial B_r} \mathcal{G}(f_l, h_l)^2 \rightarrow 0, \quad (3.18)$$

$$\text{Dir}(h_l, \partial B_r) + \text{Dir}(f_l, \partial B_r) \leq M, \quad (3.19)$$

$$\int_{\partial B_r} |\boldsymbol{\eta} \circ f_l| \leq M \|\boldsymbol{\eta} \circ f_l\|_{L^1(B_{r_1})}. \quad (3.20)$$

In case $B_l = 0$ for all l large enough, we define ψ_l dropping the last summand in (3.17) and reach the same conclusion.

Let ζ^j be the blocks of the translating sheets h_l as in Definition 3.1. We apply Lemma 3.5 to each ζ^j and find Lipschitz functions ζ_η^j satisfying the conclusion of the lemma with $\bar{\varepsilon}_1 = \bar{\varepsilon}_1(\eta, M) > 0$ (which will be chosen later). We also choose a standard radial convolution kernel φ in \mathbb{R}^m and a small parameter $\bar{\rho}$ (also to be chosen later). Then, set

$$h_{l,\eta} := \sum_{j=1}^J [\tau_{y_l^j} \circ \zeta_\eta^j] \quad \text{and} \quad \bar{h}_{l,\eta} := \sum_{i=1}^Q [(h_{l,\eta})_i - \boldsymbol{\eta} \circ h_{l,\eta} + (\boldsymbol{\eta} \circ h_l) * \varphi_{\bar{\rho}}].$$

Assume $\bar{\rho}$ is so small that $\|\boldsymbol{\eta} \circ h_l - (\boldsymbol{\eta} \circ h_l) * \varphi_{\bar{\rho}}\|_{L^2}^2 \leq \bar{\varepsilon}_1$. By very rough estimates,

$$\|\mathcal{G}(f_l, \bar{h}_{l,\eta})\|_{L^2} \leq \|\mathcal{G}(f_l, h_l)\|_{L^2} + 2\|\mathcal{G}(h_l, h_{l,\eta})\|_{L^2} + \bar{\varepsilon}_1 \leq o(1) + 3\bar{\varepsilon}_1, \quad (3.21)$$

$$\text{Dir}(\bar{h}_{l,\eta}, \partial B_r) \leq 3M + 3\bar{\varepsilon}_1, \quad (3.22)$$

and

$$\begin{aligned} \text{Dir}(\bar{h}_{l,\eta}, B_r) &= \sum_i \int_{B_r} |D(h_{l,\eta})_i - D(\boldsymbol{\eta} \circ h_{l,\eta}) + D(\boldsymbol{\eta} \circ h_l * \varphi_{\bar{\rho}})|^2 \\ &\leq \int_{B_r} (|Dh_{l,\eta}|^2 - Q|D(\boldsymbol{\eta} \circ h_{l,\eta})|^2 + Q|D(\boldsymbol{\eta} \circ h_l)|^2) \stackrel{(3.12)}{\leq} \text{Dir}(h_{l,\eta}, B_r) + Q\bar{\varepsilon}_1. \end{aligned} \quad (3.23)$$

We can then apply Lemma 3.6 to $\bar{h}_{l,\eta}$ and f_l with $\bar{\varepsilon}_2 = \bar{\varepsilon}_2(\eta, M) > 0$, and get (up to subsequences) maps g_l satisfying $g_l|_{\partial B_r} = f_l|_{\partial B_r}$ and

$$\begin{aligned} \text{Dir}(g_l, B_r) &\leq \text{Dir}(\bar{h}_{l,\eta}, B_r) + \bar{\varepsilon}_2 \text{Dir}(\bar{h}_{l,\eta}, \partial B_r) + \bar{\varepsilon}_2 \text{Dir}(f_l, \partial B_r) + C_0 \bar{\varepsilon}_2^{-1} \int_{\partial B_r} \mathcal{G}(\bar{h}_{l,\eta}, f_l)^2 \\ &\leq \text{Dir}(h_l, B_r) + Q\bar{\varepsilon}_1 + 3\bar{\varepsilon}_2(M + \bar{\varepsilon}_1) + 3C_0 \bar{\varepsilon}_2^{-1} \bar{\varepsilon}_1 \end{aligned}$$

where in the last line we have used (3.18), (3.19) and (3.21) - (3.23). An appropriate choice of the parameters ε_1 and ε_2 gives the desired bound $\text{Dir}(g_l, B_r) \leq \text{Dir}(h_l, B_r) + \eta$.

Observe next that, by construction, $\limsup_l \text{Lip}(\bar{h}_{l,\eta}) \leq C^*$, for some constant which depends on η and the two sequences, but on l . Moreover,

$$\|\mathcal{G}(\bar{h}_{l,\eta}, f_l)\|_{L^\infty(\partial B_r)} \leq \|\mathcal{G}(\bar{h}_{l,\eta}, f_l)\|_{L^2(\partial B_r)} + C\text{Lip}(f_l) + C\text{Lip}(h_{l,\eta}).$$

Thus (3.9) follows from (3.15).

Finally, (3.10) follows from the Poincaré inequality applied to $\mathcal{G}(g_l, f_l)$ (which vanishes identically on ∂B_r), and (3.11) follows from (3.16), because of (3.20) and $\|\boldsymbol{\eta} \circ \bar{h}_{l,\eta}\|_{L^1} = \|(\boldsymbol{\eta} \circ h_l) * \varphi_{\bar{\rho}}\|_{L^1} \leq \|\boldsymbol{\eta} \circ h_l\|_{L^1}$. \square

4. HARMONIC APPROXIMATION

In what follows we will always apply Proposition 2.2 with $\delta_{11} = E^{2\beta}$.

Definition 4.1 (E^β -Lipschitz approximation). Let $\beta \in (0, \frac{1}{2m})$ and T be as in Proposition 2.2 such that $32E^{(1-2\beta)/m} < 1$. The map u given by the proposition for $\delta_{11} = E^{2\beta}$ is then called the E^β -Lipschitz approximation of T in $\mathbf{C}_{3s}(x)$.

In this section we prove that, under a certain scaling of \mathbf{A} , if T is also area minimizing, the corresponding E^β -Lipschitz approximation is close to a Dir-minimizing function w . This comes with an $o(E)$ -improvement of the estimates in Proposition 2.2.

Theorem 4.2 (First harmonic approximation). *For every $\eta_1, \delta > 0$ and every $\beta \in (0, \frac{1}{2m})$, there exist constants $\varepsilon_{12}, C_{12} > 0$ with the following property. Let T be as in Assumption 1.1 in $\mathbf{C}_{4s}(x)$ and assume it is area minimizing. If $E = \mathbf{E}(T, \mathbf{C}_{4s}(x)) \leq \varepsilon_{12}$ and $s\mathbf{A} \leq E^{1/4+\delta}$, then the E^β -Lipschitz approximation f in $\mathbf{C}_{3s}(x)$ satisfies*

$$\int_{B_{2s}(x) \setminus K} |Df|^2 \leq \eta_1 E \omega_m (4s)^m = \eta_1 \mathbf{e}_T(B_{4s}(x)). \quad (4.1)$$

Moreover, there exists a Dir-minimizing function w such that

$$s^{-2} \int_{B_{2s}(x)} \mathcal{G}(f, w)^2 + \int_{B_{2s}(x)} (|Df| - |Dw|)^2 \leq \eta_1 E \omega_m (4s)^m = \eta_1 \mathbf{e}_T(B_{4s}(x)), \quad (4.2)$$

$$\int_{B_{2s}(x)} |D(\boldsymbol{\eta} \circ f) - D(\boldsymbol{\eta} \circ w)|^2 \leq \eta_1 E \omega_m (4s)^m = \eta_1 \mathbf{e}_T(B_{4s}(x)). \quad (4.3)$$

Remark 4.3 (Isoperimetric inequality). If $S \subset \mathbb{R}^{m+n}$ is an integral current of dimension $m-1$, then there is an m -dimensional integral current $R \subset \mathbb{R}^{m+n}$ such that $\partial R = S$ and $\mathbf{M}(R) \leq C\mathbf{M}(S)^{m/(m-1)}$, where the constant C is only dimensional (see [28, Theorem 30.1]). It is also well-known that, when $\text{spt}(S) \subset \Sigma$ and Σ is as in Assumption 1.1 the same inequality holds for some \bar{R} with $\text{spt}(\bar{R}) \subset \Sigma$ and $\partial \bar{R} = S$, with a dimensional constant C which depends additionally on the constant c_0 . This can be easily seen as follows: let $\mathbf{q} : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$ be the orthogonal projection and $\Lambda : \mathbb{R}^{m+n} \rightarrow \Sigma$ be the map $\Lambda(p) = (\mathbf{q}(p), \Psi(\mathbf{q}(p)))$. Λ is a global Lipschitz retraction of \mathbb{R}^{m+n} onto Σ which is the identity on Σ : thus we can simply set $\bar{R} = \Lambda_{\#}R$.

Remark 4.4 (Estimates on Ψ). Assume that T is as in Assumption 1.1 in the cylinder \mathbf{C}_4 . Then we can assume without loss of generality that the function $\Psi : \mathbb{R}^{m+\bar{n}} \rightarrow \mathbb{R}^l$ parameterizing Σ satisfies $\Psi(0) = 0$, $\|D\Psi\|_0 \leq C E^{1/2} + C\mathbf{A}$ and $\|D^2\Psi\|_0 \leq C\mathbf{A}$. In order to see that, note that

$$E = \mathbf{E}(T, \mathbf{C}_4) = \frac{1}{2\omega_m 4^m} \int_{\mathbf{C}_4} |\vec{T}(y) - \vec{\pi}_0|^2 d\|T\|(y).$$

Thus, we can fix a point $p \in \text{spt}(T) \cap \mathbf{C}_4$ such that $|\vec{T}(p) - \vec{\pi}_0| \leq C E^{1/2}$. Then, we can find an associated rotation $O \in O(m+n)$ such that $O_{\#}\vec{T}(p) = \vec{\pi}_0$ and $|O| \leq C E^{1/2}$. It follows that $\pi := O(T_p\Sigma)$ is a $(m+\bar{n})$ -dimensional plane such that $\pi_0 \subset \pi$ and $\|\pi - T_p\Sigma\| \leq C E^{1/2}$. We choose new coordinates so that π_0 remains equal to $\mathbb{R}^m \times \{0\}$ but $\mathbb{R}^{m+\bar{n}} \times \{0\}$ equals π . Since the excess E is assumed to be sufficiently small, we can write Σ as the graph of a function $\Psi : \pi \rightarrow \pi^\perp$. If $(x, \Psi(x)) = p$, then $|D\Psi(x)| \leq C\|T_p\Sigma - \mathbb{R}^{m+\bar{n}} \times \{0\}\| \leq C E^{1/2}$. However, $\|D^2\Psi\|_0 \leq C\mathbf{A}$ and so $\|D\Psi\|_0 \leq C(E^{1/2} + \mathbf{A}) \leq C E^{1/2}$. Moreover, $\Psi(0) = 0$ is achieved translating the system of reference by a vector orthogonal to $\mathbb{R}^{m+\bar{n}} \times \{0\}$ and, hence, belonging to $\{0\} \times \mathbb{R}^n$.

Proof. By rescaling and translating, it is not restrictive to assume that $x = 0$ and $s = 1$. Thus, by Remark 4.4 we can assume $\Psi(0) = 0$, $\|D\Psi\|_0 \leq C(E^{1/2} + \mathbf{A})$ and $\|D^2\Psi\|_0 \leq \mathbf{A}$. The proof of (4.1) is by contradiction. Assume there exist a constant $c_1 > 0$, a sequence of currents $(T_l)_{l \in \mathbb{N}}$ and corresponding E_l^β -Lipschitz approximations $(f_l)_{l \in \mathbb{N}}$ such that

$$E_l := \mathbf{E}(T_l, \mathbf{C}_4) \rightarrow 0 \quad \text{and} \quad \int_{B_2 \setminus K_l} |Df_l|^2 \geq c_1 E_l, \quad (4.4)$$

where $K_l := \{x \in B_3 : \mathbf{m}_{T_l}(x) < E_l^{2\beta}\}$. Set $\Gamma_l := \{x \in B_4 : \mathbf{m}_{T_l}(x) \leq 2^{-m} E_l^{2\beta}\}$ and observe that $\Gamma_l \cap B_3 \subset K_l$. From Proposition 2.2, it follows that

$$\text{Lip}(f_l) \leq C E_l^\beta, \quad (4.5)$$

$$|B_r \setminus K_l| \leq C E_l^{-2\beta} \mathbf{e}_T(B_{r+r_0(l)} \setminus \Gamma_l) \quad \text{for every } r \leq 3, \quad (4.6)$$

where $r_0(l) = 16 E_l^{(1-2\beta)/m} < \frac{1}{2}$. Then, (4.4), (4.5) and (4.6) give

$$c_1 E_l \leq \int_{B_2 \setminus K_l} |Df_l|^2 \leq C \mathbf{e}_{T_l}(B_s \setminus \Gamma_l) \quad \forall s \in [\frac{5}{2}, 3].$$

Setting $c_2 := c_1/(2C)$, we have $2c_2 E_l \leq \mathbf{e}_{T_l}(B_s \setminus \Gamma_l) = \mathbf{e}_{T_l}(B_s) - \mathbf{e}_{T_l}(B_s \cap \Gamma_l)$, thus leading to

$$\mathbf{e}_{T_l}(\Gamma_l \cap B_s) \leq \mathbf{e}_{T_l}(B_s) - 2c_2 E_l. \quad (4.7)$$

Next observe that $\omega_m 4^m E_l = \mathbf{e}_{T_l}(B_4) \geq \mathbf{e}_{T_l}(B_s)$. Therefore, by the Taylor expansion in [14, Corollary 3.3], (4.7) and $E_l \downarrow 0$, it follows that, for every $s \in [5/2, 3]$,

$$\begin{aligned} \int_{\Gamma_l \cap B_s} \frac{|Df_l|^2}{2} &\leq (1 + C E_l^{2\beta}) \mathbf{e}_{T_l}(\Gamma_l \cap B_s) \\ &\leq (1 + C E_l^{2\beta}) \left(\mathbf{e}_{T_l}(B_s) - 2c_2 E_l \right) \leq \mathbf{e}_{T_l}(B_s) - c_2 E_l. \end{aligned} \quad (4.8)$$

Our aim is to show that (4.8) contradicts the minimizing property of T_l . To construct a competitor we write $f_l(x) = \sum_i \llbracket f_l^i(x) \rrbracket \in \mathcal{A}_Q(\mathbb{R}^{\bar{n}} \times \mathbb{R}^l)$, and denote by $(f_l^i)'(x)$ the first \bar{n} components of the points $f_l^i(x)$. This induces a map $f'_l := \sum_i \llbracket (f_l^i)' \rrbracket$ taking values into $\mathcal{A}_Q(\mathbb{R}^{\bar{n}})$. Observe that, since $f_l^i(x)$ are indeed points of the manifold Σ , $f_l = \sum_i \llbracket ((f_l^i)'(x), \Psi(x, (f_l^i)'(x))) \rrbracket$.

We consider $g_l := E_l^{-1/2} f'_l$. Since $\sup_l \text{Dir}(g_l, B_3) < \infty$ and $|B_3 \setminus \Gamma_l| \rightarrow 0$, by Proposition 3.3 we can find a subsequence (not relabelled) of translating sheets h_l satisfying (3.2) - (3.3) and $\|\mathcal{G}(g_l, h_l)\|_{L^2(B_3)} \rightarrow 0$. In particular, we are in the position to apply Proposition 3.4 to g_l and h_l , with $r_0 = \frac{5}{2}$, $r_1 = 3$ and $\eta = \frac{c_2}{2}$, and find $r \in (\frac{5}{2}, 3)$ and competitor functions H_l satisfying $H_l|_{B_3 \setminus B_r} = g_l|_{B_3 \setminus B_r}$,

$$\text{Dir}(H_l, B_r) \leq \text{Dir}(g_l, B_r \cap \Gamma_l) + \frac{c_2}{2}, \quad (4.9)$$

$$\text{Lip}(H_l) \leq C^* E_l^{\beta-1/2} \quad (4.10)$$

$$\|\mathcal{G}(H_l, g_l)\|_{L^2(B_r)} \leq C \text{Dir}(g_l, B_r) + C \text{Dir}(H_l, B_r) \leq M < \infty. \quad (4.11)$$

Note that (4.10) follows from (3.10) observing that $E_l^{\beta-1/2} \uparrow \infty$: thus C^* depends on c_2 and the two chosen sequences, but not on l . From now on, although this and similar constants are not dimensional, we will keep denoting them by C , with the understanding that they do not depend on l . Note that, from (4.5) and (4.6), one gets

$$\begin{aligned} \|T_l - \mathbf{G}_{f_l}\|(\mathbf{C}_3) &= \|T_l\|(B_3 \setminus K_l \times \mathbb{R}^n) + \|\mathbf{G}_{f_l}\|((B_3 \setminus K_l) \times \mathbb{R}^n) \\ &\leq Q |B_3 \setminus K_l| + E_l + Q |B_3 \setminus K_l| + C |B_3 \setminus K_l| \text{Lip}(f_l) \\ &\leq E_l + C E_l^{1-2\beta} \leq C E_l^{1-2\beta}. \end{aligned} \quad (4.12)$$

Consider the function $\varphi(z, y) = |z|$ and the slice $\langle T_l - \mathbf{G}_{f_l}, \varphi, r \rangle$. For every l , there exists $r_l \in (r, 3)$ such that $\mathbf{M}(\langle T_l - \mathbf{G}_{f_l}, \varphi, r_l \rangle) \leq C E_l^{1-2\beta}$.

Let now $v_l := E_l^{1/2} H_l|_{B_{r_l}}$, $u_l := (v_l, \Psi(x, v_l))$ (this is a short-hand notation for the multiple valued map $x \mapsto \sum_i \llbracket (v_l(x))_i, \Psi(x, (v_l(x))_i) \rrbracket$ - cf. [13, Section 1.3.1]), and consider the current $Z_l := \mathbf{G}_{u_l} \llcorner \mathbf{C}_{r_l}$. Since $u_l|_{\partial B_{r_l}} = f_l|_{\partial B_{r_l}}$, one gets $\partial Z_l = \langle \mathbf{G}_{f_l}, \varphi, r_l \rangle$ and, hence, $\mathbf{M}(\partial(T_l \llcorner \mathbf{C}_{r_l} - Z_l)) \leq C E_l^{1-2\beta}$. By Remark 4.3 there is an integral current R_l supported in Σ such that

$$\partial R_l = \partial(T_l \llcorner \mathbf{C}_{r_l} - Z_l) \quad \text{and} \quad \mathbf{M}(R_l) \leq C E_l^{\frac{(1-2\beta)m}{m-1}}.$$

Set $S_l = T_l \llcorner (\mathbf{C}_4 \setminus \mathbf{C}_{r_l}) + Z_l + R_l$. S_l is supported in Σ and $\partial S_l = \partial T_l$. We now show that, since $\beta < \frac{1}{2m}$, for l large enough, the mass of S_l is smaller than that of T_l . To this aim we write $\text{Dir}(B_{r_l}, u_l) - \text{Dir}(B_{r_l} \cap \Gamma_l, f_l)$ as

$$\underbrace{\int_{B_{r_l}} |Dv_l|^2 - \int_{B_r \cap \Gamma_l} |Df'_l|^2}_{I_1} + \underbrace{\int_{B_{r_l}} |D\Psi(x, v_l)|^2 - \int_{B_r} |D\Psi(x, f'_l)|^2}_{I_2} + \underbrace{\int_{B_{r_l} \setminus \Gamma_l} |D\Psi(x, f'_l)|^2}_{I_3}.$$

The first term is estimated by (4.9): recalling that $v_l = E_l^{1/2} H_l$ and $f'_l = E_l^{1/2} g_l$ we achieve $|I_1| \leq \frac{c_2}{2} E_l$. For what concerns the second, we proceed as follows. First we write

$$|I_2| \leq \sum_i \int_{B_{r_l}} (D(\Psi(x, u_l(x)))_i - D(\Psi(x, f'_l(x)))_i) : (D(\Psi(x, u_l(x)))_i + D(\Psi(x, f'_l(x)))_i).$$

Next, recalling the chain rule [13, Proposition 1.12], we get

$$\begin{aligned} |D(\Psi(x, v_l(x)))_i + D(\Psi(x, f'_l(x)))_i| &\leq C \|D_x \Psi\|_0 + C \|D_u \Psi\|_0 (\text{Lip}(v_l) + \text{Lip}(f'_l)) \\ &\leq C (E_l^{1/2} + \mathbf{A}) \leq C E_l^{1/4+\delta}. \end{aligned}$$

Using the letter inequality and the chain rule once again we achieve

$$\begin{aligned} |I_2| &\leq C E_l^{1/4+\delta} \int_{B_r} \left(\sum_i |D_x \Psi(x, (v_l)_i(x)) - D_x \Psi(x, (f'_l)_i(x))| + \|D_u \Psi\|_0 (|Dv_l| + |Df'_l|) \right) \\ &\leq C E_l^{1/4+\delta} \|D^2 \Psi\|_0 \int_{B_r} \mathcal{G}(v_l, f'_l) + C E_l^{1/2+2\delta} \int_{B_r} (|Dv_l| + |Df'_l|) \\ &\leq C E_l^{1/4+\delta} \mathbf{A} E_l^{\frac{1}{2}} + C E_l^{1+2\delta} \leq C E_l^{1+2\delta}. \end{aligned} \quad (4.13)$$

Finally, $I_3 \leq C \|D\Psi_l\|_\infty^2 |B_3 \setminus \Gamma_l| \leq C E_l^{1+\beta}$. Thus, for l large enough we achieve $I_2 + I_3 \leq \frac{c_2}{4} E_l$, thereby reaching $\text{Dir}(B_{r_l}, u_l) - \text{Dir}(B_{r_l} \cap \Gamma_l, f_l) \leq \frac{3c_2}{4}$. Hence,

$$\begin{aligned} \mathbf{M}(S_l) - \mathbf{M}(T_l) &\leq \mathbf{M}(Z_l) + C \mathbf{M}(R_l) - \mathbf{M}(T_l \lfloor \mathbf{C}_r) \\ &\leq Q |B_r| + \int_{B_r} \frac{|Du_l|^2}{2} + C E_l^{1+2\beta} + C E_l^{\frac{(1-2\beta)m}{m-1}} - Q |B_r| - \mathbf{e}_{T_l}(B_r) \\ &\leq \int_{B_r \cap \Gamma_l} \frac{|Df_l|^2}{2} + \frac{3}{4} c_2 E_l + C E_l^{1+2\beta} + C E_l^{\frac{(1-2\beta)m}{m-1}} - \mathbf{e}_{T_l}(B_r) \\ &\leq -\frac{c_2 E_l}{4} + C E_l^{1+\beta} + C E_l^{\frac{(1-2\beta)m}{m-1}} < 0, \end{aligned}$$

as soon as E_l is small enough. This gives the desired contradiction and proves (4.1).

For what concerns (4.2), we argue similarly. Let $(T_l)_l$ be a sequence with vanishing $E_l := \mathbf{E}(T_l, \mathbf{C}_4)$, contradicting the second part of the statement and perform the same analysis as before. Up to subsequences, one of the following statement must be false:

- (i) $\lim_l \int_{B_2} |Dg_l|^2 = \int_{B_2} |Dh_{l_0}|^2$, for any l_0 (recall that $\int_{B_2} |Dh_l|^2$ is constant);
- (ii) h_l is Dir-minimizing in B_2 .

If (i) is false, then there is a positive constant c_2 such that

$$\int_{B_r} \frac{|Dh_l|^2}{2} \leq \int_{B_r} \frac{|Dg_l|^2}{2} - c_2 \leq \frac{\mathbf{e}_{T_l}(B_r)}{E_l} - \frac{c_2}{2},$$

for l large enough (where the last inequality is again an effect of the Taylor expansion of [14, Corollary 3.3]). Therefore we can argue exactly as in the proof of (4.1) and reach a contradiction. If (ii) is false, then h_l is not Dir-minimizing in $B_{5/2}$. This implies that one

of the ζ^j in the translating sheets h_l is not Dir-minimizing in B_2 . Indeed, in the opposite case, by [13, Theorem 3.9], $\|\mathcal{G}(\zeta^j, Q[[0]])\|_{C^0(B_2)} < \infty$ and, since $h_l = \sum_i \llbracket \tau_{y_l^i} \circ \zeta^i \rrbracket$ and $|y_l^i - y_l^j| \rightarrow \infty$ for $i \neq j$, by the maximum principle of [13, Proposition 3.5], h_l would be Dir-minimizing. Thus, we can find a competitor $\hat{\zeta}^j$ for some ζ^j with less energy in the ball B_2 . So the functions $F_l = \sum_j \llbracket \tau_{y_l^j} \circ \hat{\zeta}^j \rrbracket$ satisfy, for l large enough

$$\int_{B_r} \frac{|DF_l|^2}{2} \leq \int_{B_r} \frac{|Dh_l|^2}{2} - c_2 \leq \lim_l \int_{B_r} \frac{|Dg_l|^2}{2} - 2c_2 \leq \frac{\mathbf{e}_T(B_r)}{E_l} - \frac{c_2}{2},$$

for some $c_2 > 0$ and every $r \in [3/2, 5]$. On the other hand $F_l = h_l$ on $B_3 \setminus B_{5/2}$ and therefore $\|\mathcal{G}(F_l, g_l)\|_{L^2(B_3 \setminus B_{5/2})} \rightarrow 0$. We then argue as above with F_l in place of h_l and reach a contradiction in this case as well. \square

5. GRADIENT L^p ESTIMATE

In this section we prove Theorem 1.3. The result is a consequence of an higher integrability estimate for the gradient of Dir-minimizing functions, the $o(E)$ -improved estimate for the excess measure given in Proposition 5.3 and a very careful ‘‘covering and stopping radius’’ argument (cf. [30] for an exposition in a more elementary context).

5.1. Higher integrability of the gradient of Dir-minimizers. Most of the energy of a Dir-minimizer lies where the gradient is relatively small. We prove indeed the following a priori estimate (cf. [29] for a different proof and some improvements).

Theorem 5.1 (Higher integrability of Dir-minimizers). *There exists $p_{10} > 2$ such that, for every $\Omega' \subset \subset \Omega \subset \mathbb{R}^m$ open domains, there is a constant $C > 0$ such that*

$$\|Du\|_{L^{p_{10}}(\Omega')} \leq C \|Du\|_{L^2(\Omega)} \quad \text{for every Dir-minimizing } u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)). \quad (5.1)$$

Theorem 5.1 is a corollary of Proposition 5.2 below and a Gehring’s type lemma (cf. [21, Proposition 5.1]).

Proposition 5.2. *Let $\frac{2(m-1)}{m} < p_{11} < 2$. Then, there exists $C = C(m, n, Q, p_{11})$ such that, for every $u : \Omega \rightarrow \mathcal{A}_Q$ Dir-minimizing, it holds*

$$\left(\int_{B_r(x)} |Du|^2 \right)^{1/2} \leq C \left(\int_{B_{2r}(x)} |Du|^{p_{11}} \right)^{1/p_{11}} \quad \forall x \in \Omega, \forall r < \min \{1, \text{dist}(x, \partial\Omega)/2\}.$$

Proof. Since the estimate is invariant under translations and rescalings, it is enough to prove it for $x = 0$ and $r = 1$. We assume, therefore $\Omega = B_2$. Let $u : \Omega \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ be Dir-minimizing and let $F = \boldsymbol{\xi} \circ u : \Omega \rightarrow \mathcal{Q} \subset \mathbb{R}^N$. Denote by $\bar{F} \in \mathbb{R}^N$ the average of F on B_2 . By Fubini’s theorem and the Poincaré inequality, there exists $s \in [1, 2]$ such that

$$\int_{\partial B_s} (|F - \bar{F}|^{p_{11}} + |DF|^{p_{11}}) \leq C \int_{B_2} (|F - \bar{F}|^{p_{11}} + |DF|^{p_{11}}) \leq C \|DF\|_{L^{p_{11}}(B_2)}^{p_{11}}.$$

Consider $F|_{\partial B_s}$. Since $\frac{1}{2} > \frac{1}{p_{11}} - \frac{1}{2(m-1)}$, we can use the embedding $W^{1,p_{11}}(\partial B_s) \hookrightarrow H^{1/2}(\partial B_s)$ (see, for example, [1]). Hence, we infer that

$$\|F - \bar{F}\|_{H^{1/2}(\partial B_s)} \leq C \|DF\|_{L^{p_{11}}(B_2)}. \quad (5.2)$$

Let \hat{F} be the harmonic extension of $F|_{\partial B_s}$ in B_s . It is well known (one could, for example, use the result in [1] on the half-space together with a partition of unity) that

$$\|D\hat{F}\|_{L^2(B_s)} \leq C(m) \min_{p \in \mathbb{R}^N} \|\hat{F} - p\|_{H^{1/2}(\partial B_s)} \stackrel{(5.2)}{\leq} C \|DF\|_{L^{p_{11}}(B_2)}. \quad (5.3)$$

Consider the map ρ of [13, Theorem 2.1]. Since $\rho \circ \hat{F}|_{\partial B_s} = u|_{\partial B_s}$ and $\rho \circ \hat{F}$ takes values in \mathcal{Q} , by the minimizing property of u and the Lipschitz continuity of ξ , ξ^{-1} and ρ , we conclude:

$$\left(\int_{B_1} |Du|^2 \right)^{1/2} \leq C \left(\int_{B_s} |D\hat{F}|^2 \right)^{1/2} \leq C \left(\int_{B_2} |DF|^{p_{11}} \right)^{1/p_{11}} = C \left(\int_{B_2} |Du|^{p_{11}} \right)^{1/p_{11}}. \quad \square$$

5.2. Improved excess estimate. The higher integrability of the Dir-minimizing functions and the harmonic approximation lead to the following estimate, which we call “weak” since we will improved it in Theorem 6.1.

Proposition 5.3 (Weak excess estimate). *For every $\eta_{10} > 0$, there exists $\varepsilon_{13} > 0$ with the following property. Let T be area minimizing and assume it satisfies Assumption 1.1 in $\mathbf{C}_{4s}(x)$. If $E = \mathbf{E}(T, \mathbf{C}_{4s}(x)) \leq \varepsilon_{13}$ and $c_0 \leq \varepsilon_{13}$, where c_0 is the constant in (1.1), then*

$$\mathbf{e}_T(A) \leq \eta_{10} \mathbf{e}_T(B_{4s}(x)) + C \mathbf{A}^2 s^{m+2}, \quad (5.4)$$

for every $A \subset B_s(x)$ Borel with $|A| \leq \varepsilon_{13}|B_s(x)|$.

Proof. Without loss of generality, we can assume $s = 1$ and $x = 0$. We distinguish the two regimes: $E \leq \mathbf{A}^2$ and $\mathbf{A}^2 \leq E$. In the former, clearly $\mathbf{e}_T(A) \leq C E \leq C \mathbf{A}^2$. In the latter, we let f be the $E^{\frac{1}{4m}}$ -Lipschitz approximation of T in \mathbf{C}_3 and, arguing as for the proof of Theorem 4.2, we find a radius $r \in (1, 2)$ and a current R such that

$$\partial R = \langle T - \mathbf{G}_f, \varphi, r \rangle \quad \text{and} \quad \mathbf{M}(R) \leq C E^{(1 - \frac{1}{2m}) \frac{m}{m-1}}.$$

Therefore, by the Taylor expansion in [14, Corollary 3.3], we have:

$$\|T\|(\mathbf{C}_r) \leq \mathbf{M}(\mathbf{G}_f \llcorner \mathbf{C}_r + R) \leq \|\mathbf{G}_f\|(\mathbf{C}_r) + C E^{\frac{2m-1}{2m-2}} Q |B_r| + \int_{B_r} \frac{|Df|^2}{2} + C E^{1+\gamma}, \quad (5.5)$$

where $\gamma = \frac{1}{2m-2}$. On the other hand, using again the Taylor expansion for the part of the current which coincides with the graph of f , we deduce as well that

$$\begin{aligned} \|T\|(\mathbf{C}_r) &= \|T\|((B_r \setminus K) \times \mathbb{R}^n) + \|T\|((B_r \cap K) \times \mathbb{R}^n) \\ &\geq \|T\|((B_r \setminus K) \times \mathbb{R}^n) + Q |B_r \cap K| + \int_{B_r \cap K} \frac{|Df|^2}{2} - C E^{1+\gamma}. \end{aligned} \quad (5.6)$$

Subtracting (5.6) from (5.5), we deduce

$$\mathbf{e}_T(B_r \setminus K) \leq \int_{B_r \setminus K} \frac{|Df|^2}{2} + CE^{1+\gamma}. \quad (5.7)$$

If ε_{13} is chosen small enough, we infer from (5.7) and (4.1) in Theorem 4.2 that

$$\mathbf{e}_T(B_r \setminus K) \leq \eta \mathbf{e}_T(B_4) + CE^{1+\gamma}, \quad (5.8)$$

for a suitable $\eta > 0$ to be chosen. Let now $A \subset B_1$ be such that $|A| \leq \varepsilon_{13} \omega_m$. Combining (5.8) with the Taylor expansion, we have

$$\mathbf{e}_T(A) \leq \mathbf{e}_T(A \setminus K) + \int_A \frac{|Df|^2}{2} + CE^{1+\gamma} \leq \int_A \frac{|Df|^2}{2} + \eta \mathbf{e}_T(B_4) + CE^{1+\gamma}. \quad (5.9)$$

If ε_{13} is small enough, we can again use Theorem 4.2 and Theorem 5.1 in (5.9) to get, for a Dir-minimizing w ,

$$\mathbf{e}_T(A) \stackrel{(4.2)}{\leq} \int_A \frac{|Dw|^2}{2} + 2\eta \mathbf{e}_T(B_4) + CE^{1+\gamma} \leq (C|A|^{1-2/p_{10}} + 2\eta) \mathbf{e}_T(B_4) + CE^{1+\gamma}. \quad (5.10)$$

Hence, if ε_{13} and η are suitable chosen, (5.4) follows from (5.10). \square

5.3. Proof of Theorem 1.3. We divide the proof into two steps.

Step 1. There exist constants $\gamma \geq 2^m$ and $\varrho > 0$ such that, for every $c \in [1, (\gamma E)^{-1}]$ and $s \in [2, 4]$ with $\bar{s} = s + 2c^{-1/m} \leq 4$, we have

$$\int_{\{\gamma c E \leq \mathbf{d} \leq 1\} \cap B_s} \mathbf{d} \leq \gamma^{-\varrho} \int_{\{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\} \cap B_s} \mathbf{d} + C c^{-2/m} \mathbf{A}^2. \quad (5.11)$$

In order to prove it, let N_B be the constant in Besicovich's covering theorem [19, Section 1.5.2] and choose $N \in \mathbb{N}$ so large that $N_B < 2^{N-1}$. Let ε_{13} be as in Proposition 5.3 when we choose $\eta_{10} = 2^{-2m-N}$, and set

$$\gamma = \max\{2^m, \varepsilon_{13}^{-1}\} \quad \text{and} \quad \varrho = \min\left\{-\log_\gamma(N_B/2^{N-1}), \frac{1}{2m}\right\}.$$

Let c and s be any real numbers as above. For almost every $x \in \{\gamma c E \leq \mathbf{d} \leq 1\} \cap B_s$, there exists r_x such that

$$E(T, \mathbf{C}_{4r_x}(x)) \leq cE \quad \text{and} \quad E(T, \mathbf{C}_t(x)) \geq cE \quad \forall t \in]0, 4r_x[. \quad (5.12)$$

Indeed, since $\mathbf{d}(x) = \lim_{r \rightarrow 0} E(T, \mathbf{C}_r(x)) \geq \gamma c E \geq 2^m c E$ and

$$E(T, \mathbf{C}_t(x)) = \frac{\mathbf{e}_T(B_t(x))}{\omega_m t^m} \leq \frac{4^m E}{t^m} \leq cE \quad \text{for } t \geq \frac{4}{\sqrt[m]{c}},$$

we just choose $4r_x = \min\{t \leq 4/\sqrt[m]{c} : E(T, \mathbf{C}_t(x)) \leq cE\}$. Note also that $r_x \leq 1/\sqrt[m]{c}$. Consider the current T in $\mathbf{C}_{4r_x}(x)$. Setting $A = \{\gamma c E \leq \mathbf{d}\} \cap B_{4r_x}(x)$, we have that

$$\mathbf{E}(T, \mathbf{C}_{4r_x}(x)) \leq cE \leq \frac{E}{\gamma E} \leq \varepsilon_{13} \quad \text{and} \quad |A| \leq \frac{cE |B_{4r_x}(x)|}{\gamma c E} \leq \varepsilon_{13} |B_{4r_x}(x)|.$$

Hence, we can apply Proposition 5.3 to $T \perp \mathbf{C}_{4r_x}(x)$ to get

$$\begin{aligned} \int_{B_{r_x}(x) \cap \{\gamma c E \leq \mathbf{d} \leq 1\}} \mathbf{d} &\leq \int_A \mathbf{d} \leq \mathbf{e}_T(A) \leq 2^{-2m-N} \mathbf{e}_T(B_{4r_x}(x)) + C r_x^{m+2} \mathbf{A}^2 \\ &\leq 2^{-2m-N} (4r_x)^m \omega_m \mathbf{E}(T, \mathbf{C}_{4r_x}(x)) + C r_x^{m+2} \mathbf{A}^2 \stackrel{(5.12)}{\leq} 2^{-N} \mathbf{e}_T(B_{r_x}(x)) + C r_x^{m+2} \mathbf{A}^2. \end{aligned} \quad (5.13)$$

Thus,

$$\begin{aligned} \mathbf{e}_T(B_{r_x}(x)) &= \int_{B_{r_x}(x) \cap \{\mathbf{d} > 1\}} \mathbf{d} + \int_{B_{r_x}(x) \cap \{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\}} \mathbf{d} + \int_{B_{r_x}(x) \cap \{\mathbf{d} < \frac{cE}{\gamma}\}} \mathbf{d} \\ &\leq \int_A \mathbf{d} + \int_{B_{r_x}(x) \cap \{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\}} \mathbf{d} + \frac{cE}{\gamma} \omega_m r_x^m \\ &\stackrel{(5.12), (5.13)}{\leq} (2^{-N} + \gamma^{-1}) \mathbf{e}_T(B_{r_x}(x)) + C r_x^{m+2} \mathbf{A}^2 + \int_{B_{r_x}(x) \cap \{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\}} \mathbf{d}. \end{aligned} \quad (5.14)$$

Therefore, recalling that $\gamma \geq 2^m \geq 4$, from (5.13) and (5.14) we infer:

$$\begin{aligned} \int_{B_{r_x}(x) \cap \{\gamma c E \leq \mathbf{d} \leq 1\}} \mathbf{d} &\leq \frac{2^{-N}}{1 - 2^{-N} - \gamma^{-1}} \int_{B_{r_x}(x) \cap \{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\}} \mathbf{d} + C r_x^{m+2} \mathbf{A}^2 \\ &\leq 2^{-N+1} \int_{B_{r_x}(x) \cap \{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\}} \mathbf{d} + C r_x^{m+2} \mathbf{A}^2. \end{aligned}$$

By Besicovich's covering theorem, we choose N_B families of disjoint balls $\overline{B}_{r_x}(x)$ whose union covers $\{\gamma c E \leq \mathbf{d} \leq 1\} \cap B_s$ and, since as already noticed $r_x \leq 1/\sqrt[m]{c}$ for every x , we conclude:

$$\int_{\{\gamma c E \leq \mathbf{d} \leq 1\} \cap B_s} \mathbf{d} \leq N_B 2^{-N+1} \int_{\{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\} \cap B_{s + \frac{2}{\sqrt[m]{c}}}} \mathbf{d} + C c^{-\frac{2}{m}} \mathbf{A}^2,$$

which, for the above defined ϱ , implies (5.11).

Step 2. We iterate (5.11) in order to conclude (1.3). Denote by L the largest integer smaller than $2^{-1} \log_\gamma(E^{-1} - 1)$, $s_L = 2$ and recursively $s_k = s_{k+1} + 2\gamma^{-\frac{2k}{m}}$ for $k \in \{L, L-1, \dots, 1\}$. Notice that, since $\gamma \geq 2^m$, $s_k < 4$ for every k . Thus, we can apply (5.11) with $c = \gamma^{2k}$, $s = s_k$ and $\bar{s} = s_{k-1}$ to conclude

$$\int_{\{\gamma^{2k+1} E \leq \mathbf{d} \leq 1\} \cap B_{s_k}} \mathbf{d} \leq \gamma^{-\varrho} \int_{\{\gamma^{2k-1} E \leq \mathbf{d} \leq 1\} \cap B_{s_{k-1}}} \mathbf{d} + C \gamma^{-\frac{4k}{m}} \mathbf{A}^2 \quad \forall k \in \{1, \dots, L\}.$$

In particular, iterating this estimate we get

$$\int_{\{\gamma^{2k+1} E \leq \mathbf{d} \leq 1\} \cap B_2} \mathbf{d} \leq \gamma^{-k\varrho} \int_{\{\gamma E \leq \mathbf{d} \leq 1\} \cap B_{s_0}} \mathbf{d} + C \mathbf{A}^2 \sum_{l=0}^{k-1} \gamma^{-\left(\frac{4(k-l)}{m} + l\varrho\right)}. \quad (5.15)$$

Set $A_0 = \{\mathbf{d} < \gamma E\}$, $A_k = \{\gamma^{2k-1} E \leq \mathbf{d} < \gamma^{2k+1} E\}$ for $k = 1, \dots, L$, and $A_{L+1} = \{\gamma^{2L+1} E \leq \mathbf{d} \leq 1\}$. Since $\cup A_k = \{\mathbf{d} \leq 1\}$, for $p_1 < 1 + \frac{\varrho}{2} \leq 1 + \frac{1}{m}$, we conclude:

$$\begin{aligned} \int_{B_2 \cap \{\mathbf{d} \leq 1\}} \mathbf{d}^{p_1} &= \sum_{k=0}^{L+1} \int_{A_k \cap B_2} \mathbf{d}^{p_1} \leq \sum_k \gamma^{(2k+1)(p_1-1)} E^{p_1-1} \int_{A_k \cap B_2} \mathbf{d} \\ &\stackrel{(5.15)}{\leq} C \sum_k \gamma^{k(2(p_1-1)-\varrho)} E^{p_1} + C \sum_k \sum_{l=0}^{k-1} \gamma^{k(2(p_1-1)-\frac{4}{m})+l(\frac{4}{m}-\varrho)} E^{p_1-1} \mathbf{A}^2 \\ &\leq C E^{p_1} + C \sum_k \gamma^{k(2(p_1-1)-\varrho)} \mathbf{A}^2. \end{aligned}$$

6. ALMGREN'S APPROXIMATION THEOREM

In this section we show how Theorem 1.3 gives a simple proof of the approximation result in Theorem 1.4. The key point is the following theorem.

Theorem 6.1 (Almgren's strong excess estimate). *There are constants $\gamma_{11}, C > 0$ (depending on m, n, \bar{n}, Q) with the following property. Assume T satisfies Assumption 1.1 in \mathbf{C}_4 and is area minimizing. If $E = \mathbf{E}(T, \mathbf{C}_4) < \varepsilon_{11}$, then*

$$\mathbf{e}_T(A) \leq C (E^{\gamma_{11}} + |A|^{\gamma_{11}}) (E + \mathbf{A}^2) \quad \text{for every Borel } A \subset B_1. \quad (6.1)$$

This estimate complements (1.3) enabling to control the excess in the region where $\mathbf{d} > 1$. We call it strong Almgren's estimate because a similar formula can be found in the big regularity paper (cf. [3, Sections 3.24-3.26 & 3.30(8)]) and is a strengthened version of Proposition 5.3. To achieve (6.1) we construct a suitable competitor to estimate the size of the set K where the graph of the E^β -Lipschitz approximation f differs from T . Following Almgren, we embed \mathcal{A}_Q in a large Euclidean space, via a biLipschitz embedding $\boldsymbol{\xi}$. We then regularize $\boldsymbol{\xi} \circ f$ by convolution and project it back onto $\mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q)$. To avoid loss of energy we need a rather special "almost projection" $\boldsymbol{\rho}_\delta^*$.

Proposition 6.2. *For every $\bar{n}, Q \in \mathbb{N} \setminus \{0\}$ there are geometric constants $\delta_0, C > 0$ with the following property. For every $\delta \in]0, \delta_0[$ there is $\boldsymbol{\rho}_\delta^* : \mathbb{R}^{N(Q, \bar{n})} \rightarrow \mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^{\bar{n}}))$ such that $|\boldsymbol{\rho}_\delta^*(P) - P| \leq C \delta^{8-\bar{n}Q}$ for all $P \in \mathcal{Q}$ and, for every $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, it holds*

$$\int |D(\boldsymbol{\rho}_\delta^* \circ u)|^2 \leq \left(1 + C \delta^{8-\bar{n}Q-1}\right) \int_{\{\text{dist}(u, \mathcal{Q}) \leq \delta^{\bar{n}Q+1}\}} |Du|^2 + C \int_{\{\text{dist}(u, \mathcal{Q}) > \delta^{\bar{n}Q+1}\}} |Du|^2. \quad (6.2)$$

The proof of Proposition 6.2 is postponed to the next section. Here we show Theorem 6.1 and hence conclude the Theorems 1.4 and 1.5. Theorem 1.3 enters crucially in the argument when estimating the second summand of (6.2) for the regularization of $\boldsymbol{\xi} \circ f$.

6.1. Regularization by convolution. Here we construct the competitor.

Proposition 6.3. *Let $\beta_1 \in (0, \frac{1}{2m})$ and T be an area minimizing current satisfying Assumption 1.1 in \mathbf{C}_4 . Let f be its E^{β_1} -Lipschitz approximation. Then, there exist constants*

$\gamma_{12}, C > 0$ and a subset of radii $B \subset [1, 2]$ with $|B| > 1/2$ with the following properties. For every $\sigma \in B$, there exists a Q -valued function $g \in \text{Lip}(B_\sigma, \mathcal{A}_Q)$ such that

$$g|_{\partial B_\sigma} = f|_{\partial B_s}, \quad \text{Lip}(g) \leq C E^{\beta_1}, \quad \text{spt}(g(x)) \subset \Sigma \quad \forall x \in B_\sigma,$$

and

$$\int_{B_\sigma} |Dg|^2 \leq \int_{B_\sigma \cap K} |Df|^2 + C E^{\gamma_{12}} (E + \mathbf{A}^2). \quad (6.3)$$

Proof. By Remark 4.4 we assume that $\Psi(0) = 0$, $\|\nabla \Psi\|_0 \leq C(E^{1/2} + \mathbf{A})$ and $\|D^2 \Psi\|_0 \leq C\mathbf{A}$. Since $|Df|^2 \leq C \mathbf{d}_T \leq C E^{2\beta_1} \leq 1$ on K , by Theorem 1.3 is $q_1 = 2p_1 > 2$ such that

$$\|Df\|_{L^{q_1}(K)}^2 \leq C E^{1-1/p_1} (E + \mathbf{A}^2)^{1/p_1} \leq C(E + \mathbf{A}^2). \quad (6.4)$$

Given two (vector-valued) functions h_1 and h_2 and two radii $0 < s < r$, we denote by $\text{lin}(h_1, h_2)$ the linear interpolation in $B_r \setminus \bar{B}_s$ between $h_1|_{\partial B_r}$ and $h_2|_{\partial B_s}$. More precisely, if $(\theta, t) \in \mathbb{S}^{m-1} \times [0, \infty)$ are spherical coordinates, then

$$\text{lin}(h_1, h_2)(\theta, t) = \frac{r-t}{r-s} h_2(\theta, s) + \frac{t-s}{r-s} h_1(\theta, r).$$

Next, let $\delta > 0$ and $\varepsilon > 0$ be two parameters and let $1 < r_1 < r_2 < r_3 < 2$ be three radii, all to be chosen later. To keep the notation simple, we will write $\boldsymbol{\rho}^*$ in place of $\boldsymbol{\rho}_\delta^*$. Let $\varphi \in C_c^\infty(B_1)$ be a standard (nonnegative!) mollifier. We also use the notation $f(x) = (f_1(x), f_2(x)) \in \mathcal{A}_Q(\mathbb{R}^{\bar{n}} \times \mathbb{R}^l)$ meaning that $f(x) = \sum_i \llbracket (f_1^i(x), f_2^i(x)) \rrbracket$ with $(f_1^i(x), f_2^i(x)) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}^l$ and the maps f_1 and f_2 are then given by $f_j(x) = \sum_i \llbracket f_j^i(x) \rrbracket$. This does not create confusion in “ordering the sheets”: since the points $f^i(x)$ belong to Σ we have indeed the relation $f_2^j(x) = \Psi(x, f_1^j(x))$. We moreover set $f' := \boldsymbol{\xi} \circ f_1$. Recall the map $\boldsymbol{\rho}$ of [13, Theorem 2.1] and define:

$$g' := \begin{cases} \sqrt{E} \boldsymbol{\rho} \circ \text{lin} \left(\frac{f'}{\sqrt{E}}, \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) \right) & \text{in } B_{r_3} \setminus B_{r_2}, \\ \sqrt{E} \boldsymbol{\rho} \circ \text{lin} \left(\boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right), \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon \right) \right) & \text{in } B_{r_2} \setminus B_{r_1}, \\ \sqrt{E} \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon \right) & \text{in } B_{r_1}. \end{cases} \quad (6.5)$$

Finally set $g_1 := \boldsymbol{\xi}^{-1} \circ g'$ and $g := \sum_i \llbracket ((g_1)_i, \Psi(x, (g_1)_i)) \rrbracket$. We claim that, for $\sigma := r_3$ in a suitable set $B \subset [1, 2]$ with $|B| > 1/2$, we can choose $r_2 = r_3 - s$ and $r_1 = r_2 - s$ so that g satisfies the conclusion of the proposition. Some computations will be simplified taking into account that our choice of the parameter will imply the following inequalities:

$$\delta^{2 \cdot 8^{-\bar{n}Q}} \leq s, \quad \varepsilon \leq s \quad \text{and} \quad E^{1-2\beta_1} \leq \varepsilon^m. \quad (6.6)$$

We start noticing that clearly $g|_{\partial B_{r_3}} = f|_{\partial B_{r_3}}$. As for the Lipschitz constant, it suffices to estimate the Lipschitz constant of g' . This can be easily done observing that:

$$\begin{cases} \text{Lip}(g') \leq C \text{Lip}(f' * \varphi_\varepsilon) \leq C \text{Lip}(f') \leq C E^{\beta_1} & \text{in } B_{r_1}, \\ \text{Lip}(g') \leq C \text{Lip}(f') + C \frac{\|f' - f' * \varphi_\varepsilon\|_{L^\infty}}{s} \leq C(1 + \frac{\varepsilon}{s}) \text{Lip}(f') \leq C E^{\beta_1} & \text{in } B_{r_2} \setminus B_{r_1}, \\ \text{Lip}(g') \leq C \text{Lip}(f') + C E^{1/2} \frac{\delta^{8^{-\bar{n}Q}}}{s} \leq C E^{\beta_1} + C E^{1/2} \leq C E^{\beta_1} & \text{in } B_{r_3} \setminus B_{r_2}. \end{cases}$$

In the second inequality of the last line we have used that, since \mathcal{Q} is a cone, $E^{-1/2}f'(x) \in \mathcal{Q}$ for every x : therefore $|\boldsymbol{\rho}^*(f'/E^{1/2}) - f'/E^{1/2}| \leq C\delta^{8-\bar{n}Q}$. We pass now to estimate the Dirichlet energy of g .

Step 1. Energy in $B_{r_3} \setminus B_{r_2}$. By Section 1.5, the energy of the first component g_1 coincides with the (classical!) Dirichlet energy of g' . By Proposition 6.2, $|\boldsymbol{\rho}^*(P) - P| \leq C\delta^{8-\bar{n}Q}$ for all $P \in \mathcal{Q}$. Thus, elementary estimates on the linear interpolation give

$$\begin{aligned} \int_{B_{r_3} \setminus B_{r_2}} |Dg'|^2 &\leq \frac{CE}{(r_3 - r_2)^2} \int_{B_{r_3} \setminus B_{r_2}} \left| \frac{f'}{\sqrt{E}} - \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) \right|^2 + C \int_{B_{r_3} \setminus B_{r_2}} |Df'|^2 \\ &\quad + C \int_{B_{r_3} \setminus B_{r_2}} |D(\boldsymbol{\rho}^* \circ f')|^2 \leq C \int_{B_{r_3} \setminus B_{r_2}} |Df'|^2 + CE s^{-1} \delta^{2 \cdot 8 - \bar{n}Q}. \end{aligned} \quad (6.7)$$

As for g_2 , we compute $(Dg_2)_i(x) = D_x \Psi(x, (g_1)_i(x)) + D_u \Psi(x, (g_1)_i(x))(Dg_1)_i(x)$ and so

$$\int_{B_{r_3} \setminus B_{r_2}} |Dg_2|^2 \leq C s (\mathbf{A}^2 + E). \quad (6.8)$$

Step 2. Energy in $B_{r_2} \setminus B_{r_1}$. Here, using the same interpolation inequality and a standard estimate on convolutions of $W^{1,2}$ functions, we get

$$\begin{aligned} \int_{B_{r_2} \setminus B_{r_1}} |Dg'|^2 &\leq C \int_{B_{r_2} \setminus B_{r_1}} |Df'|^2 + \frac{C}{(r_2 - r_1)^2} \int_{B_{r_2} \setminus B_{r_1}} |f' - \varphi_\varepsilon * f'|^2 \\ &\leq C \int_{B_{r_2} \setminus B_{r_1}} |Df'|^2 + C \varepsilon^2 s^{-2} \int_{B_3} |Df'|^2 = C \int_{B_{r_2} \setminus B_{r_1}} |Df'|^2 + C \varepsilon^2 E s^{-2}. \end{aligned} \quad (6.9)$$

Similarly, for the second component we have that

$$\int_{B_{r_2} \setminus B_{r_1}} |Dg_2|^2 \leq C (\mathbf{A}^2 + E) s. \quad (6.10)$$

Step 3. Energy in B_{r_1} . Define $Z := \left\{ \text{dist} \left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon, \mathcal{Q} \right) > \delta^{\bar{n}Q+1} \right\}$ and use (6.2) to get

$$\int_{B_{r_1}} |Dg'|^2 \leq \left(1 + C \delta^{8-\bar{n}Q-1} \right) \int_{B_{r_1} \setminus Z} |D(f' * \varphi_\varepsilon)|^2 + C \int_Z |D(f' * \varphi_\varepsilon)|^2 =: I_1 + I_2. \quad (6.11)$$

We consider I_1 and I_2 separately. For I_1 we first observe the elementary inequality

$$\begin{aligned} \|D(f' * \varphi_\varepsilon)\|_{L^2}^2 &\leq \| |Df'| * \varphi_\varepsilon \|_{L^2}^2 \leq \| (|Df'| \mathbf{1}_K) * \varphi_\varepsilon \|_{L^2}^2 + \| (|Df'| \mathbf{1}_{K^c}) * \varphi_\varepsilon \|_{L^2}^2 \\ &\quad + 2 \| (|Df'| \mathbf{1}_K) * \varphi_\varepsilon \|_{L^2} \| (|Df'| \mathbf{1}_{K^c}) * \varphi_\varepsilon \|_{L^2}. \end{aligned} \quad (6.12)$$

Recalling $r_1 + \varepsilon \leq r_1 + s \leq r_2$ we estimate the first summand in (6.12) as follows:

$$\| (|Df'| \mathbf{1}_K) * \varphi_\varepsilon \|_{L^2(B_{r_1})}^2 \leq \int_{B_{r_1+\varepsilon}} (|Df'| \mathbf{1}_K)^2 \leq \int_{B_{r_3} \cap K} |Df'|^2. \quad (6.13)$$

To treat the other terms recall that $\text{Lip}(f') \leq CE^{\beta_1}$ and $|K^c| \leq CE^{1-2\beta_1}$:

$$\| (|Df'| \mathbf{1}_{K^c}) * \varphi_\varepsilon \|_{L^2(B_{r_1})}^2 \leq CE^{2\beta_1} \| \mathbf{1}_{K^c} * \varphi_\varepsilon \|_{L^2}^2 \leq CE^{2\beta_1} \| \mathbf{1}_{K^c} \|_{L^1}^2 \| \varphi_\varepsilon \|_{L^2}^2 \leq \frac{CE^{2-2\beta_1}}{\varepsilon^m}. \quad (6.14)$$

Putting (6.13) and (6.14) in (6.12) and recalling $E^{1-2\beta_1} \geq \varepsilon^m$ and $\int |Df'|^2 \leq CE$, we get

$$I_1 \leq \int_{B_{r_2} \cap K} |Df'|^2 + C \delta^{8-\bar{n}Q-1} E + C \varepsilon^{m/2} E^{3/2-\beta_1}. \quad (6.15)$$

For what concerns I_2 , first we argue as for I_1 , splitting in K and K^c , to deduce that

$$I_2 \leq C \int_Z ((|Df'| \mathbf{1}_K) * \varphi_\varepsilon)^2 + C \varepsilon^{-m/2} E^{3/2-\beta_1}. \quad (6.16)$$

Then, regarding the first summand in (6.16), we note that

$$|Z| \delta^{2\bar{n}Q+2} \leq \int_{B_{r_1}} \left| \frac{f'}{\sqrt{E}} * \varphi_\varepsilon - \frac{f'}{\sqrt{E}} \right|^2 \leq C \varepsilon^2. \quad (6.17)$$

Since $|Df'| \leq |Df|$ (and recalling that $q_1 = 2p_1 > 2$), we use (6.4) to obtain

$$\begin{aligned} \int_Z ((|Df'| \mathbf{1}_K) * \varphi_\varepsilon)^2 &\leq |Z|^{\frac{p_1-1}{p_1}} \|(|Df'| \mathbf{1}_K) * \varphi_\varepsilon\|_{L^{q_1}}^2 \leq C \left(\frac{\varepsilon}{\delta^{\bar{n}Q+1}} \right)^{\frac{2(p_1-1)}{p_1}} \| |Df'| \|_{L^{q_1}(K)}^2 \\ &\leq C \left(\frac{\varepsilon}{\delta^{\bar{n}Q+1}} \right)^{\frac{2(p_1-1)}{p_1}} (E + \mathbf{A}^2). \end{aligned} \quad (6.18)$$

Gathering all the estimates together, (6.11), (6.15), (6.16) and (6.18) give

$$\int_{B_{r_1}} |Dg'|^2 \leq \int_{B_{r_2} \cap K} |Df'|^2 + C \left(E \delta^{8-\bar{n}Q-1} + \frac{E^{3/2-\beta_1}}{\varepsilon^{m/2}} + (E + \mathbf{A}^2) \left(\frac{\varepsilon}{\delta^{\bar{n}Q+1}} \right)^{\frac{2(p_1-1)}{p_1}} \right). \quad (6.19)$$

On the other hand, for what concerns g_2 we can estimate as follows

$$\begin{aligned} \int_{B_{r_1}} |Dg_2|^2 &= \int_{B_{r_1}} |Df_2|^2 + \sum_i \int_{B_{r_1}} ((Dg_2)_i - (Df_2)_i) \cdot ((Dg_2)_i + (Df_2)_i) \\ &\leq \int_{B_{r_1} \cap K} |Df_2|^2 + \int_{B_{r_1} \setminus K} |Df_2|^2 + C (\mathbf{A} + E^{1/2}) \sum_i \int_{B_{r_1}} |(Dg_2)_i - (Df_2)_i| \end{aligned} \quad (6.20)$$

We already observed that $|Df_2| \leq C(\mathbf{A} + E^{1/2})$, leading to the estimate $\int_{K^c} |Df|^2 \leq C(\mathbf{A}^2 + E)|K| \leq C(\mathbf{A}^2 + E)E^{1-2\beta_1}$. As for the latter summand we compute

$$\begin{aligned} |(Dg_2)_i - (Df_2)_i| &\leq |D_x \Psi(x, (g_1)_i) - D_x \Psi(x, (f_1)_i)| \\ &\quad + |D_u \Psi(x, (g_1)_i(x))(Dg_1)_i| + |D_u \Psi(x, (f_1)_i(x))(Df_1)_i| \\ &\leq C \mathcal{A} \mathcal{G}(g_1, f_1) + C (\mathbf{A} + E^{1/2}) E^{\beta_1}. \end{aligned}$$

We next estimate $\|\mathcal{G}(g_1, f_1)\|_\infty \leq C \|\mathcal{G}(f', g')\|_\infty$ and

$$\begin{aligned} \|g' - f'\|_\infty &\leq C \sqrt{E} \left(\left\| \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon \right) - \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) \right\|_\infty + \left\| \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) - \frac{f'}{\sqrt{E}} \right\|_\infty \right) \\ &\leq C \text{Lip}(\boldsymbol{\rho}^*) \|f' * \varphi_\varepsilon - f'\|_{L^\infty} + C E^{1/2} \delta^{8-\bar{n}Q} \leq C E^{\beta_1} \varepsilon + C E^{1/2} \delta^{8-\bar{n}Q} \leq C E^{\beta_1}. \end{aligned}$$

We therefore conclude

$$\int_{B_{r_1}} |Dg_2|^2 \leq \int_{B_{r_1} \cap K} |Df_2|^2 + C(\mathbf{A}^2 + E)E^{\beta_1}. \quad (6.21)$$

Final estimate. Since $|Dg|^2 = |Dg'|^2 + |Dg_2|^2$, summing (6.7), (6.8), (6.9), (6.10), (6.19) and (6.21) (and recalling $\varepsilon < s$), we conclude

$$\begin{aligned} \int_{B_{r_3}} |Dg|^2 &\leq \int_{B_{r_1} \cap K} |Df|^2 + C \int_{B_{r_1+3s} \setminus B_{r_1}} |Df'|^2 + C(\mathbf{A} + E^2)(s + E^{\beta_1}) \\ &\quad + C E \left(\frac{\varepsilon^2}{s^2} + \frac{\delta^2 \cdot 8^{-\bar{n}Q}}{s} + \frac{E^{1/2-\beta_1}}{\varepsilon^{m/2}} + (1 + \mathbf{A}^2 E^{-1}) \left(\frac{\varepsilon}{\delta^{\bar{n}Q+1}} \right)^{\frac{2(p_1-2)}{p_1}} \right). \end{aligned}$$

We set $\varepsilon = E^a$, $\delta = E^b$ and $s = E^c$, where

$$a = \frac{1 - 2\beta_1}{2m}, \quad b = \frac{1 - 2\beta_1}{4m(\bar{n}Q + 1)} \quad \text{and} \quad c = \frac{1 - 2\beta_1}{8^{\bar{n}Q} 4m(\bar{n}Q + 1)}.$$

This choice respects (6.6). Assume E is small enough so that $s \leq \frac{1}{8}$. Now, if $C > 0$ is a sufficiently large constant, there is a set $B' \subset [1, \frac{7}{8}]$ with $|B'| > 1/2$ such that,

$$\int_{B_{r_1+3s} \setminus B_{r_1}} |Df'|^2 \leq C s \int_{B_2} |Df'|^2 \leq C E^{1+c} \quad \text{for every } r_1 \in B'.$$

For $\sigma = r_3 \in B = s + B'$ we then conclude, for some $\gamma(\beta_1, \bar{n}, N, Q) > 0$,

$$\int_{B_\sigma} |Dg|^2 \leq \int_{B_\sigma \cap K} |Df|^2 + C E^\gamma (E + \mathbf{A}^2). \quad \square$$

6.2. Proof of Theorem 6.1. Choose $\beta_1 = \frac{1}{4m}$ and consider the set $B \subset [1, 2]$ given in Proposition 6.3. Using the coarea formula and the isoperimetric inequality (the argument and the map φ are the same in the proof of Theorem 4.2), we find $s \in B$ and an integer rectifiable current R such that

$$\partial R = \langle T - \mathbf{G}_f, \varphi, s \rangle \quad \text{and} \quad \mathbf{M}(R) \leq C E^{\frac{2m-1}{2m-2}}.$$

Since $g|_{\partial B_s} = f|_{\partial B_s}$ and g takes values in Σ , we can use g in place of f in the estimates and, arguing as before, we get, for a suitable $\gamma > 0$:

$$\|T\|(\mathbf{C}_s) \leq Q |B_s| + \int_{B_s} \frac{|Dg|^2}{2} + C E^{1+\gamma} \stackrel{(6.3)}{\leq} Q |B_s| + \int_{B_s \cap K} \frac{|Df|^2}{2} + C E^\gamma (E + \mathbf{A}^2). \quad (6.22)$$

On the other hand, by the Taylor's expansion in [14, Corollary 3.3],

$$\begin{aligned} \|T\|(\mathbf{C}_s) &= \|T\|((B_s \setminus K) \times \mathbb{R}^n) + \|\mathbf{G}_f\|((B_s \cap K) \times \mathbb{R}^n) \\ &\geq \|T\|((B_s \setminus K) \times \mathbb{R}^n) + Q |K \cap B_s| + \int_{K \cap B_s} \frac{|Df|^2}{2} - C E^{1+\gamma}. \end{aligned} \quad (6.23)$$

Hence, from (6.22) and (6.23), we get $\mathbf{e}_T(B_s \setminus K) \leq C E^\gamma (E + \mathbf{A}^2)$.

This is enough to conclude the proof. Indeed, let $A \subset B_1$ be a Borel set. Using the higher integrability of $|Df|$ in K (and therefore possibly selecting a smaller $\gamma > 0$) we get

$$\begin{aligned} \mathbf{e}_T(A) &\leq \mathbf{e}_T(A \cap K) + \mathbf{e}_T(A \setminus K) \leq \int_{A \cap K} \frac{|Df|^2}{2} + C E^{1+\gamma} \\ &\leq C |A \cap K|^{\frac{p_1-1}{p_1}} \left(\int_{A \cap K} |Df|^{q_1} \right)^{2/q_1} + C E^{1+\gamma} \leq C |A|^{\frac{p_1-1}{p_1}} (E + \mathbf{A}^2) + C E^{1+\gamma}. \end{aligned}$$

6.3. Proofs of Theorems 1.4 and 1.5. As usual we assume, w.l.o.g., $r = 1$ and $x = 0$. Choose $\beta_{11} < \min\{\frac{1}{2m}, \frac{\gamma_{11}}{2(1+\gamma_{11})}\}$, where γ_{11} is the constant in Theorem 6.1. Let f be the $E^{\beta_{11}}$ -Lipschitz approximation of T . Clearly (1.4) follows directly from Proposition 2.2 if $\gamma_1 < \beta_{11}$. Set next $A := \{\mathbf{m}e_T > 2^{-m}E^{2\beta_{11}}\} \cap B_1$. By Proposition 2.2, $|A| \leq CE^{1-2\beta_{11}}$. Apply estimate (6.1) to A to conclude:

$$|B_1 \setminus K| \leq C E^{-2\beta_{11}} \mathbf{e}_T(A) \leq C E^{\gamma_{11}-2\beta_{11}(1+\gamma_{11})}(E + \mathbf{A}^2).$$

By our choice of γ_{11} and β_{11} , this gives (1.5) for some positive γ_1 . Finally, set $S = \mathbf{G}_f$. Recalling the strong Almgren's estimate (6.1) and the Taylor expansion in [14, Corollary 3.3], we conclude:

$$\begin{aligned} \left| \|T\|(\mathbf{C}_1) - Q\omega_m - \int_{B_1} \frac{|Df|^2}{2} \right| &\leq \mathbf{e}_T(B_1 \setminus K) + \mathbf{e}_S(B_1 \setminus K) + \left| \mathbf{e}_S(B_1) - \int_{B_1} \frac{|Df|^2}{2} \right| \\ &\leq C E^{\gamma_{11}}(E + \mathbf{A}^2) + C |B_1 \setminus K| + C \text{Lip}(f)^2 \int_{B_1} |Df|^2 \leq C E^{\gamma_1}(E + \mathbf{A}^2). \end{aligned}$$

The L^∞ bound follows from Proposition 2.2 recalling that, by Remark 4.4, we can assume $\|\nabla\Psi\|_0 \leq C(E^{1/2} + \mathbf{A})$. Finally, Theorem 1.5 is a special case of Theorem 4.2, since the map f in Theorem 1.4 is the E^{γ_1} -Lipschitz approximation of T .

7. THE ‘‘ALMOST’’ PROJECTIONS ρ_δ^*

In this section we show the existence of the maps ρ_δ^* in Proposition 6.2. Compared to the original ones introduced by Almgren, our ρ^* 's have the advantage of depending on a single parameter. Our proof is different from Almgren's and gives more explicit estimates, relying heavily on the following simple corollary of Kirszbraun's Theorem.

Lemma 7.1. *Let $f : \Omega \subset \mathbb{R}^{N_1} \rightarrow C \subset \mathbb{R}^{N_2}$ be a Lipschitz function and assume that C is closed and convex. Then, there is an extension \hat{f} of f to the whole \mathbb{R}^{N_1} which preserves the Lipschitz constant and takes values in C .*

To prove Lemma 7.1 it suffices to take the map \tilde{f} of the classical statement of Kirszbraun's theorem (see [20, Theorem 2.10.43]) which takes values in \mathbb{R}^{N_2} and compose it with the orthogonal projection π_C onto the convex closed set C , which is a 1-Lipschitz map in \mathbb{R}^{N_2} .

The proof of Proposition 6.2 consists of four parts: the first one is a detailed description of the set \mathcal{Q} , whereas the remaining three give a rather explicit construction in this order:

- (1) first we specify ρ_δ^* on \mathcal{Q} : the resulting map will be called ρ^\flat ;
- (2) then we extend it to a map ρ^\sharp on $\mathcal{Q}_{\delta^{n\mathcal{Q}+1}}$, the $\delta^{n\mathcal{Q}+1}$ -neighborhood of \mathcal{Q} ;

- (3) ρ^\sharp will satisfy $\text{Lip}(\rho^\sharp) \leq 1 + C\delta^{8-\bar{n}Q-1}$ and $|\rho^\star(P) - P| \leq C\delta^{8-\bar{n}Q}$ for every $p \in \mathcal{Q}$; we then extend it to all \mathbb{R}^N keeping its Lipschitz constant bounded.

The latter step is however immediate: taken $\xi^{-1} \circ \rho^\sharp : \mathcal{Q}_{\delta^{\bar{n}Q+1}} \rightarrow \mathcal{A}_Q$, we find a Lipschitz extension $h : \mathbb{R}^N \rightarrow \mathcal{A}_Q$ of it with $\text{Lip}(h) \leq C$, using [13, Theorem 1.7]. Our map is then $\rho_\delta^\star := \xi \circ h$. From now on we use n instead of \bar{n} to simplify the notation.

7.1. Conical simplicial structure of \mathcal{Q} . We first prove that \mathcal{Q} is the union of families $\{\mathcal{F}_i\}_{i=0}^{nQ}$ of sets, the “ i -dimensional faces” of \mathcal{Q} , with the following properties:

- (p1) $\mathcal{Q} = \cup_i \cup_{F \in \mathcal{F}_i} F$;
- (p2) $\mathcal{F} := \cup_i \mathcal{F}_i$ is a collection of finitely many disjoint sets;
- (p3) each face $F \in \mathcal{F}_i$ is a convex *open* i -dimensional cone, where open means that for every $x \in F$ there exists an i -dimensional disk D with $x \in D \subset F$;
- (p4) for each $F \in \mathcal{F}_i$, $\bar{F} \setminus F$ is the union of some elements of $\cup_{j < i} \mathcal{F}_j$.

Remark 7.2. With a small abuse of notation ∂F will denote $\bar{F} \setminus F$ for any $F \in \mathcal{F}$.

So, $\mathcal{F}_0 = \{0\}$; \mathcal{F}_1 consists of finitely many half-lines meeting at 0, i.e. of sets of type $l_v = \{\lambda v : \lambda \in]0, +\infty[\}$ for $v \in \mathbb{S}^{N-1}$; \mathcal{F}_2 consists of finitely many 2-dimensional “infinite triangles” delimited by pair of half lines $l_{v_1}, l_{v_2} \in \mathcal{F}_1$ and by $\{0\}$; and so on. To prove this statement, first of all we recall the construction of ξ (see [13, 2.1.2]). After selecting a suitable finite collection of non zero vectors $\{e_l\}_{l=1}^h$, we define the linear map $L : \mathbb{R}^{nQ} \rightarrow \mathbb{R}^N$ given by

$$L(P_1, \dots, P_Q) := \left(\underbrace{P_1 \cdot e_1, \dots, P_Q \cdot e_1}_{w^1}, \underbrace{P_1 \cdot e_2, \dots, P_Q \cdot e_2, \dots}_{w^2}, \dots, \underbrace{P_1 \cdot e_h, \dots, P_Q \cdot e_h}_{w^h} \right).$$

Then, we consider the map $O : \mathbb{R}^N \rightarrow \mathbb{R}^N$ which maps (w^1, \dots, w^h) into the vector (v^1, \dots, v^h) where each v^i is obtained from w^i ordering its components in increasing order. Note that the composition $O \circ L : (\mathbb{R}^n)^Q \rightarrow \mathbb{R}^N$ is now invariant under the action of the symmetric group \mathcal{P}_Q . Therefore, ξ is simply the induced map on $\mathcal{A}_Q = (\mathbb{R}^n)^Q / \mathcal{P}_Q$ and $\mathcal{Q} = \xi(\mathcal{A}_Q) = O(V)$ where $V := L(\mathbb{R}^{nQ})$. Moreover, since the vectors e_i 's span \mathbb{R}^n (cf. [13, 2.1.2]), the map L is injective and thus V is an nQ -dimensional subspace.

Consider the following equivalence relation \sim on V :

$$(w^1, \dots, w^h) \sim (z^1, \dots, z^h) \quad \text{if} \quad \begin{cases} w_j^i = w_k^i & \Leftrightarrow z_j^i = z_k^i \\ w_j^i > w_k^i & \Leftrightarrow z_j^i > z_k^i \end{cases} \quad \forall i, j, k, \quad (7.1)$$

where $w^i = (w_1^i, \dots, w_Q^i)$ and $z^i = (z_1^i, \dots, z_Q^i)$: if $w \sim z$, then O rearranges their components with the same permutation. We let \mathcal{E} denote the set of corresponding equivalence classes in V and $\mathcal{C} := \{L^{-1}(E) : E \in \mathcal{E}\}$. The following fact is an obvious consequence of definition (7.1):

$$L(P) \sim L(S) \quad \text{implies} \quad L(P_{\pi(1)}, \dots, P_{\pi(Q)}) \sim L(S_{\pi(1)}, \dots, S_{\pi(Q)}) \quad \forall \pi \in \mathcal{P}_Q.$$

Thus, $\pi(C) \in \mathcal{C}$ for every $C \in \mathcal{C}$ and every $\pi \in \mathcal{P}_Q$. Since ξ is injective and is induced by $O \circ L$, it follows that, for every pair $E_1, E_2 \in \mathcal{E}$, either $O(E_1) = O(E_2)$ or $O(E_1) \cap O(E_2) = \emptyset$. Therefore, the family $\mathcal{F} := \{O(E) : E \in \mathcal{E}\}$ is a partition of \mathcal{Q} .

Clearly, each $E \in \mathcal{E}$ is a convex cone. Let i be its dimension and D any i -dimensional disk $D \subset E$. Denote by x the center of D and let y be any other point of E . Then, by (7.1), the point $z = y - \varepsilon(x - y) = (1 + \varepsilon)y - \varepsilon x$ belongs as well to E for any $\varepsilon > 0$ sufficiently small. The convex envelope of $D \cup \{z\}$, which is contained in E , contains in turn an i -dimensional disk centered in y : therefore E is an open convex cone. Since $O|_E$ is a linear injective map, $F = O(E)$ is an open convex cone of dimension i . Therefore, \mathcal{F} satisfies (p1)-(p3).

Next notice that, having fixed $w \in E$, a point z belongs to $\bar{E} \setminus E$ if and only if

- (1) $w_j^i \geq w_k^i$ implies $z_j^i \geq z_k^i$ for every i, j and k ;
- (2) there exists r, s and t such that $w_s^r > w_t^r$ and $z_s^r = z_t^r$.

Thus, if d is the dimension of E , $\partial E := \bar{E} \setminus E$ (cf. Remark 7.2) is the union of some elements of $\cup_{j < d} \mathcal{E}_j$, where with \mathcal{E}_j we denote the j -dimensional elements of \mathcal{E} . Observe that, since O is continuous, we must have $\bar{F} \supset O(\bar{E})$. On the other hand, if $x \in \bar{F}$ and $x_k \rightarrow x$ is a sequence contained in F , then there is a sequence $\{y_k\} \subset E$ with $O(y_k) = x_k$. By the definition of O the sequence $\{y_k\}$ is bounded and hence, up to subsequence, we can assume that it converges to $y \in \bar{E}$: thus $O(y) = x$ and $O(\bar{E}) = \bar{F}$. On the other hand, for equivalence classes E_1, E_2 of different dimension we necessarily have $O(E_1) \cap O(E_2) = \emptyset$. Thus $O(\partial E) \cap O(E) = \emptyset$, i.e. $\partial F = O(\partial E)$, which shows (p4).

7.2. Construction of ρ^b . The main building block in the construction of ρ^b is given by the following lemma.

Lemma 7.3. *For $\tau \in]0, \frac{1}{4}[$ and any $D \in \mathbb{N} \setminus \{0\}$ consider the map $\Phi_\tau : \mathbb{R}^D \rightarrow \mathbb{R}^D$ defined by:*

$$\Phi_\tau(x) = \begin{cases} 0 & \text{if } |x| \leq \tau \\ \sqrt{\tau} \frac{|x| - \tau}{\sqrt{\tau} - \tau} \frac{x}{|x|} & \text{if } \tau \leq |x| \leq \sqrt{\tau} \\ x & \text{if } |x| \geq \sqrt{\tau}. \end{cases}$$

Then $|\Phi_\tau(x) - x| \leq \tau$ and $\text{Lip}(\Phi_\tau) \leq 1 + 2\sqrt{\tau}$.

Proof. The proofs of the two claims are straightforward computations. First $\Phi_\tau(x) = x$ if $|x| \geq \sqrt{\tau}$ and $|\Phi_\tau(x) - x| = |x| \leq \tau$ if $|x| \leq \tau$. For $\tau \leq |x| \leq \sqrt{\tau}$ we compute

$$|\Phi_\tau(x) - x| = \left| \frac{\sqrt{\tau}(|x| - \tau)}{\sqrt{\tau} - \tau} - |x| \right| = \tau \frac{\sqrt{\tau} - |x|}{\sqrt{\tau} - \tau} \leq \tau.$$

Next we show that $|D\Phi_\tau(x) \cdot v| \leq (1 + 2\sqrt{\tau})|v|$ at any point of differentiability. This inequality obviously imply the claimed Lipschitz constant estimate because Φ_τ is Lipschitz and its domain of definition is a convex set. The inequality is, moreover, obvious when $|x| < \tau$ and $|x| > \sqrt{\tau}$. For $\tau < |x| < \sqrt{\tau}$, we can compute

$$D\Phi_\tau(x) = \frac{1 - \frac{\tau}{|x|}}{1 - \sqrt{\tau}} \text{Id} + \frac{\frac{\tau}{|x|}}{1 - \sqrt{\tau}} \frac{x}{|x|} \otimes \frac{x}{|x|}.$$

The matrix is symmetric with positive eigenvalues (because $|x| > \tau$) and the maximal eigenvalue is $(1 - \sqrt{\tau})^{-1} \leq 1 + 2\sqrt{\tau}$, thereby proving our claim. \square

7.2.1. *Special coordinates, conical sections and separation.* Let S_k be the k -dimensional skeleton of \mathcal{Q} , i.e. the union of $F \in \mathcal{F}_k$ and denote by $(S_k)_\sigma$ its σ -neighborhood $\{x : \text{dist}(x, S_k) < \sigma\}$. Incidentally, $(S_k)_\sigma$ contains $(S_i)_\sigma$ for every $i < k$.

Definition 7.4 (Coordinates and conical sections). Fix any face $F \in \mathcal{F}_k$ and introduce cartesian coordinates $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ in such a way that $F \subset \mathbb{R}^k \times \{0\}$. For a positive constant \tilde{c} consider the cone $\mathcal{C}(F) := \{(y, z) \in \mathcal{Q} : (y, 0) \in F, |z| \leq \tilde{c} \text{dist}((y, 0), S_{k-1})\}$. For any $p = (y, 0) \in F$ we will denote by V_p the conical section $\{q : \exists z \text{ with } (y, z) \in \mathcal{C}(F)\}$.

Note that, if \tilde{c} is sufficiently small, we will have the following property

$$\mathcal{C}(F) \cap \mathcal{C}(G) \neq \emptyset \implies \text{either } F \subset \overline{G} \text{ or } G \subset \overline{F}.$$

For every constants $a, b > 0$, $k = 1 \dots, nQ - 1$ and $F \in \mathcal{F}_k$, we fix coordinates as in Definition 7.4 and denote by F_{ab} the sets

$$F_{ab} := \{(y, z) : |z| \leq a, (y, 0) \notin (S_{k-1})_b\}.$$

For the faces $F \in \mathcal{F}_{nQ}$ of maximal dimension and for every $a > 0$, F_a denotes the set $F_a := F \setminus (S_{nQ-1})_a$. The following lemma is an obvious corollary of the linear simplicial and conical structures of \mathcal{Q} .

Lemma 7.5. *There is a constant $\bar{c} > 0$ with the following property. Assume F and G are two distinct k dimensional faces.*

- If $k = nQ$, $a > 0$, $x \in F_a$ and $x' \in G_a$, then $|x - x'| \geq \bar{c}a$;
- If $k < nQ$, $b/a > \bar{c}^{-1}$, $x \in F_{ab}$ and $x' \in G_{ab}$, then $|x - x'| \geq \bar{c}b$.

Moreover, if $F \in \mathcal{F}_k$, $H \in \mathcal{F}_i$ with $i > k$ and $F \not\subset \partial H$ (cf. Remark 7.2), then $|x - x'| \geq \bar{c}b$ for every $x \in H$ and $x' \in F \setminus (S_{k-1})_a$.

7.2.2. *The domains $\text{Dom}(f_k)$.* Next we choose constants $c_k := \delta^{8^{-nQ+k}}$. If δ is small enough, each family $\{F_{2\sqrt{c_k}, c_{k-1}^2}\}_{F \in \mathcal{F}_k}$ with $k < nQ$ is made by pairwise disjoint sets, which are at least $\bar{c}c_{k-1}^2$ far apart, where \bar{c} is the constant of Lemma 7.5. We are ready to define the map $\boldsymbol{\rho}^b := \boldsymbol{\rho}^*|_{\mathcal{Q}}$ inductively “from the top to the bottom”. More precisely we will define a family of maps $\{f_k\}_{k \in \{0, \dots, nQ\}}$ on domains $\text{Dom}(f_k) \subset \mathcal{Q}$ starting from f_{nQ} and ending with $f_0 = \boldsymbol{\rho}^b$. We first explicitly define $\text{Dom}(f_k) := \mathcal{Q} \setminus (S_{k-1})_{c_{k-1}}$ for $k > 0$ and $\text{Dom}(f_0) = \mathcal{Q}$, and in order to simplify our notation we then agree that $c_{-1} = \delta^{8^{-nQ-1}}$ and $S_{-1} = (S_{-1})_{c_{-1}} = \emptyset$. It is obvious that

$$\text{Dom}(f_k) = \text{Dom}(f_{k+1}) \cup \bigcup_{F \in \mathcal{F}_k} F_{2\sqrt{c_k}, c_{k-1}^2} \setminus (S_{k-1})_{c_{k-1}}. \quad (7.2)$$

Indeed, if $x \in \text{Dom}(f_k) \setminus \text{Dom}(f_{k+1})$ we then must have $\text{dist}(x, S_k) < c_k$ and $\text{dist}(x, S_{k-1}) \geq c_{k-1}$. Let $q \in S_k$ be such that $|x - S_k| < c_k$. Since $\text{dist}(x, S_{k-1}) \geq c_{k-1} > c_k$, the point q must necessarily belong to a k -dimensional face F . Fix coordinates as in Definition 7.4. If $x = (y, z)$, we then obviously have $|z| < c_k \leq 2\sqrt{c_k}$. On the other hand $\text{dist}((y, 0), S_{k-1}) \geq \text{dist}(x, S_{k-1}) - |z| \geq c_{k-1} - c_k > c_{k-1}^2$. This shows that $x \in F_{2\sqrt{c_k}, c_{k-1}^2}$.

7.2.3. *The maps f_k .* On $\text{Dom}(f_{nQ})$ we define $f_{nQ} = \text{Id}$ and specify next the procedure to define f_k knowing f_{k+1} . Along the procedure we claim inductively the following.

Assumption 7.6 (Inductive step). *The map f_{k+1} has the following three properties.*

- (a_{k+1}) $\text{Lip}(f_{k+1}) \leq 1 + Cc_{k+1}^{1/2}$ and $|f_{k+1}(x) - x| \leq Cc_{k+1}$.
- (b_{k+1}) Consider $i \leq k+1$, an i -dimensional face F , the cone $\mathcal{C}(F)$ in Definition 7.4 and the corresponding coordinates. Then, f_{k+1} factorizes on $\text{Dom}(f_{k+1}) \cap \mathcal{C}(F)$ as

$$f_{k+1}(y, z) = (y, h_{k+1}^F(y, z)) \in \mathbb{R}^i \times \mathbb{R}^{N-i}. \quad (7.3)$$

- (c_{k+1}) For every $G \in \mathcal{F}_i$ with $i \geq k+1$, f_{k+1} maps $\text{Dom}(f_{k+1}) \cap \{x : \text{dist}(x, G) < \delta\}$ into \overline{G} . Moreover the restriction of f_{k+1} to G_{c_i, c_k} is the orthogonal projection onto G .

The constants involved depend on k but not on the parameter δ and since the process is iterated finitely many times, we will not keep track of such dependence. Note that f_{nQ} satisfies (a_{nQ}), (b_{nQ}) and (c_{nQ}) trivially, because it is the identity map. Given f_{k+1} we next show how to construct f_k . For every $p \in G \in \mathcal{F}_k$ with $p \notin (S_{k-1})_{c_{k-1}^2}$, set coordinates as in Definition 7.4 and consider the cone $W_p := \{(y, z) \in V_p : |z| \leq 2\sqrt{c_k}\}$. Let now Φ_τ be the map of Lemma 7.3 with $\tau = 2c_k$. The function f_k is defined in W_p by

$$f_k(x) = f_k(y, z) := (y, h_k^F(y, z)) := \begin{cases} (y, 0) & \text{for } |z| \leq \tau/2 = c_k, \\ (y, \Phi_\tau(h_{k+1}^F(y, z))) & \text{otherwise.} \end{cases} \quad (7.4)$$

If $q \in \text{Dom}(f_k)$ does not belong to any W_p as above, then we set $f_{k+1}(q) = f_k(q)$.

Observe that the definition above gives values to f_k on a set which is larger than $\text{Dom}(f_k)$: this will be useful to carry on some of the estimates, but we insist that Assumption 7.6 will only be checked on $\text{Dom}(f_k)$.

7.2.4. *Well-definition and continuity.* Consider a point $q \in \text{Dom}(f_k)$. If q is not contained in $F_{2\sqrt{c_k}, c_{k-1}^2}$ for some k -dimensional face, then by (7.2) it is contained in the domain of f_{k+1} and thus $f_k(q)$ is defined. If q is contained in $F_{2\sqrt{c_k}, c_{k-1}^2}$ for some k -dimensional face, then q belongs to some W_p as above. Let $q = (y, z)$. If $|z| \leq c_k$, then $f_k(q)$ is defined; otherwise, since $\text{dist}(q, S_k) \geq c_k$, we infer that $q \in \text{Dom}(f_{k+1})$ and $f_k(q)$ is also defined.

As for the continuity, fix $(y, z) \in W_p \cap \text{Dom}(f_k)$ with $p = (y, 0) \in F \in \mathcal{F}_k$. If $|z_k| = c_k$, then by (a_{k+1}) we have $|h_{k+1}^F(y, z)| \leq |z| + Cc_{k+1} \leq \tau/2 + C\tau^8$. For δ sufficiently small this obviously implies $|h_{k+1}^F(y, z)| \leq \tau$ and thus, by the definition of Φ_τ , $\Phi_\tau(h_{k+1}^F(y, z)) = 0$. On the other hand, if $|z| = 2\sqrt{c_k}$, then $|h_{k+1}^F(y, z)| \geq |z| - Cc_{k+1} = 2\sqrt{c_k} - Cc_k^8 \geq \sqrt{2c_k}$ and thus $\Phi_\tau(h_{k+1}^F(y, z)) = h_{k+1}^F(y, z)$. Therefore under this assumption we have $f_{k+1}(q) = f_k(q)$.

We next check that f_k maps $\text{Dom}(f_k)$ into \mathcal{Q} . This is true by induction where f_k coincides with f_{k+1} . Fix therefore a point q in some $W_p \cap \text{Dom}(f_k)$ with $p \in F \in \mathcal{F}_k$ and let G be the i -dimensional face containing q . Then, $f_{k+1}(q)$ belongs to a face \overline{G} , by Assumption 7.6. By the estimate in (a_{k+1}) and the assumption (b_{k+1}), the face G must intersect $\mathcal{C}(F)$ and thus $F \subset \overline{G}$. Observe that, by the properties of Φ_τ and by the inductive assumption (b_{k+1}), $f_k(q)$ is mapped in the segment joining $f_k(q)$ and q and thus must belong to \overline{G} .

7.2.5. *The inductive conclusions (c_k) and (b_k) .* The first claim of (c_k) is simple to prove: as noticed, if a point q belongs to $\text{Dom}(f_{k+1})$, then f_k maps it into the closure of the face containing q . If the point is not contained in $\text{Dom}(f_{k+1})$, then it must be contained in the c_k -neighborhood of some k -dimensional face F and hence it is mapped into F : when this happens F is a portion of the boundary of the face containing q . Next, fix a face $G \in \mathcal{F}_i$. If $i = k$, by the very definition of f_k , we have that the restriction of f_k to $\text{Dom}(f_k) \cap G_{c_k, c_{k-1}}$ is the orthogonal projection onto G . If $i > k$, we actually have that $f_k = f_{k+1}$ on $\mathcal{Q} \setminus (S_k)_{2\sqrt{c_k}} \supset \mathcal{Q} \setminus (S_{i-1})_{c_{i-1}}$.

Fix now an i -dimensional face L with $i \leq k$, consider coordinates $\mathbb{R}^i \times \mathbb{R}^{n-i}$ as in Definition 7.4 and the corresponding $\mathcal{C}(L)$. If $q = (y, 0) \in L$, the condition (b_k) is equivalent to say that $V_q = (\{y\} \times \mathbb{R}^{n-i}) \cap \text{Dom}(f_k)$ gets mapped into $\{y\} \times \mathbb{R}^{n-i}$. Fix a point $\tilde{q} \in V_q$. If $f_{k+1}(\tilde{q}) = f_k(\tilde{q})$ there is nothing to prove. Otherwise it turns out that there is a k -dimensional face F such that $\tilde{q} \in \mathcal{C}(F)$. But then we necessarily have $L \subset \bar{F}$. So, set coordinates $\mathbb{R}^i \times \mathbb{R}^{k-i} \times \mathbb{R}^{n-k}$ so that at the same time $L \subset \mathbb{R}^i \times \{0\} \times \{0\}$ and $F \subset \mathbb{R}^i \times \mathbb{R}^{k-i} \times \{0\}$. Thus, $(y, 0, 0)$ is the coordinate of q and (y, z, w) that of \tilde{q} . According to our definition of f_k , $f_k(q) = (y, z, w')$ for some w' , which indeed implies the desired claim.

7.2.6. *C^0 estimate.* Observe that, for every x where f_k coincides with f_{k+1} , we have $|f_k(x) - x| \leq Cc_{k+1} \leq Cc_k^8$. Instead, for any point x where f_k is newly defined, by the estimates of Lemma 7.3 and the triangle inequality we have

$$|f_k(x) - x| \leq |f_{k+1}(x) - f_k(x)| + |f_k(x) - x| \leq Cc_{k+1} + \tau \leq Cc_{k+1} + 2c_k.$$

7.2.7. *Lipschitz estimate.* We fix $x, x' \in \text{Dom}(f_k)$ and, apart from the trivial one $f_k(x) = f_{k+1}(x)$ and $f_k(x') = f_{k+1}(x')$, we distinguish three cases.

Case 1: $x, x' \in G_{2\sqrt{c_k}, c_{k-1}^2}$ **for some k -dimensional face G .** Choosing coordinates as in Definition 7.4, we set $x = (y, z)$ and $x' = (y', z')$. If both $|z|, |z'| \leq \frac{\tau}{2}$, then $|f_k(x) - f_k(x')| = |y - y'| \leq |x - x'|$. If $|z| \geq \frac{\tau}{2}$ and $|z'| \geq \frac{\tau}{2}$, then

$$\begin{aligned} |f_k(x) - f_k(x')|^2 &\leq |y - y'|^2 + (1 + 2\tau^{1/2})^2 |h_{k+1}^F(y, z) - h_{k+1}^F(y', z')|^2 \\ &\leq (1 + 2\tau^{1/2})^2 (|y - y'|^2 + |h_{k+1}^F(y, z) - h_{k+1}^F(y', z')|^2) \\ &= (1 + \sqrt{2c_k})^2 |f_k(x) - f_k(x')|^2 \leq (1 + \sqrt{2c_k})^2 (1 + C\sqrt{c_{k+1}})^2 |x - x'|^2. \end{aligned}$$

If $|z| \leq \frac{\tau}{2}$ and $|z'| > \frac{\tau}{2}$, let $\tilde{z} = \frac{\tau z}{2|z|}$ and $\tilde{x} = (y, \tilde{z})$. Observe that $f_k(\tilde{x}) = f_k(x) = (y, 0)$ and that $|\tilde{x} - x'|^2 = |y - y'|^2 + |z - \tilde{z}|^2 \leq |y - y'| + |z - z'|^2 \leq |x - x'|^2$. On the other hand we have just shown $|f_k(x) - f_k(\tilde{x})| \leq (1 + Cc_k^{1/2})|x - \tilde{x}|$.

Case 2: $x \in F_{2\sqrt{c_k}, c_{k-1}^2}$, $x' \in G_{2\sqrt{c_k}, c_{k-1}^2}$ **for distinct $F, G \in \mathcal{F}_k$.** By Lemma 7.5, $|x - x'| \geq \bar{c}_{k-1}^2 \geq \bar{c}_k^{1/4}$. On the other hand, we also have, by the C^0 estimate,

$$|f_k(x) - f_k(x')| \leq |x - x'| + Cc_k \leq (1 + Cc_k^{3/4})|x - x'|.$$

Case 3: $x \in G_{2\sqrt{c_k}, c_{k-1}^2}$, $f_k(x') = f_{k+1}(x')$. We have two possibilities.

Case 3a: x' belongs to a face H (of dimension $i > k$) with $G \not\subset \overline{H}$. Consider the closed set $\tilde{G} := G \setminus (S_{k-1})_{c_{k-1}^2}$. By Lemma 7.5 $\text{dist}(x', \tilde{G}) \geq \bar{c}c_{k-1}^2$ and thus $|x - x'| \geq \bar{c}c_{k-1}^2 - 2\sqrt{c_k} \geq \frac{\bar{c}}{2}c_{k-1}^2$. We can therefore argue as in Case 2.

Case 3b: $x' \in \overline{H}$ with $G \subset \overline{H}$. We then have two possibilities. The first is that $x \in \text{Dom}(f_{k+1})$. Since $f_k(x') = f_{k+1}(x')$, we have $|f_k(x') - x'| \leq c_{k+1} = c_k^8$. Use the coordinates of Definition 7.4 and (c_{k+1}) to conclude $f_k(x') = f_{k+1}(y', z') = (y'', z'')$ with $|z''| \geq |z'| - Cc_k^8 \geq 2\sqrt{c_k} - Cc_k^8 \geq \sqrt{2c_k}$. We can therefore write $f_k(x') = (y'', \Phi_\tau(z''))$ (because $\Phi_\tau(z'') = z''$) and, hence, recalling $f_k(x) = (y, \Phi_\tau(h_{k+1}^F(y, z)))$ and $f_{k+1}(x) = (y, h_{k+1}^F(y, z))$,

$$\begin{aligned} |f_k(x') - f_k(x)|^2 &\leq |y - y''|^2 + (1 + 2\sqrt{\tau})^2 |h_{k+1}^F(y, z) - z''|^2 \\ &\leq (1 + 2\sqrt{\tau})^2 |f_{k+1}(x) - f_{k+1}(x')|^2. \end{aligned}$$

We therefore conclude $|f_k(x') - f_k(x)| \leq (1 + C\tau^{1/2})|x' - x| \leq (1 + Cc_k^{1/2})|x' - x|$.

The second possibility is that x is not in the domain of definition of f_{k+1} . In that case x is at distance c_k from G and thus $|x - x'| \geq \sqrt{c_k}$. We then conclude that $|f_k(x) - f_k(x')| \leq |x - x'| + Cc_k \leq (1 + C\sqrt{c_k})|x - x'|$.

7.2.8. *Summary.* After nQ steps, we get a function $f_0 = \rho^\flat : \mathcal{Q} \rightarrow \mathcal{Q}$ which satisfies

$$\text{Lip}(\rho^\flat) \leq 1 + C\delta^{8-nQ-1} \quad \text{and} \quad |\rho^\flat(x) - x| \leq C\delta^{8-nQ}, \quad (7.5)$$

$$\rho^\flat(\{x : \text{dist}(x, F) \leq \delta\}) \subset \overline{F} \quad \text{for every } F \in \mathcal{F}_k, \quad (7.6)$$

$$\rho^\flat : F_{\delta, c_0^{1/8}} \rightarrow F \text{ is the orthogonal projection for every } F \in \mathcal{F}_k. \quad (7.7)$$

7.3. **The extension ρ^\sharp of ρ^\flat to $\mathcal{Q}_{\delta^{nQ+1}}$.** Next we extend the map $\rho^\flat : \mathcal{Q} \rightarrow \mathcal{Q}$ to the δ^{nQ+1} -neighborhood of \mathcal{Q} , keeping the estimate (7.5). We first observe that, since the number of all the faces is finite, when δ is small enough, there exists a constant $C = C(N)$ with the following property. Consider two distinct faces F and H in \mathcal{F}_i . If x, y are two points contained, respectively, in $F_{\delta^{i+1}} \setminus \bigcup_{j < i} \bigcup_{G \in \mathcal{F}_j} G_{\delta^{j+1}}$ and $H_{\delta^{i+1}} \setminus \bigcup_{j < i} \bigcup_{G \in \mathcal{F}_j} G_{\delta^{j+1}}$, then

$$\text{dist}(x, y) \geq C\delta^i. \quad (7.8)$$

The extension ρ^\sharp is defined inductively, but this time “from the bottom to the top”. The first extension g_0 is identically 0 on $B_\delta(0)$ (note that this is feasible because $\rho^\flat \equiv 0$ in $B_\delta(0) \cap \mathcal{Q}$). Now we come to the inductive step. Suppose we have an extension g_k of ρ^\flat , defined on the union of the δ^{l+1} -neighborhoods of the l -skeletons S_l , for $l \in \{0, \dots, k\}$, i.e.

$$\mathbf{L}_k := \mathcal{Q} \cup B_\delta \cup \bigcup_{l=1}^k \bigcup_{F \in \mathcal{F}_l} F_{\delta^{l+1}}.$$

Assume inductively that $\text{Lip}(g_k) \leq 1 + C\delta^{8-nQ-1}$ and assume that g_k maps any δ^{j+1} -neighborhood of any j -dimensional face into its closure, when $j \leq k$. Then, we define the

extension of g_k to \mathbf{L}_{k+1} in the following way. For every face $F \in \mathcal{F}_{k+1}$, we set

$$g_{k+1} := \begin{cases} g_k & \text{in } (S_k)_{\delta^{k+1}} \cap F_{\delta^{k+2}}, \\ \mathbf{p}_F & \text{in } \{x \in \mathbb{R}^N : \mathbf{p}_F(x) \in F_{\delta,1}\} \cap F_{\delta^{k+2}}, \\ g_k & \text{in } F \setminus F_{\delta,1}. \end{cases} \quad (7.9)$$

Consider now a face F as above and $U(F)$ the union of all the δ^{j+1} -neighborhoods of the j -dimensional faces which belong to \overline{F} . As defined above, g_{k+1} maps a portion of $U(F)$ into \overline{F} . We can use Lemma 7.1 to extend g_{k+1} to $U(F)$ keeping the same Lipschitz constant, which we now compute. This constant is obviously smaller than $1 + C\delta^{8-nQ-1}$ on the domain $((S_k)_{\delta^{k+1}} \cap F_{\delta^{k+2}}) \cup F$ by inductive hypothesis. The same constant is 1 on $\{x \in \mathbb{R}^N : \mathbf{p}_F(x) \in F_{\delta,1}\} \cap F_{\delta^{k+2}}$. Consider now a point $x \in \{x \in \mathbb{R}^N : \mathbf{p}_F(x) \in F_{\delta,1}\} \cap F_{\delta^{k+2}}$ and a point $y \in F \cup ((S_k)_{\delta^{k+1}} \cap F_{\delta^{k+2}})$. If $y \notin (S_k)_{c_0^{1/8}}$, then necessarily $y \in F$ and we then have

$$|g_{k+1}(x) - g_{k+1}(y)| = |\mathbf{p}_F(x) - y| = |\mathbf{p}_F(x) - \mathbf{p}_F(y)| \leq |x - y|.$$

Otherwise we have $|x - y| \geq 1 - c_0^{1/8} = 1 - \delta^{8-nQ-1}$ and we can write

$$\begin{aligned} |g_{k+1}(x) - g_{k+1}(y)| &\leq |g_{k+1}(x) - y| + Cc_0 \leq |x - y| + \delta^{k+2} + Cc_0 \\ &\leq \left(1 + \frac{\delta^{k+2} + Cc_0}{1 - Cc_0^{1/8}}\right) |x - y| \leq (1 + C\delta^{8-nQ-1})|x - y|. \end{aligned}$$

Note that, if $x \in U(F_1) \cap U(F_2)$ for two distinct F_1 and F_2 , then $x \in \mathbf{L}_k$. Thus, the map g_{k+1} is continuous. We next bound the global Lipschitz constant of g_{k+1} . Indeed consider points $x \in U(F_1)$ and $y \in U(F_2)$ for two distinct $F_i \in \mathcal{F}_{k+1}$. Since by (7.8) $|x - y| \geq C\delta^{k+1}$, we easily see that

$$\begin{aligned} |g_{k+1}(x) - g_{k+1}(y)| &\leq 2(1 + C\delta^{8-nQ-1})\delta^{k+2} + |\boldsymbol{\rho}^b(\mathbf{p}_F(x)) - \boldsymbol{\rho}^b(\mathbf{p}_{F'}(y))| \\ &\leq 2(1 + C\delta^{8-nQ-1})\delta^{k+2} + (1 + C\delta^{8-nQ-1})|\mathbf{p}_F(x) - \mathbf{p}_{F'}(y)| \\ &\leq 2(1 + C\delta^{8-nQ-1})\delta^{k+2} + (1 + C\delta^{8-nQ-1})(|x - y| + 2\delta^{k+2}) \leq (1 + C\delta^{8-nQ-1})|x - y|. \end{aligned}$$

Since the union of the $U(F_i)$ is the domain of definition of g_{k+1} , this shows $\text{Lip}(g_{k+1}) \leq 1 + C\delta^{8-nQ-1}$. Note that by construction we also have that $U(F)$ is mapped into \overline{F} , which is the other inductive hypothesis. This is the map $\boldsymbol{\rho}^\sharp$.

After making the step above nQ times we arrive to a map g_{nQ} which extends $\boldsymbol{\rho}^b$ and is defined in a δ^{nQ+1} -neighborhood of \mathcal{Q} . We denote this map by $\boldsymbol{\rho}^\sharp$.

8. PERSISTENCE OF Q -POINTS: PROOF OF THEOREM 1.6

Proof of Theorem 1.6. As usual, by scaling and translating we assume $x = 0$ and $r = 1$. According to [13, Theorem 3.9], there are constants $\bar{C}(m, n, Q), \kappa(m, n, Q) > 0$ such that

$$\sup_{x \neq y \in B_{1/2}} \frac{\mathcal{G}(w(x), w(y))}{|y - x|^\kappa} \leq \bar{C}(\text{Dir}(w))^{\frac{1}{2}} \quad \text{for any Dir-minimizer } w : B_1 \rightarrow \mathcal{A}_Q(\mathbb{R}^n). \quad (8.1)$$

The final choice of \bar{s} will be specified at the very end, but for the moment we impose

$$4\bar{C}^2(2\bar{s})^{2\kappa} \leq \frac{\hat{\delta}}{4} \quad \text{and} \quad \bar{s} < \frac{1}{4}. \quad (8.2)$$

Fix now $s < \bar{s}$ and C^* as in the statement and assume by contradiction that, no matter how small we choose $\hat{\varepsilon} > 0$, there are a current T and a submanifold Σ as in Theorem 1.4 and a point $(p, q) \in \mathbf{C}_{1/2}$ satisfying:

- (a) $E := \mathbf{E}(T, \mathbf{C}_4) < \hat{\varepsilon}$ and $\mathbf{A}^2 \leq C^*E$;
- (b) $\Theta(T, (p, q)) = Q$;
- (c) the E^{γ_1} -approximation f (which is the map of Theorem 1.4) violates (1.9), that is

$$\int_{B_s(p)} \mathcal{G}(f, Q \llbracket \boldsymbol{\eta} \circ f \rrbracket)^2 > \hat{\delta} s^m E. \quad (8.3)$$

Set $\bar{\delta} = \frac{1}{4}$ and fix $\bar{\eta} > 0$ (whose choice will be specified later). By (a), for a suitably small $\hat{\varepsilon}$ we can apply Theorem 1.5: we let w be the corresponding Dir-minimizer. If $\bar{\eta}$ and $\hat{\varepsilon}$ are suitably small, we have

$$\int_{B_s(p)} \mathcal{G}(w, Q \llbracket \boldsymbol{\eta} \circ w \rrbracket)^2 \geq \frac{3\hat{\delta}}{4} s^m E,$$

and $\sup \{\text{Dir}(f), \text{Dir}(w)\} \leq 4E$. Thus there is $\bar{p} \in B_s(p)$ with $\mathcal{G}(w(\bar{p}), Q \llbracket \boldsymbol{\eta} \circ w(\bar{p}) \rrbracket)^2 \geq \frac{3\hat{\delta}}{4} E$ and, by (8.1), we conclude

$$g(x) := \mathcal{G}(w(x), Q \llbracket \boldsymbol{\eta} \circ w(x) \rrbracket) \geq \left(\frac{3\hat{\delta}}{4} E \right)^{1/2} - (4E)^{1/2} \bar{C} \bar{s}^{\kappa} \stackrel{(8.2)}{\geq} \left(\frac{\hat{\delta}}{2} E \right)^{1/2}.$$

Setting $h(x) := \mathcal{G}(f(x), Q \llbracket \boldsymbol{\eta} \circ f(x) \rrbracket)$, we recall that we have

$$\int_{B_s(p)} |h - g|^2 \leq C \bar{\eta} E.$$

Consider therefore the set $A := \{h > (\frac{\hat{\delta}}{4} E)^{1/2}\}$. If $\bar{\eta}$ is sufficiently small, we can assume that $|B_s(p) \setminus A| < \frac{1}{8}|B_s|$. Further, define $\bar{A} := A \cap K$, where K is the set of Theorem 1.4. Assuming $\hat{\varepsilon}$ is sufficiently small we ensure $|B_s(p) \setminus \bar{A}| < \frac{1}{4}|B_s|$. Let N be the smallest integer such that $N \frac{\hat{\delta} E}{64Qs} \geq \frac{s}{2}$. Set $\sigma_i := s - i \frac{\hat{\delta} E}{64Qs}$ for $i \in \{0, 1, \dots, N\}$ and consider, for $i \leq N - 1$, the annuli $\mathcal{C}_i := B_{\sigma_i}(p) \setminus B_{\sigma_{i+1}}(p)$. If $\hat{\varepsilon}$ is sufficiently small, we can assume that $N \geq 2$ and $\sigma_N \geq \frac{s}{4}$. For at least one of such annuli we must have $|\bar{A} \cap \mathcal{C}_i| \geq \frac{1}{2}|\mathcal{C}_i|$. We then let $\sigma := \sigma_i$ be the corresponding outer radius and we denote by \mathcal{C} the corresponding annulus.

Consider now a point $x \in \mathcal{C} \cap \bar{A}$ and let T_x be the slice $\langle T, \mathbf{p}, x \rangle$. Since $\bar{A} \subset K$, for a.e. $x \in \bar{A}$ we have $T_x = \sum_{i=1}^Q \llbracket (x, f_i(x)) \rrbracket$. Moreover, there exist i and j such that $|f_i(x) - f_j(x)|^2 \geq \frac{1}{Q} \mathcal{G}(f(x), \llbracket \boldsymbol{\eta} \circ f(x) \rrbracket)^2 \geq \frac{\hat{\delta}}{4Q} E$ (recall that $x \in \bar{A} \subset A$). When $x \in \mathcal{C}$ and the points (x, y) and (x, z) belong both to $\mathbf{B}_\sigma(p)$, we must have

$$|y - x|^2 \leq 4 \left(\sigma^2 - \left(\sigma - \frac{\hat{\delta} E}{64Qs} \right)^2 \right) \leq \frac{\sigma \hat{\delta} E}{8Qs} \leq \frac{\hat{\delta} E}{8Q}.$$

Thus, for $x \in \bar{A} \cap \mathcal{C}$ at least one of the points $(x, f_i(x))$ is not contained in $\mathbf{B}_\sigma(p)$. We conclude therefore

$$\begin{aligned} \|T\|(\mathbf{C}_\sigma(p) \setminus \mathbf{B}_\sigma(p)) &\geq |\mathcal{C} \cap \bar{A}| \geq \frac{1}{2}|\mathcal{C}| = \frac{\omega_m}{2} \left(\sigma^m - \left(\sigma - \frac{\hat{\delta}E}{64Qs} \right)^m \right) \\ &\geq \frac{\omega_m}{2} \sigma^m \left(1 - \left(1 - \frac{\hat{\delta}E}{64Qs\sigma} \right)^m \right). \end{aligned} \quad (8.4)$$

Recall that, for τ sufficiently small, $(1-\tau)^m \leq 1 - \frac{m\tau}{2}$. Since $\sigma \geq \frac{s}{4}$, if $\hat{\delta}$ is chosen sufficiently small we can therefore conclude

$$\|T\|(\mathbf{C}_\sigma(p) \setminus \mathbf{B}_\sigma(p)) \geq \frac{\omega_m \sigma^m \hat{\delta}E}{256Qs\sigma} \geq \frac{\omega_m}{256Q} \hat{\delta}E \sigma^{m-2} = c_0 \hat{\delta}E \sigma^{m-2}. \quad (8.5)$$

Next, by Theorem 1.4 and Theorem 1.5,

$$\|T\|(\mathbf{C}_\sigma(p)) \leq Q\omega_m \sigma^m + CE^{1+\gamma_1} + \bar{\eta}E + \int_{B_\sigma(p)} \frac{|Dw|^2}{2}. \quad (8.6)$$

Moreover, as shown in [13, Section 3.3] (cf. [13, Proposition 3.10]), we have

$$\int_{B_\sigma(p)} |Dw|^2 \leq C \text{Dir}(w) \sigma^{m-2+2\kappa}, \quad (8.7)$$

(for some constants κ and C depending only on m , n and Q ; in fact the exponent κ is the one of (8.1)). Combining (8.5), (8.6) and (8.7), we conclude

$$\|T\|(\mathbf{B}_\sigma(p)) \leq Q\omega_m \sigma^m + \bar{\eta}E + CE^{1+\gamma_1} + CE\sigma^{m-2+2\kappa} - c_0\sigma^{m-2}\hat{\delta}E. \quad (8.8)$$

Next, by the monotonicity formula, $\rho \mapsto \exp(C\mathbf{A}^2\rho^2)\rho^{-m}\|T\|(\mathbf{B}_\rho(p))$ is a monotone function (indeed, the usual monotonicity formula of the theory of varifolds with bounded mean curvature gives the monotonicity of $\rho \mapsto \exp(C\mathbf{A}\rho)\rho^{-m}\|T\|(\mathbf{B}_\rho(p))$, cf. [28, Theorem 17.6]); the slight improvement needed in this proof follows from minor modifications of the usual argument but, since we have not been able to find a reference, we provide a proof in Lemma A.1 in the appendix). Using $\mathbf{A}^2 \leq C^*E$ and the Taylor expansion of the exponential, we conclude

$$\|T\|(\mathbf{B}_\sigma(p)) \geq Q\omega_m \sigma^m - CC^*E\sigma^{m+2}. \quad (8.9)$$

Combining (8.8) and (8.9) we conclude

$$CC^*\sigma^4 + (\bar{\eta} + CE_1^\gamma)\sigma^{2-m} + C\sigma^{2\kappa} \geq c_0\hat{\delta}. \quad (8.10)$$

Recalling that $\sigma \leq s < \bar{s}$, we can, finally, specify \bar{s} : it is chosen so that $CC^*\bar{s}^4 + C\bar{s}^{2\kappa}$ is smaller than $\frac{c_0}{2}\hat{\delta}$. Combined with (8.2) this choice of \bar{s} depends, therefore, only upon $\hat{\delta}$. (8.10) becomes then

$$(\bar{\eta} + CE^{\gamma_1})\sigma^{2-m} \geq \frac{c_0}{2}\hat{\delta}. \quad (8.11)$$

Next, recall that $\sigma \geq \frac{s}{4}$. We then choose $\hat{\varepsilon}$ so that $(\bar{\eta} + C\hat{\varepsilon}^{\gamma_1})(\frac{s}{4})^{2-m} \leq \frac{c_0}{4}\hat{\delta}$. Since $E \leq \hat{\varepsilon}$, this choice is incompatible with (8.11), thereby reaching a contradiction: for this choice of the parameter $\hat{\varepsilon}$ (which in fact depends only upon $\hat{\delta}$ and s) the conclusion of the Theorem, i.e. (1.9), must then be valid. \square

APPENDIX A. MONOTONICITY FORMULA

Lemma A.1. *There is a constant C depending only on m , n and \bar{n} with the following property. If $\Sigma \subset \mathbb{R}^{m+n}$ is a C^2 $(m + \bar{n})$ -dimensional submanifold with $\|A_\Sigma\|_\infty \leq \mathbf{A}$ and T an m -dimensional integer-rectifiable current supported in Σ which is area minimizing in Σ , then for every $\xi \in \Sigma$ the function $\rho \mapsto \exp(C\mathbf{A}^2\rho^2)\rho^{-m}\|T\|(\mathbf{B}_\rho(\xi))$ is monotone on the interval $]0, \bar{\rho}[$, where $\bar{\rho} := \min\{\text{dist}(x, \text{spt}(\partial T)), (C\mathbf{A})^{-1}\}$.*

Proof. The argument is a minor variant of the classical proof of the monotonicity formula for varifolds with bounded mean curvature due to Allard (cf. [2]), where the stronger hypothesis that T is *stationary* in a C^2 -submanifold allows a better estimate of the relevant error term. Without loss of generality assume $\xi = 0$, let $s \in]0, \bar{\rho}[$ and $\varphi \in C_c^1(]-1, 1[)$ with $\varphi \equiv 1$ in a neighborhood of 0. For each $x \in \Sigma$ let $\mathbf{p}_x : \mathbb{R}^{m+n} \rightarrow T_x\Sigma$ be the orthogonal projection onto the tangent space to Σ in x and consider the vector field $X_s(x) := \varphi\left(\frac{|x|}{s}\right)\mathbf{p}_x(x)$. Note that X_s is tangent to Σ and thus $\delta T(X_s) = 0$. In order to compute $\delta T(X_s)$, consider at $\|T\|$ -a.e. $x \in \text{spt}(T)$ an orthonormal frame e_1, \dots, e_m with $e_1 \wedge \dots \wedge e_m = \vec{T}$. It turns out that

$$\delta T(X_s) = \int \text{div}_{\vec{T}} X_s d\|T\| = \int \sum_i \langle D_{e_i} X_s, e_i \rangle d\|T\|.$$

Next, at any $x \in \Sigma$ let ν_1, \dots, ν_l ($l = n - \bar{n}$) be an orthonormal frame orthogonal to Σ . Since $\mathbf{p}_x(x) = x - \sum_j \langle x, \nu_j \rangle \nu_j$ and $\langle e_i, \nu_j \rangle = 0$, we compute:

$$\text{div}_{\vec{T}} X_s(x) = \underbrace{\sum_i \left[D_{e_i} \left(\varphi \left(\frac{|x|}{s} \right) \right) \langle x, e_i \rangle + \varphi \left(\frac{|x|}{s} \right) \langle D_{e_i} x, e_i \rangle \right]}_I - \underbrace{\varphi \left(\frac{|x|}{s} \right) \sum_{i,j} \langle x, \nu_j \rangle \langle D_{e_i} \nu_j, e_i \rangle}_{II}.$$

I is the usual expression appearing in the proof of the standard monotonicity formula for stationary varifolds. If we use the notation r for the function $x \mapsto |x|$ and $\nabla^\perp r$ for $\nabla r - \mathbf{p}_x(\nabla r)$, we find $I = m\varphi\left(\frac{r}{s}\right) + \frac{r}{s}\varphi'\left(\frac{r}{s}\right)(1 - |\nabla^\perp r|^2)$ (see for instance [10, (2.2)]). In order to bound II , we first observe that $\langle D_{e_i} \nu_j, e_i \rangle = -\langle A(e_i, e_i), \nu_j \rangle$. Next, since $r \leq (C\mathbf{A})^{-1}$, if C is chosen sufficiently large we can assume that the geodesic segment of Σ connecting 0 and x has length $\ell < 2r$. Denote by $\gamma : [0, \ell] \rightarrow \Sigma$ a parametrization by arc-length of such a segment. Then,

$$\langle x, \nu_j(x) \rangle = \int_0^\ell \langle \dot{\gamma}(\sigma), \nu_j(\gamma(\ell)) \rangle d\sigma = \int_0^\ell \underbrace{\langle \dot{\gamma}(\sigma), [\nu_j(\gamma(\ell)) - \nu_j(\gamma(\sigma))] \rangle}_{g(\sigma)} d\sigma, \quad (\text{A.1})$$

and observe that

$$|g'(\sigma)| \leq 2 \left| \frac{d}{d\sigma} \dot{\gamma}(\sigma) \right| + \left| \langle \dot{\gamma}(\sigma), \frac{d}{d\sigma} \nu_l(\gamma(\sigma)) \rangle \right| \leq 3|A(\dot{\gamma}(\sigma), \dot{\gamma}(\sigma))|.$$

Since $g(\ell) = 0$, integrating the latter inequality we conclude $|g(\sigma)| \leq 3\ell\mathbf{A} \leq 6r\mathbf{A}$, which in turn, together with (A.1), gives $|x \cdot \nu_j(x)| \leq 6r^2\mathbf{A}$.

Putting all estimates together, we achieve the inequality $|II| \leq C\varphi\left(\frac{r}{s}\right)r^2\mathbf{A}^2$. From here on we can follow the usual strategy leading to the monotonicity formula (cf. [28] or [10,

Proof of Theorem 2.1]): letting the test function φ converge from below to the indicator function of $] - 1, 1[$, after few manipulations we achieve the inequality

$$\frac{d}{ds} \frac{\|T\|(\mathbf{B}_s)}{s^m} \geq -C\mathbf{A}^2 s^2 \frac{\|T\|(\mathbf{B}_s)}{s^m},$$

which leads to the desired claim. \square

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