Regularity of area minimizing currents II: center manifold

by

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REGULARITY OF AREA MINIMIZING CURRENTS II: CENTER MANIFOLD

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Abstract. This is the second paper of a series of three on the regularity of higher codimension area minimizing integral currents. Here we perform the second main step in the analysis of the singularities, namely the construction of a center manifold, i.e. an approximate average of the sheets of an almost flat area minimizing current. Such center manifold is complemented with a Lipschitz multi-valued map on its normal bundle, which approximates the current with a high degree of accuracy. In the third and final paper these objects are used to conclude a new proof of Almgren’s celebrated dimension bound on the singular set.

0. Introduction

In this second paper on the regularity of area minimizing integer rectifiable currents (we refer to the Foreword of [6] for the precise statement of the final theorem and on overview of its proof) we address one of the main steps in the analysis of the singularities, namely the construction of what Almgren calls center manifold. Unlike the case of hypersurfaces, singularities in higher codimension currents can appear as “higher order” perturbation of smooth minimal submanifolds. In order to illustrate this phenomenon, we can consider the examples of area minimizing currents induced by complex varieties of $\mathbb{C}^n$, as explained in the Foreword of [6]. Take, for instance, the complex curve:

$$V := \{(z, w) : (z - w^2)^2 = w^5\} \subset \mathbb{C}^2.$$ 

The point $0 \in V$ is clearly a singular point. Nevertheless, in every sufficiently small neighborhood of the origin, $V$ looks like a small perturbation of the smooth minimal surface \{z = w^2\}: roughly speaking, $V = \{z = w^2 \pm w^{5/2}\}$. One of the main issues of the regularity of area minimizing currents is to understand this phenomenon of “higher order singularities”. Following the pioneering work of Almgren [2], a way to deal with it is to approximate the minimizing current with the graph of a multiple valued function on the normal bundle of a suitable, curved, manifold. Such manifold must be close to the “average of the sheets” of the current (from this the name center manifold): the hope is that such property will guarantee a singular “first order expansion” of the corresponding approximating map.

A “center manifold” with such an approximation property is clearly very far from being uniquely defined and moreover the relevant estimates are fully justified only by the concluding arguments, which will appear in [7]. In this paper, building upon the works [4, 5, 6], we
provide a construction of a center manifold $\mathcal{M}$ and of an associated approximation of the corresponding area minimizing current via a multiple valued function $F : \mathcal{M} \rightarrow A_Q(\mathbb{R}^{m+n})$.

The corresponding construction of Almgren is given in [2, Chapter 4]. Unfortunately, we do not understand this portion of Almgren’s monograph deeply enough to make a rigorous comparison between the two constructions. Even a comparison between the statements is prohibitive, since the main ones of Almgren (cf. [2, 4.30 & 4.33]) are rather involved and seem to require a thorough understanding of most of the chapter (which by itself has the size of a rather big monograph). At a first sight, our approach seems to be much simpler and to deliver better estimates.

In the rest of this introduction we will explain some of the main aspects of our construction.

0.1. Whitney-type decomposition. The center manifold is the graph of a classical function over an $m$-dimensional plane with respect to which the excess of the minimizing current is sufficiently small. To achieve a suitable accuracy in the approximation of the average of the sheets of the current, it is necessary to define the function and an appropriate scale, which varies locally. Around any given point such scale is morally the first at which the sheets of the current cease to be close. This leads to a Whitney-type decomposition of the reference $m$-plane, where the refining algorithm is stopped according to three conditions. In each cube of the decomposition the center manifold is then a smoothing of the average of the Lipschitz multiple valued approximation constructed in [4], performed in a suitable orthonormal system of coordinates, which changes from cube to cube.

0.2. $C^{3,\kappa}$-regularity of $\mathcal{M}$. The arguments of [7] require that the center manifold is at least $C^3$-regular. As it is the case of Almgren’s center manifold, we prove actually $C^{3,\kappa}$ estimates, which are a natural outcome of some Schauder estimates. It is interesting to notice that, if the current has multiplicity one everywhere (i.e., roughly speaking, is made of a single sheet), then the center manifold coincides with it and, hence, we can conclude directly a higher regularity than the one given by the usual De Giorgi-type (or Allard-type) argument. This is already remarked in the introduction of [2] and it has been proved in our paper [3] with a relatively simple and short direct argument. The interested reader might find useful to consult that reference as well, since many of the estimates of this note appear there in a much more elementary form.

0.3. Approximation on $\mathcal{M}$. Having defined a center manifold, we then give a multivalued map on its normal bundle which approximates the current with: the relevant estimates on this map and its approximation properties are then given locally for each cube of the Whitney decomposition used in the construction of the center manifold. We follow a simple principle: at each scale where the refinement of the Whitney decomposition has stopped, the image of such function coincides (on a large set) with the Lipschitz multiple valued approximation constructed in [6], i.e. the same map whose smoothed average has been used to construct the center manifold. As a result, the graph of $F$ is well centered, i.e. the average of $F$ is very close (compared to its Dirichlet energy and its $L^2$ norm) to be the manifold $\mathcal{M}$ itself. As far as we understand Almgren is not following this principle and it
seems very difficult to separate his construction of the center manifold from the one of the approximating map.

0.4. Splitting before tilting. The regularity of the center manifold $\mathcal{M}$ and the centering of the approximating map $F$ are not the only properties need to conclude our proof in [7]. Another ingredient plays a crucial role. Assume that around a certain point, at all scales larger than a given one, say $s$, the excess decays and the sheets stay very close. If at scale $s$ the excess is not decaying anymore, then the sheets must separate as well. In other words, since the tilting of the current is under control up to scale $s$, the current must in some sense "split before tilting". We borrow the terminology from a remarkable work of Rivièere [10], where first this phenomenon was investigated independently of Almgren’s monograph in the case of 2-dimensional area-minimizing currents. Rivièere’s approach relies on a clever “lower epiperimetric inequality”, which unfortunately seems limited to the 2-dimensional context.

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1. Construction algorithm and main existence theorem

The goal of this section is to specify the algorithm leading to the center manifold. This algorithm will involve a few parameters. Their choice plays an important role and for a couple of them it will be fully specified later. In fact the algorithm can be performed only when these parameters satisfy certain inequalities: these will instead be declared rather soon.

In what follows let $\Sigma \subset \mathbb{R}^{m+n}$ be a $C^{3,\varepsilon_0}$ embedded submanifold of dimension $m + \bar{n}$, and $T$ an integer rectifiable current of dimension $m$ in $\Sigma$. Moreover, set $l := n - \bar{n}$. For balls in $\mathbb{R}^{m+n}$ we will use $B_r(p)$. If $\pi \subset \mathbb{R}^{m+n}$ is a subspace, $p_\pi$ will denote the orthogonal projection onto it. $B_r(q, \pi)$ is the $m$-dimensional ball in the affine plane $q + \pi$ with center $q$ and radius $r$, i.e. $B_r(q) \cap (q + \pi)$, whereas $C_r(p, \pi)$ will be used for the cylinder $\{(x + y) : x \in B_r(p), y \in \pi^\perp\}$. The points $p$ and $q$ will be omitted if they are the origin and the plane $\pi$ will be omitted if it is $\pi_0 := \mathbb{R}^m \times \{0\}$. In what follows we also assume that $\pi$ is always oriented by an $m$-vector $\vec{\pi} := v_1 \wedge \ldots \wedge v_m$ (thereby making a distinction when the same plane is given opposite orientations).

Definition 1.1. Given a current $T$, we define the excess of $T$ in balls and cylinders as

$$E(T; B_r(x), \pi) := (\omega_m r^m)^{-1} \int_{B_r(x)} |\vec{T} - \vec{\pi}|^2 d\|T\|, \quad (1.1)$$

$$E(T; C_r(x, \pi), \pi') := (\omega_m r^m)^{-1} \int_{C_r(x, \pi)} |\vec{T} - \vec{\pi}'|^2 d\|T\|, \quad (1.2)$$
and the height functions in a set $A$ as

$$h(T, A, \pi) := \sup_{x, y \in \operatorname{spt}(T) \cap A} |p_{\pi \perp}(x) - p_{\pi \perp}(y)|,$$

with $\pi \perp$ the orthogonal complement to $\pi$. The shortened notation $E(T, C_r(x, \pi))$ will be used for $E(T, C_r(x, \pi), \pi)$, which is the cylindrical excess as defined in [6] provided $(p_\pi)_2 T \cap C_r(x, \pi) = Q[B_r(p_\pi(x), \pi)]$. With a slight abuse of notation, we also write $|\pi_2 - \pi_1|$ for $|\bar{\pi}_2 - \bar{\pi}_1|$. We say that an $m$-dimensional plane $\pi$ is optimal for $T$ in a ball $B_r(x)$ if

$$E(T, B_r(x)) := \min \{ E(T, B_r(x), \tau) : \tau \text{ satisfies (1.3)} \} = h(T, B_r(x), \pi),$$

(1.3)

and

$$h(T, B_r(x)) := \min \{ h(T, B, \tau) : \tau \text{ satisfies (1.3)} \} = h(T, B_r(x), \pi).$$

In other words, $\pi$ is optimal if it minimizes the excess and, among all the minimizers of the excess, it also minimizes the height: in particular if the minimizer $\tau$ in (1.3) were unique, then the second requirement would be redundant. In any case we do not claim any uniqueness (which in general would be false). We are now ready to summarize the main assumptions of our theorems.

**Assumption 1.2.** We assume that, for each $p \in \Sigma$, $\Sigma$ is the graph of a $C^{3,\epsilon_0}$ map $\Psi_p : T_p \Sigma \to T_p \Sigma \perp$. We denote by $c(\Sigma)$ the number $\sup_{p \in \Sigma} \|D^2 \Psi_p\|_{C^{1,\epsilon_0}}$. $T$ is an $m$-dimensional integral current supported in $\Sigma$ and area minimizing in $\Sigma$ such that

$$\Theta(0, T) = Q \quad \text{and} \quad \partial T \cap B_{6\sqrt{m}} = 0,$$

(1.4)

$$\|T\|(|B_{6\sqrt{m}}|) \leq \left(\omega_m Q(6\sqrt{m})^m + \epsilon_2^2\right) \rho^m \quad \forall \rho \leq 1,$$

(1.5)

$$E(T, B_{6\sqrt{m}}) = E(T, B_{6\sqrt{m}}, \pi_0),$$

(1.6)

$$m_0 := \max \{ c(\Sigma)^2, E(T, B_{6\sqrt{m}}) \} \leq \epsilon_2^2,$$

(1.7)

where $\epsilon_2$ is a small positive constant. We also assume $\operatorname{spt}(T) \subset B_{6\sqrt{m}}$.

**Remark 1.3.** Note that (1.7) implies $A := \|A_\Sigma\|_{C^0(\Sigma)} \leq C m_0^{1/2}$, where $A_\Sigma$ denotes the second fundamental form of $\Sigma$ and $C$ is a geometric constant. Moreover, since $D\Psi_p(p) = 0$, we also infer $\|D\Psi_p\|_{C^1} \leq C m_0^{1/2}$ in the ball of radius $6\sqrt{m}$. Similarly the oscillation of $\Psi_p$ is controlled, in $B_{6\sqrt{m}}$, by $C m_0^{1/2}$.

We will sometimes parametrize $\Sigma$ as a graph of a function $\Psi$ over a plane which is not tangent to $\Sigma$ but tilted by at most $C m_0^{1/2}$. By Lemma B.1 we then still have $\|D\Psi\|_{C^{2,\epsilon_0}} \leq C m_0^{1/2}$. We show now that, without loss of generality, we can make this assumption for the plane $\mathbb{R}^{m+n} \times \{0\}$. Indeed, by (1.7) and the monotonicity formula there is a point $p \in \Sigma \cap B_{6\sqrt{m}}$ such that the distance between $\pi_0$ and its projection $p_{T_p \Sigma}(\pi_0)$ is at most $C m_0^{1/2}$. Therefore, by possibly rotating the coordinates orthogonal to $\pi_0$, we can assume that $\mathbb{R}^{m+n} \times \{0\}$ is tilted at most $C m_0^{1/2}$ compared to $T_0 \Sigma$. 


We specify next some notation which will be recurrent in the paper when dealing with cubes of \( \pi_0 \). For each \( j \in \mathbb{N} \), \( \mathcal{C}^j \) denotes the closed cubes \( L \) of \( \pi_0 \) of the form
\[
[a_1, a_1 + 2\ell] \times \ldots \times [a_m, a_m + 2\ell] \times \{0\} \subset \pi_0 \times \pi_0^d,
\] where \( 2\ell = 2^{1-j} =: 2\ell(L) \) is the side-length of the cube, \( a_i \in 2^{1-j}\mathbb{Z} \) \( \forall i \) and we require in addition \( L \subset [-4,4]^m \). The center of \( L \) is \( x_L = (a_1 + \ell, \ldots, a_m + \ell) \) and we write \( \mathcal{C} = \bigcup_{j \in \mathbb{N}} \mathcal{C}^j \). In what follows, to avoid cumbersome notation, we will usually drop the factor \( \{0\} \) in (1.8). If \( H \in \mathcal{C}^j \), we call father of \( H \) the unique cube \( L \in \mathcal{C}^{j-1} \) which contains it and likewise we say that \( H \) is a son of \( L \). In general, if \( H \) and \( L \) are two cubes in \( \mathcal{C} \) with \( H \subset L \), then we call \( L \) an ancestor of \( H \) and \( H \) a descendant of \( L \).

**Definition 1.4.** A Whitney decomposition of \( [-4,4]^m \subset \pi_0 \) consists of a closed set \( \Gamma \subset [-4,4]^m \) and a family \( \mathcal{W} \subset \mathcal{C} \) of dyadic cubes satisfying the following properties:

1. \( \Gamma \cup \bigcup_{L \in \mathcal{W}} L = [-4,4]^m \) and \( \Gamma \) does not intersect any element of \( \mathcal{W} \);
2. the interiors of any pair of distinct cubes \( L_1, L_2 \in \mathcal{W} \) are disjoint;
3. if \( L_1, L_2 \in \mathcal{W} \) have nonempty intersection, then \( \frac{1}{2} \ell(L_1) \leq \ell(L_2) \leq 2 \ell(L_1) \).

Observe that (w1) - (w3) imply dist(\( \Gamma, L \)) \( \geq 2\ell(L) \) for every \( L \in \mathcal{W} \). However, we do not require any inequality of the form dist(\( \Gamma, L \)) \( \leq C\ell(L) \), although this would be customary for what is commonly called Whitney decomposition in the literature.

1.1. The Whitney decomposition. For every \( L \in \mathcal{C} \), set \( r_L := M_0\sqrt{m}\ell(L) \) and choose a ball \( B_L \) of radius \( 64r_L \), where the constant \( M_0 \) will be specified later, and center \( p_L := (x_L, y_L) \in \text{spt}(T) \). The existence of \( p_L \) is in fact guaranteed by Assumption 1.2 as proved in Proposition 1.7 below. However the point is not unique and we fix an arbitrary choice: it turns out that such arbitrariness does not play any role for our arguments. We are now ready to introduce the main parameters of the construction.

**Assumption 1.5.** \( C, C_k, \beta_2, \delta_2, M_0 \) are positive constants and \( N_0 \) a natural number for which we assume always
\[
\beta_2 = 4\delta_2 = \min \left\{ \frac{1}{2m}, \frac{\gamma_1}{100} \right\}, \quad \text{where } \gamma_1 \text{ is the constant of [6, Theorem 1.4]},
\]
\[
M_0 \geq C(m, n, Q), \quad 2^{-N_0-1} < \frac{1}{16\sqrt{m}} \quad \text{and} \quad M_0 2^{7-N_0} \leq 1.
\]

**Remark 1.6** (Choice of the parameters). Along the various statements of the main propositions, we will make several further assumptions upon the parameters involved. However, we stress that their choice is consistent and finally made in the following order:

- \( m, n, \bar{n}, Q \) and \( \varepsilon_0 \) are given: the dependence of the constants upon these parameters will usually not be mentioned;
- \( \beta_2 \) and \( \delta_2 \), as already seen, have an explicit dependence upon \( \gamma_1 \);
- \( M_0 \) is chosen large enough, so to fulfill several inequalities and, more importantly, the assumption of Proposition 3.3, which depends only on \( \delta_2 \);
- \( N_0 \) is chosen so to satisfy (1.10) (and hence depends on \( M_0 \)) but also so large that Proposition 3.6 holds;
• $C_\varepsilon$ is then chosen, depending upon all the previous parameters, so large that the statements of Proposition 1.7 and Theorem 1.12 hold;
• $C_h$ is chosen large enough, depending also on $C_\varepsilon$, in particular such that Propositions 1.7 and 3.1 hold;
• $\varepsilon_2$ is the last to be chosen and will have to satisfy several smallness conditions depending upon all the other parameters.

Next we identify five families of cubes $\mathcal{I}$ and $\mathcal{W} = \mathcal{W}_e \cup \mathcal{W}_h \cup \mathcal{W}_n$, using the convention that $\mathcal{I}^j = \mathcal{I} \cap \mathcal{I}^j$ and an analogous one for all the other families. We start with $j = N_0$ and set $\mathcal{W}^{N_0-1} := \emptyset$. A cube $L \in \mathcal{I}^j$ belongs to

(EX) $\mathcal{W}_e^j$ if $E(T, B_L) > C_\varepsilon m_0 \ell(L)^{2-2\delta_2}$;

(HT) $\mathcal{W}_h^j$ if $L \not\in \mathcal{W}_e^j$ and $h(T, B_L) > C_h m_0^{1/2m} \ell(L)^{1+\beta_2}$;

(NN) $\mathcal{W}_n^j$ if $L \not\in \mathcal{W}_e^j \cup \mathcal{W}_h^j$ but it intersects an element of $\mathcal{W}^{j-1}$;

(S) $\mathcal{I}^j$ if none of the above occurs.

We then subdivide each element of $\mathcal{I}^j$ into $2^m$ cubes of equal side-length, clearly belonging to $\mathcal{I}^{j+1}$, and we iterate the selection procedure explained above for each of them. Finally set

$$\Gamma := [-4, 4]^m \setminus \bigcup_{L \in \mathcal{W}} L = \bigcap_{j \geq N_0} \bigcup_{L \in \mathcal{I}^j} L.$$

**Proposition 1.7** (Whitney decomposition). Let Assumptions 1.2 and 1.5 hold. If $\varepsilon_2$ is sufficiently small, the balls $B_L$ are well-defined and $(\Gamma, \mathcal{W})$ is a Whitney decomposition of $\pi_0$. Moreover, for any fixed $M_0$ and $N_0$, there is $C^* := C^*(M_0, N_0)$ such that, if $C_h \geq C^* C_\varepsilon \geq (C^*)^2$, then $\mathcal{W}^j = \emptyset$ for all $j \leq N_0 + 6$. As a consequence, under such assumption the following estimates hold:

$$E(T, B_J) \leq C_\varepsilon m_0 \ell(J)^{2-2\delta_2} \quad \text{and} \quad h(T, B_J) \leq C_h m_0^{1/2m} \ell(J)^{1+\beta_2} \quad \forall J \in \mathcal{I},$$

$$E(T, B_L) \leq C_\varepsilon m_0 \ell(L)^{2-2\delta_2} \quad \text{and} \quad h(T, B_L) \leq C_h m_0^{1/2m} \ell(L)^{1+\beta_2} \quad \forall L \in \mathcal{W},$$

where the latter constant $C$ depends only on $M_0, N_0, \beta_2, \delta_2, C_\varepsilon, C_h$.

1.2. **Construction algorithm.** We fix next two important functions $\vartheta, \varrho : \mathbb{R}^m \to \mathbb{R}$, and set, as usual, $\varrho_\lambda := \lambda^{-m} \varrho(\frac{x}{\lambda})$.

**Assumption 1.8.** $\varrho \in C^\infty_c(B_1)$ is radial, $\int \varrho = 1$ and $\int |x|^2 \varrho(x) \, dx = 0$.

$\vartheta \in C^\infty_c([-\frac{17}{16}, \frac{17}{16}]^m, [0, 1])$ is identically 1 on $[-1, 1]^m$.

**Definition 1.9** ($\pi$-approximations). Fix next any $L \in \mathcal{I}^j \cup \mathcal{W}^j$ and an $m$-plane $\pi$.

(a1) If [6, Theorem 1.4] can be applied to $T$ in the cylinder $C_{32r_L}(p_L, \pi)$, we then call the resulting map $f : B_{32r_L}(p_L, \pi) \to \mathcal{A}_Q(\pi^\perp)$ the $\pi$-approximation of $T$ in $C_{32r_L}(p_L, \pi)$.

(a2) The map $h : B_{7r_L}(p_L, \pi) \to \pi^\perp$ given by $h := (\eta \circ f) * \varrho_\ell(L)$ will be called the smoothed average of the $\pi$-approximation.

**Definition 1.10** (Interpolating functions). For each $L$ as in Definition 1.9, we select an optimal plane $\hat{\pi}_L$. We then denote by $\pi_L$ a $m$-dimensional plane contained in $T_{p_L} \Sigma$ which minimizes $|\hat{\pi}_L - \pi_L|$. The $\pi_L$-approximation (if it exists) is denoted by $f_L$. If $h$ is its
The constant $\Phi$ setting associate a Whitney region manifold of Definition 1.13 (Whitney regions).

We denote by $\bar{\Psi}$ smoothed average and $\bar{h} := \text{pr}_{\bar{\Psi}}(h)$, then the map $x \mapsto h_L(x) := \bar{\Psi}_{p_L}(x, \bar{h}(x))$ is called the tilted interpolating function relative to $L$. Moreover, if there is a map $g_L : B_{4r_L}(p_L, \bar{\pi}_0) \to \pi_0^+$ such that $G_{g_L} = G_{p_L} \subset C_{4r_L}(p_L, \bar{\pi}_0)$ (notation as in [5]), then $g_L$ is the interpolating function relative to $L$.

**Definition 1.11** (Glued interpolations). For each $j$ consider the family of cubes $\mathcal{P}_j := \mathcal{P}^j \cup \bigcup_{i=0}^{j} \mathcal{W}^i$. For each $L \in \mathcal{P}_j$, define $\vartheta_L(y) := \vartheta(\frac{y - \bar{L}}{L})$ and set

$$\hat{\varphi}_j := \sum_{L \in \mathcal{P}_j} \frac{\vartheta_L g_L}{\sum_{L \in \mathcal{P}_j} \vartheta_L}.$$  \hspace{1cm} (1.13)

We denote by $\hat{\varphi}_j(y)$ the first $\bar{n}$ components of $\hat{\varphi}_j(y)$ and set $\varphi_j(y) = (\hat{\varphi}_j(y), \Psi(y, \hat{\varphi}_j(y)))$. This latter map will be called glued interpolation.

**Theorem 1.12** (Existence of the center manifold). Assume that the hypotheses of Proposition 1.7, Assumptions 1.2 and 1.5 hold. Let $\kappa := \min\{\varepsilon_0/2, \beta_2/4\}$. If $\varepsilon_2$ is sufficiently small (depending on all the other parameters), then

(i) $\varphi_j$ is well-defined, $\|D\varphi_j\|_{C^{2,\kappa}} \leq Cm_0^{1/2}$ and $\|\varphi_j\|_{C^0} \leq Cm_0^{1/2m}$.

(ii) $\{\varphi_j\}$ is a stabilizing sequence: i.e., if $L \in \mathcal{W}^i$ and $H$ is a cube concentric to $L$ with $\ell(H) = \frac{3}{4}\ell(L)$, then $\varphi_j = \varphi_k$ on $H$ for any $j, k \geq i + 2$.

(iii) $\varphi_j$ converges to a map $\varphi$ and $\mathcal{M} := \text{Gr}(\varphi)$ is a $C^{3,\kappa}$ submanifold of $\Sigma$.

The constant $C$ in (i) depends only upon $M_0, N_0, \beta_2, \delta_2, \gamma_1, C_{\varepsilon}$ and $C_h$.

**Definition 1.13** (Whitney regions). The manifold $\mathcal{M}$ in Theorem 1.12 is called a center manifold of $T$ relative to $\pi_0$ and $(\Gamma, \mathcal{W})$ the Whitney decomposition associated to $\mathcal{M}$. Setting $\tilde{\Phi}(y) := (y, \varphi(y))$, we call $\tilde{\Phi}(\Gamma)$ the contact set. Moreover, to each $L \in \mathcal{W}$ we associate a Whitney region $\mathcal{L}$ on $\mathcal{M}$ as follows:

(WR) $\mathcal{L} := \tilde{\Phi}(H \cap [-\frac{7}{2}, \frac{7}{2}]^m)$, where $H$ is the cube concentric to $L$ with $\ell(H) = \frac{15}{16}\ell(L)$.

2. The $\mathcal{M}$-normal approximation and related estimates

In what follows we assume that the hypotheses of Theorem 1.12 are fulfilled. In order to simplify the notation, for any $V \subset \mathcal{M}$ we will denote by $|V|$ its $\mathcal{H}^m$-measure and we write $\int_V f$ for the integration with respect to $\mathcal{H}^m$. $B_r(q)$ denotes the geodesic balls in $\mathcal{M}$. Moreover, we refer to [5] for all the relevant notation pertaining the differentiation of (multiple valued) maps defined on $\mathcal{M}$, induced currents, differential-geometric tensors and so on.

**Assumption 2.1**. We fix the following notation and assumptions.

(U) $U := \{x \in \mathbb{R}^{m+n} : \exists! y \in \mathcal{M}$ with $|x - y| < 1$ and $(x - y) \perp \mathcal{M}\}$.

(P) $p : U \to \mathcal{M}$ is the map such that $p(x)$ is the point $y$ in (U).

(R) Having fixed all the other parameters, we assume $\varepsilon_2$ to be so small that $p$ extends to $C^{2,\kappa}(\bar{U})$ and $p^{-1}(y) = y + B_1(0, \pi_y)$ for every $y \in \mathcal{M}$, where $\pi_y$ is the $n$-plane perpendicular to $T_y\mathcal{M}$.

(L) We denote by $\partial_l U := p^{-1}(\partial \mathcal{M})$ the lateral boundary of $U$. 
Theorem 2.4

(Local estimates for the normal approximation.

Moreover, for every the following estimates hold on every Whitney region $L$

Under the hypotheses of Theorem 1.12 and of Assumption 2.1, it holds:

Corollary 2.2. Under the hypotheses of Theorem 1.12 and of Assumption 2.1, it holds:

(i) $\text{spt}(\partial (T\mathcal{L} U)) \subset \partial U$, $\text{spt}(T([-\frac 72, \frac 72]^m \times \mathbb{R}^n)) \subset U$ and $p_2(T\mathcal{L} U) = Q[\mathcal{M}]$;

(ii) $\text{spt}((T, p, \Phi(q))) \subset \{ y : |\Phi(q) - y| \leq C m_0^{1/2m} \ell(L)^{1+\beta_2} \}$ for every $q \in L \in \mathscr{W}$;

(iii) $(T, p, p) = Q[p]$ for every $p \in \Phi(T)$.

The main goal of this paper is to couple the center manifold of Theorem 1.12 with a good approximating map defined on it.

Definition 2.3 ($\mathcal{M}$-normal approximation). An $\mathcal{M}$-normal approximation of $T$ is given by a pair $(\mathcal{K}, F)$ such that

(A1) $F : \mathcal{M} \to \mathcal{A}_Q(U)$ is Lipschitz and takes the special form $F(x) = \sum_i \|x + N_i(x)\|$, with $N_i(x) \perp T_x \mathcal{M}$ and $x + N_i(x) \in \Sigma$ for every $x$ and $i$.

(A2) $\mathcal{K} \subset \mathcal{M}$ is closed, contains $\Phi(T \cap [-\frac 72, \frac 72]^m)$ and $T_F \perp \mathcal{P}^{-1}(\mathcal{K}) = T \perp \mathcal{P}^{-1}(\mathcal{K})$.

The map $N = \sum_i \|N_i\| : \mathcal{M} \to \mathcal{A}_Q(U)$ is the normal part of $F$.

We are now ready to state the key theorem which establishes the existence of an $\mathcal{M}$-normal approximation.

Theorem 2.4 (Local estimates for the $\mathcal{M}$-normal approximation). Let $\gamma_2 := \frac{\gamma_1}{4}$, with $\gamma_1$ the constant of [6, Theorem 1.4]. Under the hypotheses of Theorem 1.12 and Assumption 2.1, if $\varepsilon_2$ is suitably small, then there is an $\mathcal{M}$-normal approximation $(\mathcal{K}, F)$ such that the following estimates hold on every Whitney region $\mathcal{L}$ associated to a cube $L \in \mathscr{W}$:

$$\text{Lip}(N|_{\mathcal{L}}) \leq C m_0^{\gamma_2} \ell(L)^{\gamma_2} \quad \text{and} \quad \|N|_{\mathcal{L}}\|_{C^0} \leq C m_0^{1/2m} \ell(L)^{1+\beta_2},$$

$$|\mathcal{L} \setminus \mathcal{K}| + \|T_F - T\|((\mathcal{P}^{-1}(\mathcal{L})) \leq C m_0^{1+\gamma_2} \ell(L)^{m+2+\gamma_2},$$

$$\int_{\mathcal{L}} |D N|^2 \leq C m_0^{\ell(L)^{m+2-2\delta_2}}. \quad (2.3)$$

Moreover, for every $a > 0$ and every Borel $\mathcal{V} \subset \mathcal{L}$, it holds

$$\int_{\mathcal{V}} |\eta \circ N| \leq C m_0 \left( \ell(L)^{3+\beta_2/3} + a \ell(L)^{2+\gamma_2} \right) |\mathcal{V}| + \frac{C}{a} \int_{\mathcal{V}} \mathcal{G}(N, Q[\eta \circ N])^{2+\gamma_2}. \quad (2.4)$$

The constant $C$ depends on all the parameters introduced so far except $\varepsilon_2$.

Remark 2.5 (Global estimates). As a simple consequence of (2.1) - (2.3) and the structure of the Whitney decomposition, if we denote by $\mathcal{M}'$ the domain $\Phi([-\frac 72, \frac 72]^m)$, then we obtain the following global estimates

$$\text{Lip}(N|_{\mathcal{M}'}) \leq C m_0^{\gamma_2} \quad \text{and} \quad \|N|_{\mathcal{M}'}\|_{C^0} \leq C m_0^{1/2m},$$

$$|\mathcal{M}' \setminus \mathcal{K}| + \|T_F - T\|((\mathcal{P}^{-1}(\mathcal{M}'))) \leq C m_0^{1+\gamma_2},$$

$$\int_{\mathcal{M}'} |D N|^2 \leq C m_0. \quad (2.7)$$
3. ADDITIONAL CONCLUSIONS UPON $\mathcal{M}$ AND THE $\mathcal{M}$-NORMAL APPROXIMATION

3.1. Height bound and separation. We now analyze more in detail the consequences of the various stopping conditions for the cubes in $\mathcal{W}$. We first deal with $L \in \mathcal{W}_h$.

**Proposition 3.1** (Separation). There is a dimensional constant $C^2 > 0$ with the following property. Assume the parameters $\beta_2, \delta_2, N_0, M_0, C_e$ and $C_h$ fulfill the assumptions of Theorems 1.12 and 2.4, and in addition $C_h^{2m} \geq C^2 C_e$. If $\varepsilon_2$ is sufficiently small, then the following conclusions hold for every $L \in \mathcal{W}_h$:

1. $\Theta(T, p) \leq Q - \frac{1}{2}$ for every $p \in B_{16r_L}(p_L)$.
2. $L \cap H = \emptyset$ for every $H \in \mathcal{W}_n$ with $\ell(H) \leq \frac{1}{2} \ell(L)$;
3. $\mathcal{G}(N(x), Q[\eta \circ N(x)]) \geq \frac{1}{4} C_h m_0^{1/2m} \ell(L)^{1+\beta_2}$ for every $x \in \mathcal{L}$.

A simple corollary of the previous proposition is the following.

**Corollary 3.2** (Domains of influence). The cubes in $\mathcal{W}_n$ can be partitioned in disjoint families $\mathcal{W}_n(L)$, where $L$ is running in $\mathcal{W}_e$. More precisely, for such $H$ and $L$ there is a chain $L = L_0, L_1, \ldots, L_s$ such that $L_i \in \mathcal{W}_n, L_i \cap L_{i-1} \neq \emptyset, \ell(L_i) = 2 \ell(L_{i-1})$ and $L_s \cap H \neq \emptyset$. In particular, $H \subset B_{3\sqrt{m}(L)(x_L)}$ for every $H \in \mathcal{W}_n(L)$.

3.2. Splitting before tilting I. The following is the main consequence of the splitting before tilting phenomenon.

**Proposition 3.3.** (Splitting I) Assume the parameters satisfy the assumptions of Theorems 1.12 and 2.4. If $M_0 \geq C(\delta_2)$ and $\varepsilon_2$ is chosen sufficiently small, then the following estimates hold for every $L \in \mathcal{W}_e$ ($\mathcal{L}$ its associated Whitney region) and every set $\Omega := \Phi(B_{\ell(L)/4}(q, \pi_0))$, where $q \in \pi_0$ is an arbitrary point with $\text{dist}(L, q) \leq 4 \sqrt{m} \ell(L)$:

$$C_e m_0 \ell(L)^{m+2-2\beta_2} \leq \ell(L)^m \mathbf{E}(T, B_L) \leq C \int_{\Omega} |DN|^2, \quad (3.1)$$

$$\int_{\mathcal{L}} |DN|^2 \leq C \ell(L)^m \mathbf{E}(T, B_L) \leq C \ell(L)^{-2} \int_{\Omega} |N|^2. \quad (3.2)$$

The constant $C$ depends on all the parameters involved except $\varepsilon_2$.

3.3. Persistence of $Q$ points. We next state two important properties triggered by the existence of $p \in \text{spt}(T)$ with $\Theta(p, T) = Q$, both related to the splitting before tilting.

**Proposition 3.4.** (Splitting II) Assume the parameters satisfy the assumptions of Theorem 1.12. For every $\alpha, \bar{\alpha}, \hat{\alpha} > 0$, there exists $\varepsilon_3 > 0$ (depending upon all the parameters $\beta_2, \delta_2, M_0, N_0, C_e, C_h$ and also $\alpha, \bar{\alpha}$ and $\hat{\alpha}$) such that the following holds. Assume $s \in ]0, 1[$,

$$\sup \left\{ \ell(L) : L \in \mathcal{W}, L \cap B_{3s}(0, \pi_0) \neq \emptyset \right\} \leq s, \quad (3.3)$$

$$H^{m-2+\alpha}_\infty(\{\Theta(T, \cdot) = Q\} \cap B_{3s}) \geq \bar{\alpha}s^{m-2+\alpha}, \quad (3.4)$$

and $\min \{s, m_0\} \leq \varepsilon_3$. Then,

$$\sup \left\{ \ell(L) : L \in \mathcal{W}_e \text{ and } L \cap B_{3s}(0, \pi_0) \neq \emptyset \right\} \leq \hat{\alpha}s.$$
Proposition 3.5. (Persistence of Q-points) Assume the parameters satisfy the hypotheses of the Theorems 1.12 and 2.4. For every \( \eta_2 > 0 \) there is \( s, \ell > 0 \), with the following property. If \( L \in \mathcal{W}_\varepsilon, \ell(L) \leq \ell, \Theta(T, p) = Q \) and \( \text{dist}(p_{\pi_0}(p), L) \leq 4\sqrt{m} \ell(L) \), then
\[
\int_{B_\ell(L)(p(p))} \mathcal{G}^2(N, Q[\eta \circ N]) \leq \frac{\eta_2}{\omega m \ell^2} \int_{B_\ell(L)(p(p))} |DN|^2.
\]

3.4. Comparison between different center manifolds. We list here a final key property of center manifolds and \( \mathcal{M} \)-normal approximations. Once again this is also a consequence of the splitting before tilting phenomenon. In what follows we use the notation \( \iota_{0, r} \) for the map \( z \mapsto \frac{z}{r} \).

Proposition 3.6 (Comparing center manifolds). Assume the parameters satisfy the hypotheses of Theorems 1.12, 2.4 and 3.3. If \( N_0 \) is larger than a geometric constant, there is \( c_s > 0 \) (which depends on all the parameters of Assumption 1.5 except for \( \varepsilon_2 \)) with the following property. If \( \varepsilon_2 \) is sufficiently small, \( c_s := \frac{1}{16\sqrt{m}} \) and \( r \in [0, 1] \) is a radius such that:

(a) \( \ell(L) \leq c_s r \) for every \( \rho > r \) and every \( L \in \mathcal{W}_\varepsilon \) with \( L \cap B_\rho(0, \pi_0) \neq \emptyset \);

(b) \( E(T, B_{6\sqrt{m}}) < \varepsilon_2 \) for every \( \rho > r \);

(c) there is \( L \in \mathcal{W}_\varepsilon \) such that \( \ell(L) \geq c_s r \) and \( L \cap B_r(0, \pi_0) \neq \emptyset \);

then

(i) the current \( T' := (\iota_{0, r})_\#T \lVert B_{6\sqrt{m}} \) and the submanifold \( \Sigma' := \iota_{0, r}(\Sigma) \) satisfy the assumptions of Theorems 1.12 and 2.4 for some plane \( \pi \) in place of \( \pi_0 \);

(ii) for the center manifold \( \mathcal{M}' \) of \( T' \) relative to \( \pi \) and the \( \mathcal{M}' \)-normal approximation \( N' \) as in Theorem 2.4, it holds
\[
\int_{\mathcal{M}' \cap B_2} |N'|^2 \geq c_s \max \left\{ E(T, B_{6\sqrt{m}}), c(\Sigma')^2 \right\}.
\]

4. Center manifold’s construction

In this section we lay down the technical preliminaries to prove Theorem 1.12, state the related fundamental estimates and show how the theorem follows from them.

4.1. Consistency of the construction algorithm. The main preliminary technical work consists in ensuring that all the building blocks of the construction algorithm are in fact well-defined. Along the way we also conclude some pieces of information which settle some of the estimates in Theorem 2.4.

Lemma 4.1 (Well-definition of \( B_L \) and \( \mathcal{W} \)). Let Assumptions 1.2 and 1.5 hold. If \( \varepsilon_2 \) is sufficiently small, the balls \( B_L \) are well-defined and \( (T, \mathcal{W}) \) is a Whitney decomposition of \( \pi_0 \). Moreover, for any fixed \( M_0 \) and \( N_0 \) there is \( C^* := C^*(M_0, N_0) \) such that, if \( C_h \geq C^* C_e \geq (C^*)^2 \), then \( \mathcal{W}^j = \emptyset \) for all \( j \leq N_0 + 6 \).

Proof. We start noticing that, from Assumption 1.2, it follows that
\[
(p_{\pi_0})_T \circ C_{11\sqrt{m}/2} = Q \left[ B_{11\sqrt{m}/2} \right] \quad \text{and} \quad h(T, C_{5\sqrt{m}}, \pi_0) \leq C_0 m_0^{1/2m}.
\]

(4.1)
To this regard, we can argue by contradiction: if the first statement were false, we would have a sequence of currents \( T_k \) in \( B_{6\sqrt{m}} \) and of submanifolds \( \Sigma \) satisfying Assumption 1.2 with \( \varepsilon_2(k) \downarrow 0 \) and \( (p_{\pi_0})_T L C_{11\sqrt{m}/2} \neq Q [B_{11\sqrt{m}/2}] \). It is then easy to see that

\[
T_k \to T_\infty := Q [B_{6\sqrt{m}}] \quad \text{and} \quad \text{spt}(T_k) \to \text{spt}(T_\infty) \quad \text{locally in the Hausdorff sense.}
\]

Since \( \partial T_k \) vanishes in \( B_{6\sqrt{m}} \), \( T_k \subseteq C_{11\sqrt{m}/2} \) has no boundary in \( C_{11\sqrt{m}/2} \) for \( k \) large enough, thereby implying that \( (p_{\pi_0})_T L C_{11\sqrt{m}/2} = Q_k [B_{11\sqrt{m}/2}] \) for some integer \( Q_k \). Since \( T_k \to T_\infty \), we deduce that \( Q_k = Q \) for \( k \) large enough, giving the desired contradiction. The height bound now follows easily from Theorem A.1 because of \( (p_{\pi_0})_T L C_{11\sqrt{m}/2} = Q [B_{11\sqrt{m}/2}] \) and \( \Theta(T, 0) = Q \): in particular, the latter assumption together with Theorem A.1(iii) implies that there is one single open set \( S_1 \) as in Theorem A.1(i), which in turn must contains the origin.

By the slicing theory of currents (see [12, Section 28] or [8, 4.3.8]) and by (4.1), there is a set \( A \subset B_{5\sqrt{m}} \) of full measure such that, for every \( x \in A \), there exist \( k_i(x) \in \mathbb{N} \) with \( \sum_i k_i = Q \), and points \( (x, y_i(x)) \in \text{spt}(T) \) with \( |y_i(x)| \leq C_0 m_0^{1/2m} \), for which it holds

\[
\langle T, p_{\pi_0}, x \rangle = \sum_{i=1}^{N(x)} k_i(x) \delta_{(x, y_i(x))} \quad \forall x \in A.
\]

By the monotonicity formula and the density of \( \varepsilon \) with \( \varepsilon \), it is then easy to see that \( |\pi_0| \) is well-defined for every cube \( L \in \mathcal{C} \) and its center \( p_L \) satisfies \( |p_L| \leq 4\sqrt{m} + C_0 m_0^{1/2m} \).

Fix now \( L \in \mathcal{W}^j \) with \( N_0 \leq j < N_0 + 6 \). Since \( r_L \leq 2^{-7} \sqrt{m} \) by Assumption 1.5, we have that \( B_L \subset B_{5\sqrt{m}} \) if \( \varepsilon_2 \) is small enough, and

\[
E(T, B_L, \pi_0) \leq \frac{6^m}{(64M_0^2 - N_0 - 6)^m} E(T, B_{6\sqrt{m}}, \pi_0) \leq \frac{6^m}{(64M_0^2 - N_0 - 6)^m} m_0.
\]

For a suitable \( C^*(M_0, N_0) \) the inequality \( C_{e} \geq C^* \) implies

\[
E(T, B_L) \leq E(T, B_L, \pi_0) \leq C_{e} m_0 \ell(L)^{2 - 2\delta_2}.
\]

Let now \( \hat{\pi}_L \) be an optimal plane: since the center \( p_L \) of \( B_L \) belongs to \( \text{spt}(T) \), the monotonicity formula guarantees \( |T||(B_L) \geq c_0 r_L^n \) (cp. [12, Section 17] or [6, Appendix A]). We then conclude

\[
|h(T, B_L)| \leq C (|\hat{\pi}_L - \pi_0| \ell(L) + h(T, B_L, \pi_0)) \leq C C_{e}^{1/2} m_0^{1/2} \ell(L)^{2 - \delta_2} + h(T, C_{5\sqrt{m}}, \pi_0)
\]

\[
\geq C(N_0)(C_{e}^{1/2} + 1) m_0^{1/2m} \ell(L)^{1 + \delta_2}.
\]

Thus, if \( C^*(M_0, N_0) \) is chosen sufficiently large and \( C_{h} \geq C^* C_{e} \geq (C^*)^2 \), neither the condition (EX) nor (HT) apply to \( L \). Therefore, \( \mathcal{W}^{N_0} = \emptyset \). Similarly, at the successive
Recalling that \( spt(\mathcal{T}) \) also for \( H, L \) step, none of the cube in \( S^{12} \) satisfies the conditions (EX), (HT) or (NN), because \( \mathcal{W}^{N_0} = \emptyset \). Proceeding in this way, we conclude that \( \mathcal{W}^{j} = \emptyset \) for every \( j \leq N_0 + 6 \). \( \square \)

Next we show some basic estimates for the tilting of the optimal planes and the height functions for cubes in \( \mathcal{W} \cup \mathcal{S} \).

**Proposition 4.2** (Tilting of optimal planes). Having fixed the parameters \( M_0, N_0, \beta_2, \delta_2, C_e \) and \( C_h \), if \( \varepsilon_2 \) is sufficiently small, then the following holds for every \( H, L \in \mathcal{W} \cup \mathcal{S} \) with \( H \subset L \):

(i) \( B_H \subset B_L \);
(ii) \( |\bar{\pi}_L - \pi_L| \leq \bar{C}m_0^{1/2}L^{1-\delta_2} \);
(iii) \( |\bar{\pi}_H - \pi_L| \leq \bar{C}m_0^{1/2}L^{1-\delta_2} \);
(iv) \( |\pi_L - \pi_0| \leq \bar{C}m_0^{1/2} \);
(v) \( h(T, C_{36r_L}(p_L, \pi_0)) \leq Cm_0^{1/2}L^{1+\beta_2} \) and \( \text{spt}(T) \cap C_{36r_L}(p_L, \pi_0) \subset B_L \);
(vi) \( h(T, C_{36r_L}(p_L, \pi_H)) \leq Cm_0^{1/2}L^{1+\beta_2} \) and \( \text{spt}(T) \cap C_{36r_L}(p_L, \pi_H) \subset B_L \).

where \( \bar{C} \) is given by \( C_e^{1/2} \) times a geometric constant. In addition, (iii) and (vi) hold true also for \( H, L \in \mathcal{W}^{j} \cup \mathcal{S} \) with \( H \cap L \neq \emptyset \) and, in particular, Proposition 1.7 follows.

**Proof.** We prove that, for every \( i \geq N_0 \), given a sequence of cubes \( H_i \subset \ldots \subset H_{N_0} \), with \( H_j \in \mathcal{W}^{j} \cup \mathcal{S}^j \), then (i) - (vi) hold for every \( H = H_j \) and \( L = H_k \) with \( j \geq k \) and \( j, k \in \{N_0, \ldots, i\} \). We proceed by induction on \( i \). The basic step \( i = N_0 \) follows easily from Lemma 4.1 and the definition of the Whitney Decomposition. We start observing that for (i) and (iii) there is nothing to prove, since the only case is \( H = L = H_{N_0} \). Next, since \( \mathcal{W}^{N_0} = \emptyset \) by Lemma 4.1, the cube \( H_{N_0} \) does not satisfy condition (EX). Therefore, using the monotonicity formula, \( \|T\|(B_{H_{N_0}}) \geq c_0 r_{H_{N_0}}^{m} \) and there exists at least a point \( p \in \text{spt}(T) \cap B_{H_{N_0}} \) such that

\[
|\bar{T}(p) - \pi_{H_{N_0}}|^2 \leq E(T, B_{H_{N_0}}) \frac{C r_{H_{N_0}}^{m}}{\|T\|(B_{H_{N_0}})} \leq Cm_0 \ell(H_{N_0})^{2-2\delta_2}. \tag{4.3}
\]

Since \( \bar{T}(p) \) is an \( m \)-vector of \( T \Sigma \), this implies that \( |p_{T \Sigma}(\pi_{H_{N_0}}) - \pi_{H_{N_0}}| \leq Cm_0^{1/2} \ell(H_{N_0})^{1-\delta_2} \). Recalling that \( |p_{T \Sigma}(\pi_{H_{N_0}}) - p_{T \Sigma}(\pi_{H_{N_0}})| \leq Cr_{H_{N_0}} A \leq Cm_0^{1/2} \ell(H_{N_0}) \), we conclude (ii). Next, (iv) follows simply from (4.2) and (ii), while (v) from \( \text{spt}(T) \cap C_{36r_{H_{N_0}}}(p_{H_{N_0}}, \pi_0) \subset C_{5\sqrt{m}} \) and (4.1). Finally, for what concerns (vi), we notice that by (ii), (iii) and (v), if \( \varepsilon_2 \) is sufficiently small, then \( \text{spt}(T) \cap C_{36r_{H_{N_0}}}(p_{H_{N_0}}, \pi_{H_{N_0}}) \subset C_{5\sqrt{m}} \). From (4.1) it follows then that \( \text{spt}(T) \cap C_{36r_{H_{N_0}}}(p_{H_{N_0}}, \pi_{H_{N_0}}) \subset B_{H_{N_0}} \). Since \( H_{N_0} \notin \mathcal{W} \), we then conclude

\[
h(T, C_{36r_{H_{N_0}}}(p_{H_{N_0}}, \pi_{H_{N_0}}), \pi_{H_{N_0}}) \leq h(T, B_{H_{N_0}}) + C\ell(H_{N_0})|\pi_{H_{N_0}} - \pi_{H_{N_0}}| \leq Cm_0^{1/2} \ell(H_{N_0})^{1+\beta_2} + Cm_0^{1/2} \ell(H_{N_0})^{2-\delta_2}.
\]

Now we pass to the inductive step: we assume to have proved the conclusions for \( i \) and show that they hold also for \( i + 1 \). For what concerns (i), it is enough to prove that \( B_{H_{i+1}} \subset B_{H_{i}} \), because the other inclusions follows by the inductive hypothesis. To this
Therefore, using (iv) applied to $H_i$ and $|x_{H_i} - x_{H_{i+1}}| \leq \sqrt{m} \ell(H_i)$, we deduce that $|p_L - p_H| \leq C \ell(L)$ for some geometric constant $C > 0$. Therefore, if $M_0$ is taken sufficiently large, we infer $B_{H_{i+1}} \subset B_{H_i}$. We show now (ii) for $L = H_{i+1}$. Note that, by (i),

$$E(T, B_{H_{i+1}}) \leq C E(T, B_{H_i}) \leq CC_0 \ell(H_{i+1})^{2-2\delta_2},$$

(4.4)

for some geometric constant $C > 0$. Therefore, we can argue as for $H_{N_0}$ and conclude as above. For (iii) and (iv), we start considering the case $H = H_1$ and $L = H_{i-1}$, for some $l \in \{N_0, \ldots, i + 1\}$. Note that, by the inclusion in (i), we can argue again by monotonicity formula (used to estimate $\|T\|(B_{H_i})$ and $\|T\|(B_{H_{i-1}})$ from below) and infer that

$$|\hat{\pi}_{H_{i-1}} - \hat{\pi}_{H_i}|^2 \leq (E(T, B_{H_{i-1}}) + E(T, B_{H_i})) \frac{C \rho_{H_{i-1}}}{\|T\|(B_{H_i})} \leq CC_0 \ell(H_i)^{2-2\delta_2}.$$  

(4.5)

Therefore, using (ii) we conclude (iii) for generic $H$ and $L$ by the estimate $\sum_{i-j} \ell(H_i)^{1-\delta_2} \leq C \ell(H_j)^{1-\delta_2}$. As for (iv) it follows from (iii) and the case $|\pi_{H_{N_0}} - \pi_0| \leq C \ell_0^{\delta_2}$. Coming now to (v), by inductive hypothesis it is enough to show it for $H = H_{i+1}$. To this aim, notice that, by (v) for $H_i$, we conclude $\text{spt}(T) \cap C_{36r_{H_{i+1}}}(p_{H_i}, \pi_0) \subset B_{H_i}$. Next, since $|x_{H_{i+1}} - x_{H_i}| \leq \sqrt{m} \ell(H_i)$ and $r_{H_{i+1}} = \frac{1}{2} r_{H_i}$, we obviously have $C_{36r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0) \subset C_{36r_{H_i}}(p_{H_i}, \pi_0)$, provided $M_0$ is larger than a geometric constant. Thus:

$$h(T, C_{36r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0)) \leq h(T, B_{H_i}) + C r_{H_i} |\hat{\pi}_{H_i} - \pi_0|$$

(iv)

$$\leq C_0 \ell_{0}^2 \ell(H_i)^{1+\delta_2} + C \ell_0^{\delta_2} \ell(H_i) \leq C \ell_0^{\delta_2} \ell(H_i),$$

where we used $H_i \in \mathcal{H}^i$. The inclusion $\text{spt}(T) \cap C_{36r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0) \subset B_{H_{i+1}}$ is an obvious corollary of the bound and of the fact that the center of the ball $B_{H_{i+1}}$ (i.e. the point $p_{H_{i+1}}$) belongs to $\text{spt}(T) \cap C_{36r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0)$.

Next we show (vi) for $H = H_{i+1}$ and $L$ an ancestor of $H$ (included the case $L = H_{i+1}$). First we consider the case $L = H_{N_0}$. By $|\pi_{H_{N_0}} - \pi_0| \leq C \ell_0^{\delta_2}$ and a simple geometric argument, it is easy to see that, provided $\varepsilon_2$ is sufficiently small, $C_{36r_{L}}(p_{L}, \pi_{H}) \cap B_{6\sqrt{m}} \cap C_{5\sqrt{m}}$ and thus, by (4.1) we conclude

$$h(T, C_{36r_{L}}(p_{L}, \pi_{H})) \leq h(T, C_{5\sqrt{m}}, \pi_0) + C r_{L} |\pi_{H} - \pi_0| \leq C \ell_0^{\delta_2} \ell(L)^{1+\delta_2}$$

(recalling that $\ell(L) = 2^{-N_0}$!)

Otherwise we have $L = H_i$ with $l > N_0$ and we can set $J := H_{l+1}$. We have already observed that $|p_j - p_{L}| \leq C \ell(J)$ for a geometric constant. Moreover we have $|\pi_H - \pi_J| \leq C \ell_0^{\delta_2} \ell(J)^{1-\delta_2}$. If $\varepsilon_2$ is sufficiently small, a simple geometric argument shows that

$$C_{36r_{L}}(p_{L}, \pi_{H}) \cap B_{6\sqrt{m}} \subset C_{36r_{J}}(p_{J}, \pi_{J}).$$

On the other hand by inductive hypothesis we have $C_{36r_{J}}(p_{J}, \pi_{J}) \cap \text{spt}(T) \subset B_{J}$ and, since $J \not\in \mathcal{W}$, we easily conclude

$$h(T, C_{36r_{L}}(p_{L}, \pi_{H})) \leq h(T, B_{J}) + C r_{J} (|\hat{\pi}_{J} - \pi_{J}| + |\pi_{J} - \pi_{L}|) \leq C \ell_0^{\delta_2} \ell(J)^{1+\delta_2},$$

(4.1)
where the constant $C$ depends only on $\bar{C}$ and $C_h$. Since $\ell(J) = 2\ell(L)$, this concludes the proof of the bound. As above, the inclusion $\text{spt}(T) \cap C_{36r_L}(p_L, \pi_H) \subset B_L$ follows from the bound.

We pass to the last claim of the proposition. If $H, L \in \mathcal{C}^j \cup \mathcal{F}^j$ are such that $H \cap L \neq \emptyset$, then $|x_L - x_H| \leq 2\sqrt{m} \ell(H)$ and by (v) it follows that $|p_L - p_H| \leq C \ell(H)$ for some geometric constant $C > 0$. This in turn implies that $B_{36r_L}(p_L) \subset B_H \cap B_L$ and therefore, by the monotonicity formula, we conclude (iii):

$$|\hat{\pi}_H - \hat{\pi}_L|^2 \leq \left(E(T, B_H) + E(T, B_L)\right) \frac{C r_H^m}{\|T\|(B_{36r_L}(p_L))} \leq C m_L \ell(H)^{2-2\delta_2}.$$

For (vi), assume first $L \in \mathcal{C}^j$ with $j > N_0$. Note that by (iii) we can argue as above and conclude that $\text{spt}(T) \cap C_{36r_L}(p_L, \pi_H) \subset B_J$ for the father $J$ of $L$. The proof of (vi) follows the same pattern. When $L \in \mathcal{C}^{N_0}$ the argument is entirely equal to the one above and we leave it to the reader.

Thanks to Lemma 4.1, Proposition 1.7 now follows straightforwardly: indeed, (1.11) is an immediate consequence of the definition and (1.12) follows from (i) and (iii). □

Next, we prove that the building blocks for the construction of the center manifold are well-defined. For later purposes, we introduce the following notation.

**Definition 4.3.** Let $H \in \mathcal{C}^j \cup \mathcal{F}^j$ and let $L$ be either an ancestor of $H$ (including $H$ itself) or an element of $\mathcal{C}^j \cup \mathcal{F}^j$ with $H \cap L \neq \emptyset$. The $\pi_H$-approximation of $T$ in the cylinder $C_{32r_L}(p_L, \pi_H)$, derived by [6, Theorem 1.4] (if it can be applied) will be denoted by $f_{HL}$. Similarly, setting $\hat{h} := p_{r_L}((\eta \circ f_{HL}) \ast \rho_{r(L)})$, then the map $x \mapsto h_{HL}(x) := \Psi_{p_H}(x, \hat{h}(x))$ is the tilted interpolating function and, if it exists, the map $g_{HL} : B_{4r_L}(p_L, \pi_0) \to \pi_0^\perp$ such that $G_{g_{HL}} = G_{h_{HL},L} C_{4r_L}(p_L, \pi_0)$ is the interpolating function relative to the cubes $H$ and $L$.

If $H = L$, then $h_{HL} = h_H$ and $g_{HL} = g_H$ are the interpolating functions of Definition 1.10.

**Proposition 4.4** (Consistency of the center manifold algorithm). Having fixed $M_0$, $N_0$, $\beta_2$, $\delta_2$, $C_\epsilon$ and $C_h$, the following facts are true provided $\varepsilon_2$ is sufficiently small. Let $H \in \mathcal{C}^j \cup \mathcal{F}^j$ and let $L$ be either an ancestor of $H$ (including $H$ itself) or an element of $\mathcal{C}^j \cup \mathcal{F}^j$ with $H \cap L \neq \emptyset$. Then,

(i) $(p_{\pi_H})_{(T \subset C_{34r_L}(p_L, \pi_H))} = Q \left[B_{34r_L}(p_{\pi_H}(p_L), \pi_H)\right]$;

(ii) the $\pi_H$-approximation $f_{HL}$ and the interpolating function $g_{HL}$ are well-defined.

**Proof.** To prove (i), we join $\pi_H = (1, \pi_0)$ with a continuous one-parameter family of planes $\pi(t)$ with the property that $|\pi(t) - \pi_0| \leq C|\pi_H - \pi_0|$, with $C > 0$ some geometric constant. If $\varepsilon_2$ is suitably small, by Proposition 4.2 we have $\text{spt}(T) \cap C_{34r_L}(p_L, \pi_t) \subset C_{36r_L}(p_L, \pi_0)$ for every $t \in [0,1]$. We consider then the currents $S(t) := (p_{\pi(t)})_{(T \subset C_{34r_L}(p_L, \pi(t))}$ and note that $S(t) = Q(t) \left[B_{34r_L}(p_{\pi(t)}(p_L), \pi(t))\right]$, where $Q(t)$ is an integer for every $t$ by the Constancy Theorem. On the other hand $t \mapsto S(t)$ is weakly continuous in the space of currents and thus $Q(t)$ must be constant. Since $Q(0) = Q$ by Lemma 4.1, this proves the desired claim.
For what concerns (ii), by Proposition 4.2 it follows that, for $\varepsilon_2$ smaller than a geometric constant,

$$\text{spt}(T \mathcal{L} C_{32r_L}(p_L, \pi_H)) \subset B_L \subset B_{r_L/2}.$$  \hspace{1cm} (4.6)

We then conclude

$$E(T, C_{32r_L}(p_L, \pi_H)) \leq CE(T, B_L, \pi_H) \leq CE(T, B_L) + C|\pi_H - \hat{\pi}_L|^2 \leq Cm_0 \ell(L)^{2-2\delta_2}.$$  

If $\varepsilon_2$ is sufficiently small, then $E(T, C_{32r_L}(p_L, \pi_H)) < \varepsilon_1$, where $\varepsilon_1$ is the constant of [6, Theorem 1.4]. On the other hand (4.6) implies also that $\partial(T \mathcal{L} C_{32r_L}(p_L, \pi_H))$ vanishes in $C_{32r_L}(p_L, \pi_H)$. Therefore, the current $T \mathcal{L} C_{32r_L}(p_L, \pi_H)$ and the submanifold $\Sigma$ satisfy all the assumptions of [6, Theorem 1.4] in the cylinder $C_{32r_L}(p_L, \pi_H)$ and therefore the approximation $f_{HL}$ is well-defined. By [6, Theorem 1.4] and the properties of $\Psi_{p_H}$, we have

$$\text{Lip}(h_{HL}) \leq CL\text{Lip}(\eta \circ f_{HL}) \leq C \left(E(T \mathcal{L} C_{32r_L}(p_L, \pi_H))\right)^{71} \leq Cm_0^{71} \ell(L)^{71},$$

and

$$|h_{HL} - p_{x_{HL}}(p_L)| \leq C|\eta \circ f_{HL} - p_{x_{HL}}(p_L)| \leq CG(f_{HL}, Q[p_{x_{HL}}(p_L)])$$

$$\leq C(h(T, C_{32r_L}(p_L, \pi_H)) + (E(T, C_{32r_L}(p_L, \pi_H)))^{1/2} + A\ell_L)r_L$$

$$\leq Cm_0^{1/2m}r_L,$$

where the constant $C$ does not depend on $\varepsilon_2$. If $\varepsilon_2$ is then smaller than a suitable constant, we can apply Lemma B.1 to conclude that the interpolating function $g_{HL}$ is well-defined.

\[\square\]

4.2. Key estimates and proof of Theorem 1.12. We are now ready to state the key construction estimates and show how Theorem 1.12 follows easily from them.

Proposition 4.5 (Construction estimates). Assume the parameters satisfy the requirements of Lemma 4.1 and $\kappa = \min\{\beta_2/4, \varepsilon_0/2\}$. Then, the following holds for any pair of cubes $H, L \in \mathcal{P}^j$:

(i) $\|g_H\|_{C^0} \leq Cm_0^{1/2m}$ and $\|Dg_H\|_{C^{2,\kappa}} \leq Cm_0^{1/2}$;

(ii) if $H \cap L \neq \emptyset$, then $\|g_H - g_L\|_{C(B_{r_L}(x_L))} \leq Cm_0^{1/2}\ell(H)^{3+\kappa-\delta_2}$ for every $i \in \{0, \ldots, 3\}$;

(iii) $|D^3g_H(x_H) - D^3g_L(x_L)| \leq Cm_0^{1/2}|x_H - x_L|^{\kappa}$;

(iv) $\|g_H - y_H\|_{C^0} \leq Cm_0^{1/2m}\ell(H)$ and $|\pi_H - T_{(x_H, y_H)}(g_{gh})| \leq Cm_0^{1/2}\ell(H)^{1-\delta_2}$ for all $x \in H$;

(v) if $L'$ is the cube concentric to $L \in \mathcal{P}^j$ with $\ell(L') = \frac{9}{8}\ell(L)$, then

$$\|\varphi_i - g_L\|_{L^1(L')} \leq Cm_0\ell(L)^{m+3+\beta_2/3} \text{ for all } i \geq j.$$  

The constant $C$ depends upon $\beta_2, \delta_2, M_0, N_0, C_e$ and $C_h$ but not on $\varepsilon_2$.

Using the estimates in the above proposition, we can prove the main existence result for the center manifold.
Proof of Theorem 1.12. The well-definition of the glued interpolations \( \varphi_j \) follows from Proposition 4.4. Define \( \chi_H := \vartheta_H / (\sum_{L \in \mathcal{P}_j} \partial_L) \) and observe that
\[
\sum_i \chi_H = 1 \quad \text{and} \quad \| \chi_H \|_{C^0} \leq C(i, m, n) \ell(H)^{-i} \quad \forall i \in \mathbb{N}.
\]
(4.7)
Set \( \mathcal{P}_j(H) := \{ L \in \mathcal{P}_j : L \cap H \neq \emptyset \} \setminus \{ H \} \). By construction \( \ell(H) \leq 2 \ell(L) \) for every \( L \in \mathcal{P}_j \) and the cardinality of \( \mathcal{P}_j(H) \) is bounded by a constant \( C(m) \). The estimate \( |\varphi_j| \leq Cm_0^{1/2m} \) follows then easily from Proposition 4.5(i). For \( i \in \{ 1, \ldots, 3 \} \) and \( x \in H \), write
\[
D^i \varphi_j(x) = D^i \left( g_H \chi_H + \sum_{L \in \mathcal{P}_j(H)} g_L \chi_L \right)(x) = D^i g_H(x) + D^i \sum_{L \in \mathcal{P}_j(H)} (g_L - g_H) \chi_L(x). \tag{4.8}
\]
Using the Leibnitz rule, (4.7) and the estimates of Proposition 4.5(i) - (ii), we get
\[
\| D^i \varphi_j \|_{C^0(H)} \leq \| g_H \|_{C^0} + \sum_{0 \leq l \leq i} \sum_{L \in \mathcal{P}_j(H)} \| g_L - g_H \|_{C^l(\text{spt}(\chi_L))} \| \ell(L) \|_{1-i} \leq Cm_0^{1/2} (1 + \ell(H)^{3+\kappa-i}),
\]
(assuming \( M_0 \) is larger than the geometric constant \( 2\sqrt{m} \), we have \( \text{spt}(\chi_L) \subset B_{r_1}(x_L) \), implying that Proposition 4.5(ii) can be applied). On the other hand observe that, by interpolation between \( \| g_H - g_L \|_{C^0} \leq Cm_0^{1/2} \ell(H)^{3+\kappa} \) and \( \| D^3 g_H - D^3 g_L \|_{C^0} \leq Cm_0^{1/2} \), we obtain
\[
\| g_H - g_L \|_{C^{j,\kappa}} \leq Cm_0^{1/2} \ell(H)^{3+\kappa-j-\kappa} \quad \text{for any} \quad j + \kappa \leq 3 + \kappa.
\]
Thus,
\[
[D^3 \varphi_j]_{\kappa, H} \leq \sum_{0 \leq l \leq 3} \sum_{L \in \mathcal{P}_j(H)} \ell(H)^{l-3} \| D^l (g_L - g_H) \|_{C^0(H)} + [D^l (g_L - g_H)]_{\kappa, H} \\
+ [D^3 g_H]_{\kappa, H} \leq Cm_0^{1/2}.
\]
Fix now \( x, y \in [-4, 4]^m \), let \( H, L \in \mathcal{P}_j \) be such that \( x \in H \) and \( y \in L \). If \( H \cap L \neq \emptyset \), then
\[
|D^3 \varphi_j(x) - D^3 \varphi_j(y)| \leq C([D^3 \varphi_j]_{\kappa, H} + [D^3 \varphi_j]_{\kappa, L}) |x - y|^\kappa. \tag{4.9}
\]
If \( H \cap L = \emptyset \), say \( \ell(H) \leq \ell(L) \), then
\[
\max \{|x - x_H|, |y - x_L|\} \leq \ell(L) \leq 2|x - y|.
\]
Moreover, by construction \( \varphi_j \) is identically equal to \( g_H \) in a neighborhood of its center \( x_H \). Thus, we can estimate
\[
|D^3 \varphi_j(x) - D^3 \varphi_j(y)| \leq |D^3 \varphi_j(x) - D^3 \varphi_j(x_H)| + |D^3 g_H(x_H) - D^3 g_L(x_L) + |D^3 \varphi_j(x_L) - D^3 \varphi_j(y)|| \\
\leq Cm_0^{1/2} (|x - x_H|^\kappa + |x_H - x_L|^\kappa + |y - x_L|^\kappa) \leq Cm_0^{1/2} |x - y|^\kappa, \tag{4.10}
\]
where we used (4.9) and Proposition 4.5(iii).
We have then proved that \( |D^3 \varphi_j|_{C^2, \kappa} \leq Cm_0^{1/2} \). Since \( \varphi_j(x) = (\varphi_j(x), \Psi(x, \varphi_j(x))) \), where \( \varphi_j(x) \) denote the first \( \bar{n} \) components of \( \varphi_j(x) \), Theorem 1.12(i) follows easily from the chain rule.
Let $L \in \mathcal{W}^j$ and fix $j \geq i + 2$. Observe that, by the inductive procedure defining $\mathcal{J}^j \cup \mathcal{W}^j$, we have $\mathcal{P}^j(H) = \mathcal{W}^{i+2}(H) \subset \mathcal{W}^j$. Moreover, by Assumption 1.8, $\text{spt}(\vartheta_L) \cap \text{spt}(\vartheta_H) = \emptyset$ for all $L \notin \mathcal{P}^j(H)$. Thus, Theorem 1.12(ii) follows.

Finally, to prove (iii), it suffices to show that $\{\varphi_j\}$ is a Cauchy sequence in $C^0$ (the convergence up to subsequence follows straightforwardly from (i)). To this aim, let $x \in [-4, 4]^m$ and assume that $x \in L \cap H$ with $L \in \mathcal{P}^j$ and $H \in \mathcal{P}^{j+1}$. Without loss of generality, we can make the choice of $H$ and $L$ in such a way that either $H = L$ or $H$ is a son for $L$. Now, if $\ell(L) \geq 2^{-j+2}$, then by (ii) we have $\varphi_j(x) = \varphi_{j+1}(x)$. Otherwise, from (i) and Proposition 4.5(iv), we can conclude that:

$$
|\hat{\varphi}_j(x) - \hat{\varphi}_{j+1}(x)| \leq |\hat{\varphi}_j(x) - \hat{\varphi}_j(x_H)| + |g_H(x_H) - g_L(x_L)| + |\hat{\varphi}_{j+1}(x) - \hat{\varphi}_{j+1}(x_L)|
\leq C(\|\hat{\varphi}_j\|_{C^1} + \|\hat{\varphi}_{j+1}\|_{C^1})2^{-j} + \|g_H - y_H\|_{C^0} + \|g_L - y_L\|_{C^0} + |y_H - y_L|
\leq Cm_0^{1/2}2^{-j} + |p_H - p_L|.
$$

(4.11)

Since $B_H \subset B_L$ by Proposition 4.2(ii), we conclude $|\hat{\varphi}_j(x) - \hat{\varphi}_{j+1}(x)| \leq C2^{-j}$, where the constant $C$ depends upon the various parameters, but not on $j$. Given that $\Psi$ is Lipschitz, we get $\|\varphi_j - \varphi_{j+1}\|_{C^0} \leq C2^{-j}$ and conclude. □

5. Proof of the three key construction estimates

5.1. Elliptic PDE for the average. This section contains the most important computation, namely the derivation via a first variation argument of a suitable elliptic system for the average of the $\pi$-approximations. In order to simplify the notation we introduce the following definition.

Definition 5.1 (Tangential parts). Having fixed $H \in \mathcal{P}^j$ and $\pi := \pi_H \subset T_{p_H} \Sigma$, we let $\mathcal{T}$ be the orthogonal complement of $\pi$ in $T_{p_H} \Sigma$. For any given point $q \in \mathbb{R}^{m+n}$, any set $\Omega \subset \pi$ and any map $\xi : q + \Omega \to \pi^\perp$, we denote by $\bar{\xi}$ the map $p_{\mathcal{T}} \circ \xi$, and call it the tangential part of $\xi$. Analogous notation will be used for multiple-valued maps.

Proposition 5.2 (Elliptic system). Let $H \in \mathcal{W}^j \cup \mathcal{J}^j$ and $L$ be either an ancestor of $H$ or another element $L \in \mathcal{W}^j \cap \mathcal{J}^j$ with $H \cap L \neq \emptyset$ (possibly also $H$ itself). Set $\pi := \pi_H$, $r := r_L$, $p := p_L$, $B := B_{2r}(p, \pi)$. Let $f : B \to A_0(\pi^\perp)$ be the $\pi$-approximation of $T$ in $C_{8r}(p, \pi)$ and $h$ its smoothed average, according to Definition 1.9. Then, there is a matrix $L$, which depends on $\Sigma$ and $H$ but not on $L$, such that $|L| \leq CA^2 \leq Cm_0$ and

$$
\left| \int (D(\eta \circ \bar{f}) : D\zeta + (p_{\pi}(x - p_H))^T \cdot L \cdot \zeta) \right| \leq Cm_0 r^{m+1 + \beta_2} (r \|\zeta\|_{C^1} + \|\zeta\|_{C^0}),
$$

(5.1)

for every test function $\zeta \in C_c^\infty(B, \mathcal{T})$. Moreover,

$$
\|\bar{h} - \eta \circ \bar{f}\|_{L^1(B, (B_{2r}(p, \pi)))} \leq Cm_0 r^{m+3 + \beta_2}.
$$

(5.2)

The constant $C$ depends on all parameters except $\varepsilon_2$ (in particular it does not depend on $H$ and $L$).
Proof. We fix a system of coordinates \((x, y, z) \in \pi \times \Xi \times (T_{pH} \Sigma)^\perp\) so that \(pH = (0, 0, 0)\).

Also, in order to simplify the notation, although the domains of the various maps are subsets of \(p_L + \pi\), we will from now on consider them as functions of \(x\) (i.e. we shift their domains to \(p_x(\Omega)\)). We also drop the subscript \(p_H\) for the map \(\Psi_{pH}\) of Assumption 1.2.

Recall that \(\Psi(0, 0) = 0\), \(D\Psi(0, 0) = 0\) and \(\|\Psi\|_{C^{3, \alpha}} \leq Cm_0^{1/2}\).

Given a test function \(\zeta\) and any point \(q = (x, y, z) \in \Sigma\), we consider the vector field \(\chi(q) = (0, \zeta(x), D_y \Psi(x, y) \cdot \zeta(x))\). Observe that \(\chi\) is tangent to \(\Sigma\) and therefore \(\delta T(\chi) = 0\).

Thus,

\[
|\delta G_f(\chi)| \leq |\delta G_f(\chi) - \delta T(\chi)| \leq C \int_{C^{3, \alpha} (p_L, \pi)} |D\chi| d\|G_f - T\|. \tag{5.3}
\]

Observe also that \(|\chi| \leq C|\zeta|\) and \(|D\chi| \leq C|\zeta| + C|D\zeta|\). Set now \(E := E(T, C_{32r}(p_L, \pi))\) and apply [6, Theorem 1.4] to conclude that

\[
|Df| \leq C \varepsilon_1 \leq Cm_0^{\gamma_1} r^{\gamma_1}, \tag{5.4}
\]

\[
|f| \leq C(h(T, C_{32r} (p_L, \pi)) + (E_1^{1/2} + rA) \leq Cm_0^{1/2} r^{1 + \beta_2}, \tag{5.5}
\]

\[
\int_B |Df|^2 \leq C r^m E \leq Cm_0 r^{m + 2 - 2\delta_2}. \tag{5.6}
\]

Concerning (5.5) observe that the statement of [6, Theorem 1.4] bounds indeed \(\text{osc} \ f\).

However, in our case we have \(p_H = (0, 0, 0) \in \text{spt}(T)\) and \(\text{spt}(T) \cap \text{Gr}(f) = \emptyset\). Thus we conclude \(|f| \leq \text{osc} \ f(\bar{f}) + h(T, C_{32r} (p_L, \pi))\).

Writing \(f = \sum_i \|f_i\|\) and \(\bar{f} = \sum_i \|f_i\|\), since \(\text{Gr}(f) \subset \Sigma\), we have \(f = \sum_i \|f_i(\Psi(x, f_i))\|\).

From [5, Theorem 4.1] we can infer that

\[
\delta G_f = \int_B \sum_i \left( \left( D_{x y} \Psi(x, f_i) \cdot \zeta \right) \left( \left( D_{x y} \Psi(x, f_i) \cdot \zeta \right) + D_y \Psi(x, f_i) \cdot D_x \zeta \right) \right) + \int_B \sum_i D\zeta : Df_i + \text{Err}. \tag{5.7}
\]

To avoid cumbersome notation we use \(\| \cdot \|_0\) for \(\| \cdot \|_{C^0}\) and \(\| \cdot \|_1\) for \(\| \cdot \|_{C^1}\). Recalling [5, Theorem 4.1], the error term \(\text{Err}\) in (5.7) satisfies the inequality

\[
|\text{Err}| \leq C \int |D\chi| |Df|^3 \leq \|\chi\|_1 \int |Df|^3 \leq C \|\chi\|_1 m_0^{1+\gamma_1} r^{m+2-2\delta_2+\gamma_1}. \tag{5.8}
\]

The second integral in (5.7) is obviously \(Q \int_B D\zeta : D(\eta \circ \bar{f})\). We therefore expand the product in the first integral and estimate all terms separately. We will greatly profit from the Taylor expansion \(D\Psi(x, y) = D_x D\Psi(0, 0) \cdot x + D_y D\Psi(0, 0) \cdot y + O(m_0^{1/2} ||x|^2 + |y|^2))\).

In particular we gather the following estimates:

\[
|D\Psi(x, f_i)| \leq C m_0^{1/2} r \quad \text{and} \quad D\Psi(x, f_i) = D_x D\Psi(0, 0) \cdot x + O(m_0^{1/2+1/2m} r^{1 + \beta_2}),
\]

\[
|D^2\Psi(x, f_i)| \leq C m_0^{1/2} \quad \text{and} \quad D^2\Psi(x, f_i) = D^2\Psi(0, 0) + O(m_0^{1/2} r).
\]
We are now ready to compute

\[
\int \sum_i (A) : (D) = \int \sum_i (D_{xy} \Psi(0, 0) \cdot \zeta) : D_x \Psi(x, \tilde{f}_i) + O\left( m_0 r^2 \int |\zeta| \right)
\]

\[
= \int \sum_i (D_{xy} \Psi(0, 0) \cdot \zeta) : D_{xx} \Psi(0, 0) \cdot x + O\left( m_0 r^{1+\beta_2} \int |\zeta| \right) .
\tag{5.9}
\]

Obviously the first integral in (5.9) has the form \( \int x^t \cdot L_{AD} \cdot \zeta \). Next, we estimate

\[
\int \sum_i (A) : (E) = O\left( m_0^{1+\gamma_1} r^{1+\gamma_1} \int |\zeta| \right) ,
\tag{5.10}
\]

\[
\int \sum_i (B) : ((D) + (E)) = \left( m_0^{1+\gamma_1} r^{1+\gamma_1} \int |\zeta| \right) ,
\tag{5.11}
\]

\[
\int \sum_i (C) : (E) = O\left( m_0^{1+\gamma_1} r^{2+\gamma_1} \int |D\zeta| \right) .
\tag{5.12}
\]

Finally we compute

\[
\int \sum_i (C) : (D) = \int \sum_i ((D_{xy} \Psi(0, 0) \cdot x) \cdot D_x \zeta) : D_x \Psi(x, \tilde{f}_i) + O\left( m_0 r^{2+\beta_2} \int |D\zeta| \right)
\]

\[
= \int \sum_i (D_{xy} \Psi(0, 0) \cdot x) \cdot D_x \zeta) : (D_{xx} \Psi(0, 0) \cdot x) + O\left( m_0 r^{2+\beta_2} \int |D\zeta| \right) .
\]

Integrating by parts the last integral we reach

\[
\int \sum_i (C) : (D) = \int x^t \cdot L_{CD} \cdot \zeta + O\left( m_0 r^{2+\beta_2} \int |D\zeta| \right) .
\tag{5.13}
\]

Set next \( L := L_{AD} + L_{CD} \). Clearly \( L \) is a quadratic function of \( D^2 \Psi(0, 0) \), i.e. a quadratic function of the tensor \( A_\Sigma \) at the point \( p_H \). In order to summarize all our estimates we introduce some simpler notation. We define \( f = \eta \circ \tilde{f} \), \( \ell := \ell(L) \) and recall that \( K \) is the closed set of \([6, \text{Theorem 1.4}]\), on which \( G_f \) and \( T \) coincide: \( G_f \mathcal{L}(K \times \pi^+) = T \mathcal{L}(K \times \pi^+) \).

Let \( \mu \) be the measure on \( B \) given by

\[
\mu(E) := |E \setminus K| + \|T\|((E \setminus K) \times \mathbb{R}^n) .
\]

We can then summarize (5.3) and (5.7) - (5.13) into the following estimate:

\[
\left| \int (Df : D\zeta + x^t \cdot L \cdot \zeta) \right| \leq C m_0 r^{1+\beta_2} \int (r |D\zeta(x)| + |\zeta(x)|) \, dx \\
+ C \int (|D\zeta(x)| + |\zeta(x)|) (|Df(x)|^2 \, dx + d\mu(x)) .
\tag{5.14}
\]
From (5.4) and (5.6), we infer that

\[
\int |Df|^3 \leq C r^m \text{Lip}(f) E \leq C m_0^{1+\gamma_1} r^{m+2-2\delta_2+\gamma_1}, \tag{5.15}
\]

\[
\mu(B) \leq C E^{\gamma_1} (E + r^2 A^2) r^m \leq C m_0 r^{m+2-2\delta_2+\gamma_1}. \tag{5.16}
\]

Therefore (5.1) follows from (5.14) and our choice of the parameters in Assumption 1.5.

We next come to (5.2). Fix a smooth radial test function \( \zeta \) and set \( \zeta(\cdot) := \zeta(z - \cdot) e_i \), where \( e_{m+1}, \ldots, e_{m+n} \) is on orthonormal base of \( x \). Observe that, if in addition we assume \( \int \zeta = 0 \), then \( \int y \zeta(z - y) dx = 0 \). Under these assumptions, from (5.14) we get

\[
\left| \int \langle Df(y), D\zeta(z-y) \rangle dy \right| \leq C \int |Df|^3(y) |D\zeta + |\zeta|| (z-y) dy
\]

\[
+ C \int (|D\zeta| + |\zeta|)(z-y) d\mu(y) + C m_0 r^{1+\beta_1} \int (r|D\zeta| + |\zeta|). \tag{5.17}
\]

Recall the standard estimate on convolutions \( \|a * \mu\|_{L^1} \leq \|a\|_{L^1} \mu(B) \), and integrate (5.17) in \( z \); by (5.15) and (5.16) we reach

\[
\|Df * D\zeta\|_{L^1} \leq C m_0 r^{m+1+\beta_1} \int (r|D\zeta| + |\zeta|) \quad \forall \zeta \in C_c^\infty(B_0) \text{ with } \int \zeta = 0. \tag{5.18}
\]

By a simple density argument, (5.18) holds also when \( \zeta \in W^{1,1} \) is supported in \( B_0 \) and \( \int \zeta = 0 \). Observe next

\[
\bar{h}(x) - f(x) = \int \varrho(y)(f(x-y) - f(x)) dy = \int \varrho(y) \int_0^1 Df(x-\sigma y) \cdot (-y) d\sigma dy
\]

\[
= \int \int_0^1 \varrho \left( \frac{w}{\sigma} \right) Df(x-w) \cdot \frac{-w}{\sigma^{m+1}} dw = \int \int_0^1 \varrho \left( \frac{w}{\sigma} \right) \sigma^{-m-1} d\sigma dw. \tag{5.19}
\]

Note that \( \Upsilon \) is smooth on \( \mathbb{R}^m \setminus \{0\} \) and unbounded in a neighborhood of 0. However,

\[
\|\Upsilon\|_{L^1} = \int \int_0^1 |w| \varrho \left( \frac{w}{\sigma} \right) \ell^{-m} \sigma^{-m-1} d\sigma dw = \ell \int \int_0^1 |u| \varrho(u) d\sigma du \leq Cr. \tag{5.19}
\]

Observe also that \( \Upsilon(w) = w \psi(|w|) \). Therefore \( \Upsilon \) is a gradient. Since \( \Upsilon(w) \) vanishes outside a compact set, integrating along rays from \( \infty \), we can compute a potential for it:

\[
\zeta(w) = \int_{|w|}^\infty \tau \int_0^1 \varrho \left( \frac{w}{\sigma} \right) \sigma^{-m-1} d\sigma d\tau = |w|^2 \int_{|w|}^\infty t \int_0^1 \varrho \left( \frac{w}{\sigma} \right) \sigma^{-m-1} d\sigma dt. \tag{5.20}
\]
Then, \( \zeta \) is a \( W^{1,1} \) function, supported in \( B_I(0), \int \zeta = 0 \) by Assumption 1.8 and (5.17). Summarizing, \( \tilde{h}^i - f^i = (Df^i) \ast D\zeta \) for a convolution kernel for which (5.18) holds. Since

\[
\|\varsigma\|_{L^1} \leq \int \int_1^\infty \int_0^1 t|w|^2 |\varrho(\frac{w}{t\sigma})| \ell^{-m}\sigma^{-m-1}d\sigma\ dt\ dw
\]

\[
= \ell^2 \int_1^\infty \int_0^1 \int |u|^2 |\varrho(u)| du \sigma d\sigma t^{-m-1}dt \leq C r^2,
\]

we then conclude from (5.17) and (5.18)

\[
\int |\tilde{h} - f| \leq C m_0 r^{m+\beta_2} \int (r|D\varsigma| + |\varsigma|) \leq C m_0 r^{m+3+\beta_2}.
\]

5.2. \( C^k \) estimates for \( h_{HL} \) and \( g_{HL} \). We fix cubes \( H, L \) as in Proposition 5.2 and the maps \( h_{HL} \) and \( g_{HL} \) of Definition 4.3.

**Lemma 5.3.** Assume \( H \) and \( L \) are as in Proposition 5.2. Set \( B' := B_{5r_H}(p_H, \pi_H) \) and \( B := B_{4r_H}(p_H, \pi_0) \). Then,

\[
\|h_{HL} - h_H\|_{C^j(B')} + \|g_{HL} - g_H\|_{C^j(B)} \leq C m_0 \ell(L)^{3+2k-j} \quad \forall j \in \{0, \ldots, 3\},
\]

\[
\|h_{HL} - h_H\|_{C^3(B')} + \|g_{HL} - g_H\|_{C^3(B)} \leq C m_0 \ell(L)^{3}.
\]

As a consequence Proposition 4.5(i) and (iv) hold.

**Proof.** Consider a triple of cubes \( H, J \) and \( L \) where:

(a) either \( J = H \) and \( L \) is the father of \( J \);
(b) or \( J = H \), and \( L \in \mathcal{J}^j \cup \mathcal{J}^j \) adjacent to \( H \);
(c) or \( J \) is an ancestor of \( H \) and \( L \) the father of \( J \).

In order to simplify the notation let \( \pi := \pi_H \) and \( r := r_J \). By Proposition 4.2(i), up to choose \( M_0 \) larger than a geometric constant, we can assume that \( B^\circ := B_{6r}(p_J, \pi) \subset B' = B_{13r/J}(p_J, \pi) \subset B := B_{7r_L}(p_L, \pi) \). Consider the \( \pi \)-approximations \( f_{HL} \) and \( f_{HJ} \), respectively in \( C_{\delta r_L}(p_L, \pi) \) and \( C_{\delta r_J}(p_J, \pi) \), and introduce the corresponding maps

\[
f_L := p_{\pi}(\eta \circ f_{HL}) \quad \text{and} \quad f_J := p_{\pi}(\eta \circ f_{HJ}),
\]

\[
h_{HL} := f_L \ast \varrho_{\ell(L)} \quad \text{and} \quad h_{HJ} = f_J \ast \varrho_{\ell(J)}.
\]

If \( \mathfrak{I} \) is an affine function on \( \mathbb{R}^m \) and \( \zeta \) a radial convolution kernel, then \( \zeta \ast \mathfrak{I} = (\int \zeta) \mathfrak{I} \) because \( \mathfrak{I} \) is an harmonic function. This means that \( \int \langle (\zeta \ast \varrho), 1 \rangle = \int \langle \zeta, 1 \rangle \) for any test function \( \zeta \) and any radial convolution kernel \( \varrho \) with integral 1. Similarly \( \int \langle (\zeta \ast \partial^l \varrho), 1 \rangle = \int \langle \zeta, \partial^l \mathfrak{I} \rangle \) for any partial derivative \( \partial^l \) of any order. Consider now a ball \( B \) concentric to \( B^\circ \) and contained in \( B^2 \) in such a way that, if \( \zeta \in C_c^\infty(B) \), then \( \zeta \ast \varrho_{\ell(L)} \) and \( \zeta \ast \varrho_{\ell(J)} \) are both supported in \( B^2 \). Set \( \xi := h_{HL} - h_{HJ} \) and (assuming \( p_{\pi}(x_H) \) is the origin of our system of
coordinates) compute:
\[
\int \langle \zeta, \Delta \xi \rangle = - \int D(h_{HL} - h_{HJ}) : D\xi = \int Df_J : D(\zeta * g_{\ell(J)}) - \int Df_L : D(\zeta * g_{\ell(L)})
\]
\[
= \int (Df_J : D(\zeta * g_{\ell(J)}) + x^t \cdot L \cdot (\zeta * g_{\ell(J)})) - \int (Df_L : D(\zeta * g_{\ell(L)}) + x^t \cdot L \cdot (\zeta * g_{\ell(L)}) ) ,
\]
where the last line holds for any matrix \( L \) because \( x \mapsto x^t \cdot L \) is a linear function. In particular, we can use the matrix of Proposition 5.2 to achieve
\[
\int \langle \zeta, \Delta \xi \rangle \leq Cm_0 r^{m+1+\beta_2} \left( r\|\zeta * g_{\ell(L)}\|_1 + r\|\zeta * g_{\ell(J)}\|_1 + \|\zeta * g_{\ell(J)}\|_0 + \|\zeta * g_{\ell(L)}\|_0 \right) ,
\]
where \( \| \cdot \|_0 \) and \( \| \cdot \|_1 \) denote the \( C^0 \) and \( C^1 \) norms respectively. Recalling the inequality \( \|\psi * \zeta\|_0 \leq \|\psi\|_\infty \|\zeta\|_{L^1} \) and taking into account that \( \ell(L) \) and \( \ell(J) \) are both comparable to \( r \) (up to a constant depending only on \( M_0 \) and \( m \)), we achieve \( \int \langle \zeta, \Delta \xi \rangle \leq Cm_0 r^{1+\beta_2} \|\xi\|_{L^1} \).

Taking the supremum over all possible test functions with \( \|\xi\|_{L^1} \leq 1 \), we obviously conclude \( \|\Delta \xi\|_{L^\infty(\bar{B})} \leq Cm_0 r^{1+\beta_2} \). Observe that a similar estimate could be achieved for any partial derivative \( D^k \xi \) simply using the identity
\[
\int D(D^k(a * \xi)) : Db = - \int Da : (Db * D^k \xi) .
\]

Summarizing we conclude
\[
\|\Delta D^k(f_{HL} - f_{HJ})\|_{C^0(\bar{B})} \leq \|\Delta D^k \xi\|_{\infty} \leq Cm_0 r^{1+\beta_2 - k} , \tag{5.24}
\]
where the constant \( C \) depends upon all the parameters and on \( k \in \mathbb{N} \), but not on \( \varepsilon_2; m_0, H, J \) or \( L \). By [6, Theorem 1.4] (cp. also the proof of Proposition 4.4), it holds \( \text{osc}(f_{HL}) + \text{osc}(f_{HJ}) \leq Cm_0^{1/2} r \) and
\[
\mathcal{H}^m\left( \{ f_{HL} \neq f_{HJ} \} \right) \leq C E(T, C_{32rL}(p_L, \pi_H)) r^m \leq C m_0^{1+\gamma_1} r^{m+2+\gamma_1/2} .
\]
Therefore, taking into account (5.2), we conclude \( \|h_{HL} - h_{HJ}\|_{L^1} \leq Cm_0 r^{m+3+\beta_2} \). Thus, we appeal to Lemma C.1 and use the latter estimate together with (5.24) (in the case \( k = 0 \)) to get \( \|h_{HL} - h_{HJ}\|_{C^k(B')} \leq Cm_0 r^{3+\beta_2 - k} \) for \( k = \{0, 1\} \) and for every concentric smaller ball \( B' \subset B \) (where the constant depends also on the ratio between the corresponding radii). This implies \( \|D(h_{HL} - h_{HJ})\|_{L^1(B')} \leq Cm_0 r^{m+2+\beta_2} \) and hence we can use again Lemma C.1 (based on the case \( k = 1 \) of (5.24)) to conclude \( \|h_{HL} - h_{HJ}\|_{C^2(B')} \leq Cm_0 r^{1+\beta_2} \).

Iterating another two times we can then conclude \( \|h_{HL} - h_{HJ}\|_{C^4(B')} \leq Cm_0 r^{3+\beta_2 - k} \) for \( k \in \{0, 1, 2, 3, 4\} \). By interpolation, since \( \kappa \leq \beta_2 / 4 \), \( \|h_{HL} - h_{HJ}\|_{C^{3+\kappa}} \leq Cm_0 \ell(L)^{3\kappa} \).

Fix now \( L = L_j \subset L_{j-1} \subset \ldots \subset L_{N_0} \) be the chain of fathers with \( L_i \in \mathcal{P}_3 \). Summing the corresponding estimates, we get
\[
\|h_{HL} - h_{HL_{N_0}}\|_{C^{3,\kappa}} \leq C \sum_{i=N_0}^{j-1} \|h_{HL_{i+1}} - h_{HL_i}\|_{C^{3,\kappa}} \leq Cm_0 \sum_i 2^{-3\kappa i} \leq Cm_0 . \tag{5.25}
\]
Observe next that \( \bar{h}_{HL_N_0} = f_{HL_N_0} \ast p_{2^{-N_0}} \) and that
\[
\|Df_{HL_N_0}\|_{L^2}^2 \leq \text{Dir}(f_{HL_N_0}) \leq C\text{E}(T, C^{32r_{L_N_0}}(p_{L_N_0}, \pi_H)) \leq Cm_0 + C|\pi_H - \pi_0|^2 \leq Cm_0.
\]
Thus, by standard convolution estimates, \( \|D\bar{h}_{L_N_0}\|_{C^k} \leq Cm_0^{1/2} \) (where the constant \( C \) depends on \( k \in \mathbb{N} \) and on he various parameters). The latter estimate combined with (5.25) leads to \( \|D\bar{h}_{HL}\|_{C^{2,\kappa}} \leq Cm_0^{1/2} \). Moreover, we infer \( \|\bar{h}_{HL}\|_{C^0} \leq Cm_0^{1/2m} \), appealing again to (5.25) and using this time \( \|\bar{h}_{HL_N_0}\| \leq Cm_0^{1/2m} \). Since \( h_{HL} = \Psi(x, \bar{h}_{HL}) \) and \( h_{HJ} = \Psi(x, \bar{h}_{HJ}) \), we deduce the corresponding estimates for \( h_{HL} \) and \( h_{HJ} \) from the chain rule.

Now we pass to prove Proposition 4.5(i) and (iv). Since \( h_{HH} = h_H \) and \( g_{HH} = g_H \), the first claim of (i) follows then from Lemma B.1. Coming to (iv), the estimate on \( g_H - y_H \) is a straightforward consequence of the height bound, [6, Theorem 1.4] and Lemma B.1 (applied to \( h_H \)). Note that, together with (4.1), this implies the second claim in (i). Next, observe that
\[
\|Dh_H\|_{L^2}^2 \leq C\|D(\eta \circ f_H)\|_{L^2}^2 \leq C\text{Dir}(f_H) \leq Cm_0 \ell(H)^{2-2\delta_2}.
\]
Thus, there is at least one point \( q \in \text{Gr}(h_H) \) such that \( |T_q G_{h_H} - \pi_H| \leq Cm_0^{1/2} \ell(H)^{1-\delta_2} \). Since \( \|D^2h_H\|_0 \leq Cm_0^{1/2} \), we then conclude that \( |T_q G_{h_H} - \pi_H| \leq Cm_0^{1/2} \ell(H)^{1-\delta_2} \) holds indeed for any point \( q' \in \text{Gr}(h_H) \). Since \( \text{Gr}(g_H) \) is a subset of \( \text{Gr}(h_H) \) (with the same orientation!), the second inequality of Proposition 4.5(iv) follows.

### 5.3. Tilted \( L^1 \) estimate
In order to achieve Proposition 4.5(ii) and (iii), we need to compare tilted interpolating functions coming from different coordinates. To this aim, we set the following terminology.

**Definition 5.4** (Distant relation). Four cubes \( H, J, L, M \) make a distant relation between \( H \) and \( L \) if \( J, M \in \mathcal{S}^j \cup \mathcal{W}^j \) have nonempty intersection, \( H \) is a descendant of \( J \) (or \( J \) itself) and \( L \) a descendant of \( M \) (or \( M \) itself).

**Lemma 5.5** (Tilted \( L^1 \) estimate). Let \( H, J, L, M \) be a distant relation between \( H \) and \( L \), and let \( h_{HJ}, h_{LM} \) be the maps given in Definition 4.3. Consider the map \( \hat{h}_{LM} : B_{4r_J}(p_J, \pi_H) \to \pi_{H}^\perp \) such that \( G_{\hat{h}_{LM}} = G_{h_{LM}} \downarrow C_{4r_J}(p_J, \pi_H) \) (the existence is ensured by Lemma B.1). Then,
\[
\|h_{HJ} - \hat{h}_{LM}\|_{L^1(B_{2r_J}(p_J, \pi_H))} \leq Cm_0 \ell(J)^{m+3+\delta_2/3}.
\]  

**Proof.** First observe that Lemma B.1 can be applied because
\[
|\pi_L - \pi_H| \leq |\pi_H - \pi_J| + |\pi_J - \pi_M| + |\pi_M - \pi_L| \leq Cm_0^{1/2} \ell(J)^{1-\delta_2}.
\]
Set \( \pi := \pi_H \) and \( \chi \) for its orthogonal complement in \( T_{p_H} \Sigma \), and similarly \( \bar{\pi} = \pi_L \) and \( \bar{\chi} \) its orthogonal in \( T_{p_L} \Sigma \). After a translation we also assume \( p_J = 0 \), and write \( r = r_J = r_M, \ell = \ell(J) = \ell(M) \) and \( E := E(T, C_{32r}(p_J, \bar{\pi})) \), \( \bar{E} := E(T, C_{32r}(p_M, \bar{\pi})) \). Recall that \( \max\{E, \bar{E}\} \leq Cm_0 \ell^{2-2\delta_2} \). We fix also the maps \( \Psi : T_0 \Sigma \to T_0 \Sigma^\perp \) and \( \bar{\Psi} : T_{p_L} \Sigma \to T_{p_L} \Sigma^\perp \).
whose graphs coincide with the submanifold $\Sigma$. Observe that $|\pi - \tilde{\pi}| + |x_0 - \tilde{x}| \leq C m_0^{1/2} r^{1-\delta_2}$, $\|\Psi\|_{C^3} + \|\tilde{\Psi}\|_{C^3} \leq C m_0^{1/2}$ and

$$\|D\Psi\|_{C^0(B_{sr})} + \|D\tilde{\Psi}\|_{C^0(B_{sr})} \leq C m_0^{1/2} r^{1-\delta_2}. $$

Consider the map $\hat{f} : B_{4r}(0, \pi) \to A_Q(\pi^\perp)$ such that $G_{\hat{f}} = G_{f_{LM}} \cup C_{4r}(0, \pi)$, which exists by [5, Proposition 5.2]. Recalling the estimates therein and those of [6, Theorem 1.4], if we set $f = f_{HJ}$ we have

$$\text{Lip}(f) + \text{Lip}(\hat{f}) \leq C m_0^{1/2} r^{1-\gamma_1}, \quad |f| + |\hat{f}| \leq C m_0^{1/2m} r^{1+\beta_2}, \quad (5.27)$$

$$\text{Dir}(f) + \text{Dir}(\hat{f}) \leq C m_0 r^{m+2-2\delta_2}. \quad (5.28)$$

Consider next the projections $A$ and $\hat{A}$ onto $\pi$ of the closed sets $\text{Gr}(f) \setminus \text{spt}(T)$ and $\text{Gr}(\hat{f}) \setminus \text{spt}(T)$. We know from [6, Theorem 1.4] that

$$|A \cup \hat{A}| \leq C \left[ \|G_f - T\|((C_{32}(0, \pi)) + \|G_{\hat{f}} - T\|((C_{32}(p_M, \pi))) \right] \leq C m_0 r^{m+2+\gamma_1}. \quad (5.29)$$

Define next $f = \Psi(x, p_{x}(\eta \circ f))$, $h := h_{HJ} = \Psi(x, p_{x}(\eta \circ \tilde{g}_0))$, $f_M = \tilde{\Psi}(x, p_{x}(\eta \circ f_{LM}))$ and $h_{LM} = \tilde{\Psi}(x, p_{x}(\eta \circ f_{LM} \circ \tilde{g}_0))$. We define that $h : B_{4r}(0, \pi) \to \pi^\perp$ such that $G_h = G_{h_{LM}} \cup C_{4r}(0, \pi)$ and $f$ such that $G_f = G_f \cup C_{4r}(0, \pi)$. We use Proposition 5.2, the Lipschitz regularity of $\Psi$ and $\tilde{\Psi}$ and Lemma B.1 to conclude

$$\|h - \hat{f}\|_{L^1} \leq C \|h_{LM} - f_M\|_{L^1} \leq C m_0 r^{m+3+\beta_2}. $$

Likewise $\|h - f\|_{L^1} \leq C m_0 r^{m+3+\beta_2}$. We therefore need to estimate $\|f - \hat{f}\|_{L^1}$. Define next the map $g = \Psi(x, p_{x}(\eta \circ f))$ and observe that the $\|g - f\|_{L^1} \leq C \|\eta \circ f - \eta \circ f\|_{L^1}$. On the other hand, since the two maps $f$ and $\hat{f}$ differ only on $A \cup \hat{A}$, we can estimate the latter with $C|A \cup \hat{A}|(\|f\| + |\hat{f}|) \leq C m_0^{1+1/2m} r^{3m+\gamma_1+\beta_2}$. It thus suffices to estimate $\|g - \hat{f}\|_{L^1}$. This estimate is indeed independent of the rest and we prove it in the next lemma.

**Lemma 5.6.** Consider two triples of planes $(\pi, \varpi, \tilde{\varpi})$ and $(\tilde{\pi}, \tilde{\varpi}, \tilde{\tilde{\varpi}})$, where

- $\pi, \tilde{\pi}$ are $m$-dimensional;
- $\varpi$ and $\tilde{\varpi}$ are $n$-dimensional and orthogonal, respectively, to $\pi$ and $\tilde{\pi}$;
- $\omega$ and $\tilde{\omega}$ are $l$-dimensional and orthogonal, respectively, to $\pi \times \varpi$ and $\tilde{\pi} \times \tilde{\varpi}$.

Assume $|\pi - \tilde{\pi}|, |\varpi - \tilde{\varpi}| \leq C m_0^{1/2} r^{1-\delta_2}$ and let $\Psi : \pi \times \varpi \to \varpi$, $\tilde{\Psi} : \tilde{\pi} \times \tilde{\varpi} \to \tilde{\omega}$ be two maps whose graphs coincide and such that $\|\Psi\|_{C^3} + \|\tilde{\Psi}\|_{C^3} \leq C m_0^{1/2}$, $|\Psi(0)| + |\tilde{\Psi}(0)| \leq C m_0^{1/2}$ and

$$\|D\Psi\|_{C^0(B_{sr})} + \|D\tilde{\Psi}\|_{C^0(B_{sr})} \leq C m_0^{1/2} r^{1-\delta_2}. $$

Let $\tilde{u} : B_{sr}(0, \tilde{\pi}) \to A_Q(\tilde{\varpi})$ be a map with

$$\text{Lip}(\tilde{u}) \leq C m_0^{1/2} r^{\gamma_1}, \quad \|\tilde{u}\|_{0} \leq C m_0^{1/2m} r^{1+\beta_2} \quad \text{and} \quad \text{Dir}(\tilde{u}) \leq C m_0 r^{m+2+\gamma_1}. \quad (5.30)$$

Consider the maps $\tilde{g}(x) = \sum_i \|(\tilde{u}_i(x), \tilde{\Psi}(x, \tilde{u}_i(x)))\|$, $\tilde{g}(x) = (\eta \circ \tilde{u}(x), \tilde{\Psi}(\eta \circ \tilde{u}(x)))$. Let $u : B_{4r}(0, \pi) \to A_Q(\pi)$ be such that the map $g(x) := \sum_i \|(u_i(x), \Psi(x, u_i(x)))\|$ satisfies
\[ G_g = G_{\hat{g}} \mathcal{L} C_{4r}(0, \pi) \] and \( \hat{g} : B_{4r}(0, \pi) \rightarrow \mathbb{R} \times \mathbb{R} \) be such that \( G_g = G_{\hat{g}} \mathcal{L} C_{4r}(0, \pi) \). If \( g(x) := (\eta \circ u(x), \Psi(x, \eta \circ u(x))) \), then
\[ \| g - \hat{g} \|_{L^1} \leq C m_0 r^{m+3+3/2} . \] (5.31)

**Proof.** We start fixing the following terminology: we say that \( R \) is of type \( A \) if there are two orthonormal vectors \( e_1, e_2 \) and an angle \( \theta \) such that \( R(e_1) = \cos \theta e_1 + \sin \theta e_2, R(e_2) = \cos \theta e_2 - \sin \theta e_1 \) and \( R(v) = v \) for every \( v \perp \text{span} \left( e_1, e_2 \right) \). We then say that:

- \( R \) is of type \( A \) with respect to \( (\pi, \kappa, \omega) \) if \( e_1 \in \pi \) and \( e_2 \in \omega \);
- \( R \) is of type \( B \) with respect to \( (\pi, \kappa, \omega) \) if \( e_1 \in \pi \) and \( e_2 \in \kappa \);
- \( R \) is of type \( C \) with respect to \( (\pi, \kappa, \omega) \) if \( e_1 \in \pi \) and \( e_2 \in \omega \).

The lemma is based on the following claim, whose proof is postponed to the end.

**Claim.** There is a number \( N \) depending only on \( (m, n, l) \) and a constant \( C \) such that \( \left( (\pi, \kappa, \omega) =: (\pi_0, \kappa_0, \omega_0) \right) \) for every \( v \) is of type \( A \) with respect to \( (\pi, \kappa, \omega) \).

For the rest of the proof we will then focus on proving (5.31) under the assumption that the triple \( (\pi, \kappa, \omega) \) is obtained from \((\tilde{\pi}, \tilde{\kappa}, \tilde{\omega})\) applying a small 2d-rotation of type \( A, B \) or \( C \) with respect to the latter triple.

We then iterate the estimate \( N \) times and achieve a slight variant of (5.31) in the case of two general triples:
\[ \| g - \hat{g} \|_{L^1(B_{2-N, \pi})} \leq C m_0 r^{m+3+3/2} . \] (5.32)

Since \( N \) is just a geometric constant, a simple covering argument will then conclude (5.31).

**Type A.** In this case we show the stronger bound \( \| g - \hat{g} \|_{C^0} \leq C \, m_0 r^{3+\beta/2} \). Use the notation \( (z, w) \in \pi \times \omega \) and \( (\hat{z}, \hat{w}) \in \tilde{\pi} \times \tilde{\omega} \) for the same point. In what follows we will drop the \( \cdot \) when writing the usual products between matrices. We then have \( \hat{z} = U z + V w \) and \( \hat{w} = W z + Z w \), where the orthogonal matrix
\[ L := \begin{pmatrix} U & V \\ W & Z \end{pmatrix} \] has the property that \( | L - \text{Id} | \leq C m_0^{1/2} r^{4-\delta} \). Clearly, \( \Psi \) and \( \Psi \) are related by the identity
\[ W z + Z \Psi(x, z) = \Psi(x, U z + V \Psi(x, z)) . \] (5.33)

Fix \( x \) and \( g(x) = \sum_i \left[ (u_i(x), \Psi(x, u_i(x))) \right] =: \sum_i \left[ (z_i, \Psi(x, z_i)) \right] \). We then have
\[ g(x) = (a, b) := \left( \frac{1}{Q} \sum z_i, \Psi \left( \frac{1}{Q} \sum z_i \right) \right) \text{ in } \pi \times \omega, \] and
\[ \hat{g}(x) = L^{-1} \left( U \frac{1}{Q} \sum z_i + V \frac{1}{Q} \sum \Psi(x, z_i), \Psi \left( x, U \frac{1}{Q} \sum z_i + V \frac{1}{Q} \sum \Psi(x, z_i) \right) \right) =: L^{-1}(c, d) . \]
Since $L$ is orthogonal, we have
\[
|\mathbf{g}(x) - \mathbf{g}(x)| = |L(a, b) - (c, d)|
= \left| \left( V \left( \Psi \left( x, \frac{1}{Q} \sum z_i \right) - \frac{1}{Q} \sum_i \Psi(x, z_i) \right) , W \frac{1}{Q} \sum_i z_i + Z \Psi \left( x, \frac{1}{Q} \sum z_i \right) \right) - \tilde{\Psi} \left( x, U \frac{1}{Q} \sum z_i + V \frac{1}{Q} \sum \Psi(x, z_i) \right) \right|.
\]

Thus,
\[
|\mathbf{g}(x) - \mathbf{g}(x)| \leq \left( 1 + \text{Lip}(\tilde{\Psi}) \right) |V| \left| \frac{1}{Q} \sum \Psi(x, z_i) - \Psi \left( x, \frac{1}{Q} \sum z_i \right) \right|.
\]

Observe that $|V| \leq |L - \text{Id}| \leq C m_0^{1/2} r^{1-\delta_2}$. On the other hand, with a simple Taylor expansion around the point $(x, \frac{1}{Q} \sum z_i)$ we easily achieve
\[
\left| \frac{1}{Q} \sum \Psi(x, z_i) - \Psi \left( x, \frac{1}{Q} \sum z_i \right) \right| \leq C \| D \Psi \|_0 \sum_i \left| z_i - \frac{1}{Q} \sum z_i \right| \leq C m_0^{1/2 + 1/2m, 2+\beta_2 - \delta_2}.
\]

**Type B.** In this case $\Psi = \tilde{\Psi}$ and, given its Lipschitz regularity, it suffices to estimate $\| \eta \circ u - \rho_x(\mathbf{g}) \|_{L^1}$. We fix an orthonormal base $e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n}$, where the first $m$ vectors span $\pi$ and the remaining span $\mathbf{x}$. We also assume that the rotation $R$ acts on the plane spanned by $\{e_m, e_{m+1}\}$ and set $v = R(e_m) = a e_m + b e_{m+1}$ and $v_{m+1} = R(e_{m+1})$. We then define two systems of coordinates: given $q \in \mathbb{R}^m \times \mathbb{R}^n$, we write
\[
q = \sum_i z_i(q)e_i + t(q)e_m + \tau(q)e_{m+1} + \sum_j y_j(q)e_{j+m} = \sum_i z_i(q)e_i + s(q)v_m + \sigma(q)v_{m+1} + \sum_j y_j(q)e_{j+m}.
\]

The first will be called $(t, \tau)$-coordinates and the second $(s, \sigma)$-coordinates.

We fix for the moment $x \in \mathbb{R}^{m-1}$ with $|x| \leq 4r$ and focus our attention on the interval $\tilde{I}_x = \{ s : |(x, s)| \leq 6r \}$. We restrict the map $\tilde{u}$ to this interval and, by [4, Proposition 1.2] we know that there is a Lipschitz selection such that $\tilde{u}(x, s) = \sum_i [\tilde{\theta}_i(s)]$. In the $(s, \sigma)$-coordinates: $\text{Gr}(\tilde{\theta}_i) = \{(x, s, \tilde{\theta}_i(s), \ldots, \tilde{\theta}_i^n(s)) : s \in \tilde{I}_x \}$. In the $(t, \tau)$ coordinates we can choose functions $\tilde{\theta}_i$, also defined on an appropriate interval, whose graphs coincide with the ones of the $\theta_i$. We then obviously must have $u(x, t) = \sum_i [\tilde{\vartheta}_i(t)]$ on the domain of definition of $g$. The coordinate functions $\tilde{\vartheta}^i_1$ and $\tilde{\vartheta}^i_2$ are linked by the following relations
\[
\begin{cases}
\Phi_i(t) = a t + b \vartheta_i^1(t), \\
\vartheta_i^1(\Phi_i(t)) = -bt + a \vartheta_i^1(t), \\
\vartheta_i^l(\Phi_i(t)) = \vartheta_i^l(t),
\end{cases}
\quad \text{for } l = 2, \ldots, n.
\]

(5.34)
For $\Phi_i$ holds $\text{Lip}(\Phi_i) \leq (1 + C m_0^{1/2} r^{1-\beta_2}) \leq 2$. Likewise we can assume that $\text{Lip}(\Phi_i^{-1}) \leq 2$. Consider now $\tilde{v}(s) = \eta \circ \tilde{u}(x, s) = \frac{1}{Q} \sum_i \theta_i(s)$ and the corresponding $t \mapsto \hat{v}(t) = p_{x_c} (g(x, t))$, linked to $\eta \circ \tilde{u}(x, \cdot)$ through a relation as in (5.34) with a corresponding map $\Phi$:

$$
\begin{align*}
\Phi(t) &= a t + b \hat{v}(t), \\
\frac{1}{Q} \sum_i \theta_i^n(\Phi(t)) &= \hat{v}^1(\Phi(t)) = -b t + a \hat{v}^1(t), \\
\frac{1}{Q} \sum_i \theta_i^l(\Phi(t)) &= \hat{v}^l(\Phi(t)) = \hat{v}^l(t),
\end{align*}
$$

(5.35) for $l = 2, \ldots, \tilde{n}$.

Moreover, write $v(t) = \frac{1}{Q} \sum_i \hat{v}(t) = \eta \circ u(x, t)$. We can then write

$$
\eta \circ u(x, t) - p_{x_c}(g(x, t)) = v(t) - \hat{v}(t) = Q^{-1} \sum_i (\theta_i(t) - \hat{v}(t)) = Q^{-1} \sum_i \left( a^{-1} \theta_i^n(\Phi_i(t)) - a^{-1} \theta_i^n(\Phi(t)), \ldots, \theta_i^l(\Phi_i(t)) - \theta_i^l(\Phi(t)), \ldots \right).
$$

(5.36)

This implies that

$$
|\eta \circ u(x, t) - p_{x_c}(g(x, t))| = |v(t) - \hat{v}(t)| \leq C \sum_i \left| \int_{\Phi_i(t)}^{\Phi(t)} D\theta(\tau) d\tau \right|.
$$

(5.37)

Next we compute

$$
\Phi_i(t) - \Phi(t) = b(\theta_i^1(t) - \hat{v}^1(t)) = b(\theta_i^1(t) - v^1(t)) + b(v^1(t) - \hat{v}^1(t))
$$

(5.38)

Since $|b| \leq C m_0^{1/2} r^{1-\beta_2}$, the terms in (5.38) can be estimated respectively as follows:

$$
|b||v_i^1(t) - v^1(t)| = |b||u_i(x, t) - (\eta \circ u)^1(t)| \leq C m_0^{1/2+1/2m} r^{2-\beta_2} \leq C m_0^{1/2+1/2m} r^{2+2\beta_2/3},
$$

$$
|v^1(t) - \hat{v}^1(t)| \leq \|D\theta\|_{L^\infty} \sum_{i=1}^Q |\Phi_i(t) - \Phi(t)| \leq C m_0^{\gamma_1} r^{\gamma_1} \sum_{i=1}^Q |\Phi_i(t) - \Phi(t)|.
$$

Combining the last two inequalities with (5.38), we therefore conclude, for $\varepsilon_2$ small enough,

$$
\sum_{i=1}^Q |\Phi_i(t) - \Phi(t)| \leq C m_0^{1/2+1/2m} r^{2+2\beta_2/3} =: \rho.
$$

(5.39)

With this estimate at our disposal we can integrate (5.37) in $t$ to conclude

$$
\int_{I_x} |v(t) - \hat{v}(t)| \leq C \int_{I_x} \int_{\Phi(t)-C\rho}^{\Phi(t)+C\rho} |D\theta|(|\tau|) d\tau dt \leq C \int_{I_x} \int_{s-C\rho}^{s+C\rho} |D\tilde{g}|(x, \tau) d\tau ds,
$$

where in the latter inequality we have used the change of variables $s = \Phi(t)$ and the fact that both the Lipschitz constants of $\Phi$ and its inverse are under control. Integrating over
Define the maps $\omega$ $\in$ Lip$(\tilde{\Psi})$ achieve $-\tilde{\Psi}$. Using a Taylor expansion for $\tilde{\Psi}$ we conclude $g$ and recalling that $28$ CAMILLO DE LELLIS AND EMANUELE SPADARO

Proof of the Claim. We first show that, if $\omega = \tilde{\omega}$, or $\kappa = \tilde{\kappa}$ or $\pi = \tilde{\pi}$, then the claim can be achieved with small $2d$-rotations all of the same type, namely of type B, C and A, respectively. Assume for instance that $\omega = \tilde{\omega}$. Let $\omega$ be the intersection of $\pi$ and $\tilde{\pi}$ and $\omega'$ be the intersection of $\kappa$ and $\tilde{\kappa}$. Pick a vector $e \in \pi$ which is not contained in $\tilde{\pi}$ and
is orthogonal to \( \omega \). Let \( \tilde{e} := \frac{p_{x}(\alpha)}{|p_{x}(\alpha)|} \). Then, \( \tilde{e} \) is necessarily orthogonal to \( \omega \) and the angle between \( \tilde{e} \) and \( e \) is controlled by \(|\pi - \tilde{\pi}|\). There is therefore a small 2d-rotation \( R \) such that \( R(e) = \tilde{e} \). It turns out that \( R \) keeps \( \varpi \) and \( \omega \) fixed. So the new triple \((R(\varpi), R(\omega), R(\varpi))\) has the property that \( R(\varpi) = \varpi = \tilde{\varpi} \) and the dimension of \( R(\varpi) \cap \tilde{\pi} \) is larger than that of \( \pi \cap \tilde{\pi} \). This procedure can be repeated and after \( N \leq m \) times it leads to a triple of planes \((\pi_{N}, \varpi_{N}, \varpi_{N})\) with \( \varpi_{N} = \tilde{\varpi} \) and \( \pi_{N} = \tilde{\pi} \). This however implies necessarily \( \tilde{\varpi} = \varpi_{N} \).

Assume therefore that \( \varpi \) and \( \tilde{\varpi} \) do not coincide. Let \( \omega := (\varpi \times \pi) \cap (\tilde{\varpi} \times \tilde{\pi}) \). There is then a unit vector \( \tilde{e} \in \tilde{\varpi} \) or a unit vector \( \tilde{e} \in \pi \) which does not belong to \( \varpi \times \varpi \) and which is orthogonal to \( \omega \). Assume for the moment that we are in the first case, and consider the vector \( e := \frac{p_{x}(\alpha)}{|p_{x}(\alpha)|} \). The vector \( e \) forms an angle with the plane \( \varpi \) bounded by \( C|\varpi - \tilde{\varpi}| \).

Therefore there is a rotation \( R \) with \(|R - Id| \leq C|\varpi - \tilde{\varpi}| \) of the plane \( \varpi \times \varpi \) with the property that \( R(\varpi) \) contains \( e \) and keep fixed \( \omega \), which is orthogonal to \( e \). By the previous step, \( R \) can be written as composition \( R_{N_{1}} \circ \ldots \circ R_{1} \) of \( \beta \) small rotations of type B keeping \( \varpi \) fixed. Since \( e \perp R(\varpi) \), we can then find a small 2d-rotation \( S \) of type A with respect to \((R(\varpi), R(\varpi), \varpi)\) acting on the plane span \((\tilde{e}, e)\) for which \( S(R(\varpi)) \ni \tilde{e} \). \( S \) keeps then \( \omega \) fixed. An analogous argument works if the vector \( e \in \tilde{\pi} \). We therefore conclude that, after applying a finite number of rotations \( R_{1}, \ldots, R_{N_{1}}, R_{N_{1}+1} \), of the three types above, the dimension of \( R_{N_{1}+1} \circ R_{N_{1}} \circ \ldots \circ R_{1}(\varpi \times \varpi) \cap \tilde{\pi} \times \tilde{\varpi} \) is larger than that \( \varpi \times \varpi \cap \tilde{\pi} \times \tilde{\varpi} \) (where the number \( N_{1} \) is smaller than a geometric constant depending only on \( m \) and \( n \)). Obviously, after at most \( m + n \) iterations of this argument, we are reduced to the situation \( \pi \times \varpi = \tilde{\pi} \times \tilde{\varpi} \).

5.4. Proof of Proposition 4.5. We are finally ready to complete the proof of Proposition 4.5. Recall that (i) and (iv) have already been shown in Lemma 5.3. In order to show (ii) fix two cubes \( H, L \in \mathcal{P}^{j} \) with nonempty intersection. If \( \ell(H) \leq \ell(L) \), then we can apply Lemma 5.5 to conclude

\[
\|h_{H} - \hat{h}_{L}\|_{L^{1}(B_{2r_{H}(p_{H}, \pi_{H}))}} \leq Cm_{0} \ell(H)^{m+3+2\beta/3} \leq Cm_{0} \ell(H)^{m+3+\kappa}.
\] (5.42)

If \( \ell(H) = \frac{1}{2} \ell(L) \), then let \( J \) be the father of \( H \). Obviously, \( J \cap L \neq \emptyset \). You can therefore apply Lemma 5.5 above to infer \( \|h_{HJ} - \hat{h}_{L}\|_{L^{1}(B_{2r_{HJ}(p_{J}, \pi_{H})})} \leq Cm_{0} \ell(J)^{m+3+2\beta/3} \). On the other hand, by Lemma 5.3, \( \|h_{H} - h_{HJ}\|_{L^{1}(B_{2r_{H}(p_{H}, \pi_{H}))}} \leq Cm_{0} \ell(H) - h_{HJ}\|_{0} \leq Cm_{0} \ell(J)^{m+3+\kappa} \). Thus we conclude (5.42) as well. Note that \( G_{gL} \in L_{C\ell_{\epsilon}}(x_{H}, \pi_{0}) = G_{h_{L}} \in L_{C\ell_{\epsilon}}(x_{H}, \pi_{0}) \) and that the same property holds with \( g_{H} \) and \( h_{H} \). We can thus appeal to Lemma B.1 to conclude

\[
\|g_{H} - g_{L}\|_{L^{1}(B_{r_{H}(p_{H}, \pi_{0}))}} \leq Cm_{0} \ell(H)^{m+3+\kappa}.
\] (5.43)

However, recall also that \( [D^{3}(g_{H} - g_{L})]_{\pi} \leq Cm_{0}^{1/2} \). We can then apply Lemma C.2 to conclude (ii).

Now, if \( L \in \mathcal{P}^{j} \) and \( i \geq j \), consider the subset \( \mathcal{P}^{i}(L) \) of all cubes in \( \mathcal{P}^{i} \) which intersect \( L \). If \( L' \) is the cube concentric to \( L \) with \( \ell(L') = \frac{9}{8} \ell(L) \), we then have by definition of \( \varphi_{i} \):

\[
\|\varphi_{i} - g_{L}\|_{L^{1}(L')} \leq C \sum_{H \in \mathcal{P}^{i}(L)} \|g_{H} - g_{L}\|_{L^{1}(B_{r_{H}(p_{L}, \pi_{0}))}} \leq Cm_{0} \ell(H)^{m+3+\kappa},
\] (5.44)
which is the claim of (v).

As for (iii), observe first that the argument above applies also when \( L \) is the father of \( H \). Iterating then the corresponding estimates, it is easy to see that

\[
|D^3 g_H(x_H) - D^3 g_J(x_J)| \leq C m_0^{1/2} \ell(J)^\kappa \quad \text{for any ancestor } J \text{ of } H. \tag{5.45}
\]

Fix now any pair \( H, L \in \mathcal{P}^3 \). Let \( H_i, L_i \) be the “first ancestors” of \( H \) and \( L \) which are adjacent, i.e. among all ancestors of \( H \) and \( L \) with the same side-length \( \ell(H_i) = \ell(L_i) =: \ell \) and nonempty intersection, we assume the side-length is the smallest possible. We can therefore use the estimates obtained so far to conclude

\[
|D^3 g_H(x_H) - D^3 g_L(x_L)| \leq |D^3 g_H(x_H) - D^3 g_{H_i}(x_{H_i})| + |D^3 g_{H_i}(x_{H_i}) - D^3 g_{L_i}(x_{L_i})| + |D^3 g_{L_i}(x_{L_i}) - D^3 g_L(x_L)| \leq C m_0^{1/2} \ell^\kappa.
\]

A simple geometric consideration shows that \( |x_L - x_H| \geq c_0 \ell \), where \( c_0 \) is a dimensional constant, thus completing the proof.

6. Existence and Estimates for the \( \mathcal{M} \)-Normal Approximation

We start proving the corollary of Theorem 1.12.

6.1. Proof of Corollary 2.2. The first two statements of (i) follow immediately from Theorem 1.12(i) and Proposition 4.2(v). Coming to the third claim of (i), we extend the function \( \varphi \) to the entire plane \( \pi_0 \) by increasing its \( C^{3,\kappa} \) norm by a constant geometric factor. Let \( \varphi_t(x) := t \varphi(x) \) for \( t \in [0,1] \), \( \mathcal{M}_t := \text{Gr}(\varphi_t|_{-1,4^m}) \) and set

\[
U_t := \{ x + y : x \in \mathcal{M}_t, y \perp T_x \mathcal{M}_t, |y| < 1 \}.
\]

For \( \varepsilon_2 \) sufficiently small the orthogonal projection \( \pi_t : U_t \to \mathcal{M}_t \) is a well-defined \( C^{2,\kappa} \) map for every \( t \in [0,1] \), which depends smoothly on \( t \). It is also easy to see that \( \partial T \mathcal{L} U_t = \emptyset \). Thus, \( (\pi_t)_* (T \mathcal{L} U_t) = Q(t) [\mathcal{M}_t] \) for some integer \( Q(t) \). On the other hand these currents depend continuously on \( t \) and therefore \( Q(t) \) must be a constant. Since \( \mathcal{M}_0 = [ -4, 4^{m} \times \{0\} ] \subset \pi_0 \) and \( \mathcal{P}_0 = \mathcal{P}_\varepsilon \), we conclude \( Q(0) = Q \).

For what concerns (ii), consider \( q \in L \in \mathcal{W} \), set \( p := \Phi(q) \) and \( \pi := T_p \mathcal{M} \), whereas \( \pi_L \) is as in Definition 1.10. Let \( J \) be the cube concentric to \( L \) and with side-length \( \frac{17}{16} \ell(L) \). By the definition of \( \varphi \), Theorem 1.12(ii) and Proposition 4.5, we have that, denoting by \( \tilde{\varphi} \) and \( \bar{g}_L \) the first \( \tilde{n} \) components of the corresponding maps,

\[
\| \varphi - \bar{g}_L \|_{C^0(J)} \leq C \sum_{H \in \mathcal{W}, H \cap L \neq \emptyset} \| g_L - g_H \|_{C^0} \leq C m_0^{1/2} \ell(L)^{3+\kappa}.
\]

So, since \( \varphi = (\varphi, \Psi(x, \varphi)) \) and \( g_H = (\bar{g}_H, \Psi(x, \bar{g}_H)) \), we conclude \( \| g_L - \varphi \|_{C^0(J)} \leq C m_0^{1/2} \ell(L)^{3+\kappa} \). On the other hand the graph of \( g_L \) coincides with the graph of the tilted interpolating function \( h_L \). Consider in \( C := C_{32r_L}(p_L, \pi_L) \) the \( \pi_L \)-approximation \( f_L \) used in the construction algorithm and recall that, by [6, Theorem 1.4],

\[
\text{osc} (f_L) \leq C (h(T, C_{32r_L}(p_L, \pi_L), \pi_L) + ((E(T, C_{32r_L}(p_L, \pi_L))^{1/2} + r_L A)r_L) \leq C m_0^{1/2m} \ell(L)^{1+\beta_2}.
\]
Setting \( p_L = (z_L, w_L) \in \pi_L \times \pi_L^L \) and recalling that \( p_L \in \text{spt}(T) \), we easily conclude that 
\[
\| \eta \circ f_L - w_L \|_{C^0} \leq Cm_0^{1/2m} \ell(L)^{1+\beta_2}. 
\]
This implies \( \| h_L - w_L \|_{C^0} \leq Cm_0^{1/2m} \ell(L)^{1+\beta_2} \). Putting all these estimates together, we easily conclude that, for any point \( p \in \text{spt}(T) \cap C_\tau \pi_L(p_L, \pi_L) \) the distance to the graph of \( h_L \) is at most \( Cm_0^{1/2m} \ell(L)^{1+\beta_2} \).

Finally, we show (iii). Fix a point \( p \in \Gamma \). By construction, there is an infinite chain \( L_{N_0} \supset L_{N_0+1} \supset \ldots \supset L_j \supset \ldots \) of cubes \( L_j \in \mathcal{S}^j \) such that \( p = \bigcap_j L_j \). Set \( \pi_j := \pi_{L_j} \). From Proposition 4.2 we infer that the planes \( \pi_j \) converge to a plane \( \pi \) with a rate \( |\pi_j - \pi| \leq Cm_0^{1/2m} \ell(L)^{1+\beta_2} \). Moreover, the rescaled currents \( (t_{p_{L_j}, 2^{-j}})T \) (where the map \( t_{q, r} \) is given by \( t_{q, r}(z) = \frac{z - q}{r} \)) converge to \( Q[\pi] \). Since \( |\Phi(p) - p_L| \leq C\sqrt{m} \ell^{-j} \) for some constant \( C \) independent of \( j \), we easily conclude that \( \Theta(T, \Phi(p)) = Q \) and \( Q[\pi] \) is the unique tangent cone to \( T \) at \( \Phi(p) \). We next show that \( p^{-1}(\Phi(p)) = \{ \Phi(p) \} \). Indeed, assume there were \( q \neq \Phi(p) \) such that \( p(q) = \Phi(p) \) and let \( j \) be such that \( 2^{-j-1} \leq |\Phi(p) - q| \leq 2^{-j} \). Provided \( \varepsilon_2 \) is sufficiently small, Proposition 4.2(v) guarantees that \( j \geq N_0 \). Consider the cube \( L_j \) in the chain above and recall that \( h(T, C_32r_{L_j}(p_{L_j}, \pi_j)) \leq Cm_0^{1/2m} \ell^{-j(1+\beta_2)} \). Hence,
\[
2^{-j-1} \leq |q - \Phi(p)| = |p_\pi(q - \Phi(p))| \leq C|q - \Phi(p)||\pi - \pi_j| + h(T, C_32r_{L_j}(p_{L_j}, \pi_j)) \\
\leq Cm_0^{1/2m} \ell^{-j(1+\beta_2)} \ell^{-j} + Cm_0^{1/2m} \ell^{-j(1+\beta_2)} \leq C\varepsilon_2^{1/2m} \ell^{-j},
\]
which, for an appropriate choice of \( \varepsilon_2 \) (depending only on the various other parameters \( \beta_2, \delta_2, \gamma_1, C_\varepsilon, C_h, M_0, N_0 \)) is a contradiction.

6.2. Construction of the \( M \)-normal approximation and first estimates. We set \( F(p) = Q[p] \) for \( p \in \Phi(\Gamma) \). For every \( L \in \mathcal{W}^j \) consider the \( \pi_L \)-approximating function \( f_L : C_{8r_{L_j}}(p_{L_j}, \pi_L) \to A_Q(\pi_L^L) \) of Definition 1.9 and \( K_L \subset B_{8r_{L_j}}(p_{L_j}, \pi_L) \) the maximal (closed) set such that \( G_{f_L, K_L} = T\mathcal{L}(K_L \times \pi_L^L) \). We then denote by \( \mathcal{D}(L) \) the portions of the supports of \( T \) and \( \text{Gr}(f_L) \) which differ:
\[
\mathcal{D}(L) := (\text{spt}(T) \cup \text{Gr}(f_L)) \cap [(B_{8r_{L_j}}(p_{L_j}, \pi_L) \setminus K_L) \times \pi_L^L].
\]
Observe that, by [6, Theorem 1.4] and Assumption 1.5, we have
\[
\mathcal{H}^n(\mathcal{D}(L)) \leq CE^{\gamma_1}(E + \ell(L)^2 \mathbf{A}^2)\ell(L)^m \leq Cm_0^{1+\gamma_2} \ell(L)^{m+2+\gamma_2}, \tag{6.1}
\]
where \( E = E(T, C_32r_{L_j}(p_{L_j}, \pi_L)) \). Let \( \mathcal{L} \) be the Whitney region in Definition 1.13 and set \( \mathcal{L}' := \Phi(J) \) where \( J \) is the cube concentric to \( L \) with \( \ell(J) = \frac{9}{8} \ell(L) \). Observe that our choice of the constants is done in such a way that,
\[
L \cap H = \emptyset \quad \iff \quad \mathcal{L}' \cap \mathcal{H}' = \emptyset, \tag{6.2}
\]
\[
\Phi(\Gamma) \cap \mathcal{L}' = \emptyset \quad \forall L \in \mathcal{W}. \tag{6.3}
\]
We then apply [5, Theorem 5.1] to obtain maps \( F_L, N_L : \mathcal{L}' \to A_Q(U) \) with the following properties:
- \( F_L(p) = \sum_i [p + (N_L)_i(p)] \).
- \((N_L)_i(p) \perp T_p M \) for every \( p \in \mathcal{L}' \).
- \( T_{F_L} = T_{F_L}(p^{-1}(\mathcal{L}')) \).
For each $L$ consider the set $\mathcal{W}(L)$ of elements in $\mathcal{W}$ which have a nonempty intersection with $L$. We then define the set $\mathcal{K}$ in the following way:

$$\mathcal{K} = \mathcal{M} \setminus \left( \bigcup_{L \in \mathcal{W}} \left( \mathcal{L}' \cap \bigcup_{M \in \mathcal{W}(L)} \mathcal{p}(\mathcal{D}(M)) \right) \right). \quad (6.4)$$

Observe that, by (6.3), $\mathcal{K}$ contains necessarily $\Gamma$. Moreover, recall that $\text{Lip}(\mathcal{p}) \leq C$, that the cardinality $\mathcal{W}(L)$ is bounded by a geometric constant and that each element of $\mathcal{W}(L)$ has side-length at most twice that of $L$. Thus (6.1) implies

$$|\mathcal{L} \setminus \mathcal{K}| \leq |\mathcal{L}' \setminus \mathcal{K}| \leq \sum_{H \in \mathcal{W}(J)} \sum_{J \in \mathcal{W}(L)} \mathcal{p}(\mathcal{D}(H)) \leq Cm_0^{1+\gamma_2} \ell(L)^{m+2+\gamma_2}. \quad (6.5)$$

On $\Gamma$ we define $F(p) = Q[p]$. By (6.2), if $J$ and $L$ are such that $\mathcal{J}' \cap \mathcal{L}' \neq \emptyset$, then $J \in \mathcal{W}(L)$ and therefore $F_L = F_J$ on $\mathcal{K} \cap (\mathcal{J}' \cap \mathcal{L}')$. We can therefore define a unique map on $\mathcal{K}$ by simply setting $F(p) = F_L(p)$ if $p \in \mathcal{K} \cap \mathcal{L}'$. Our resulting map has obviously the Lipschitz bound of (2.1) in each $\mathcal{L} \cap \mathcal{K}$. Moreover, $T_F = T_L \mathcal{p}^{-1}(\mathcal{K})$, which implies two facts. First, by Corollary 2.2(ii) we also have that $N(p) := \sum_i [F_i(p) - p]$ enjoys the bound $\|N|_{\mathcal{L} \cap \mathcal{K}}\|_{C^0} \leq Cm_0^{1/2m} \ell(L)^{1+\beta_2}$. Secondly,

$$\|T\|_{(\mathcal{L} \setminus \mathcal{K})} \leq Q \sum_{M \in \mathcal{W}(L)} \sum_{H \in \mathcal{W}(M)} \mathcal{H}^m(\mathcal{D}(H)) \leq Cm_0^{1+\gamma_2} \ell(L)^{m+2+\gamma_2}. \quad (6.6)$$

Hence, $F$ and $N$ satisfy the bounds (2.1) on $\mathcal{K}$. We next extend them to the whole center manifold and conclude (2.2) from (6.6) and (6.5). The extension is achieved in three steps:

- we first extend the map $F$ to a map $\tilde{F}$ taking values in $\mathcal{A}_Q(U)$;
- we then modify $\tilde{F}$ to achieve the form $\tilde{F}(x) = \sum_i [x + \tilde{N}_i(x)]$ with $\tilde{N}_i(x) \perp T_x \mathcal{M}$ for every $x$;
- we finally modify $\tilde{F}$ to reach the desired extension $F(x) = \sum_i [x + N_i(x)]$, with $N_i(x) \perp T_x \mathcal{M}$ and $x + N_i(x) \in \Sigma$ for every $x$.

**First extension.** We use on $\mathcal{M}$ the coordinates induced by its graphical structure, i.e. we work with variables in flat domains. Note that the domain parametrizing the Whitney region for $L \in \mathcal{W}$ is then the cube concentric to $L$ and with side-length $\frac{17}{16} \ell(L)$. The multi-valued map $N$ is extended to a multi-valued $\tilde{N}$ inductively to appropriate neighborhoods of the skeleta of the Whitney decomposition (a similar argument has been used in [4, Section 1.2.2]). The extension of $F$ will obviously be $\tilde{F}(x) = \sum_i [\tilde{N}_i(x) + x]$. The neighborhoods of the skeleta are defined in this way:

1. if $p$ belongs to the 0-skeleton, we let $L \in \mathcal{W}$ be (one of) the smallest cubes containing it and define $U^p := B_{\ell(L)/16}(p)$;
2. if $\sigma = [p, q] \subset L$ is the edge of a cube $L \in \mathcal{W}$, we then define $U^\sigma$ to be the neighborhood of size $\frac{1}{4} \frac{\ell(L)}{16}$ of $\sigma$ minus the closure of the unions of the $U^r$’s, where $r$ runs in the 0-skeleton;
(3) we proceed inductively till the $m - 1$-skeleton: given a $k$-dimensional facet $\sigma$ of a cube $L$, $U^\sigma$ is its neighborhood of size $4^{-k} \frac{\ell(L)}{16}$ minus the closure of the union of all $U^\tau$'s, where $\tau$ runs among all facets of dimension at most $k - 1$.

Denote by $\bar{U}$ the closure of the union of all these neighborhoods and let $\{V_i\}$ be the connected components of the complement. For each $V_i$ there is a $L_i \in \mathcal{W}$ such that $V_i \subset L_i$. Moreover, $V_i$ has distance $c_0 \ell(L)$ from $\partial L_i$, where $c_0$ is a geometric constant. It is also clear that if $\tau$ and $\sigma$ are two distinct facets of the same cube with the same dimension, then the distance between any pair of points $x, y$ with $x \in U^\tau$ and $y \in U^\sigma$ is at least $c_0 \ell(L)$.

Cp. with Figure 1.

![Figure 1. The sets $U^p$, $U^\sigma$ and $V_i$.](image)

At a first step we extend $N$ to a new map $\tilde{N}$ separately on each $U^p$, where $p$ are the points in the 0-skeleton. Fix $p \in L$ and let $\text{St}(p)$ be the union of all cubes which contain $p$. Observe that the Lipschitz constant of $N|_{U^p \cup (\mathcal{K} \cap \text{St}(p))}$ is smaller than $C m_0^{\gamma_2} \ell(L)^{\gamma_2}$ and that $|N| \leq C m_0^{1/2m} \ell(L)^{1+\beta_2}$. We can therefore extend the map $N$ to $U^p$ at the price of slightly enlarging this Lipschitz constant and this height bound, using [4, Theorem 1.7]. Being the $U^p$ disjoint, the resulting map, for which we use the symbol $\tilde{N}$, is well-defined.

It is obvious that this map has the desired height bound in each Whitney region. We therefore want to estimate its Lipschitz constant. Consider $L \in \mathcal{W}$ and $H$ concentric to $L$ with side-length $\ell(H) = \frac{17}{16} \ell(L)$. Let $x, y \in H$. If $x, y \in \mathcal{K}$, then there is nothing to check. If $y \in U^p$ for some $p$ and $x \not\in \bigcup_q U^q$, then $x \in \text{St}(p)$ and $G(\tilde{N}(x), \tilde{N}(y)) \leq C m_0^{\gamma_2} \ell(L)^{\gamma_2} |x - y|$. The same holds when $x, y \in U^p$. The remaining case is $x \in U^p$ and $y \in U^q$ with $p \neq q$. Observe however that this would imply that $p, q$ are both vertices of $L$. Given that $L \setminus \mathcal{K}$ has much smaller measure than $L$ there is at least one point $z \in L \cap \mathcal{K}$. It is then obvious that

$$G(\tilde{N}(x), \tilde{N}(y)) \leq G(\tilde{N}(x), \tilde{N}(z)) + G(\tilde{N}(z), \tilde{N}(y)) \leq C m_0^{\gamma_2} \ell(L)^{\gamma_2} \ell(L),$$

and, since $|x - y| \geq c_0 \ell(L)$, the desired bound readily follows. The map is also Lipschitz in any neighborhood of a point $x \in \Gamma$. Thus we can extend it to the closure of its domain,
which indeed, by the property of the Whitney decomposition, is simply the union of $\mathcal{K}$ and the closures of the $U^p$’s.

This procedure can now be iterated over all skeleta inductively on the dimension $k$ of the corresponding skeleton, up to $k = m - 1$: in the argument above we simply replace points $p$ with $k$-dimensional faces $\sigma$, defining $\text{St}(\sigma)$ as the union of the cubes which contain $\sigma$. In the final step we then extend over the domains $V_i$’s: this time $\text{St}(V_i)$ will be defined as the union of the cubes which intersect the cube $L_i \supset V_i$. The correct height and Lipschitz bounds follow from the same arguments. Since the algorithm is applied $m + 1$ times, the original constants have been enlarged by a geometric factor.

**Second extension: orthogonality.** For each $x \in \mathcal{M}$ let $p^\perp(x, \cdot) : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ be the orthogonal projection on $(T_x \mathcal{M})^\perp$ and set $\tilde{N}(x) = \sum_i \|p^\perp(x, \tilde{N}_i(x))\|$. Obviously $|\tilde{N}(x)| \leq |\tilde{N}(x)|$, so the $L^\infty$ bound is trivial. We now want to show the estimate on the Lipschitz constant. To this aim, fix two points $p, q$ in the same Whitney region associated to $L$ and parameterize the corresponding geodesic segment $\sigma \subset \mathcal{M}$ by arc-length $\gamma : [0, d(p, q)] \to \sigma$, where $d(p, q)$ denotes the geodesic distance on $\mathcal{M}$. Use [4, Proposition 1.2] to select $Q$ Lipschitz functions $N_i' : \sigma \to U$ such that $\tilde{N}|_\gamma = \sum N_i'$ and $\text{Lip}(N_i') \leq \text{Lip}(\tilde{N})$. Fix a frame $\nu_1, \ldots, \nu_n$ on the normal bundle of $\mathcal{M}$ with the property that $\|D\nu_i\|_{c^0} \leq C$ (which is possible since $\mathcal{M}$ is the graph of a $C^{3,\kappa}$ function, cp. [5, Appendix A]). We have $\tilde{N}(\gamma(t)) = \sum_i [\tilde{N}_i(t)]$, where

$$\tilde{N}_i(t) = N_i'(\gamma(t)) - \sum [\nu_j(\gamma(t)) \cdot N_i''(\gamma(t))] \nu_j(t).$$

Hence we can estimate

$$\left| \frac{d\tilde{N}_i}{dt} \right| \leq CL \text{Lip}(N_i') + C \sum_j \|D\nu_j\| \|N_i'\|_{c^0} \leq C m_0^{1/2} \ell(L)^{\gamma_2} + C m_0^{1/2} \ell(L)^{1+\beta_2} \leq C m_0^{1/2} \ell(L)^{\gamma_2}.$$

Integrating this inequality we find

$$G(\tilde{N}(p), \tilde{N}(q)) \leq \sum_{i=1}^Q |\tilde{N}_i(d(p, q)) - \tilde{N}_i(0)| \leq C m_0^{1/2} \ell(L)^{\gamma_2} d(p, q).$$

Since $d(p, q)$ is comparable to $|p - q|$, we achieve the desired Lipschitz bound.

**Third extension and conclusion.** For each $x \in \mathcal{M} \subset \Sigma$ consider the orthogonal complement $\nu_x$ of $T_x \mathcal{M}$ in $T_x \Sigma$. Let $\mathcal{T}$ be the fiber bundle $\bigcup_{x \in \mathcal{M}} \nu_x$ and observe that, by the regularity of both $\mathcal{M}$ and $\Sigma$ there is a global $C^{2,\kappa}$ trivialization (argue as in [5, Appendix A]). It is then obvious that there is a $C^{2,\kappa}$ map $\Xi : \mathcal{T} \to \mathbb{R}^{m+n}$ with the following property: for each $(x, v)$, $q := x + \Xi(x, v)$ is the only point in $\Sigma$ which is orthogonal to $T_x \mathcal{M}$ and such that $p_{\nu_x}(q - x) = v$. We then set $\hat{N}(x) = \sum_i [\Xi(x, p_{\nu_x}(\tilde{N}_i(x)))]$. Obviously, $\hat{N}(x) = \tilde{N}(x)$ for $x \in \mathcal{K}$, simply because in this case $x + \tilde{N}_i(x)$ belongs to $\Sigma$.

In order to show the Lipschitz bound, denote by $\Omega(x, q)$ the map $\Xi(x, p_{\nu_x}(q))$. $\Omega$ is a $C^{2,\kappa}$ map. Thus

$$|\Omega(x, q) - \Omega(x, p)| \leq C |q - p|. \quad (6.7)$$
Moreover, since \( \Omega(x,0) = 0 \) for every \( x \), we have \( D_2 \Omega(x,0) = 0 \). We therefore conclude that \(|D_2 \Omega(x,q)| \leq C|q|\) and hence that

\[
|\Omega(x,q) - \Omega(y,q)| \leq C|q||y - x|. \tag{6.8}
\]

Thus, fix two points \( x, y \in \mathcal{L} \) and let assume that \( \mathcal{G}(\hat{N}(x), \hat{N}(y))^2 = \sum_i |\hat{N}_i(x) - \hat{N}_i(y)|^2 \) (which can be achieved by a simple relabeling). We then conclude

\[
\mathcal{G}(N(x), N(y))^2 \leq 2 \sum_i |\Omega(x, \hat{N}_i(x)) - \Omega(x, \hat{N}_i(y))|^2 + 2 \sum_i |\Omega(x, \hat{N}_i(y)) - \Omega(y, \hat{N}_i(y))|^2
\]

\[
\leq C \mathcal{G}(\hat{N}(x), \hat{N}(y))^2 + C \sum_i |\hat{N}_i(y)|^2 |x - y|^2
\]

\[
\leq C \mathbf{m}^{2\gamma_2} \ell(L)^{2\gamma_2} |x - y|^2 + C \mathbf{m}_0^{1/2} \ell(L)^{1 + \beta_2} |x - y|^2. \tag{6.9}
\]

This proves the desired Lipschitz bound. Finally, using the fact that \( \Omega(x,0) = 0 \), we have \(|\Omega(x,v)| \leq C|v|\) and the \( \ell^\infty \) bound readily follows.

6.3. **Estimates** (2.3) and (2.4). First consider the cylinder \( \mathbf{C} := \mathbf{C}_{srL}(p_L, \pi_L) \). Denote by \( \tilde{\mathbf{M}} \) the unit \( m \)-vector orienting \( TM \) and by \( \tilde{\tau} \) the one orienting \( TG_{h_L} = TG_{g_L} \). Recalling that \( g_L \) and \( \varphi \) coincide in a neighborhood of \( x_L \), by Proposition 4.5(iv) we have

\[
\sup_{p \in M \cap \mathbf{C}} |\tilde{\tau}(x_L, g_L(x_L)) - \tilde{\mathbf{M}}(p)| \leq C \|D^2 \varphi\|_{C^0} \ell(L) \leq C \mathbf{m}_0^{1/2} \ell(L).
\]

Recalling moreover that \( \|Dh_L\|_L^2 \leq C \text{Dir}(f_L) \leq C \mathbf{m}_0 \ell(L)^{2 - 2\delta_2} \), we also conclude the existence of at least one point \( q \in \mathbf{C} \cap M \) such that \( |\tilde{\mathbf{M}}(q) - \pi_L| \leq C \mathbf{m}_0^{1/2} \ell(L)^{1 - \delta_2} \). Since \( \|D^2 h_L\| \leq C \mathbf{m}_0^{1/2} \) we then conclude \( |\tilde{\tau}(x_L, g_L(x_L)) - \tau(q)| \leq C \mathbf{m}_0^{1/2} \ell(L) \), which intern implies This in turn implies that \( \sup_{\mathbf{C} \cap M} |\tilde{\mathbf{M}} - \pi_L| \leq C \mathbf{m}_0^{1/2} \ell(L)^{1 - \delta_2} \). Therefore, we can estimate

\[
\int_{\mathbf{p}^{-1}(\mathcal{L})} |\tilde{T}_F(p) - \tilde{\mathbf{M}}(p)(x)|^2 d\|T_F\|(x)
\]

\[
\leq C \int_{\mathbf{p}^{-1}(\mathcal{L})} |\tilde{T}(x) - \tilde{\mathbf{M}}(p)(x)|^2 d\|T\|(x) + C \mathbf{m}_0^{1+\gamma_2} \ell(L)^{m+2+\gamma_2}
\]

\[
\leq C \int_{\mathbf{p}^{-1}(\mathcal{L})} |\tilde{T}(x) - \tilde{\pi}_{L}|^2 d\|T\|(x) + C \mathbf{m}_0 \ell(L)^{m+2-2\delta_2} \tag{6.10}
\]

In turn, since \( \mathbf{p}^{-1}(\mathcal{L}) \cap \text{spt}(T) \subset \mathbf{C} \), the integral in (6.10) is smaller than \( C \ell(L)^m \mathbf{E}(T, \mathbf{C}, \pi_L) \). By [5, Proposition 3.4] we then conclude

\[
\int_{\mathcal{L}} |DN|^2 \leq C \int_{\mathbf{p}^{-1}(\mathcal{L})} |\tilde{T}_F(p) - \tilde{\mathbf{M}}(p)(x)|^2 d\|T_F\|(x) + C \|A_M\|_{C^0}^2 \int_{\mathcal{L}} |N|^2
\]

\[
\leq C \mathbf{m}_0 \ell(L)^{m+2-2\delta_2} + C \mathbf{m}_0 \ell(L)^{m+2+2\beta_2},
\]

where we have used \( \|A_M\|_{C^0} \leq C \|D^2 \varphi\|_{C^0} \leq C \mathbf{m}_0^{1/2} \).
We finally come to (2.4). First observe that, by (2.1) and (2.2),

\[
\int_{\mathcal{L}\setminus\mathcal{K}} |\eta \circ N| \leq C m_0^{1/2m} \ell(L)^{1+\beta_2} |\mathcal{L}\setminus\mathcal{K}| \leq C m_0^{1+\gamma_2+1/2m} \ell(L)^{m+3+\beta_2+\gamma_2}. \tag{6.11}
\]

Fix now \( p \in \mathcal{K} \). Recalling that \( F_L(x) = \sum_i [p + (N_L)_i(p)] \) is given by [5, Theorem 5.1] applied to the map \( f_L \), we can use [5, Theorem 5.1(5.4)] to conclude

\[
|\eta \circ N_L(p)| \leq C |\eta \circ f_L(p_{\pi_L}(p))| + C \text{Lip}(N_L|\mathcal{L}) |T_p \mathcal{M} - \pi_L| |N_L|(p)
\]

\[
\leq C |\eta \circ f_L(p_{\pi_L}(p))| + C m_0^{1/2+\gamma_2} \ell(L)^{1+\gamma_2-\delta_2} \mathcal{G}(N_L(p), Q [\eta \circ N_L(p)]) + Q |\eta \circ N_L|(p).
\]

For \( \varepsilon_2 \) sufficiently small (depending only on \( \beta_2, \gamma_2, M_0, N_0, C_e, C_h \)), we then conclude that

\[
|\eta \circ N_L(p)| \leq C |\eta \circ f_L(p_{\pi_L}(p))| + C m_0^{1/2+\gamma_2} \ell(L)^{1+\gamma_2-\delta_2} \mathcal{G}(N_L(p), Q [\eta \circ N_L(p)])
\]

\[
\leq C |\eta \circ f_L(p_{\pi_L}(p))| - p_{\pi_L}^+ (p) + C a m_0^{1+\gamma_2} \ell(L)^{(1+2\gamma_2-\delta_2)/1+\gamma_2}
\]

\[
+ \frac{C}{a} \mathcal{G}(N_L(p), Q [\eta \circ N_L(p)])^{2+\gamma_2}.
\tag{6.12}
\]

Our choice of \( \delta_2 \) makes the exponent \( (1+2\gamma_2-\delta_2)/1+\gamma_2 \) larger than \( 2+\gamma_2 \). Let next \( \varphi' : \pi_L \rightarrow \pi_L^+ \) such that \( \mathcal{G}_{\varphi'} = \mathcal{M} \). Applying Lemma B.1 we conclude that

\[
\int_{K \cap \mathcal{V}} |\eta \circ f_L(p_{\pi_L}(p)) - p_{\pi_L^+}^+(p)| \leq \int_{p_{\pi_L}^+ (K \cap \mathcal{V})} |\eta \circ f_L(x) - \varphi'(x)| \leq C \int_H |g_L(x) - \varphi(x)|,
\]

where \( H \) is a cube concentric to \( L \) with side-length \( \ell(H) = \frac{2}{3} \ell(L) \). From Proposition 4.5(v) we get \( \|\varphi - g_L\|_{L^1(H)} \leq C m_0 \ell(L)^{m+3+\beta_2/3} \) and (2.4) follows integrating (6.12).

7. Separation and splitting before tilting

7.1. Vertical separation. In this section we prove Proposition 3.1 and Corollary 3.2.

Proof of Proposition 3.1. Let \( J \) be the father of \( L \). By Proposition 4.2, Theorem A.1 can be applied to the cylinder \( C := C_{3l_{r_j}}(p_J, \pi_J) \). Moreover, \( |p_J - p_L| \leq C \ell(J) \), where \( C \) is a geometric constant, and \( r_J = 2l_{r_j} \). Thus, if \( M_0 \) is larger than a geometric constant, we have \( B_L \subset C_{3r_{r_j}}(p_J, \pi_J) \). Denote by \( q_L, q_J \) the projections \( p_{\pi_L^+} \) and \( p_{\pi_J^+} \) respectively. Since \( L \in \mathcal{W}_h \), there are two points \( p_1, p_2 \in \text{spt}(T) \cap B_L \) such that \( |q_L(p_1 - p_2)| \geq C h m_0^{1/2m} \ell(L)^{1+\beta_2} \).

On the other hand, recalling Proposition 4.2, \( |\pi_J - \pi_L| \leq C e^{1/2} \ell(L)^{1-\delta_2} \), where \( C \) is a geometric constant. Thus,

\[
|q_J(p_1 - p_2)| \geq |q_L(p_1 - p_2)| - C_2 |\tilde{\pi}_L - \pi_L| |p_1 - p_2| \geq C h m_0^{1/2m} \ell(L)^{1+\beta_2} - C C e^{1/2} m_0^{1/2} \ell(L)^{2-\delta_2},
\]

where the constant \( C \) depends upon \( M_0, N_0 \) and the dimensions. Hence, if \( \varepsilon_2 \) is sufficiently small, we actually conclude

\[
|q_J(p_1 - p_2)| \geq \frac{15}{16} C h m_0^{1/2m} \ell(L)^{1+\beta_2}.
\tag{7.1}
\]
Set \( E := E(T, C_{36r_J}(p_J, \pi_J)) \) and apply Theorem A.1 to \( C \): the union of corresponding “stripes” \( S_i \) the set \( \text{spt}(T) \cap C_{36r_J}(1-CE^{1/2m|\log E|})(p_J, \pi_J) \), where \( C \) is a geometric constant. We can therefore assume that they contain \( \text{spt}(T) \cap C_{34r_J}(p_J, \pi_J) \). The width of these stripes is bounded as follows:

\[
\sup \{ |q_J(x - y)| : x, y \in S_i \} \leq C E^{1/2m} r_J \leq C C_e^{1/2m} m_0^{1/2m} \ell(L)^{1+(2-2\delta)/2m},
\]

where \( C \) is a dimensional constant. So, if \( C^e \) is chosen large enough, we actually conclude that \( p_1 \) and \( p_2 \) must belong to at least two different stripes, say \( S_1 \) and \( S_2 \). By Theorem A.1(iii) we conclude that all points in \( C_{34r_J}(p_J, \pi_J) \) have density \( \Theta \) strictly smaller than \( Q - \frac{1}{2} \), thereby implying (S1). Moreover, by choosing \( C^e \) appropriately, we achieve that

\[
|q_J(x - y)| \geq \frac{7}{8} C_h m_0^{1/2m} \ell(L)^{1+\delta_2} \quad \forall x \in S_1, y \in S_2.
\]

(7.2)

Assume next there is \( H \in \mathcal{U}_n \) with \( \ell(H) \leq \frac{1}{2} \ell(L) \) and \( H \cap L \neq \emptyset \). From our construction it follows that \( \ell(H) = \frac{1}{2} \ell(L) \), \( B_H \subset C_{34r_J}(p_J, \pi_J) \) and \( |\pi_H - \pi_J| \leq C C_e^{1/2m_0^{1/2m}} \ell(L)^{1-\delta_2} \), for a geometric constant \( C \) (see again Proposition 4.2). We then conclude

\[
|p_{\pi_H}^+(x - y)| \geq \frac{3}{4} C_h m_0^{1/2m} \ell(L)^{1+\delta_2} \geq \frac{3}{2} C_h m_0^{1/2m} \ell(L)^{1+\delta_2} \quad \forall x \in S_1, y \in S_2.
\]

(7.3)

Now, recalling Proposition 4.2, if \( \varepsilon_2 \) is sufficiently small, \( C_{32r_H}(p_H, \pi_H) \) and \( \text{spt}(T) \subset B_H \). Moreover, by Theorem A.1(ii),

\[
(p_{\pi_J})_z(T \cup (S_i \cap C_{32r_H}(p_H, \pi_J))) = Q_i [B_{32r_H}(p_H, \pi_J)] \quad \text{for } i = 1, 2, Q_i \geq 1.
\]

A simple argument already used several other times allows to conclude that instead

\[
(p_{\pi_H})_z(T \cup (S_i \cap C_{32r_H}(p_H, \pi_H))) = Q_i [B_{32r_H}(p_H, \pi_H)] \quad \text{for } i = 1, 2, Q_i \geq 1.
\]

Thus, \( B_H \) must necessarily contain two points \( x, y \) with \( |p_{\pi_H}^+(x - y)| \geq \frac{3}{2} C_h m_0^{1/2m} \ell(H)^{1+\delta_2} \). Given that \( |\pi_H - \pi_J| \leq C C_e^{1/2m_0^{1/2m}} \ell(H)^{1-\delta_2} \), for some dimensional constant \( C \), we conclude that

\[
|p_{\pi_H}^+(x - y)| \geq \frac{5}{4} C_h m_0^{1/2m} \ell(H)^{1+\delta_2} \text{, i.e. the cube } H \text{ satisfies the stopping condition (HT), which has “priority over the condition (NN)” and thus it cannot belong to } \mathcal{U}_n. \text{ This shows (S2).}
\]

Coming to (S3), observe that, for each \( p \in \mathcal{K} \cap \mathcal{L} \), the support of \( p + N(p) \) must contain at least one point \( p + N_1(p) \in S_1 \) and at least one point \( p + N_2(p) \in S_2 \). Now,

\[
|N_1(p) - N_2(p)| \geq \frac{7}{8} C_h m_0^{1/2m} \ell(L)^{1+\delta_2} \ell(L) |T_p \mathcal{M} - \pi_J|.
\]

(7.4)

Recalling, however, Proposition 4.5 and that \( \mathcal{M} \) and \( \text{Gr}(g_H) \) coincide on a nonempty open set, we easily conclude that \( |T_p \mathcal{M} - \pi_J| \leq C m_0^{1/2m} \ell(L)^{1-\delta_2} \) and, via (7.4),

\[
\mathcal{G}(N(p), Q [\eta \circ N(p)]) \geq \frac{1}{2} |N_1(p) - N_2(p)| \geq \frac{3}{8} C_h m_0^{1/2m} \ell(L)^{1+\delta_2}.
\]

Next observe that, since \( |\mathcal{L} \setminus \mathcal{K}| \leq C m_0^{1+\gamma_2} \ell(L)^{m+2+\gamma_2} \), for every point \( p \in \mathcal{L} \) there exists \( q \in \mathcal{K} \cap \mathcal{L} \) which has geodesic distance to \( p \) at most \( C m_0^{1/2m+\gamma_2/2m+\gamma_2/m} \). Given the
Lipschitz bound for $N$ and the choice $\beta_2 \leq \frac{1}{2m}$, we then easily conclude (S3):
\[
\mathcal{G}(N(q), Q \llbracket \eta \circ N(q) \rrbracket) \geq \frac{3}{8} C_h m_0^{1/2m} \ell(L)^{1+\beta_2} - Cm_0^{1/m} \ell(L)^{1+2/m} \geq \frac{1}{4} C_h m_0^{1/2m} \ell(L)^{1+\beta_2} .
\]

Proof of Corollary 3.2. The proof is straightforward. Consider any $H \in \mathcal{W}^j$. By definition it has a nonempty intersection with some cube $J \in \mathcal{W}^{j-1}$: this cube cannot belong to $\mathcal{W}_h$ by Proposition 3.1. It is then either an element of $\mathcal{W}_e$ or an element $H_{j-1} \in \mathcal{W}^{j-1}$. Proceeding inductively, we then find a chain $H = H_j, H_{j-1}, \ldots, H_t =: L$, where $H_t \cap H_{t-1} \neq \emptyset$ for every $l, H_t \in \mathcal{W}_n^l$ for every $l > i$ and $L = H_i \in \mathcal{W}_c^i$. This chain is not unique, however we can choose one for each element $H \in \mathcal{W}_n$. We then define $\mathcal{W}_n(L) = \{H \in \mathcal{W}_n: \text{the chain of } H \text{ ends with } L\}$. Observe that, if $H \in \mathcal{W}_n(L)$ and $H = H_j, H_{j-1}, \ldots, H_t = L$ is the corresponding chain, then
\[
|x_H - x_L| \leq \sum_{i=1}^{j-1} |x_{H_i} - x_{H_{i+1}}| \leq \sqrt{m} \ell(L) \sum_{i=1}^{\infty} 2^{l-i} \leq 2\sqrt{m} \ell(L) .
\]
It then follows easily that $H \subset B_{3\sqrt{m} \ell(L)}(L)$. \hfill \Box

7.2. Unique continuation for Dir-minimizers. Proposition 3.3 is based on a De Giorgi-type decay estimate for Dir-minimizing $Q$-valued maps which are close to a classical harmonic function with multiplicity $Q$. The argument involves a unique continuation-type result for Dir-minimizers.

Lemma 7.1 (Unique continuation for Dir-minimizers). For every $\eta \in (0, 1)$ and $c > 0$, there exists $\gamma > 0$ with the following property. If $w : \mathbb{R}^m \supset B_{2r} \to A_Q(\mathbb{R}^n)$ is Dir-minimizing, $\text{Dir}(w, B_r) \geq c$ and $\text{Dir}(w, B_{2r}) = 1$, then
\[
\text{Dir}(w, B_s(q)) \geq \gamma \text{ for every } B_s(q) \subset B_{2r} \text{ with } s \geq \eta r .
\]

Proof. We start showing the following claim:

(UC) if $\Omega$ is a connected open set and $w \in W^{1,2}(\Omega, A_Q(\mathbb{R}^n))$ is Dir-minimizing in any open $\Omega' \subset \subset \Omega$, then either $w$ is constant or $\int_{\Omega} |Dw|^2 > 0$ on any open $J \subset \Omega$.

We prove (UC) by induction on $Q$. If $Q = 1$, this is the classical unique continuation for harmonic functions. Assume now it holds for all $Q' < Q$ and we prove it for $Q$-valued maps. Assume $w \in W^{1,2}(\Omega, A_Q(\mathbb{R}^n))$ and $J \subset \Omega$ is an open set on which $|Dw| \equiv 0$. Without loss of generality, we can assume $J$ connected and $w|_J \equiv T$ for some $T \in A_Q$. Let $J'$ be the interior of $\{w = T\}$ and $K := J' \cap \Omega$. We prove now that $K$ is open, which in turn by connectedness of $\Omega$ concludes (UC). We distinguish two cases.

Case (a): the diameter of $T$ is positive. Since $w$ is continuous, for every $x \in K$ there is $B_{\rho}(x)$ where $w$ separates into $\llbracket w_1 \rrbracket + \llbracket w_2 \rrbracket$ and each $w_i$ is a $Q_i$-valued Dir-minimizer. Since $J' \cap B_{\rho}(x) \neq \emptyset$, each $w_i$ is constant in a (nontrivial) open subset of $B_{\rho}(x)$. By inductive hypothesis each $w_i$ is constant in $B_{\rho}(x)$ and therefore $w = T$ in $B_{\rho}(x)$, that is $B_{\rho}(x) \subset J' \subset K$. 
Case (b): $T = Q[p]$ for some $p$. Let $\tilde{J}$ be the interior of $\{ w = Q[\eta \circ w] \}$ and $\tilde{K} := \tilde{J} \cap \Omega$. By [4, Definition 0.10], $\Omega \cap \partial \tilde{J}$ is contained in the singular set of $w$. By [4, Theorem 0.11], $\mathcal{H}^{m-2+\varepsilon}(\Omega \cap \partial J^c) = 0$ for every $\varepsilon > 0$. Since $\tilde{J}$ is an open set, either $\Omega \cap \partial \tilde{J}$ is empty or it has positive $\mathcal{H}^{m-1}$ measure. We hence conclude that $\Omega \cap \partial \tilde{J} = \emptyset$, i.e. $\tilde{J} = \Omega$. This implies $w = Q[\eta \circ w]$, with $\eta \circ w$ harmonic function (cf. [4, Lemma 3.23]). Being $\eta \circ w |_{\tilde{J}} \equiv p$, by the classical unique continuation $\eta \circ w \equiv p$ on $\Omega$.

We now come to proof of the proposition. Without loss of generality, we can assume $r = 1$. Arguing by contradiction, there exists sequences $\{w_k\}_{k \in \mathbb{N}} \subset W^{1,2}(B_2, \mathcal{A}_Q(\mathbb{R}^n))$ and $\{B_{s_k}(q_k)\}_{k \in \mathbb{N}}$ with $s_k \geq \eta$ and such that $\text{Dir}(w_k, B_{s_k}(q_k)) \leq \frac{1}{k}$. By the compactness of Dir-minimizers (cp. [4, Proposition 3.20]), a subsequence (not relabeled) converges to $w \in W^{1,2}(B_2, \mathcal{A}_Q(\mathbb{R}^n))$ Dir-minimizing in every open $\Omega' \subset B_2$. Up to subsequences, we can also assume that $q_k \to q$ and $s_k \to s \geq \eta > 0$. Thus, $B_s(q) \subset B_2$ and $\text{Dir}(w, B_s(q)) = 0$. By (UC) this implies that $w$ is constant. On the other hand, by [4, Proposition 3.20] $\text{Dir}(w, B_1) = \lim_k \text{Dir}(w_k, B_1) \geq c > 0$ gives the desired contradiction. $\square$

Next we show that if the energy of a Dir-minimizer $w$ does not decay appropriately, then $w$ must split. In order to simplify the exposition, in the sequel we fix $\lambda > 0$ such that

$$(1 + \lambda)^{3n+2} < 2^{d_2}.$$  \hspace{1cm} (7.5)

**Proposition 7.2** (Decay estimate for Dir-minimizers). For every $\eta > 0$, there is $\gamma > 0$ with the following property. Let $w : \mathbb{R}^m \supset B_{2r} \to \mathcal{A}_Q(\mathbb{R}^n)$ be Dir-minimizing in every $\Omega' \subset B_{2r}$ such that

$$\int_{B_{(1+\lambda)r}} \mathcal{G}(Dw, Q[D(\eta \circ w)(0)])^2 \geq 2^{d_2-m-2} \text{Dir}(w, B_{2r}).$$  \hspace{1cm} (7.6)

Then, if we set $\tilde{w} = \sum_i [w_i - \eta \circ w]$, it holds

$$\gamma \text{Dir}(w, B_{(1+\lambda)r}) \leq \text{Dir}(\tilde{w}, B_{(1+\lambda)r}) \leq \frac{1}{\gamma} r^2 \int_{B_s(q)} |\tilde{w}|^2 \quad \forall B_s(q) \subset B_{2r} \text{ with } s \geq \eta r. \hspace{1cm} (7.7)$$

**Proof.** By a simple scaling argument we can assume $r = 1$ and we argue by contradiction. Let $w_k$ be a sequence of local Dir-minimizers which satisfy (7.6), $\text{Dir}(w_k, B_2) = 1$ and

(a) either $\int_{B_{s_k}(q_k)} |\tilde{w}_k|^2 \to 0$ for some sequence of balls $B_{s_k}(q_k) \subset B_{2r}$ with $s_k \geq \eta$;

(b) or $\text{Dir}(\tilde{w}_k, B_{1+\lambda}) \to 0$.

Up to subsequences, $w_k$ converges locally in $W^{1,2}$ to $w$ locally Dir-minimizing. If (a) holds, we can appeal to Lemma 7.1 and conclude that $\tilde{w} = \sum_i [w_i - \eta \circ w]$ vanishes identically on $B_2$. This means in particular that $\text{Dir}(\tilde{w}_k, B_{1+\lambda}) \to \text{Dir}(\tilde{w}, B_{1+\lambda}) = 0$, i.e. (b) holds.

Therefore, we can assume to be always in case (b). Let next $u_k := \eta \circ w_k$. From (7.6) we get

$$\int_{B_{1+\lambda}} Q[|Du_k - D\tilde{w}_k(0)|^2 = \int_{B_{1+\lambda}} (\mathcal{G}(Dw_k, Q[Du_k(0)])^2 - |D\tilde{w}_k|^2) \geq 2^{d_2-m-2} \int_{B_2} |Dw_k|^2 - \int_{B_{1+\lambda}} |D\tilde{w}_k|^2. \hspace{1cm} (7.8)$$
As \( k \uparrow \infty \), by (b) and \( \text{Dir}(w_k, B_2) = 1 \), we then conclude

\[
\int_{B_{1+\lambda}} |Du - Du(0)|^2 \geq 2^{d_2-m-2} \geq 2^{d_2-m-2} \int_{B_2} |Du|^2.
\]  

(7.9)

Since \( (1 + \lambda)^{m+2} < 2^{d_2} \), (7.9) violates the decay estimate for classical harmonic functions: 
\[
\int_{B_{1+\lambda}} |Du - Du(0)|^2 \leq 2^{d_2-m-2}(1 + \lambda)^{m+2} \int_{B_2} |Du|^2,
\]
thus concluding the proof. \( \square \)

### 7.3. Splitting before tilting I: Proof of Proposition 3.3.

Given \( L \in \mathcal{W}_e^j \), let us consider its ancestors \( H \in \mathcal{W}^{j-1} \) and \( J \in \mathcal{W}^{j-6} \). Set \( \ell = \ell(L) \), \( C := C_{8r_J}(p_J, \pi_H) \) and let \( f : h_{8r_J}(p_J, \pi_H) \rightarrow A_Q(\pi_H^1) \) be the \( \pi_H \)-approximation Proposition 4.4, and let \( K \subset B_{8r_J}(p_J, \pi_H) \) denote the set such that \( \mathbf{G}_{f/K} = T \cup K \times \pi_H^1 \). Observe that \( B_L \subset bC \) (provided \( \varepsilon_2 \) is sufficiently small, depending on all the other parameters). The following are simple consequences of Proposition 4.2:

\[
E := E(T, C_{32r_J}(p_J, \pi_H)) \leq Cm_0 \ell^{2-2\beta_2},
\]  

(7.10)

\[
h(T, C, \pi_H) \leq Cm_0^{1/2m} \ell^{1+\beta_2},
\]  

(7.11)

\[
cm_0 \ell^{2-2\beta_2} \leq E,
\]  

(7.12)

where (7.12) follows from \( B_L \subset C \) and \( L \in \mathcal{W}_e^j \). In particular the positive constants \( c \) and \( C \) do not depend on \( \varepsilon_2 \). We divide the proof of Proposition 3.3 in three steps.

**Step 1: decay estimate for** \( f \). Let \( 2p := 64r_H - \hat{C}m_0^{1/2m} \ell^{1+\beta_2} \); since \( p_H \in \text{spt}(T) \), it follows from (7.11) that, upon chosen \( C \) appropriately, \( \text{spt}(T) \cap C_{2p}(p_H, \pi_H) \subset B_H \subset C \). Observe in particular that \( \hat{C} \) does not depend on \( \varepsilon_2 \), although it depends upon the other parameters. In particular, setting \( B = B_{2p}(x, \pi_H) \) with \( x = p_{\pi_H}(p_H) \), using the Taylor expansion in [5, Corollary 3.3] and the estimates in [6, Theorem 1.4], we get

\[
\text{Dir}(\tilde{B}, f) \leq 2|B|E(T, C_{2p}(x_H, \pi_H)) + Cm_0^{1+\gamma_1} \ell^{m+2+\gamma_1}
\]

\[
\leq 2\omega_m\beta^mE(T, B_H) + Cm_0^{1+\gamma_1} \ell^{m+2+\gamma_1}.
\]  

(7.13)

Consider next the cylinder \( C : C_{64r_L}(p_L, \pi_H), x' := p_{\pi_H}(p_L) \) and \( B' = B_{64r_L}(x', \pi_H) \). Set \( A := \int_{B'} D(\eta \circ f), \bar{A} : \pi_H \rightarrow \pi_H^1 \) the linear map \( x \mapsto A \cdot x \) and \( \pi \) for the plane corresponding to \( \mathbf{G}_{\bar{A}} \). Using [5, Theorem 3.5], we can estimate

\[
\frac{1}{2} \int_{B'} G(Df, Q[A])^2 \geq |B'|E(T, C', \pi) - Cm_0^{1+\gamma_1} \ell^{m+2+\gamma_1/2}
\]

\[
\geq |B'|E(T, B_L, \pi) - Cm_0^{1+\gamma_1} \ell^{m+2+\gamma_1/2}
\]

\[
\geq \omega_m(64r_L)^mE(T, B_L) - Cm_0^{1+\gamma_1} \ell^{m+2+\gamma_1/2}
\]  

(7.14)

where we have used the obvious inclusion \( B_L \subset C' \). Next, considering that \( B_H \supset B_L \) and that, by \( L \in \mathcal{W}_e^j \), \( E(T, B_L) \geq C_e m_L^{2-\beta_2} \), we conclude from (7.13) and (7.14) that

\[
\text{Dir}(\tilde{B}, f) \leq 2\omega_m\beta^m(1 + m_0^{\gamma_1})E(T, B_H).
\]  

(7.15)

\[
\int_{B'} G(Df, Q[A])^2 \geq 2\omega_m64r_L^m(1 - Cm_0^{\gamma_1/2})E(T, B_L).
\]  

(7.16)
Since \( |x - x'| \leq |p_H - p_L| \leq C \ell(H) \), where \( C \) is a geometric constant (cp. Proposition 4.2), the ball \( \bar{B} = B_\rho(x, \pi_H) \), with radius \( \rho := 64r_L + C \ell(H) = 32r_H + C \ell(H) \) contains the ball \( B' \). Moreover, if \( \lambda \) is the constant in (7.5) and \( M_0 \) is chosen sufficiently large (thus fixing a lower bound for \( M_0 \) which depends only on \( \delta_2 \)) we reach

\[
\sigma \leq \left(\frac{1}{2} + \frac{\lambda}{4}\right) 64r_H \leq \left(1 + \frac{\lambda}{2}\right) \rho + \bar{C}m_0^{1/2m}\ell^{1+\beta_2}.
\]

In particular, choosing \( \varepsilon_2 \) sufficiently small we conclude \( \sigma \leq (1 + \lambda)\rho \). Now, recalling that \( H \in \mathcal{H} \) and \( L \in \mathcal{H}_e \), i.e. \( E(T, B_H) \leq 2^{2-2\varepsilon_2}E(T, B_L) \), we can then combine (7.13) and (7.14) to conclude

\[
\text{Step 2: harmonic approximation. From now on, to simplify our notation, we use } B_s(y) \text{ in place of } B_s(y, \pi_H). \text{ Set } p := \mathbf{p}_\pi_H(p_J). \text{ From (7.12) we infer that } 8r_JA \leq 8r_Jm_0^{1/2} \leq \varepsilon_2^{3/8} \text{ for } \varepsilon_2 \text{ sufficiently small. Therefore, for every positive } \bar{\eta}, \text{ we can apply [6, Theorem 1.5] to the cylinder } C \text{ and achieve a Dir-minimizing } w : B_{8r_J}(p, \pi_H) \to A_Q(\pi_{1/2}) \text{ such that}
\]

\[
(8r_J)^{-2} \int_{B_{8r_J}(p)} \mathcal{G}(f, w)^2 + \int_{B_{8r_J}(p)} (|Df| - |Dw|)^2 \leq \bar{\eta} E(8r_J)^m, \tag{7.18}
\]

\[
\int_{B_{8r_J}(p)} |D(\eta \circ f) - D(\eta \circ w)|^2 \leq \bar{\eta} E(8r_J)^m. \tag{7.19}
\]

Choosing \( \bar{\eta} \) sufficiently small (depending on all the parameters but \( \varepsilon_2 \)) and recalling that, by the harmonicity of \( \eta \circ w \), \( D(\eta \circ w)(x) = \int_{B_{1+\lambda}\rho}(x) D(\eta \circ w) \), we then get from (7.17):

\[
\int_{B_{1+\lambda}\rho}(x) \mathcal{G}(Dw, Q[D(\eta \circ w)(0)])^2 = \int_{B_{1+\lambda}\rho}(x) |Dw|^2 - Q[D(\eta \circ w)(x)]^2 |B_{1+\lambda}\rho(x)|
\]

\[
> 2^{2\beta_2-m-2} \text{Dir}(w, B_{2\rho}(x)). \tag{7.20}
\]

We can now apply Proposition 7.2 and conclude that (possibly choosing \( \bar{\eta} \) even smaller)

\[
m_0 \ell^{m+2-2\beta_2} \leq C \ell^{m} E(T, B_L) \leq \bar{C} \int_{B_{\ell/8}(q)} \mathcal{G}(Df, Q[D(\eta \circ f)])^2
\]

\[
\leq \bar{C} \ell^{-2} \int_{B_{\ell/8}(q)} \mathcal{G}(f, Q[\eta \circ f])^2, \tag{7.21}
\]

for any ball \( B_{\ell/8}(q) = B_{\ell/8}(q, \pi_H) \subset B_{8r_J}(p, \pi_H) = B_{8r_J}(p, \pi_H) \), where \( C, \bar{C} \) and \( \bar{C} \) are constants which depend on all the parameters except \( \varepsilon_2 \) and \( \bar{\eta} \).
Step 3: Estimate for the $\mathcal{M}$-normal approximation. Now, consider any ball $B_{\ell/4}(q, \pi_0)$ with dist$(L, q) \leq 4\sqrt{m} \ell$ and let $\Omega := \Phi(B_{\ell/4}(q, \pi_0))$. Observe that $p_{\pi_H}(\Omega)$ must contain a ball $B_{\ell/8}(q', \pi_H) \subset B_{8r_f}(p, \pi_H)$, because of the estimates on $\varphi$ and $|\pi_0 - \pi_H|$. Moreover, $p^{-1}(\Omega) \cap \text{spt}(T) \supset C_{\ell/8}(q', \pi_H) \cap \text{spt}(T)$ and, for an appropriate geometric constant $C$, $\Omega$ cannot intersect a Whitney region $\mathcal{L}'$ corresponding to an $L'$ with $\ell(L') \geq C\ell(L)$. In particular, Theorem 2.4 implies that

$$
\|T_F - T\|((p^{-1}(\Omega)) + \|T_F - G_f\|(p^{-1}(\Omega)) \leq Cm_0^{1+\gamma_2} \ell^{m+2+\gamma_2}.
$$

(7.22)

Let now $F'$ be the map such that $T_{F'} L((p^{-1}(\Omega)) = G_f L((p^{-1}(\Omega))$. The region over which $F$ and $F'$ differ is contained in the projection onto $\Omega$ of $(\text{Im}(F) \setminus \text{spt}(T)) \cup (\text{Im}(F') \setminus \text{spt}(T))$ and therefore its $\mathcal{H}^\mathcal{M}$ measure is bounded as in (7.22). Together with the height bound on $N$ and $f$ ($|N| + |N'| \leq Cm_0^{1/2m} \ell^{1+\beta_2}$), this implies

$$
\int_\Omega |N|^2 \geq \int_\Omega |N'|^2 - Cm_0^{1+1/m + \gamma_2} \ell^{m+4+2\beta_2 + \gamma_2}.
$$

(7.23)

On the other hand, let $\varphi' : B_{8r_f}(p, \pi_H) \to \pi_H$ be such that $G_{\varphi'} = [\mathcal{M}]$ and $\Phi'(z) = (z, \varphi'(z))$; then, applying [5, Theorem 5.1 (5.3)], we conclude

$$
N'(\varphi'(z)) \geq \frac{1}{2\sqrt{Q}} G(f(z), Q[f'(p)]) \geq \frac{1}{4\sqrt{Q}} G(f(z), Q[f \circ f(z)]),
$$

which in turn implies

$$
m_0 \ell^{m+2-2\beta_2} \leq C\ell^{-2} \int_{B_{\ell/8}(q', \pi_H)} G(f, Q[f \circ f])^2 \leq C\ell^{-2} \int_\Omega |N'|^2
$$

$$
\leq C\ell^{-2} \int_\Omega |N|^2 + Cm_0^{1+\gamma_2+1/2m} \ell^{m+2+2\beta_2 + \gamma_2}.
$$

(7.24)

For $\varepsilon_2$ sufficiently small, this leads to the second inequality of (3.2), while the first one comes from Theorem 2.4 and $E(T, B_L) \geq C_{\varepsilon}m_0^{\ell^2-2\beta_2}$.

We next complete the proof showing (3.1). Since $D(\eta \circ f)(z) = \eta \circ Df(z)$ for a.e. $z$, we obviously have

$$
\int_{B_{\ell/8}(q', \pi_H)} G(Df, Q[D(\eta \circ f)])^2 \leq \int_{B_{\ell/8}(q', \pi_H)} G(Df, Q[D\varphi'])^2.
$$

(7.25)

Let now $\tilde{G}_f$ be the orienting tangent $m$-vector to $G_f$ and $\tau$ the one to $\mathcal{M}$. For a.e. $z$ we have the inequality

$$
2 \sum_{i} |\tilde{G}_f(f_i(z)) - \tilde{\tau}(\varphi'(z))|^2 \geq 2G(Df(z), Q[D\varphi'(z)])^2,
$$

where $f_i(z)$ and $\varphi'(z)$ are, respectively, the $i$th components of $f(z)$ and $\varphi'(z)$.
and, hence,
\[
\int_{B_{\ell/s}(q', \pi_H)} \mathcal{G}(Df, Q[D\varphi'])^2 \leq C \int_{C_{\ell/s}(q', \pi_H)} |\tilde{G}_f(z) - \tilde{T}(\varphi'(p_{\pi_H}(z)))|^2 d\|G_f\|(z)
\]
\[
\leq C \int_{C_{\ell/s}(q', \pi_H)} |\tilde{T}(z) - \tilde{T}(\varphi'(p_{\pi_H}(z)))|^2 d\|T\|(z) + C m_0^{1+\gamma_1 \ell^{m+2+\gamma_1}}. \tag{7.26}
\]

Now, thanks to the height bound and to the fact that \(|\tilde{T} - \pi_H| \leq C m_0^{1/2} \ell^{1-\delta_2}\) in the cylinder \(C = C_{\ell/s}(q', \pi_H)\), we have the inequality
\[
|\varphi(z) - \varphi'(p_{\pi_H}(z))| \leq C m_0^{1/2+\gamma_2} \ell^{2+\beta_2-\delta_2} \leq C m_0^{1/2+\gamma_2} \ell^{2+\gamma_2/2} \quad \forall z \in \text{spt}(T) \cap \hat{C}.
\]

Using \(\|\varphi\|_{C^2} \leq C m_0^{1/2}\) we then easily conclude from (7.26) that
\[
\int_{B_{\ell/s}(p, \pi_H)} \mathcal{G}(Df, Q[D\varphi])^2 \leq C \int_{\hat{C}} |\tilde{T}(z) - \tilde{T}(\varphi(z))|^2 d\|T\|(z) + C m_0^{1+\gamma_1 \ell^{m+2+\beta_2/2}}
\]
\[
\leq C \int_{p^{-1}(\Omega)} |\tilde{T}_F(z) - \tau(\varphi(z))|^2 d\|T_F\|(z) + C m_0^{1+\gamma_2 \ell^{m+2+\gamma_2}},
\]
where we used (7.22).

Since \(|DN| \leq C m_0^{2\gamma_2} \ell^{2\gamma_2}, |N| \leq C m_0^{1/2} \ell^{1+\beta_2}\) on \(\Omega\) and \(\|A_M\|^2 \leq C m_0\), applying now [5, Proposition 3.4] we conclude
\[
\int_{p^{-1}(\Omega)} |\tilde{T}_F(x) - \tau(\varphi(x))|^2 d\|T_F\|(x) \leq (1 + C M_0^{2\gamma_2} \ell^{2\gamma_2}) \int_{\Omega} |DN|^2 + C m_0^{1+1/2} \ell^{m+2+3\beta_2}.
\]

Thus, putting all these estimates together we achieve
\[
m_0 \ell^{m+2-2\delta_2} \leq C (1 + C m_0^{2\gamma_2} \ell^{2\gamma_2}) \int_{\Omega} |DN|^2 + C m_0^{1+\gamma_2 \ell^{m+2+\gamma_2}}. \tag{7.27}
\]

Since the constant \(C\) might depend on the various other parameters but not on \(\varepsilon_2\), we conclude that for a sufficiently small \(\varepsilon_2\) we have
\[
m_0 \ell^{m+2-2\delta_2} \leq C \int_{\Omega} |DN|^2. \tag{7.28}
\]

But \(E(T, B_L) \leq C m_0 \ell^{2-2\delta_2}\) and thus (3.1) follows.

8. Persistence of \(Q\)-points

8.1. Proof of Proposition 3.4. We argue by contradiction. Assuming the proposition does not hold, there are sequences \(T_k\)'s and \(\Sigma_k\)'s satisfying the Assumption 1.2 and radii \(s_k\) for which

(a) either \(m_0(k) := \max\{E(T_k, B_{6\sqrt{m}}, c(\Sigma_k))\} \rightarrow 0\) and \(\bar{s} = \lim_k s_k > 0\); or \(s_k \downarrow 0\);
(b) the sets \(\Delta_k := \{\Theta(x, T_k) = Q\} \cap B_{3s_k}\) satisfy \(\mathcal{H}_{\infty}^{m-2+\alpha}(\Lambda_k) \geq \bar{s} s_k^{m-2+\alpha}\);
(c) denoting by \(\mathcal{W}(k)\) and \(\mathcal{A}(k)\) the families of cubes in the Whitney decompositions related to \(T_k\) with respect to \(\pi_0\), sup \(\{\ell(L) : L \in \mathcal{W}(k), L \cap B_{3s}(0, \pi_0) \neq \emptyset\} \leq s_k\);
(d) there exists \(L_k \in \mathcal{W}(k)\) with \(L_k \cap B_{3s}(0, \pi_0) \neq \emptyset\) and \(\bar{s} s_k < \ell(L_k) \leq s_k\).
It is not difficult to see that $E(T_k, B_{\sqrt{m} s_k}, \pi_{H_k}) \leq C m_0(k) s_k^{2-2\delta_2}$, where the constant $C$ is independent of $k$. Indeed, this is obvious in the case $\lim_k s_k > 0$, and it follows in the case $s_k \downarrow 0$ from the fact that, for $k$ large enough, there is an ancestor $J_k$ of $L_k$ with $B_{\sqrt{m} s_k} \subset B_{J_k}$ and $\ell(J_k) \leq C s_k$.

Consider now the ancestors $H_k$ and $J_k$ of $L_k$, and the corresponding Lipschitz approximation $f_k$ as in Section 7.3. By Proposition 4.2, it follows that

$$E(T, B_{\sqrt{m} s_k}, \pi_{H_k}) \leq C m_0(k) s_k^{2-2\delta_2}$$

and

$$h(T, B_{\sqrt{m} s_k}, \pi_{H_k}) \leq C m_0(k)^{1/2m} s_k^{1+\beta_2}.$$ 

Let moreover $g_k$ be the $\pi_{H_k}$-approximation of $T_k$ in the cylinder $C_{s_k}(0, \pi_{H_k})$, given by [6, Theorem 1.4] applied to the cylinder $C_k := C_{s_k}(0, \pi_{H_k})$ (since either $m_0(k) \downarrow 0$ or $s_k \downarrow 0$, the latter theorem is applicable for $k$ large enough). Observe also that $A_k^2 := \|A_{s_k} \circ C_k\|^2 \leq C s_k^2 m_0(k)$ and by Proposition 3.3 (3.1) and [6, Theorem 1.4], we easily conclude

$$E_k := E(T_k, C_k, \pi_{H_k}) \geq c_0 E(T, B_{L_k}) \geq c_0 C e m_0(k) \ell(L_k)^2-2\delta_2 \geq c_0(\hat{\alpha}) m_0(k) s_k^{2-2\delta_2}. \quad (8.1)$$

Observe now that on $B_{s_k}$ the two maps $g_k$ and $f_k$ coincide on a set $K_k$ with the property that $|B_{s_k} \setminus K_k| \leq C m_0(k)^{1+\gamma_1} s_k^{m+2+\frac{2\gamma}{n}}$. Moreover, by Proposition 3.3 (cp. Step 1 in Section 7.3), there exists a ball $B_{s_k}' \subset \pi_{H_k}$ contained in $B_{4s_k}$ and with radius at least $\ell(L_k)/8$ such that

$$\int_{B_{s_k}'} \mathcal{G}(f_k, Q[\eta \circ f_k])^2 \geq c_1(\hat{\alpha}) m_0(k) \ell(L_k)^{m+4-2\delta_2} \geq c_1(\hat{\alpha}) m_0(k) s_k^{m+4-2\delta_2}. \quad (8.2)$$

Combining the latter inequality with (8.1) we conclude

$$\int_{B_{s_k}'} \mathcal{G}(g_k, Q[\eta \circ g_k])^2 \geq c_2(\hat{\alpha}) s_k^{m+2} E_k. \quad (8.3)$$

Note that by (8.1), we have that $A_k^2 s_k^2 \leq C^* E_k$, for some $C^*$ independent of $k$. In particular, since either $s_k \downarrow 0$ or $m_0(k) \downarrow 0$, it turns out that, for $k$ large enough, $A_k s_k \leq E_k^{3/8}$. We can then apply [6, Theorem 1.5] to find a sequence of Dir-minimizing functions $w_k$ on $B_{4s_k}$ such that

$$(4 s_k)^{-2} \int_{B_{s_k}'} \mathcal{G}(g_k, w_k)^2 + \int_{B_{s_k}'} (|Dg_k| - |Dw_k|)^2 = o(E_k)s_k^m. \quad (8.4)$$

Up to rotations (so to get $\pi_{H_k} = \pi_0$) and dilations (of a factor $s_k$) of the system of coordinates, we then end up with a sequence of $C^3, \epsilon_0$ $(m + n)$-dimensional submanifolds $\Gamma_k$ of $\mathbb{R}^{m+n}$, area-minimizing currents $S_k$ in $\Gamma_k$, functions $h_k$ and $\tilde{w}_k$ with the following properties:

(1) the excess $E_k := E(S_k, C_{32}(0, \pi_0))$ and the height $h(S_k, C_{32}(0, \pi_0), \pi_0)$ converge to 0;

(2) $A_k^2 := \|A_{r_k}\|^2 \leq C^* E_k$ and hence it also converges to 0;

(3) Lip$(h_k) \leq C E_k^{\gamma_1}$;

(4) $\|G_{h_k} - T_k\|(C_s(0, \pi_0)) \leq C E_k^{1+\gamma_1}$;
(5) $\bar{w}_k$ are Dir-minimizing in $B_4(0, \pi_0)$ and
\[
\int \left( (|Dh_k| - |D\bar{w}_k|)^2 + \mathcal{G}(h_k, \bar{w}_k) \right)^2 = o(E_k). 
\]
(8.5)

(6) for some positive constant $c$ independent of $k$ it holds
\[
\int_{B_3} \mathcal{G}(h_k, \eta \circ h_k)^2 \geq cE_k; 
\]
(8.6)

(7) $\Xi_k := \{ \Theta(S_k, y) = Q \} \cap B_3$ has the property that $\mathcal{H}^{m-2+\alpha}(\Xi_k) \geq \bar{\alpha} > 0$ and $0 \in \Xi_k$.
Consider the projections $\bar{\Xi}_k := p_{\pi_0}(\Xi_k)$. We are therefore in the position of applying [6, Theorem 1.6] to conclude that, for every fixed $\rho \in (0, \frac{1}{2})$,
\[
\max_{x \in \bar{\Xi}_k} \int_{B_{\rho}(x)} \mathcal{G}(h_k, Q[\eta \circ h_k])^2 = o(E_k) \quad \text{for} \quad k \to +\infty.
\]

Up to subsequences we can assume that $\bar{\Xi}_k$ (and hence also $\Xi_k$) converges, in the Hausdorff sense, to a compact set $\bar{\Xi}$, which is nonempty. Moreover, the maps $x \mapsto u_k(x) = E_k^{-1/2} \sum_i [(\bar{w}_k)_i(x) - \eta \circ \bar{w}_k(x)]$ are easily recognized to converge to a Dir-minimizing function $u$ with $\eta \circ u = 0$, vanishing identically on $\bar{\Xi}$. From (8.5) and (8.6) it follows that $u$ does not vanish identically. Then, by [4, Theorem 0.11] we conclude that $\mathcal{H}^{m-2+\alpha}(\bar{\Xi}) = 0$, otherwise $\bar{\Xi}$ must have nonempty interior, which together with Lemma 7.1 implies $\bar{\Xi} = B_3$ and contradicts $u \not\equiv 0$. On the other hand, $\mathcal{H}^{m-2+\alpha}(\Xi_k) \geq \limsup_k \mathcal{H}^{m-2+\alpha}(\Xi_k) \geq \bar{\alpha} > 0$ gives the desired contradiction.

8.2. Proof of Proposition 3.5. We fix the notation as in Section 7.3 and notice that
\[
E := E(T, C_{32r_I}(p_J, \pi_H)) \leq C m_0 \ell(L)^2 - 2\delta_2 \leq C m_0 \ell(L)^2 - 2\delta_2.
\]
By Proposition 3.3 we have
\[
\int_{B_{\ell(L)}(p(p))} |DN|^2 \geq \tilde{c}_1 m_0 \ell(L)^{m+2-2\delta_2}. 
\]
(8.7)

Next, let $p := (x, y) \in \pi_H \times \pi_{1,2}$, fix a $\tilde{n} > 0$, to be chosen later, and note that (7.12) allows us to apply [6, Theorem 1.6]: there exists then $s > 0$ such that
\[
\int_{B_{2\tilde{n}}(x, \pi_H)} \mathcal{G}(f, Q[\eta \circ f])^2 \leq \tilde{n}^m \ell(L)^{m+2} E.
\]
(8.8)

Observe that, no matter how small $\tilde{n}$ is chosen, such estimate holds when $s$ and $E$ are appropriately small: the smallness of $E$ is then achieved choosing $\ell$ as small as needed.

Now consider the graph $\text{Gr}(\eta \circ f) \subseteq C_{2\tilde{n}}(\ell(x, \pi_H))$ and project it down onto $\mathcal{M}$. Since $\mathcal{M}$ is a graph over $\pi_H$ of a function $\hat{\phi}$ with $|D\hat{\phi}|_{C^{2+\kappa}} \leq C m_0^{1/2}$ and since the Lipschitz constant of $\eta \circ f$ is controlled by $C m_0^{1/3}$, provided $\varepsilon_2$ is smaller than a geometric constant we have that $\Omega := p \left( \text{Gr}(\eta \circ f) \subseteq C_{2\tilde{n}}(\ell(x, \pi_H)) \right)$ contains a ball $B_{\ell(L)}(p(p))$.

Consider now the map $\mathcal{F}(q) = \sum_i [q + N_i(q)]$ such that $T_{\ell(L)}^{-1}(\Omega) = G_L \ell p^{-1}(\Omega)$ given by [5, Theorem 5.1]. Consider also the map $\xi : B_{2\tilde{n}}(\ell(x, \pi_H)) \ni z \mapsto p((z, \eta \circ f(z)) \in \Omega$. This map is biLipschitz with controlled constant, again assuming that $\varepsilon_2$
is sufficiently small. Let now \( \hat{n} : \Omega \to \mathbb{R}^{m+n} \) with the property that \( \hat{n}(q) \perp T_q\mathcal{M} \) and \( \xi(x) + \hat{n}(\xi(x)) = (x, \eta \circ f(x)) \). Applying the estimate of [5, Theorem 5.1 (5.5)] we then get
\[
\mathcal{G}(N' \langle \xi(x) \rangle, Q \langle \eta \circ N' \langle \xi(x) \rangle \rangle) \leq \mathcal{G}(N \langle \xi(x) \rangle, Q \langle \hat{n}(\xi(x)) \rangle) \leq 2\sqrt{Q} \mathcal{G}(f(x), Q \langle \eta \circ f(x) \rangle).
\]
Integrating the latter inequality, changing variable and using \( B_{\delta(t)}(p(q)) \subset \Omega \), we then obtain
\[
\int_{B_{\delta(t)}(p(q))} \mathcal{G}(N', Q \langle \eta \circ N' \rangle)^2 \leq C \hat{\eta} s^m \ell(L)^{m+2} E \leq C \hat{\eta} m_0 \hat{s}^m \ell(L)^{m+4-2\delta_2}.
\]
Next, recalling the height bound and the fact that \( N \) and \( N' \) coincide outside a set of measure \( m_0^{1+\gamma_1} \ell(L)^{m+2+\gamma_2} \), we infer
\[
\int_{B_{\delta(t)}(p(q))} \mathcal{G}(N, Q \langle \eta \circ N \rangle)^2 \leq C_1 \hat{\eta} m_0 \hat{s}^m \ell(L)^{m+4-2\delta_2} + C_2 m_0^{1+\gamma_1} \ell(L)^{m+4+\gamma_2+2\beta_2}.
\] (8.9)
Since the constants \( \tilde{c}_1 \), \( C_1 \) and \( C_2 \) in (8.7) and (8.9) are independent of \( \ell(L) \) and \( \hat{\eta} \), we fix \( \eta \) (and consequently \( \hat{s} \)) so small that \( C_1 \hat{\eta} \leq \tilde{c}_1 \frac{m}{2} \). We therefore achieve from (8.9)
\[
\int_{B_{\delta(t)}(p(q))} \mathcal{G}(N, Q \langle \eta \circ N \rangle)^2 \leq \frac{c_1}{2} \eta_2 m_0 \ell(L)^{4-2\delta_2} + C_2 m_0^{1+\gamma_1} \hat{s}^{-m} \ell(L)^{4+\gamma_2+2\beta_2}.
\] (8.10)
Having now fixed \( \hat{s} \) we choose \( \ell \) so small that \( C_3 \hat{s}^{-m} \ell^{2\delta_2+\gamma_2+2\beta_2} \leq \tilde{c}_1 \eta_2 / 2 \). For these choices of the parameters, under the assumptions of the proposition we then infer
\[
\int_{B_{\delta(t)}(p(q))} \mathcal{G}(N, Q \langle \eta \circ N \rangle)^2 \leq \eta_2 \tilde{c}_1 m_0 \ell(L)^{4-2\delta_2}.
\] (8.11)
The latter estimate combined with (8.7) gives the desired conclusion.

9. Comparison between different center manifolds

Proof of Proposition 3.6. We first verify (i). Observe that
\[
E(T', B_{6\sqrt{m}}) = E(T, B_{6\sqrt{m}}) \leq \liminf_{\rho \to 0} E(T, B_{6\sqrt{m}}) \leq \varepsilon_2.
\]
Moreover, since \( \Sigma' \) is a rescaling of \( \Sigma, c(\Sigma')^2 = r^2 c(\Sigma)^2 \leq r^2 m_0 \). Therefore, (1.7) is fulfilled by \( \Sigma' \) and \( T' \) as well; (1.6) follows trivially upon substituting \( \pi_0 \) with the optimal \( \pi; (1.4) \) is scaling invariant; whereas \( \partial T' \cap B_{6\sqrt{m}} = (\pi_0, r)^2 \partial T \cap B_{6\sqrt{m}} = 0 \).

We now come to (ii). From now we assume \( N_0 \) to be so large that \( 2^{-N_0} \) is much smaller than \( c_s \). In this way we know that \( r \) must be much smaller than \( 1 \). We have that \( \ell(L) = c_s r \), otherwise condition (a) would be violated. Moreover, we can exclude that \( L \in \mathcal{W}_\rho \). Indeed, in this case there must be a cube \( J \in \mathcal{W} \) with \( \ell(J) = 2\ell(L) \) and nonempty intersection with \( L \). It then follows that, for \( \rho := r + 2\sqrt{m} \ell(L) = (1 + 2\sqrt{m} c_s) r \), \( B_{\rho}(0, \pi_0) \) intersects \( J \). Again upon assuming \( N_0 \) sufficiently large, such \( \rho \) is necessarily smaller than \( 1 \). On the other hand, since \( 2\sqrt{m} c_s < 1 \) we then have \( c_s \rho < 2 c_s r \leq 2 \ell(L) = \ell(J) \).

Next observe that \( E(T, B_{6\sqrt{m}}) \leq C m_0 \rho^{2-2\delta_2} \) for some constant \( C \) and for every \( \rho \geq r \). Indeed, if \( \rho \) is smaller than a threshold \( r_0 \) but larger than \( r \), then \( B_{6\sqrt{m}} \) is contained in the ball \( B_J \) for some ancestor \( J \) of \( L \) with \( \ell(J) \leq 2 \ell(L) = \hat{C} \rho \), where the constant \( \hat{C} \) and the threshold...
Finally, since \( r_0 \) depend upon the various parameters. Then, \( E(T, B_{6\sqrt{\rho m}}) \leq C E(T, B_J) \leq C m_0 \rho^{2-\delta_2} \). If instead \( \rho \geq r_0 \), we then use simply \( E(T, B_{6\sqrt{\rho m}}) \leq C (r_0) E(T, B_{6\sqrt{\rho m}}) \leq C (r_0) m_0 \). This estimate also has the consequence that, if \( \pi(\rho) \) is an optimal \( m \)-plane in \( B_{6\sqrt{\rho m}} \), then \(|\pi_L - \pi(\rho)| \leq C m_0^{1/2} \rho^{1-\delta_2} \).

We next consider the notation introduced in Section 7.3, the corresponding cubes \( L \subset H \subset J \) and the \( \pi_H \)-approximation \( f \) introduced there. If \( L \in \mathcal{W}^e \), then by (7.21) we get
\[
\int_{B_{t_0/s}(x, \pi H)} \mathcal{G}(f, Q \lfloor \eta \circ f \rfloor)^2 \geq \tilde{c} m_0^{1/m} \ell^{m+2\beta_2} - C m_0^{1/m} \ell^{2+2\beta_2} |K| \geq \tilde{c} m_0^{1/m} \ell^{m+2\beta_2} - C m_0^{1+\gamma_1+1/m} \ell^{(m-2\delta_2)(1+\gamma)} + 2 \beta_2 \geq \tilde{c} m_0 r^{m+2-\delta_2}. \tag{9.1}
\]

where \( x = p_{\pi_H}(x_H) \). On the other hand, if \( L \in \mathcal{W}^h \), we can argue as in the proof of Proposition 3.1 and use Theorem A.1 to conclude the existence of at least two stripes \( S_1 \) and \( S_2 \), at distance \( \tilde{c} m_0^{1/2m} \ell^{1+\beta_2} \) with the property that any slice \((T, \pi_H, z)\) with \( z \in B_{t_0/s}(x, \pi H) \) must intersect both of them. Since for \( x \in K \) such slice coincides with \( f(x) \), we then have
\[
|\mathcal{H}(x, \pi H, f)|^2 \geq \tilde{c} m_0^{1/m} \ell^{m+2\beta_2} - C m_0^{1/m} \ell^{2+2\beta_2} |K| \geq \tilde{c} m_0^{1/m} \ell^{m+2\beta_2} - C m_0^{1+\gamma_1+1/m} \ell^{(m-2\delta_2)(1+\gamma)} + 2 \beta_2 \geq \tilde{c} m_0 r^{m+4-\delta_2}. \tag{9.2}
\]

Rescale next through \( t_{\rho r} \) and consider \( T' := (t_{\rho r})_T T \). We also rescale the graph of the corresponding \( \pi_H \)-approximation \( f \) to the graph of a map \( g \), which then has the following properties. If \( B \subset \pi_H \) is the rescaling of the ball \( B_{t_0/s}(x, \pi_H) \), then \( B \subset B_{3/2} \) and the radius of \( B \) is given by \( c_\rho /8 \). On \( B \) we have the estimate
\[
\int_B \mathcal{G}(g, Q \lfloor \eta \circ g \rfloor)^2 = r^{-m-2} \int_{B_{t_0/s}(x, \pi H)} \mathcal{G}(f, Q \lfloor \eta \circ f \rfloor)^2 \geq \tilde{c} m_0 r^{2-\delta_2}. \tag{9.3}
\]

The Lipschitz constant of \( g \) is the same of that of \( f \) and hence controlled by \( C m_0^{\gamma_1+\gamma} \). On the other hand, we have
\[
\tilde{m}_0 := \max \{ E(T', B_{\sqrt{m_0}}, c(S)^2) \} \leq \max \{ C m_0 r^{2-\delta_2}, C(S)^2 r^2 \} \leq C m_0 r^{2-\delta_2}. \tag{9.4}
\]

Moreover, denoting by \( \tilde{C} \) the cylinder \( C_4(0, \pi H) \), we have that
\[
\| G_g - T' \| (\tilde{C}) \leq C m_0^{1+\gamma_1/2}. \tag{9.5}
\]

Finally, since \( |\pi - \pi_H| \leq C m_0^{1/2-1-\delta_2} \) and because \( M' \) is the graph of a function \( \varphi' \) with \( \| D \varphi' \|_{C^{2, \alpha}} \leq C m_0^{1/2-\alpha} \) and \( \| \varphi' \|_{C^0} \leq C m_0^{1/2m} \), by (9.4) we can actually conclude that \( M' \) is the graph over \( \pi_H \) of a map \( \tilde{\varphi} : \pi_H \rightarrow \pi_H \), with \( \| D \tilde{\varphi} \|_{C^{2, \alpha}} \leq C m_0^{1/2-\delta_2} \). Similarly, the \( M' \)-approximating map \( x \mapsto T'(x) := \sum_i \lfloor x + N_i(x) \rfloor \) coincides with \( T' \) over a subset \( K' \subset M' \) with \( |M' \setminus K'| \leq \tilde{m}_0^{1+\gamma_2} \) \( \leq C m_0^{1+\gamma_2} r^{(2-2\delta_2)(1+\gamma_2)} \).

Consider now the projection \( p' \) over \( M' \) and hence define the set \( \mathcal{J} := p'(\text{sp}(T') \cap \text{Gr}(g) \cap \text{Im}(F')) \). It turns out that \( |B_2 \setminus \mathcal{J}| \leq m_0^{1+\gamma_2} r^{(2-2\delta_2)(1+\gamma_2)} \). On the other hand, if \( G : B_2 \rightarrow A_Q(\mathbb{R}^{m+n}) \) is the map with \( T_G = G_g \circ p'^{-1}(B_2) \) given by [5, Theorem 5.1], we
then have that $G \equiv F$ on $\mathcal{J}$. Consider therefore a point $p \in \mathcal{J}$ with $p = (x, \varphi(x))$ and consider that for this point we have by [5, Theorem 5.1 (5.3)]
\[ \mathcal{G}(g(x), Q [\eta \circ g(x)]) \leq \mathcal{G}(g(x), Q [\varphi(x)]) \leq C \mathcal{G}(G(p), Q [p]). \]
Therefore, using (9.4) we can easily estimate
\[ \int_{B_2 \cap M'} |N'|^2 \geq \int_{B_2} (G(p), Q [p])^2 - C \hat{m}_0^{1/m} B_2 \setminus J \]
\[ \geq \hat{c}_1 m_0 r^{2 - 2\delta_2} - C m_0^{1/m + 1 + \gamma_0} r^{2(2 - 2\delta_2)(1 - \gamma_2)} \geq \hat{c}_2 m_0 r^{2 - 2\delta_2}, \tag{9.6} \]
where all the constants are independent of $\varepsilon_2$ and the latter is supposed to be sufficiently small. Thus finally, by (9.4) we conclude
\[ \int_{B_2 \cap M'} |N'|^2 \geq \hat{c}_s \hat{m}_0 = \hat{c}_s \max \{ E(T', B_{6\sqrt{m}}), c(\Sigma')^2 \}. \]
\[ \square \]

### APPENDIX A. HEIGHT BOUND REVISITED

In this section we prove a strengthened version of the so-called “height bound” (see [8, Lemma 5.3.4]), which appeared first in [1]. Our proof follows closely that of [11].

**Theorem A.1.** Let $Q$, $m$, $\bar{n}$ and $n$ be positive integers. Then there are $\varepsilon > 0$ and $C$ with the following property. Assume:

- (h1) $\Sigma \subset \mathbb{R}^{m+n}$ is an $(m + \bar{n})$-dimensional $C^2$ submanifold with $A := \| A_{Ec} \|_0 \leq \varepsilon$;
- (h2) $R$ is an integer rectifiable $m$-current with $\text{spt}(R) \subset \Sigma$ and area-minimizing in $\Sigma$;
- (h3) $\partial R \subset C_r(x_0) = 0$, $(\mathbf{p}_{\pi_0})_\varepsilon R \subset C_r(x_0) = Q \left[ B_r(\mathbf{p}_{\pi_0}(x_0), \pi_0) \right]$ and $E := E(R, C_r(x_0)) < \varepsilon$.

Then there are $k \in \mathbb{N}$, points $\{ y_1, \ldots, y_k \} \subset \mathbb{R}^n$ and positive integers $Q_1, \ldots, Q_k$ such that:

- (i) having set $\sigma := C E^{1/2m}$, the open sets $S_i := \mathbb{R}^m \times (y_i + \gamma - r \sigma, r \sigma^{n})$ are pairwise disjoint and $\text{spt}(R) \cap C_r(1 - \sigma \log E)(x_0) \subset \bigcup_i S_i$;
- (ii) $(\mathbf{p}_{\pi_0})_\varepsilon \left[ R \subset C_r(1 - \sigma \log E)(x_0) \cap S_i \right] = Q_i \left[ B_r(\mathbf{p}_{\pi_0}(x_0), \pi_0) \right] \forall i \in \{ 1, \ldots, k \}$.
- (iii) for every $p \in \text{spt}(R) \cap C_r(1 - \sigma \log E)(x_0)$ we have $\Theta(R, p) < \min \{ k_i \} + \frac{1}{2}$.

**Remark A.2.** Obviously, $\sum_i Q_i = Q$ and hence $1 \leq k \leq Q$. Most likely the bound on the radius of the inner cylinder could be improved to $1 - \sigma$. However this would not give us any advantage in the rest of the paper and hence we do not pursue the issue here.

**Proof.** We first observe that, without loss of generality, we can assume $x_0 = 0$ and $r = 1$. Moreover, (iii) follows from (i) and (ii) through the monotonicity formula. Indeed, let $p \in \text{spt}(R)$ be such that $B_\rho(p) := B_{E^{1/2m}}(p) \subset C_{1 - \sigma \log E}(x_0) = C'$. $p$ must be contained in one of the $S_i$, say $S_1$. Consider the current $R_1 = R \subset C_r(1 - \sigma \log E)(x_0)$, where $R$ must be area-minimizing in $\Sigma$, $\Theta(R_1, p) = \Theta(R, p)$ and that $E(R_1, C') \leq E$. On the other hand, if $\| A_{Ec} \|$ is smaller than a geometric constant, the monotonicity formula implies
\[ M(R_1 \subset C_r(p)) \geq M(R_1 \subset B_r(p)) \geq \omega_m(\Theta(R, p) - \frac{1}{4}) \rho^m = \omega_m(\Theta(R, p) - \frac{1}{4}) E^{1/2}. \]
On the other hand, \( M(R_1 \mathbf{C}_p) \leq \omega_m k_1 p^m + E \). Thus, if \( E \) is smaller than a geometric constant, we ensure \( \Theta(R, p) \leq k_1 + \frac{1}{2} \). This means that, having proved (i) and (ii) for \( \sigma = CE^{1/2m} \), (iii) would hold if we redefine \( \sigma \) as \( (C + 1)E^{1/2m} \).

The proof of (i) and (ii) is by induction on \( Q \). The starting step \( Q = 1 \) is Federer’s classical statement (cp. with [8, Lemma 5.3.4] and [11, Lemma 2]) and though its proof can be easily concluded from what we describe next, our only concern will be to prove the inductive step. Hence, from now on we assume that the theorem holds for all multiplicities up to \( Q - 1 \geq 1 \) and we prove it for \( Q \). Indeed, we will show a slightly weaker assertion, i.e. the existence of numbers \( a_1, \ldots, a_k \in \mathbb{R} \) such that the conclusions (i) and (ii) apply when we replace \( S_i \) with \( \Sigma_i = \mathbb{R}^{m+1} \times ]a_i - \sigma, a_i + \sigma[ \). The general statement is obviously a simple corollary. To simplify the notation we use \( \bar{p} \) in place of \( p_{s_0} \).

**Step 1.** Let \( r \geq \frac{1}{2} \) and \( a - b > 2\eta = 2C_0 E^{1/2m} \), where \( C_0 \) is a constant depending only on \( m \) and \( n \), which will be determined later. We denote by \( W_r(a, b) \) the open set \( B_r \times \mathbb{R}^{n-1} \times [a, b[ \). In this step we show

\[
\| R \| (W_r(a, b)) \leq \frac{2q-1}{2q} \omega_mr^m \implies \text{spt}(R) \cap W_{r-\eta}(a + \eta, b - \eta) = \emptyset . \tag{A.1}
\]

Without loss of generality, we assume \( a = 0 \). For each \( \tau \in [0, \frac{1}{2}] \), consider the currents \( R_\tau := R \llcorner W_r(\tau, b - \tau) \) and \( S_\tau := \bar{p}_x R_\tau \). It follows from the slicing theory that \( S_\tau \) is a locally integral current for a.e. \( \tau \). There are then functions \( f_\tau \in BV_{loc}(B_r) \) which take integer values and such that \( S_\tau = f_\tau \llcorner B_r \). Since \( \| f_\tau \|_1 = M(S_\tau) \leq \| R \| (W_r(0, b)) \leq \frac{2q-1}{2q} \omega_mr^m \), \( f_\tau \) must vanish on a set of measure at least \( \frac{\omega_m}{2q} r^m \). By the relative Poincaré inequality,

\[
M(S_\tau)^{1-m} = \| f_\tau \|_{L^1}^{1-m} \leq C \| Df_\tau \| (B_r) = C \| \partial (\bar{p}_x R_\tau) \| (B_r) \leq C \| \partial R_\tau \| (C_r) .
\]

We introduce the slice \( \langle R, \tau \rangle \) relative to the map \( x_{m+n} : \mathbb{R}^{m+n} \to \mathbb{R} \) which is the projection on the last coordinate factor. Then the usual slicing theory gives that

\[
(M(S_\tau))^{1-m} \leq C \| \partial R_\tau \| (C_r) = CM(\langle R, \tau \rangle - \langle R, b - \tau \rangle) \quad \text{for a.e. } \tau . \tag{A.2}
\]

Let now \( \bar{\tau} \) be the supremum of \( \tau \)'s such that \( M(S_\tau) \geq \sqrt{E} \forall \tau < \tau \). If \( M(S_0) < \sqrt{E} \), we then set \( \bar{\tau} := 0 \). If \( \bar{\tau} > 0 \), observe that, for a.e. \( \tau \in [0, \bar{\tau}] \), we have

\[
\bar{E} \frac{m-1}{2m} \leq (M(S_\tau))^{1-m} \leq C \left( M(\langle R, \tau \rangle - \langle R, b - \tau \rangle) \right) . \tag{A.3}
\]

Integrate (A.3) between 0 and \( \bar{\tau} \) to conclude

\[
\bar{\tau} \bar{E} \frac{m-1}{2m} \leq C \int_0^{\bar{\tau}} M(\langle R, \tau \rangle - \langle R, \ell - \tau \rangle) \, d\tau = \int_{W_r(0, \tau) \cup W_r(b - \bar{\tau}, b)} |\bar{R}_\llcorner dx_{m+n}| \, d\| R \| . \tag{A.4}
\]

We then achieve \( \bar{\tau} \bar{E} \frac{m-1}{2m} \leq C \sqrt{E} \), i.e. \( \bar{\tau} \leq C E^{1/2m} \), applying Cauchy-Schwartz and recalling

\[
\int_{C_1} |\bar{R}_\llcorner dx_{m+n}|^2 \, d\| R \| \leq E(R, C_1) = E .
\]
(\(\bar{C}\) depends only on \(m\) and \(n\)). Set \(C_0 := \bar{C} + 2\) and recall that \(\eta = C_0 E^{1/2m}\). Observe also that there must be a sequence of \(\tau_k \downarrow \bar{\tau}\) with \(\text{M}(S_{\tau_k}) < \sqrt{E}\). Therefore,

\[
\|R\|(W_{r}(\bar{\tau}, b - \bar{\tau})) \leq \liminf_{k \to \infty} \|R\|(W_{e}(\tau_k, b - \tau_k)) \leq \liminf_{k \to \infty} \text{M}(S_{\tau_k}) + E \leq 2\sqrt{E}. \tag{A.5}
\]

Assume now the existence of \(p \in \text{spt}(T) \cap W_{r-\eta}(\eta, b - \eta)\). By the properties of area-minimizing currents, \(\Theta(T, p \geq 1\). Set \(\rho := 2E^{1/2m}\) and \(B' := B_\rho(p) \subset W_{r}(\bar{\tau}, \ell - \bar{\tau})\). By the monotonicity formula, \(\|R\|(B') \geq c2^m \omega_m \sqrt{E}\), where \(c\) depends only on \(A\) (recall that \(\rho \leq 1\) and approaches 1 as \(A\) approaches 0). Thus, for \(\varepsilon_2\) sufficiently small, this would contradict (A.5). We have therefore proved (A.1).

**Step 2.** We are now ready to conclude the proof of (i) and (ii). Assume

\[
\max \{\|R\|(W_1(0, \infty)), \|R\|(W_1(-\infty, 0))\} \leq \frac{1}{2} \text{M}(R). \tag{A.6}
\]

Divide the interval \([0, 1]\) into \(Q + 1\) intervals \([a_i, a_{i+1}]\) and let \(W^i := W_1(a_i, a_{i+1})\). For each \(i\) consider \(S^i := \text{p}_i(T \setminus W^i)\). Observe that there must be one \(i\) for which \(\text{M}(S^i) \leq (1 - \frac{1}{2Q}) \omega_m\).Otherwise we would have

\[
\omega_m Q + E \geq \text{M}(R) \geq \sum \text{M}(S^i) \geq \omega_m \left( Q + \frac{Q-1}{2Q} \right),
\]

which is obviously a contradiction if \(E\) is sufficiently small.

It follows from Step 1 that there must be an \(i\) so that \(\text{spt}(T)\) does not intersect \(W_1-\eta(a_i + \eta, a_{i+1} - \eta)\). Consider \(R_1 := R \setminus W_1(\eta, a_i + \eta)\) and \(R_2 := R \setminus W_1(\eta, (a_{i+1} - \eta, \infty)\).

By the constancy theorem \(\text{p}_i R_1 = k_i \ll [B_1-\eta]\), where both \(k_i\)'s are integers. Indeed, having assumed that \(E\) is sufficiently small, each \(k_i\) must be nonnegative and their sum is \(Q\). There are now two possibilities.

(a) Both \(k_i\)'s are positive. In this case \(R_1\) and \(R_2\) satisfy again the assumptions of the Theorem with \(1 - \eta\) in place of 1. After a suitable rescaling we can apply the inductive hypothesis to both currents and hence get the desired conclusion.

(b) One \(k_i\) is zero. In this case \(\text{M}(R_i) \leq E\) and it cannot be \(R_1\), since \(\text{M}(R_1) \geq \frac{1}{2} \text{M}(R)\) by (A.6). Thus it is \(R_2\) and, arguing as at the end of Step 1, we conclude

\[
R \setminus W_1(\eta, a_{i+1} + 2\eta, \infty) = 0.
\]

In case (b) we repeat the argument splitting \([-1, 0]\) into \(Q + 1\) intervals. Once again, either we can “separate” the current into two pieces and apply the inductive hypothesis, or we conclude that \(\text{spt}(R \setminus \mathbf{C}_{1-4\eta}) \subset W_1-4\eta(-1 - \eta, 1 + \eta) =: W_1-4\eta(a_0, b_0)\). If this is the case, we apply once again the argument above and either we “separate” \(R^1 := R \setminus \mathbf{C}_{1-6\eta} \times \mathbb{R}^n\) into two pieces, or we conclude that \(\text{spt}(R^1) \subset W_1-6\eta(\mathbf{a}_1, \mathbf{b}_1)\).

\[
b_1 - a_1 \leq (b_0 - a_0) \left( 1 - \frac{1}{Q+1} + \eta \right) \leq 2 \left( 1 - \frac{1}{Q+2} \right)
\]

(provided \(\varepsilon_2\) is smaller than a geometric constant). We now iterate this argument at most \(c_0 \log E\) times, stopping if at any step we “separate” the current and can apply the inductive hypothesis, or if the resulting current is contained in \(W_1-(4+2k)\eta(a_k, b_k)\) for some \(a_k, b_k\) with \(b_k - a_k \leq c_1 E^{1/2m}\). The constant \(c_1\) is chosen larger than 1 and in such a way
that, if $\ell > c_1 E^{1/2m}$, then $\ell \frac{Q}{Q+1} + \eta \leq \frac{Q+1}{Q+2}\ell$. Observe that, since $\eta = C_0 E^{1/2m}$, $c_1$ depends only upon $Q$, $m$ and $n$.

If the procedure does not stop until we reach $c_0 |\log E|$ iterations, then we conclude that the current $\mathbb{R} \mathcal{C}_1 - (4 + 2c_0 |\log E|) \eta$ is supported in $W_{1 - (4 + 2c_0 |\log E|)}(a, b)$ where

$$b - a \leq (2 + \eta) \left( \frac{Q+1}{Q+2} \right)^{c_0 |\log E|}.$$

$c_0$ is chosen so that this last number is smaller than $c_1 E^{1/2m}$, i.e. so that

$$c_0 |\log E| |\log \frac{Q+1}{Q+2} \leq -2(2 + \eta) - \log c_1 + \frac{1}{2m} |\log E|. \tag{A.7}$$

Observe that both $\log E$ and $|\log \frac{Q+1}{Q+2}$ are negative. Moreover, since $c_1$ is a geometric constant, if $\varepsilon_2$ is chosen small enough, $|\log E| \geq 2m(2 + \eta) + \log c_1$. Thus (A.7) is surely fulfilled when $c_0 \geq \frac{1}{m} |\log \frac{Q+1}{Q+2}|^{-1}$.

\section*{Appendix B. Changing Coordinates for Classical Functions}

\textbf{Lemma B.1.} There are constants $c_0, C > 0$ with the following properties. Assume that

(i) $|x - x_0| \leq c_0$, $r \leq 1$;

(ii) $p = (q, u) \in \mathcal{X} \times \mathcal{X}^\perp$ and $f, g : B^m_\rho(q, \mathcal{X}) \rightarrow \mathcal{X}^\perp$ are Lipschitz functions such that

$$\text{Lip}(f), \text{Lip}(g) \leq c_0 \quad \text{and} \quad |f(q) - u| + |g(q) - u| \leq c_0 r.$$

Then there are two maps $f', g' : B_5r(p, x_0) \rightarrow \mathcal{X}_0^\perp$ such that

(a) $G_{f'} = G_f \mathcal{L} C_5r(p, x_0)$ and $G_{g'} = G_g \mathcal{L} C_5r(p, x_0)$;

(b) $\|f' - g'\| L^1(B_5r(p, x_0)) \leq C \|f - g\| L^1(B_7r(p, x))$;

(c) if $f \in C^4(B_7r(p, x))$ (resp. $C^{3,\kappa}(B_7r(p, x))$) then $f' \in C^4(B_5r(p, x_0))$ (resp. $C^{3,\kappa}(B_5r(p, x_0))$) with the estimates

$$\|f' - u'\| C^3 \leq \Phi (|x - x_0|, \|f - u\|_{C^3}), \tag{B.1}$$

$$\|D^4 f'\| C^3 \leq \Lambda (|x - x_0|, \|f - u\|_{C^3}) \left(1 + \|D^4 f\| C^0\right), \tag{B.2}$$

$$\text{resp.} \quad \|D^3 f'\| \kappa \leq \Lambda_\kappa (|x - x_0|, \|f - u\|_{C^0}) \left(1 + \|D^3 f\| \kappa\right), \tag{B.3}$$

where $\Phi$, $\Lambda$ and $\Lambda_\kappa$ are smooth functions with $\Phi(\cdot, 0) \equiv 0$;

(d) $\|f' - g'\| W^{1,2}(B_5r(p, x_0)) \leq C (1 + \|D f\| C^0) \|f - g\| W^{1,2}(B_7r(p, x))$.

All the conclusions of the Lemma still hold if we replace the exterior radius $7r$ and interior radius $5r$ with $p$ and $s$: the corresponding constants (and functions $\Phi$ and $\lambda$) will then depend also on the ratio $\frac{s}{p}$.

\textbf{Proof.} The case of two general radii $s$ and $p$ follows easily from that of $p = 7r$ and $s = 5r$ and a simple covering argument. In what follows, given a pair of points $x \in \mathcal{X}, y \in \mathcal{X}^\perp$ we use the notation $(x, y)$ for the vector $x + y$. By translation we can assume that $(q, u) = (0, 0)$. Consider then the maps $F, G : B_7r(0, \mathcal{X}) \rightarrow \mathcal{X}_0^\perp$ and $I, J : B_7r(0, \mathcal{X}) \rightarrow \mathcal{X}_0$ given by

$$F(x) = p_{x_0}^{-1}((x, f(x))) \quad \text{and} \quad G(x) = p_{x_0}^{-1}((x, g(x))),$$

$$I(x) = p_{x_0}((x, f(x))) \quad \text{and} \quad J(x) = p_{x_0}((x, g(x))).$$
Obviously, if $c_0$ is sufficiently small, $I$ and $J$ are injective Lipschitz maps. Hence, $G_{\kappa_0}(f)$ and $G_{\kappa_0}(g)$ coincide, in the new coordinates, with the graphs of the functions $f'$ and $g'$ defined respectively in $D := I(B_{r_0}(0, \kappa))$ and $\tilde{D} := J(B_{r_0}(0, \kappa))$ by $f' = F \circ I^{-1}$ and $g' = G \circ J^{-1}$. If $c_0$ is chosen sufficiently small, then
\[
\text{Lip}(I), \text{Lip}(J), \text{Lip}(I^{-1}), \text{Lip}(J^{-1}) \leq 1 + Cc_0, \tag{B.4}
\]
and
\[
|I(q) - q'|, |J(q) - q| \leq Cc_0 r, \tag{B.5}
\]
where the constant $C$ is only geometric. Clearly, (B.4) and (B.5) easily imply that $B_{5r}(0, \kappa_0) \subset D \cap \tilde{D}$ when $c_0$ is smaller than a geometric constant, thereby implying (a). Conclusion (c) is a simple consequence of the inverse function theorem. Finally we claim (B.4), (b) follows.

From which, using the change of variables formula for biLipschitz homeomorphisms and (B.4), (b) follows.

In order to prove (B.6), consider any $x' \in B_r(q')$, set $x := I^{-1}(x')$ and
\[
p_1 := (x, f(x)) \in \kappa \times \kappa^\perp, \quad p_2 := (x, g(x)) \in \kappa \times \kappa^\perp \quad \text{and} \quad p_3 := (x', g'(x')) \in \kappa_0 \times \kappa_0^\perp.
\]
Obviously $|f'(x') - g'(x')| = |p_1 - p_3|$ and $|f(x) - g(x)| = |p_1 - p_2|$. Note that, $g(x) = f(x)$ if and only if $g'(x') = f'(x')$, and in this case (B.6) follows trivially. If this is not the case, the triangle with vertices $p_1$, $p_2$, and $p_3$ is non-degenerate. Let $\theta_i$ be the angle at $p_i$. Note that, $\text{Lip}(g) \leq c_0$ implies $|\frac{\pi}{2} - \theta_2| \leq Cc_0$ and $|\kappa - \kappa_0| \leq c_0$ implies $|\theta_1| \leq Cc_0$, for some dimensional constant $C$. Since $\theta_3 = \pi - \theta_1 - \theta_2$, we conclude as well $|\frac{\pi}{2} - \theta_3| \leq Cc_0$. Therefore, if $c_0$ is small enough, we have $1 \leq 2 \sin \theta_3$, so that, by the Sinus Theorem,
\[
|f'(x') - g'(x')| = |p_1 - p_3| = \frac{\sin \theta_2}{\sin \theta_3} |p_1 - p_2| \leq 2 |p_1 - p_2| = 2 |f(x) - g(x)|,
\]
thus concluding the claim.

We finally come to (d). The estimate $\|f' - g'\|_{L^2} \leq C\|f - g\|_{L^2}$ is an obvious consequence of (B.6). Given next a point $p$ in the graph of $f$, resp. in the graph of $g$, we denote by $\sigma(p)$, resp. $\tau(p)$, the oriented tangent plane to the corresponding graphs. Observe that the points are described by the pairs $(x', f(x'))$ and $(x', g(x'))$, in the coordinates $\kappa \times \kappa^\perp$, and by $(I^{-1}(x'), f(I^{-1}(x'))) \text{ and } (J^{-1}(x'), g(J^{-1}(x')))$, in the coordinates $\kappa_0 \times \kappa_0^\perp$. Thus
\[
|\nabla f'(x') - \nabla g'(x')| \leq C|\sigma(p) - \tau(q)| \leq C|\nabla f(I^{-1}(x')) - \nabla g(J^{-1}(x'))|
\]
\[
\leq C|\nabla f(I^{-1}(x')) - \nabla f(J^{-1}(x'))| + C|\nabla f(J^{-1}(x')) - \nabla g(J^{-1}(x'))|
\]
\[
\leq C|\nabla f|c_0|I^{-1}(x') - J^{-1}(x')| + C|\nabla f(J^{-1}(x')) - \nabla g(J^{-1}(x'))| + C|\nabla f(J^{-1}(x')) - \nabla g(J^{-1}(x'))|.
\]

Integrating this last inequality in $x'$ and changing variables we then conclude
\[
\|\nabla f' - \nabla g'\|_{L^2} \leq C\|\nabla f - \nabla g\|_{L^2} + C\|D^2 f\|c_0\|f' - g'\|_{L^2},
\]
which, together with the $L^2$ estimate, gives (d).
APPENDIX C. TWO INTERPOLATION INEQUALITIES

Lemma C.1. Let $A > 0$ and $\psi \in C^2(B_\rho, \mathbb{R}^n)$ satisfy $\|\psi\|_{L^1} \leq A \rho^{m+1}$ and $\|\Delta \psi\|_{L^\infty} \leq \rho^{-1} A$. Then, for every $r < \rho$ there is a constant $C > 0$ (depending only on $m$ and $\frac{r}{\rho}$) such that

$$\rho^{-1} \|\psi\|_{L^\infty(B_r)} + \|D\psi\|_{L^\infty(B_r)} \leq C A.$$  \hfill (C.1)

Proof. By a simple covering argument we can, w.l.o.g., assume $\rho = 3r$. Moreover, if we apply the scaling $\psi_r(x) := r^{-1}\psi(rx)$ we see that $\|\psi_r\|_{L^1(B_3)} = (\rho/3)^{-m-1}\|\psi\|_{L^1(B_\rho)}$, $\|\psi_r\|_{L^\infty} = (\rho/3)^{-1}\|\psi\|_{L^\infty}$, $\|D\psi_r\|_{L^\infty} = \|D\psi\|_{L^\infty}$ and $\|\Delta \psi_r\|_{L^\infty} = (\rho/3)^3\|\Delta \psi\|_{L^\infty}$. We can therefore assume $r = 1$. Consider the harmonic function $\zeta : B_2 \to \mathbb{R}$ with boundary data $\psi|_{\partial B_2}$,

$$\begin{cases} \Delta \zeta = 0 & \text{in } B_2, \\ \zeta = \psi & \text{on } \partial B_2. \end{cases}$$

Set $u := \psi - \zeta$ and note that $u = 0$ on $\partial B_2$, $\|\Delta u\|_{C^0(B_2)} \leq A$. Hence, using the Poincaré inequality, we can estimate the $L^1$-norm of $u$ in the following way:

$$\|u\|_{L^1} \leq \|D\psi\|_{L^\infty} \leq C \|D\psi\|_{L^\infty} \leq C \left( \int_{B_2} |\Delta u\, u| \right)^{1/2} \leq C \|\Delta u\|_{C^0}^{1/2} \|u\|_{L^1}^{1/2} \leq C A.$$  

Choose now $a \in [0, 1]$ and $s \in [1, \infty]$ such that $\frac{1}{m} + a \left( \frac{1}{s} - \frac{2}{m} \right) + 1 - a < 0$ (which exist because for $s \to \infty$ and $a \to 1$ the expression converges to $-\frac{1}{m}$). By a classical interpolation inequality, (see [9])

$$\|D\psi\|_{L^\infty} \leq C \|D^2\psi\|_{L^\alpha(\partial B_2)} \|u\|_{L^{1-a}}^{1-a} + C \|u\|_{L^1}.$$  

Using the $L^\alpha$-estimate for the Laplacian, we deduce

$$\|D\psi\|_{L^\infty} \leq C \|\Delta u\|_{L^\alpha(\partial B_2)} \|u\|_{L^{1-a}}^{1-a} + C \|u\|_{L^1} \leq C \|\Delta u\|_{C^0} \|u\|_{L^{1-a}}^{1-a} + \|u\|_{L^1} \leq C A.$$  \hfill (C.2)

From (C.2) and $u|_{\partial B_2} = 0$ it follows trivially $\|u\|_{L^\infty} \leq A$. To infer (C.1), we observe that, by $\|\zeta\|_{L^1(B_2)} \leq \|u\|_{L^1(B_2)} + \|\psi\|_{L^1(B_2)} \leq C A$ and the harmonicity of $\zeta$,

$$\|\zeta\|_{L^\infty(B_2)} + \|D\zeta\|_{L^\infty(B_2)} \leq C \|\zeta\|_{L^1(B_2)} \leq C A.$$  

\hfill \Box

Lemma C.2. For every $m, r < s$ and $\kappa$ there is a positive constant $C$ (depending on $m$, $\kappa$ and $\frac{r}{s}$) with the following property. Let $f$ be a $C^{3,\kappa}$ function in the ball $B_3 \subset \mathbb{R}^m$. Then

$$\|D^j f\|_{C^0(B_3)} \leq C r^{-m-j} \|f\|_{L^1(B_3)} + C r^{\kappa + 3-j} [D^3 f]_{\kappa, B_3} \quad \forall j \in \{0, 1, 2, 3\}.$$  \hfill (C.3)

Proof. A simple covering argument reduces the lemma to the case $s = 2r$. Moreover, define $f_r(x) := f(rx)$ to see that we can assume $r = 1$. So our goal is to show

$$\sum_{j=0}^3 |D^j f(y)| \leq C \|f - g\|_{L^1} + C [D^3 f]_{\kappa} \quad \forall y \in B_1, \forall f \in C^{3,\kappa}(B_2).$$  \hfill (C.4)

By translating it suffices then to prove the estimate

$$\sum_{j=0}^3 |D^j f(0)| \leq C \|f\|_{L^1(B_1)} + C [D^3 f]_{\kappa, B_1} \quad \forall f \in C^{3,\kappa}(B_1).$$  \hfill (C.5)
Consider now the space of polynomials $R$ in $m$ variables of degree at most 3, which we write as $R = \sum_{j=0}^{3} A_j x^j$. This is a finite dimensional vector space, on which we can define the norms $|R| := \sum_{j=0}^{3} |A_j|$ and $\|R\| := \int_{B_1} |R(x)| \, dx$. These two norms must then be equivalent, so there is a constant $C$ (depending only on $m$), such that $|R| \leq C\|R\|$ for any such polynomial. In particular, if $P$ is the Taylor polynomial of third order for $f$ at the point 0, we conclude

$$\sum_{j=0}^{3} |D^j f(0)| = |P| \leq C\|P\| = C \int_{B_1} |P(x)| \, dx \leq C\|f\|_{L^1(B_1)} + C\|f - P\|_{L^1(B_1)} \leq C\|f\|_{L^1} + C[D^3 f]_{\kappa}. \quad \Box$$

References


