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by

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THE QUALITATIVE BOUNDARY BEHAVIOR OF BLOW-UP SOLUTIONS OF THE SUPER-LIOUVILLE EQUATIONS

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Abstract. Continuing our work on the boundary value problem for super-Liouville equation, we study the qualitative behavior of boundary blow-ups. The boundary condition is derived from the chirality conditions in the physics literature, and is geometrically natural. In technical terms, we derive a new Pohozaev type identity and provide a new alternative, which also works at the boundary, to the classical method of Brézis-Merle.

1. Introduction

Motivated by the supersymmetric extension of Liouville theory in the recent physics literature, we have constructed a corresponding variational problem that can be studied with the tools of nonlinear analysis (see [JWZ, JWZZ1, JWZZ2]). On one hand, this functional possesses an interesting and rich geometric structure, and on the other hand, the powerful tools of geometric analysis that have been developed since the 1980s allow for a very detailed and precise investigation of the properties of the solutions. In this paper, we carry this program further. In technical terms, we introduce a new argument for the blow-up analysis that is based on the removability for a local singularity at the boundary. This argument can not only reprove the classical results on the blow-up behavior of the Liouville equation, but also naturally extends to the boundary situation. In fact, for both physical and geometrical reasons, the boundary behavior is of particular interest. Physically, it corresponds to chirality conditions, and geometrically, it incorporates reflection principles.

Our functional couples the standard Liouville functional with a spinor field term and is therefore called the super-Liouville functional. It is naturally defined on a compact Riemann surface M with or without boundary. The important point is that this generalization preserves a fundamental property of the Liouville functional on Riemann surfaces, namely its conformal invariance. As is well known, conformal invariance is both a key feature in quantum field theory, see the nonlinear sigma model or string theory, and in geometric analysis, see the theory of two-dimensional harmonic maps, minimal surfaces, pseudoholomorphic curves, and the like.

When the domain M is a closed surface, the super-Liouville functional is

$$E(u,\psi) = \int_M \left\{ \frac{1}{2} |\nabla u|^2 + K_g u + \langle (\not{D} + e^u)\psi, \psi \rangle - e^{2u} \right\} dv,$$

and the Euler-Lagrange system is

$$\begin{cases} -\Delta u = 2e^{2u} - e^u \langle \psi, \psi \rangle - K_g, & \text{in } M, \\ \not{D}\psi = -e^u \psi, & \text{in } M. \end{cases} \tag{1}$$

Here M is a Riemann surface with conformal metric g and with a spin structure, K_g is the Gaussian curvature in M, ΣM is the spinor bundle on M with a natural Hermitian product $\langle \cdot, \cdot \rangle$ induced by g, u is a real-valued function on M and ψ is a spinor on M. The Dirac operator $\not{D}$ is defined by $\not{D}\psi := \sum_{\alpha=1}^2 e_\alpha \cdot \nabla_{e_\alpha} \psi$, where $\{e_1, e_2\}$ is a local orthonormal basis on TM, ∇ is the spin

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connection on ΣM and $\cdot$ denotes Clifford multiplication in the spinor bundle ΣM. The Clifford multiplication between e_i and $\psi, \varphi \in \Gamma(\Sigma M)$ satisfies

$$e_i \cdot e_j \cdot \psi + e_j \cdot e_i \cdot \psi = -2\delta_{ij}\psi, \quad \langle \psi, \varphi \rangle = \langle e_i \cdot \varphi, e_i \cdot \psi \rangle. \tag{2}$$

We refer to [LM, Jo] for more geometric background of spinors and its calculus.

As one knows in geometric analysis, because of conformal invariance, the key for understanding this functional is the blow-up behavior for limits of sequences of solutions. This has been achieved in [JWZ, JWZZ1].

The purpose of the present paper is to continue the investigation of the more general situation of surfaces with boundary, extending [JWZZ2]. When the domain M has a nonempty boundary ∂M, the super-Liouville functional becomes

$$E_B(u,\psi) = \int_M \left\{ \frac{1}{2}|\nabla u|^2 + K_g u + \langle (\not{D} + e^u)\psi, \psi \rangle - e^{2u} \right\} dv + \int_{\partial M} (h_g u - ce^u)\, d\sigma,$$

which has been introduced in [JWZZ2]. Here h_g is the geodesic curvature on ∂M and c is a given constant. In fact, there exists a rich physics literature on this topic, see e.g. [Po, ARS, FH].

To continue the discussion about surfaces with boundary, let us first recall the chirality boundary condition (introduced in [GHHP]. See also [HMR]) for the Dirac operator $\not{D}$, which turns out to be a natural boundary condition for ψ.

We now have to set up the details. Let M be a compact Riemann surface with $\partial M \neq \emptyset$ and with a fixed spin strcuture, admitting a chirality operator G, which is an endomorphism of the spinor bundle ΣM satisfying:

$$G^2 = I, \quad \langle G\psi, G\varphi \rangle = \langle \psi, \varphi \rangle, \quad \nabla_X(G\psi) = G\nabla_X\psi, \quad X \cdot G\psi = -G(X \cdot \psi),$$

for any $X \in \Gamma(TM), \psi, \varphi \in \Gamma(\Sigma M)$. Here I denotes the identity endomorphism of ΣM. Usually, we take $G = \gamma(\omega_2)$, the Clifford multiplication by the complex volume form $\omega_2 = ie_1e_2$, where $\{e_1, e_2\}$ is a local orthonormal frame on M.

Denote by $S = \Sigma M|_{\partial M}$ the restricted spinor bundle with induced Hermitian product. Let $\overrightarrow{n}$ be the outward unit normal vector field on ∂M. One can verify that $\overrightarrow{n}G : \Gamma(S) \to \Gamma(S)$ is a self-adjoint endomorphism satisfying

$$(\overrightarrow{n}G)^2 = I, \qquad \langle \overrightarrow{n}G\psi, \varphi \rangle = \langle \psi, \overrightarrow{n}G\varphi \rangle.$$

Hence, we can decompose $S = V^+ \bigoplus V^-$, where $V^\pm$ is the eigensubbundle corresponding to the eigenvalue ± 1. One verifies that the orthogonal projection onto the eigensubbundle $V^\pm$:

$$\begin{aligned} \mathbf{B}^\pm : L^2(S) &\to L^2(V^\pm) \\ \psi &\mapsto \frac{1}{2}(I \pm \overrightarrow{n}G)\psi, \end{aligned}$$

defines a local elliptic boundary condition for the Dirac operator $\not{D}$ (see [HMR]). We say that a spinor $\psi \in W^{1,4/3}(\Sigma M)$ satisfies the chirality boundary conditions $\mathbf{B}^\pm$ if

$$\mathbf{B}^\pm \psi|_{\partial M} = 0.$$

Recall that, on a surface M with nonempty boundary $\partial M \neq \emptyset$, the Dirac operator $\not{D}$ is in general not formally self-adjoint. In fact, we have

$$\int_M \langle \psi, \not{D}\varphi \rangle dv = \int_M \langle \not{D}\psi, \varphi \rangle dv - \int_{\partial M} \langle \overrightarrow{n} \cdot \psi, \varphi \rangle dv, \quad \forall \psi, \varphi \in \Gamma(\Sigma M).$$

When imposing the chirality boundary conditions $\mathbf{B}^\pm$, $\not{D}$ becomes self-adjoint (see e.g. [HMR]). To see this, suppose that $\psi, \varphi \in W^{1,\frac{4}{3}}(\Gamma(\Sigma M))$ satisfy the chirality boundary conditions $\mathbf{B}^\pm$, then

$$\langle \overrightarrow{n} \cdot \psi, \varphi \rangle = 0, \quad \text{on } \partial M.$$

In particular, there holds

$$\int_{\partial M} \langle \overrightarrow{n} \cdot \psi, \varphi \rangle = 0.$$

It may help the reader if we recall that on a surface the (usual) Dirac operator $\not{D}$ can be seen as the (doubled) Cauchy-Riemann operator. Consider $\mathbb{R}^2$ with the Euclidean metric ds^2+dt^2. Let $e_1=\frac{\partial}{\partial s}$ and $e_2=\frac{\partial}{\partial t}$ be the standard orthonormal frame. A spinor field on $\mathbb{R}^2$ is simply a map $\Psi:\mathbb{R}^2\to\Delta_2=\mathbf{C}^2$, and e_1 and e_2 acting on spinor fields can be identified by multiplication with matrices

$$e_1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\quad e_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Here, W.L.O.G., we keep the representations of e_1 and e_2 consistent with that in [JWZZ2].

If $\Psi:=\begin{pmatrix}f\\ g\end{pmatrix}:\mathbb{R}^2\to\mathbf{C}^2$ is a spinor field, then the Dirac operator is

$$\not{D}\Psi=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial s}\\ \frac{\partial g}{\partial s}\end{pmatrix}+\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial t}\\ \frac{\partial g}{\partial t}\end{pmatrix}=2i\begin{pmatrix}\frac{\partial g}{\partial z}\\ \frac{\partial f}{\partial \bar{z}}\end{pmatrix},$$

where $\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial s}-i\frac{\partial}{\partial t}\right),\quad \frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial s}+i\frac{\partial}{\partial t}\right).$

On the upper-half Euclidean plane $\mathbb{R}^2_+$, the chirality operator is $G=ie_1e_2=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\overrightarrow{n}=-e_2$. A simple calculation gives that

$$\mathbf{B}^{\pm}=\frac{1}{2}(I\pm\overrightarrow{n}\cdot G)=\frac{1}{2}\begin{pmatrix}1 & \pm 1\\ \pm 1 & 1\end{pmatrix}.$$

Write $\psi=\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}$ via the standard chirality decomposition, then the chirality boundary condition

$$\mathbf{B}^{\pm}\psi|_{\partial\mathbb{R}^2_+}=0$$

becomes

$$\psi_+=\mp\psi_-\quad \text{on } \partial\mathbb{R}^2_+.$$

The Euler-Lagrange system for $E_B(u,\psi)$ is (see [JWZZ2])

$$\begin{cases}-\Delta u = 2e^{2u}-e^u\langle\psi,\psi\rangle-K_g, & \text{in } M^o,\\ \not{D}\psi = -e^u\psi, & \text{in } M^o,\\ \frac{\partial u}{\partial n} = ce^u-h_g, & \text{on } \partial M,\\ \mathbf{B}^{\pm}\psi = 0, & \text{on } \partial M.\end{cases}\tag{3}$$

In [JWZZ2], some analytic foundations for the above boundary value problem have been laid, such as the regularity for weak solutions, the small energy compactness theorem, a removable global singularity theorem, and the fundamental blow-up analysis of solutions.

In this paper, we will continue to investigate the blow-up behavior for solutions of this boundary value problem, including the energy identity for solutions and the blow-up values at the blow-up points. Therefore, we can extend the full blow-up theory for the Liouville equation (see [BM, LSh, Ly, BCLT, JLW] and the references therein) to the super-Liouville equation, especially to the nonempty boundary case.

To begin with, we recall the following main result of [JWZZ2]:

Theorem 1.1 (Theorem 4.1, [JWZZ2])**.** *Let (M,g) be a compact Riemann surface with nonempty boundary $\partial M\neq\emptyset$ and with a fixed spin structure. Let (u_n,ψ_n) be a sequence of solutions to* (3) *with uniformly bounded energy,*

$$\int_M(e^{2u_n}+|\psi_n|^4)dv+\int_{\partial M}e^{u_n}d\sigma\le C\tag{4}$$

for some positive constant C.

Define the blow-up set of (u_n, ψ_n) *as follows*

$$\begin{array}{rcl}\Sigma_1 &=& \{x \in M, \text{ there is a sequence } y_n \to x \text{ such that } u_n(y_n) \to +\infty\} \\ \Sigma_2 &=& \{x \in M, \text{ there is a sequence } y_n \to x \text{ such that } |\psi_n(y_n)| \to +\infty\}.\end{array}$$

Then, we have $\Sigma_2 \subset \Sigma_1$. *Moreover,* (u_n, ψ_n) *admits a subsequence, denoted still by* (u_n, ψ_n), *satisfying*

a) $|\psi_n|$ *is bounded in* $L^\infty_{loc}(M\backslash\Sigma_2)$.

b) *For* u_n, *one of the following alternatives holds:*

  i) u_n *is bounded in* $L^\infty(M)$.

  ii) $u_n \to -\infty$ *uniformly on* M.

  iii) Σ_1 *is finite, nonempty and either*

$$u_n \text{ is bounded in } L^\infty_{loc}(M\backslash\Sigma_1) \tag{5}$$

or

$$u_n \to -\infty \text{ uniformly on compact subsets of } M\backslash\Sigma_1. \tag{6}$$

Our first main result in this paper is to extend the energy identity for the spinor part ψ_n on closed surfaces (see Theorem 1.2 in [JWZZ1]) to the case of surfaces with boundary.

Theorem 1.2. *Notations and assumptions as in Theorem 1.1. Assume that the constant c in* (3) *is nonnegative and write* $\Sigma_1 = \{x_1, x_2, \cdots, x_K\}$. *Then there are finitely many solutions* $(u^{i,k}, \psi^{i,k})$ *to* (1) *on* S^2 *for* $i = 1, 2, \cdots, I; k = 1, 2, \cdots, K_i$ *and finitely many solutions* $(u^{j,l}, \psi^{j,l})$ *to* (3) *on a spherical cap* $S^2_{c'}$ *(where* c' *is the geodesic curvature of the boundary* $\partial S^2_{c'}$*) for* $j = 1, 2, \cdots, J; l = 1, 2, \cdots, L_j$. *After selection of a subsequence,* ψ_n *converges in* C^∞_{loc} *to some* ψ *on* $M\backslash\Sigma_1$ *and the following energy identity holds:*

$$\lim_{n\to\infty}\int_M |\psi_n|^4 dv = \int_M |\psi|^4 dv + \sum_{i=1}^{I}\sum_{k=1}^{K_i}\int_{S^2}|\psi^{i,k}|^4 dv + \sum_{j=1}^{J}\sum_{l=1}^{L_j}\int_{S^2_{c'}}|\psi^{j,l}|^4 dv. \tag{7}$$

In view of the energy identity in (7), we know that the neck energy of the spinors ψ_n is converging to zero. As an application of this energy identity, we shall complete the qualitative picture of the blow-up process of (u_n, ψ_n) on surfaces with boundary. In this paper, we develop a new method, which provides an alternative to the argument in [BM], as it can also deal with the interior situation for both the Liouville equation and the super-Liouville equation (see Section 4). Our next result is

Theorem 1.3. *Notations and assumptions as in Theorem 1.1. Assume that the constant c in* (3) *is nonnegative and the blow-up set* $\Sigma_1 \neq \emptyset$. *Then*

$$u_n \to -\infty \quad \text{uniformly on compact subset of } M\backslash\Sigma_1.$$

Furthermore,

$$\int_M (2e^{2u_n} - e^{u_n}|\psi_n|^2)\phi dv + \int_{\partial M} ce^{u_n}\phi d\sigma \to \sum_{x_i\in\Sigma_1}\alpha_i\phi(x_i), \quad \forall\phi \in C^\infty(M) \tag{8}$$

with $\alpha_i \geq 4\pi$ *for* $x_i \in \Sigma_1 \cap M^o$ *and* $\alpha_j \geq 2\pi$ *for* $x_j \in \Sigma_1 \cap \partial M$.

To continue our discussion, we recall that the Pohozaev type identity plays an important role in the blow-up theory of the Liouville equation. The corresponding Pohozaev type identity for the super-Liouville equations on closed Riemann surfaces was derived in [JWZZ1]. In this paper, we extend it to the case of surfaces with boundary. More precisely, we have

Proposition 1.4. *For* $R > 0$, *denote* $B^+_R = \{(s,t) \in \mathbb{R}^2 | s^2 + t^2 < R^2, t > 0\}$. *Let* (u, ψ) *be a solution of*

$$\begin{cases} -\Delta u &=& 2e^{2u} - e^u|\psi|^2, & \text{in } B^+_R, \\ \not{D}\psi &=& -e^u\psi, & \text{in } B^+_R, \\ \frac{\partial u}{\partial n} &=& ce^u, & \text{on } \partial B^+_R \cap \partial\mathbb{R}^2_+, \\ \mathbf{B}^{\pm}\psi &=& 0, & \text{on } \partial B^+_R \cap \partial\mathbb{R}^2_+. \end{cases}$$

Then the following Pohozaev type identity holds

$$
\begin{aligned}
& R\int_{\partial B_R^+\cap\mathbb{R}^2_+}(|\frac{\partial u}{\partial\nu}|^2-\frac{1}{2}|\nabla u|^2)d\sigma \\
= & \int_{B_R^+}2e^{2u}dv-\int_{B_R^+}e^u|\psi|^2dv+\int_{\partial B_R^+\cap\partial\mathbb{R}^2_+}ce^u ds \\
& -R\int_{\partial B_R^+\cap\mathbb{R}^2_+}e^{2u}d\sigma-cse^{u(s,0)}|_{s=-R}^{s=R} \\
& +\frac{1}{4}\int_{\partial B_R^+\cap\mathbb{R}^2_+}\langle\frac{\partial\psi}{\partial\nu},(x+\bar{x})\cdot\psi\rangle d\sigma+\frac{1}{4}\int_{\partial B_R^+\cap\mathbb{R}^2_+}\langle(x+\bar{x})\cdot\psi,\frac{\partial\psi}{\partial\nu}\rangle d\sigma \qquad (9)
\end{aligned}
$$

where ν is the outward normal vector to $\partial B_R^+\cap\mathbb{R}^2_+$. For $x=(s,t)\in\mathbb{R}^2$, $\bar{x}=(s,-t)$ is its reflection point about $\partial\mathbb{R}^2_+$ and $x\cdot\psi=(se_1+te_2)\cdot\psi$.

Further exploring the qualitative behavior of the blow-up solutions (u_n,ψ_n), we shall calculate the blow-up value at a blow-up point. Let $p\in\Sigma_1$, then the blow-up value at p is defined as

$$m(p)=\lim_{r\to 0}\lim_{n\to\infty}\int_{B_r^M(p)}(2e^{2u_n}-e^{u_n}|\psi_n|^2)dv+c\int_{B_r^M(p)\cap\partial M}e^{u_n}d\sigma.$$

When $p\in\Sigma_1\cap M^o$, it follows from Theorem 1.5 in [JWZZ1] that $m(p)=4\pi$. In this paper, we will calculate the value of $m(p)$ for $p\in\Sigma_1\cap\partial M$. More precisely, using the Pohozaev type identity in Proposition 1.4 and the asymptotic behavior of (u_n,ψ_n) at a blow-up point obtained in Theorem 1.3, we can show:

Theorem 1.5. *Notations and assumptions as in Theorem 1.1. Assume that the constant c in* (3) *is nonnegative. Let $p\in\Sigma_1\cap\partial M$, then $m(p)=2\pi$.*

Finally, from equation (3) and the Gauss-Bonnet formula, we have

$$\int_M(2e^{2u_n}-e^{u_n}|\psi_n|^2)dv+\int_{\partial M}ce^{u_n}d\sigma=2\pi\chi(M), \qquad (10)$$

where $\chi(M)$ is the Euler characteristic of a surface M with boundary ∂M. As an application of Theorem 1.2, Theorem 1.3 and Theorem 1.5, we conclude from (10) that

Theorem 1.6. *Notations and assumptions as in Theorem 1.1. Assume that the constant c in* (3) *is nonnegative. For the blow-up set Σ_1 we have*
(1) If $\chi(M)=1$, then the blow-up set Σ_1 contains at most one point.
(2) If $\chi(M)\le 0$, then the blow-up set is empty, that is, there is no blow-up in this case.

Remark 1.7. In the case $\chi(M)=1$, i.e., M is a closed disc, the solution space is not compact.

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2. Some basic geometric and analytic properties

In this section, we shall recall some basic geometric and analytic properties for solutions of the super-Liouville boundary value problem (3) established in [JWZ] and [JWZZ2], which form the fundamental tools for the blow-up analysis. In addition, we show the smoothness up to the boundary of weak solutions and prove a result about the removability of local singularities at the boundary. At the end of this section, we shall derive the Pohozaev type identity near a boundary point $p \in \partial M$ for solutions - Proposition 1.4.

First of all, we recall that our problem (3) is conformally invariant.

Proposition 2.1 (Proposition 2.2, [JWZZ2]). *Let (u, ψ) be a solution of (3). For any conformal diffeomorphism $\varphi : M \to M$, set*

$$\begin{aligned} \widetilde{u} &= u \circ \varphi - \phi \\ \widetilde{\psi} &= e^{-\frac{\phi}{2}} \psi \circ \varphi \end{aligned} \tag{11}$$

where e^{ϕ} is the conformal factor of φ, i.e., $\varphi^*(g) = e^{2\phi} g$, then $(\tilde{u}, \tilde{\psi})$ is also a solution of (3). Moreover, the functional $E_B(u, \psi)$ is conformally invariant.

Set

$$W_B^{1,\frac{4}{3}}(\Gamma(\Sigma M)) = \left\{ \psi | \psi \in W^{1,\frac{4}{3}}(\Gamma(\Sigma M)), \mathbf{B}^{\pm} \psi|_{\partial M} = 0 \right\}.$$

$(u, \psi) \in W^{1,2}(M) \times W_B^{1,\frac{4}{3}}(\Gamma(\Sigma M))$ is called a weak solution of (3) if

$$\begin{aligned} \int_M \nabla u \nabla \phi dv &= \int_M (2e^{2u} - e^u |\psi|^2 - K_g) \phi dv - \int_{\partial M} (ce^u - h_g) \phi d\sigma \\ \int_M \langle \psi, \not{D} \xi \rangle dv &= - \int_M e^u \langle \psi, \xi \rangle dv \end{aligned}$$

for any function $\phi \in C^{\infty}(M)$ and any spinor $\xi \in C^{\infty} \cap W_B^{1,\frac{4}{3}}(\Gamma(\Sigma M))$.

A weak solution with bounded energy is shown to be smooth in the interior and regular at the boundary:

Proposition 2.2 (Proposition 4.1, [JWZ]; Proposition 3.1, [JWZZ2]). *Let (u, ψ) be a weak solution of (3) with $\int_M e^{2u} + |\psi|^4 dv + \int_{\partial M} e^u d\sigma < \infty$. Then $u \in C^{\infty}(M^o) \cap W^{2,p}(\overline{M})$ for any $p > 2$ and $\psi \in C^{\infty}(\Gamma(\Sigma M^o)) \cap W^{2,q}(\Gamma(\Sigma \overline{M}))$ for any $q > 1$.*

Here, we prove that any weak solution is smooth up to the boundary. To do this, we recall the following classical elliptic estimates for the Laplacian and the Dirac operator under appropriate boundary constraints:

Theorem 2.3 ([ADN, We]). *Let $k \in \mathbb{N}_0$ and $1 < p < \infty$. Let $U \subset \mathbb{R}^m, m \geq 2$ be an open domain and $T \subset \partial U$ a smooth boundary portion. Suppose that $u \in W^{1,p}$ weakly solves*

$$\begin{cases} -\Delta u &= f \in W^{k,p}(U) \quad \text{in } U \\ \frac{\partial u}{\partial \vec{n}} &= g \in W_{\partial}^{k+1,p}(T) \quad \text{on } T \subset \partial U \end{cases}$$

Then for any $V \subset\subset U \cup T$, we have $u \in W^{k+2,p}(V)$ and there exists some $C = C(p, k, V, T) > 0$ such that

$$\|u\|_{W^{k+2,p}(V)} \leq C(\|f\|_{W^{k,p}(U)} + \|g\|_{W_{\partial}^{k+1,p}(T)} + \|u\|_{L^p(U)}).$$

Here $W_{\partial}^{k,p}(T) := \{g \in L^1(T) : g = G|_T \text{ for some } G \in W^{k,p}(U)\}$ with norm

$$\|g\|_{W_{\partial}^{k,p}(T)} = \inf_{G \in W^{k,p}(U)\ \ G|_T = g} \|G\|_{W^{k,p}(U)}.$$

Theorem 2.4. *Let $k \in \mathbb{N}_0$, $1 < p < \infty$, $U \subset \mathbb{R}^2$ any open set with $T \subset \partial U$ a smooth boundary portion. Suppose $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \in W^{1,p}(U, \mathbf{C}^2)$ weakly solves the following boundary value problem:*

$$\begin{cases} \not{D} \psi &= F \in W^{k,p}(U) \quad \text{in } U \\ \mathbf{B}^{\pm} \psi &= 0 \quad \text{on } T \subset \partial U \end{cases}$$

Then for any $V \subset\subset U \cup T$, we have $\psi \in W^{k+1,p}(V)$ and there exists some $C = C(p, V, T) > 0$ such that

$$\|\psi\|_{W^{k+1,p}(V)} \leq C(\|F\|_{W^{k,p}(U)} + \|\psi\|_{L^p(U)}).$$

Remark 2.5. Theorem 2.4 can be seen as a classical result, as it can be proved by reducing the Dirac equations with chirality boundary condition to the classical $\overline{\partial}$ equations for functions with a vanishing imaginary (or real) part on the boundary (see e.g. Theorem 4.6, [SZ]).

Based on the regularity results in Proposition 2.2, we can apply a bootstrapping argument by using Theorem 2.3 and Theorem 2.4 to the system (3) to conclude the following

Proposition 2.6. *Let (u, ψ) be a weak solution of (3) with $\int_M e^{2u} + |\psi|^4 dv + \int_{\partial M} e^u d\sigma < \infty$. Then $u \in C^\infty(\overline{M})$ and $\psi \in C^\infty(\Gamma(\Sigma\overline{M}))$.*

Lemma 2.7 (Small energy compactness. Lemma 3.5, [JWZZ2])**.** *Let $0 < \epsilon_1 < \frac{\pi}{2}$ and $0 < \epsilon_2 < \pi$. Let (u_n, ψ_n) be a sequence of solutions satisfying*

$$\begin{cases} -\Delta u_n &= 2e^{2u_n} - e^{u_n} \langle \psi_n, \psi_n \rangle, & \text{in } B_r^+, \\ \not{D}\psi_n &= -e^{u_n}\psi_n, & \text{in } B_r^+, \\ \dfrac{\partial u_n}{\partial n} &= ce^{u_n}, & \text{on } \partial B_r^+ \cap \{t=0\} \\ \mathbf{B}^{\pm}\psi_n &= 0, & \text{on } \partial B_r^+ \cap \{t=0\} \end{cases}$$

and

$$\int_{B_r^+} e^{2u_n} dx < \epsilon_1, \quad |c| \int_{\partial B_r^+ \cap \{t=0\}} e^{u_n} ds < \epsilon_2, \quad \int_{B_r^+} |\psi_n|^4 dx < C.$$

Then $||u_n^+||_{L^\infty(\overline{B}^+_{\frac{r}{4}})}$ and $||\psi_n||_{L^\infty(\overline{B}^+_{\frac{r}{8}})}$ are uniformly bounded.

It follows from Lemma 2.7 that the blow-up set Σ_1 can also be defined by

$$\Sigma_1 = \bigcap_{r>0} \left\{ x \in M | \liminf_{n\to\infty} \int_{B_r^M(x)} e^{2u_n} dv \geq \epsilon_1 \text{ or } \liminf_{n\to\infty} |c| \int_{B_r^M(x) \cap \partial M} e^{u_n} d\sigma \geq \epsilon_2 \right\}$$

The next lemma is about the decay at a singularity for a solution.

Lemma 2.8 (Lemma 5.3, [JWZZ2])**.** *There exist $0 < \epsilon_1 < \frac{\pi}{2}$ and $0 < \epsilon_2 < \pi$ such that if (v, ϕ) is a solution of*

$$\begin{cases} -\Delta v &= 2e^{2v} - e^{v} \langle \phi, \phi \rangle, & \text{in } B_1^+, \\ \not{D}\phi &= -e^{v}\phi, & \text{in } B_1^+, \\ \frac{\partial v}{\partial n} &= ce^{v}, & \text{on } (\partial \mathbb{R}^2_+ \cap \partial B_1^+) \backslash \{0\}, \\ \mathbf{B}^{\pm}\phi &= 0, & \text{on } (\partial \mathbb{R}^2_+ \cap \partial B_1^+) \backslash \{0\}, \end{cases}$$

with energy conditions

$$\int_{B_1^+} e^{2v} dx \leq \epsilon_1 < \pi, \ \int_{B_1^+} |\phi|^4 dx \leq C, \ |c| \int_{\partial B_1^+ \cap \{t=0\}} e^{v} ds \leq \epsilon_2 < \pi.$$

Then for any $x \in \overline{B}^+_{\frac{1}{2}}$ we have

$$|\phi(x)||x|^{\frac{1}{2}} + |\nabla\phi(x)||x|^{\frac{3}{2}} \leq C(\int_{B^+_{2|x|}} |\phi|^4 dx)^{\frac{1}{4}}. \tag{12}$$

Furthermore, if we assume that $e^{2v} = O(\frac{1}{|x|^{2-\epsilon}})$, then, for any $x \in \overline{B}^+_{\frac{1}{2}}$, we have

$$|\phi(x)||x|^{\frac{1}{2}} + |\nabla\phi(x)||x|^{\frac{3}{2}} \leq C|x|^{\frac{1}{4C}}(\int_{B_1^+} |\phi|^4 dx)^{\frac{1}{4}}, \tag{13}$$

for some positive constant C. Here ϵ is any sufficiently small positive number.

A global singularity at the boundary for a solution on $\mathbb{R}^2_+$ with bounded energy is shown to be removable.

Proposition 2.9 (Proposition 5.4 and Theorem 5.5 [JWZZ2]). *Let (u, ψ) be a solution of (3) on $\mathbb{R}^2_+$ with bounded energy*

$$\int_{\mathbb{R}^2_+} (e^{2u} + |\psi|^4)dx + \int_{\partial\mathbb{R}^2_+} e^u ds < \infty$$

and $c \geq 0$. Then (u, ψ) extends conformally to a solution on a spherical cap $S^2_{c'}$, where c' is the geodesic curvature of $\partial S^2_{c'}$. Moreover, we have

$$\int_{\mathbb{R}^2_+} (2e^{2u} - e^u|\psi|^2)dx + \int_{\partial\mathbb{R}^2_+} ce^u ds = 2\pi.$$

However, a local singularity at the boundary of a solution is in general not removable. To see this, we shall give an example. Set

$$u(x) = \log \frac{\sqrt{2}(1+\beta)|x|^\beta}{1 + |x^{1+\beta} - x_0|^2}$$

for some $x_0 = (s_0, t_0)$ with $t_0 = \frac{-c}{\sqrt{2}}$ and for $\beta > -1$. Then u is a solution of

$$\begin{cases} -\Delta u &= 2e^{2u}, & \text{in } \mathbb{R}^2_+ \\ \frac{\partial u}{\partial n} &= ce^u, & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\} \end{cases}$$

and $(u, 0)$ is a solution of (3) on $\mathbb{R}^2_+$ with finite energy. It is easy to see that $x = 0$ is a local singularity at the boundary which is not removable when $\beta \neq 0$.

Given a solution (u, ψ) of (3) on $\mathbb{R}^2_+$, we can associate to it the following quadratic differential:

$$T(z)dz^2 = \left\{ (\partial_z u)^2 - \partial_z^2 u + \frac{1}{4}\langle \psi, dz \cdot \partial_{\bar{z}}\psi\rangle + \frac{1}{4}\langle d\bar{z} \cdot \partial_z \psi, \psi\rangle \right\} dz^2. \tag{14}$$

Proposition 2.10 (Proposition 5.2 , [JWZZ2]). *$T(z)$ is holomorphic in $\mathbb{R}^2_+$ and it is real on the boundary $\partial\mathbb{R}^2_+$.*

In the above example, a simple computation gives that $\int_{B_r^+(0)} |T(z)|dz = +\infty$ for the holomorphic quadratic differential $T(z)dz^2$ associated to the solution $(u, 0)$.

Our observation for the removability of local singularities at the boundary is:

Proposition 2.11 (Removability of a local singularity at the boundary). *Let (u, ψ) be a solution of*

$$\begin{cases} -\Delta u &= 2e^{2u} - e^u \langle \psi, \psi \rangle, & \text{in } B_1^+, \\ \not{D}\psi &= -e^u \psi, & \text{in } B_1^+, \\ \frac{\partial u}{\partial n} &= ce^u, & \text{on } (\partial\mathbb{R}^2_+ \cap \partial B_1^+)\backslash\{0\}, \\ \mathbf{B}^{\pm}\psi &= 0, & \text{on } (\partial\mathbb{R}^2_+ \cap \partial B_1^+)\backslash\{0\}, \end{cases}$$

with energy condition

$$\int_{B_1^+} (e^{2u} + |\psi|^4)dx + \int_{\partial B_1^+ \cap \partial\mathbb{R}^2_+} e^u ds \leq C.$$

If $c \geq 0$ and the quadratic differential $T(z)dz^2$ satisfies

$$\int_{B_1^+} |T(z)|dz \leq C,$$

then (u, ψ) is smooth on $\overline{B}^+_{\frac{1}{2}}$.

Proof: By conformal invariance, we may assume for convenience that

$$\int_{B_1^+} e^{2u} dx \leq \epsilon_1, \quad |c| \int_{\partial B_1^+ \cap \partial\mathbb{R}^2_+} e^u ds \leq \epsilon_2,$$

where ϵ_1 and ϵ_2 are as in Lemma 2.8. Since u is a solution of

$$\begin{cases} -\Delta u &= 2e^{2u} - e^u|\psi|^2, & \text{in } B_1^+ \\ \frac{\partial u}{\partial n} &= ce^u, & \text{on } \partial B_1^+ \cap \partial \mathbb{R}^2_+ \backslash \{0\} \end{cases}$$

with bounded energy $\int_{B_1^+}(e^{2u}+|\psi|^4)dx+\int_{\partial B_1^+\cap\partial\mathbb{R}^2_+} e^u ds < \infty$. By applying the standard potential analysis, we know that there is a constant γ such that

$$\lim_{|x|\to 0}\frac{u}{-\log|x|} = \gamma.$$

Since $\int_{B_1^+}(e^{2u}+|\psi|^4)dx < \infty$, we obtain that $\gamma \le 1$. Furthermore, by a similar argument as in Proposition 5.4 of [JWZZ2], we can improve this to $\gamma < 1$.

Define $v(x)$ by

$$v(x) = -\frac{1}{\pi}\int_{B_1^+}(\log|x-y|)(2e^{2u(y)} - e^{u(y)}|\psi(y)|^2)dy - \frac{1}{\pi}\int_{\partial B_1^+\cap\partial\mathbb{R}^2_+} c(\log|x-y|)e^{u(y)}dy$$

and set $w = u - v$. Then v satisfies

$$\begin{cases} -\Delta v &= 2e^{2u} - e^u|\psi^2|, & \text{in } B_1^+ \\ \frac{\partial v}{\partial n} &= ce^u, & \text{on } \partial B_1^+ \cap \partial \mathbb{R}^2_+ \end{cases}$$

and hence w satisfies

$$\begin{cases} -\Delta w &= 0, & \text{in } B_1^+ \\ \frac{\partial w}{\partial n} &= 0, & \text{on } \partial B_1^+ \cap \partial \mathbb{R}^2_+ \backslash \{0\} \end{cases}$$

Therefore, by extending w to $B_1\backslash\{0\}$ evenly, we obtain a harmonic w in $B_1\backslash\{0\}$. Furthermore, one can verify that

$$\lim_{|x|\to 0}\frac{v(x)}{-\log|x|} = 0.$$

Consequently,

$$\lim_{|x|\to 0}\frac{w(x)}{-\log|x|} = \lim_{|x|\to 0}\frac{u-v}{-\log|x|} = \gamma.$$

Since the extended function w is harmonic in $B_1\backslash\{0\}$, there holds

$$w = -\gamma\log|x| + w_0,$$

where w_0 is a smooth harmonic function in B_1. Therefore, we get

$$u = -\gamma\log|x| + w_0 + v \quad \text{near } 0$$

Since $T(z)$ is holomorphic in B_1^+ and is real on $\partial B_1^+\cap\partial\mathbb{R}^2_+$. We can extend $T(z)$ to a holomorphic function in B_1. Then, by Lemma 2.8, we get

$$T(z) = \frac{\gamma^2 - 2\gamma}{4z^2} + o(\frac{1}{z^2}).$$

Since $\int_{B_1^+}|T(z)|dz \le C$ and hence $\int_{B_1}|T(z)|dz \le 2C$, we have $\gamma(\gamma-2) = 0$. Consequently, $\gamma = 0$. Finally, we can apply standard elliptic theory to conclude that (u,ψ) is smooth on $\overline{B}^+_{\frac{1}{2}}$. □

Now, we come to the Pohozaev type identity for solutions near a boundary point.

Proof of Proposition 1.4: Following the typical procedure of deriving Pohozaev type identities, we multiply the first equation by $x\cdot\nabla u$ and integrate over B_R^+ to get

$$-\int_{B_R^+}\Delta u x\cdot\nabla u dv = \int_{B_R^+} 2e^{2u}x\cdot\nabla u dv - \int_{B_R^+} e^u|\psi|^2 x\cdot\nabla u dv.$$

By a direct calculation, we have

$$
\begin{aligned}
&\int_{B_R^+} \Delta u x\cdot\nabla u dv \\
= \;& R\int_{\partial B_R^+\cap\mathbb{R}^2_+} (|\frac{\partial u}{\partial \nu}|^2 - \frac{1}{2}|\nabla u|^2) d\sigma + \int_{\partial B_R^+\cap\partial\mathbb{R}^2_+} \frac{\partial u}{\partial n}(x\cdot\nabla u) ds \\
= \;& R\int_{\partial B_R^+\cap\mathbb{R}^2_+} (|\frac{\partial u}{\partial \nu}|^2 - \frac{1}{2}|\nabla u|^2) d\sigma + \int_{\partial B_R^+\cap\partial\mathbb{R}^2_+} ce^u(x\cdot\nabla u) ds \\
= \;& R\int_{\partial B_R^+\cap\mathbb{R}^2_+} (|\frac{\partial u}{\partial \nu}|^2 - \frac{1}{2}|\nabla u|^2) d\sigma - \int_{\partial B_R^+\cap\partial\mathbb{R}^2_+} ce^{u(x)} ds + cse^{u(s,0)}|_{s=-R}^{s=R},
\end{aligned}
$$

$$
\int_{B_R^+} 2e^{2u} x\cdot\nabla u dv = R\int_{\partial B_R^+\cap\mathbb{R}^2_+} e^{2u} d\sigma - \int_{B_R^+} 2e^{2u} dv,
$$

and

$$
\int_{B_R^+} e^u|\psi|^2 x\cdot\nabla u dv = R\int_{\partial B_R^+\cap\mathbb{R}^2_+} e^u|\psi|^2 d\sigma - \int_{B_R^+} e^u x\cdot\nabla(|\psi|^2) dv - 2\int_{B_R^+} e^u|\psi|^2 dv.
$$

Combining the above three identities, we have

$$
\begin{aligned}
& R\int_{\partial B_R^+\cap\mathbb{R}^2_+} (|\frac{\partial u}{\partial \nu}|^2 - \frac{1}{2}|\nabla u|^2) d\sigma \\
= \;& -R\int_{\partial B_R^+\cap\mathbb{R}^2_+} e^{2u} d\sigma + \int_{B_R^+} 2e^{2u} dv + R\int_{\partial B_R^+\cap\mathbb{R}^2_+} e^u|\psi|^2 d\sigma - 2\int_{B_R^+} e^u|\psi|^2 dv \\
& -\int_{B_R^+} e^u x\cdot\nabla(|\psi|^2) dv + \int_{\partial B_R^+\cap\partial\mathbb{R}^2_+} ce^u ds - cse^{u(s,0)}|_{s=-R}^{s=R}. \qquad (15)
\end{aligned}
$$

On the other hand, recall that the orthonormal basis $\{e_1, e_2\}$ on $\mathbb{R}^2_+$ satisfies the Clifford multiplication relation (2). It is easy to verify that

$$
\langle \psi, e_i\cdot\psi\rangle + \langle e_i\cdot\psi, \psi\rangle = 0, \quad i = 1, 2 \qquad (16)
$$

By the chirality boundary condition of ψ, namely, $\mathbf{B}^{\pm}\psi|_{\partial B_R^+\cap\partial\mathbb{R}^2_+} = 0$, we extend (u,ψ) to the lower half disk B_R^- as follows:

$$
\begin{cases} u(\bar{x}) & := & u(x), \\ \psi(\bar{x}) & := & ie_1\cdot\psi(x), \end{cases} \qquad (17)
$$

where $\bar{x}\in B_R^-$ is the reflection point of x about $\partial\mathbb{R}^2_+$. Then, applying the same argument as in the proof of Lemma 3.4 in [JWZZ2], we know that ψ solves

$$
\not{D}\psi = -A(x)\psi \quad \text{in } B_R,
$$

where

$$
A(x) = \begin{cases} e^{u(x)}, & x\in B_R^+, \\ e^{u(\bar{x})}, & x\in B_R^-. \end{cases}
$$

Using the Schrödinger-Lichnerowicz formula (see e.g. [Jo, LM])

$$
\not{D}^2 = -\Delta + \frac{1}{2}K_g
$$

and noticing that $K_g = 0$ in our case of a flat domain B_R, we have

$$
-\Delta\psi = -dA(x)\cdot\psi + A^2(x)\psi \quad \text{in } B_R. \qquad (18)
$$

Then, we multiply (18) by $x \cdot \psi$ (where $\cdot$ denotes the Clifford multiplication) and integrate over B_R to get

$$\int_{B_R} \langle \Delta\psi, x\cdot\psi\rangle dv = \int_{B_R} \langle dA(x)\cdot\psi, x\cdot\psi\rangle dv - \int_{B_R} A^2(x)\langle\psi, x\cdot\psi\rangle dv,$$

and

$$\int_{B_R} \langle x\cdot\psi, \Delta\psi\rangle dv = \int_{B_R} \langle x\cdot\psi, dA(x)\cdot\psi\rangle dv - \int_{B_R} A^2(x)\langle x\cdot\psi, \psi\rangle dv.$$

On the other hand, by partial integration, we have

$$\begin{aligned}
& \int_{B_R} \langle \Delta\psi, x\cdot\psi\rangle dv \\
= & \int_{B_R} div\langle \nabla\psi, x\cdot\psi\rangle dv - \int_{B_R} \sum_{\alpha=1}^{2} \langle \nabla_{e_\alpha}\psi, e_\alpha\cdot\psi\rangle dv - \int_{B_R} \langle \nabla\psi, x\cdot\nabla\psi\rangle \\
= & \int_{\partial B_R} \langle \frac{\partial\psi}{\partial\nu}, x\cdot\psi\rangle d\sigma + \int_{B_R} \langle \not{D}\psi, \psi\rangle dv - \int_{B_R} \langle \nabla\psi, x\cdot\nabla\psi\rangle \\
= & \int_{\partial B_R} \langle \frac{\partial\psi}{\partial\nu}, x\cdot\psi\rangle d\sigma - \int_{B_R} A(x)|\psi|^2 dv - \int_{B_R} \langle \nabla\psi, x\cdot\nabla\psi\rangle, \\
= & \int_{\partial B_R^+\cap\mathbb{R}^2_+} \langle \frac{\partial\psi}{\partial\nu}, (x+\bar{x})\cdot\psi\rangle d\sigma - 2\int_{B_R^+} e^u|\psi|^2 dv - \int_{B_R} \langle \nabla\psi, x\cdot\nabla\psi\rangle,
\end{aligned}$$

and similarly,

$$\int_{B_R} \langle x\cdot\psi, \Delta\psi\rangle = \int_{\partial B_R^+\cap\mathbb{R}^2_+} \langle x\cdot\psi, \frac{\partial\psi}{\partial\nu}\rangle d\sigma - 2\int_{B_R^+} e^u|\psi|^2 dv - \int_{B_R} \langle x\cdot\nabla\psi, \nabla\psi\rangle.$$

Moreover, we have

$$\begin{aligned}
& \int_{B_R} \langle dA(x)\cdot\psi, x\cdot\psi\rangle dv + \int_{B_R} \langle x\cdot\psi, dA(x)\cdot\psi\rangle dv \\
= & \int_{B_R} \sum_{\alpha,\beta=1}^{2} \langle \nabla_{e_\alpha}A(x) e_\alpha\cdot\psi, e_\beta\cdot\psi\rangle x_\beta dv + \int_{B_R} \sum_{\alpha,\beta=1}^{2} \langle e_\beta\cdot\psi, \nabla_{e_\alpha}A(x) e_\alpha\cdot\psi\rangle x_\beta dv \\
= & \ 2\int_{B_R} \sum_{\alpha=1}^{2} \langle \nabla_{e_\alpha}A(x) e_\alpha\cdot\psi, e_\alpha\cdot\psi\rangle x_\alpha dv \\
= & \ 2\int_{B_R} x\cdot\nabla(A(x))|\psi|^2 dv \\
= & -2\int_{B_R} A(x) x\cdot\nabla(|\psi|^2) dv - 4\int_{B_R} A(x)|\psi|^2 dv + 2R\int_{\partial B_R} A(x)|\psi|^2 d\sigma \\
= & -4\int_{B_R^+} e^u x\cdot\nabla(|\psi|^2) dv - 8\int_{B_R^+} e^u|\psi|^2 dv + 4R\int_{\partial B_R^+\cap\mathbb{R}^2_+} e^u|\psi|^2 d\sigma.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
& R\int_{\partial B_R^+\cap\mathbb{R}^2_+} e^u|\psi|^2 d\sigma - \int_{B_R^+} e^u x\cdot\nabla(|\psi|^2) dv \\
= & \frac{1}{4}\int_{\partial B_R^+\cap\mathbb{R}^2_+} \langle \frac{\partial\psi}{\partial\nu}, (x+\bar{x})\cdot\psi\rangle d\sigma + \frac{1}{4}\int_{\partial B_R^+\cap\mathbb{R}^2_+} \langle (x+\bar{x})\cdot\psi, \frac{\partial\psi}{\partial\nu}\rangle d\sigma \\
& + \int_{B_R^+} e^u|\psi|^2 dv. \qquad (19)
\end{aligned}$$

Combining (15) and (19), we obtain the Pohozaev type identity (9). □

3. The energy identity for spinors

The energy identity for the spinor part ψ_n of a sequence of solutions (u_n, ψ_n) to the super-Liouville equation on closed Riemann surfaces was derived in [JWZZ1]. In this section, we shall prove an analogue for the super-Liouville boundary value problem, i.e. Theorem 1.2.

First, we derive a local estimate for the spinor part on a portion of an annulus:

Lemma 3.1. *For $x_0 \in B^+_{\frac{1}{2}} \cup (\partial B^+_{\frac{1}{2}} \cap \partial \mathbb{R}^2_+)$ and for $0 < r_1 < 2r_1 < \frac{r_2}{2} < r_2 < \frac{1}{2}$, consider the upper portion of an annulus*

$$A^+_{r_1,r_2}(x_0) = \left\{ x \in \mathbb{R}^2_+ |\ r_1 \le |x - x_0| \le r_2 \right\} \subset B^+_1.$$

Let (u, ψ) be a solution of

$$\begin{cases} -\Delta u &= 2e^{2u} - e^v \langle \psi, \psi \rangle, & \text{in } A^+_{r_1,r_2}(x_0), \\ \not{D}\psi &= -e^u \psi, & \text{in } A^+_{r_1,r_2}(x_0), \\ \frac{\partial u}{\partial n} &= ce^u, & \text{on } (\partial \mathbb{R}^2_+ \cap \partial A^+_{r_1,r_2}(x_0)), \\ \mathbf{B}^{\pm}\psi &= 0, & \text{on } (\partial \mathbb{R}^2_+ \cap \partial A^+_{r_1,r_2}(x_0)). \end{cases}$$

Then we have

$$\begin{aligned} &\Big(\int_{A^+_{2r_1,\frac{r_2}{2}}(x_0)} |\nabla \psi|^{\frac{4}{3}}\Big)^{\frac{3}{4}} + \Big(\int_{A^+_{2r_1,\frac{r_2}{2}}(x_0)} |\psi|^4\Big)^{\frac{1}{4}} \qquad (20) \\ \le\ & \Lambda\Big(\int_{A^+_{r_1,r_2}(x_0)} e^{2u}\Big)^{\frac{1}{2}}\Big(\int_{A^+_{r_1,r_2}(x_0)} |\psi|^4\Big)^{\frac{1}{4}} + C\Big(\int_{A^+_{r_1,2r_1}(x_0)} |\psi|^4\Big)^{\frac{1}{4}} + C\Big(\int_{A^+_{\frac{r_2}{2},r_2}(x_0)} |\psi|^4\Big)^{\frac{1}{4}} \end{aligned}$$

for two universal positive constants $\Lambda > 0$ and $C > 0$.

Proof : Let $\bar{x_0}$ be the reflection point of x_0 about $\partial \mathbb{R}^2_+$ and let

$$A^-_{r_1,r_2}(x_0) = \left\{ x \in \mathbb{R}^2_- |\ r_1 \le |x - \overline{x}_0| \le r_2 \right\}$$

be the reflection domain of $A^+_{r_1,r_2}(x_0)$ about $\partial \mathbb{R}^2_+$. By the chirality boundary condition of ψ, we can extend (u, ψ) to $A^-_{r_1,r_2}(x_0)$ as in (17):

$$\begin{aligned} u(\bar{x}) &:= \quad u(x), \quad \bar{x} \in A^-_{r_1,r_2}(x_0), \\ \psi(\bar{x}) &:= \quad ie_1 \cdot \psi(x), \quad \bar{x} \in A^-_{r_1,r_2}(x_0), \end{aligned}$$

where $\bar{x}$ is the reflection point of x about $\partial \mathbb{R}^2_+$. Set

$$\Omega_{r_1,r_2}(x_0) := \overline{A^+_{r_1,r_2}(x_0)} \cup \overline{A^-_{r_1,r_2}(x_0)} \subset B_1.$$

Then, we apply the same argument as in the proof of Lemma 3.4 of [JWZZ2] to deduce that

$$\not{D}\psi = -A(x)\psi \quad \text{in } \Omega^\circ_{r_1,r_2}(x_0), \qquad (21)$$

where

$$A(x) = \begin{cases} e^{u(x)}, & x \in A^+_{r_1,r_2}(x_0), \\ e^{u(\bar{x})}, & x \in A^-_{r_1,r_2}(x_0). \end{cases}$$

Now choose a cut-off function $\eta \in [0, 1]$ on B_1 satisfying

$$\begin{aligned} &\eta \in C^\infty_0(\Omega^\circ_{r_1,r_2}(x_0)); \quad \eta \equiv 1 \text{ in } \Omega_{2r_1,\frac{r_2}{2}}(x_0) \\ &|\nabla \eta| \le \frac{4}{r_1} \text{ in } \Omega_{r_1,2r_1}(x_0); \quad |\nabla \eta| \le \frac{4}{r_2} \text{ in } \Omega_{\frac{r_2}{2},r_2}(x_0), \end{aligned}$$

Similar to the arguments in the proof of Lemma 3.1 in [JWZZ1], we can apply the local interior L^p estimates for the Dirac operator $\not{D}$ to $\eta\psi$, use equation (21) and the following facts:

$$|\Omega_{r_1,2r_1}(x_0)| \le Cr_1^2, \quad |\Omega_{\frac{r_2}{2},r_2}(x_0)| \le Cr_2^2$$

to conclude that

$$
\begin{aligned}
&(\int_{\Omega_{2r_1,\frac{r_2}{2}}(x_0)}|\nabla\psi|^{\frac{4}{3}})^{\frac{3}{4}}+(\int_{\Omega_{2r_1,\frac{r_2}{2}}(x_0)}|\psi|^4)^{\frac{1}{4}} \qquad (22)\\
\leq\ &\Lambda(\int_{\Omega_{r_1,r_2}(x_0)}|A(x)|^2)^{\frac{1}{2}}(\int_{\Omega_{r_1,r_2}(x_0)}|\psi|^4)^{\frac{1}{4}}+C(\int_{\Omega_{r_1,2r_1}(x_0)}|\psi|^4)^{\frac{1}{4}}+C(\int_{\Omega_{\frac{r_2}{2},r_2}(x_0)}|\psi|^4)^{\frac{1}{4}}
\end{aligned}
$$

for some universal positive constants $\Lambda>0$ and $C>0$.

By definition of $\Omega_{r_1,r_2}(x_0)$, $A(x)$ and the extended spinor ψ, we have

$$
\begin{aligned}
\int_{\Omega_{r_1,r_2}(x_0)}|A(x)|^2 &\leq 2\int_{A^+_{r_1,r_2}(x_0)}e^{2u},\\
\int_{\Omega_{r_1,r_2}(x_0)}|\psi|^4 &\leq 2\int_{A^+_{r_1,r_2}(x_0)}|\psi|^4.
\end{aligned}
$$

Using the above inequalities, we can conclude from (22) that (20) holds. □

Now we apply the analytic properties in Section 2 and Lemma 3.1 to prove Theorem 1.2.

Proof of Theorem 1.2: We will follow closely the arguments for the case of closed Riemann surfaces [JWZZ1], which is similar to Ding-Tian's scheme for the energy identity of harmonic maps from surfaces [DT].

Since the blow-up set Σ_1 is finite, we can find small geodesic balls $D^M_{\delta_i}$ for each blow-up point x_i such that $D^M_{2\delta_i}\bigcap D^M_{2\delta_j}=\emptyset$ for $i\neq j, i,j=1,2,\cdots,K$, and on $M\backslash\bigcup_{i=1}^P D^M_{\delta_i}$, ψ_n converges strongly to some limit ψ in L^4 and $\int_M|\psi|^4<\infty$. Then, it suffices to prove that for each fixed blow-up point $p\in\Sigma_1$, there are solutions (u^k,ξ^k) of (1) on S^2, $k=1,2,\cdots,K$, and solutions (u^l,ξ^l) of (3) on $S^2_{c'}$, $l=1,2,\cdots,L$, such that

$$\lim_{\delta_i\to 0}\lim_{n\to\infty}\int_{D^M_\delta}|\psi_n|^4dv=\sum_{k=1}^K\int_{S^2}|\xi^k|^4dv+\sum_{l=1}^L\int_{S^2_{c'}}|\xi^l|^4dv, \qquad (23)$$

where D^M_δ is a geodesic ball of the blow-up point p.

By the energy identity for ψ_n on closed surfaces (Theorem 1.2, [JWZZ1]), to prove the theorem, it suffices to consider the case that $p\in\partial M$ and there is only one bubble at p. Then, what we need to show is that either there exists a bubbling solution (u,ξ) of (1) on S^2, such that

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int_{D^M_\delta}|\psi_n|^4dv=\int_{S^2}|\xi|^4dv, \qquad (24)$$

or there exists a bubbling solution (u,ξ) of (3) on $S^2_{c'}$ such that

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int_{D^M_\delta}|\psi_n|^4dv=\int_{S^2_{c'}}|\xi|^4dv, \qquad (25)$$

Since the considered problem is conformally invariant, w.l.o.g., we assume that $p=0$ and $D^M_\delta=\{x=(s,t)\in\mathbb{R}^2_+||x|<\delta,t\geq 0\}$. Write $D^+_\delta=\{x=(s,t)\in\mathbb{R}^2_+||x|<\delta,t>0\}$ and $D^+_\delta(x_0)=\{x=(s,t)\in\mathbb{R}^2_+||x-x_0|<\delta,t>0\}$.

We rescale each (u_n,ψ_n) near the blow-up point p. Choose $x_n=(s_n,t_n)\in\overline{D^+_\delta}$ such that $u_n(x_n)=\max_{\overline{D^+_\delta}}u_n(x)$. Then we have $x_n\to p=0$ and $u_n(x_n)\to+\infty$. Let $\lambda_n=e^{-u_n(x_n)}\to 0$ and define

$$
\begin{cases}
\widetilde{u}_n(x) = u_n(\lambda_n x+x_n)+\ln\lambda_n\\
\widetilde{\psi}_n(x) = \lambda_n^{\frac{1}{2}}\psi_n(\lambda_n x+x_n)
\end{cases}
$$

for any x such that $\lambda_n x + x_n \in D^+_{\frac{\delta}{2}}(x_n)$. Setting $\Omega_n = D_{\frac{\delta}{2\lambda_n}} \cap \{t > -\frac{t_n}{\lambda_n}\}$, then $(\widetilde{u}_n(x), \widetilde{\psi}_n(x))$ satisfies

$$\begin{cases} -\Delta \widetilde{u}_n(x) &= 2e^{2\widetilde{u}_n(x)} - e^{\widetilde{u}_n(x)}|\widetilde{\psi}_n(x)|^2, & \text{in } \Omega_n, \\ \not{D}\widetilde{\psi}_n(x) &= -e^{\widetilde{u}_n(x)}\widetilde{\psi}_n(x), & \text{in } \Omega_n, \\ \frac{\partial \widetilde{u}_n(x)}{\partial n} &= ce^{\widetilde{u}_n(x)}, & \text{on } \partial\Omega_n \cap \{t = -\frac{t_n}{\lambda_n}\}, \\ \mathbf{B}^{\pm}\widetilde{\psi}_n(x) &= 0, & \text{on } \partial\Omega_n \cap \{t = -\frac{t_n}{\lambda_n}\}, \end{cases}$$

with energy conditions

$$\int_{\Omega_n} (e^{2\widetilde{u}_n(x)} + |\widetilde{\psi}_n(x)|^4) dv + \int_{\partial\Omega_n \cap \{t = -\frac{t_n}{\lambda_n}\}} e^{\widetilde{u}_n(x)} d\sigma < C.$$

By passing to a subsequence, we assume w.l.o.g. that

$$\lim_{n\to\infty} \frac{t_n}{\lambda_n} = \lambda \in [0, +\infty]$$

Since $\widetilde{u}_n(0,0) = 0$ and $\widetilde{u}_n(x) \le 0$, it follows from Theorem 1.1 (and the corresponding techniques used in [JWZZ2]) that

$$\sup_n \Big(||\widetilde{u}_n||_{L^\infty(\Omega_{R,n})} + ||\widetilde{\psi}_n||_{L^\infty(\Omega_{R,n})} \Big) < \infty, \quad \text{for any } R > 0,$$

where $\Omega_{R,n} = \overline{D}_R \cap \{t \ge -\frac{t_n}{\lambda_n}\}$. Now there are two cases:

Case I: $\lambda = \infty$

By passing to a subsequence, $(\widetilde{u}_n, \widetilde{\psi_n})$ converges in $C^\infty_{loc}(\mathbb{R}^2)$ to some $(\widetilde{u}, \widetilde{\psi})$ satisfying

$$\begin{cases} -\Delta \widetilde{u} &= 2e^{2\widetilde{u}} - e^{\widetilde{u}}|\psi|^2, & \text{in } \mathbb{R}^2, \\ \not{D}\widetilde{\psi} &= -e^{\widetilde{u}}\widetilde{\psi}, & \text{in } \mathbb{R}^2, \end{cases} \tag{26}$$

with the energy condition $\int_{\mathbb{R}^2} (e^{2\widetilde{u}} + |\widetilde{\psi}|^4) dx + \int_{\partial\mathbb{R}^2} e^{\widetilde{u}} < \infty$. By the removability of a global singularity (see Proposition 6.3 and Theorem 6.4 in [JWZ]), there holds

$$\int_{\mathbb{R}^2} (2e^{2\widetilde{u}} - e^{\widetilde{u}}|\widetilde{\psi}|^2) dx = 4\pi$$

and we get a bubbling solution of (1) on S^2.

Case II: $0 \le \lambda < \infty$

By passing to a subsequence, we have

$$\lim_{n\to\infty} \Big(||\widetilde{u}_n - \widetilde{u}||_{C^k(\Omega_{R,n}\cap\mathbb{R}^2_\lambda)} + ||\widetilde{\psi}_n - \widetilde{\psi}||_{C^k(\Omega_{R,n}\cap\mathbb{R}^2_\lambda)} \Big) = 0, \quad \text{for any } k \in \mathbb{N}_0 \text{ and any } R > 0,$$

where $(\widetilde{u}, \widetilde{\psi})$ is a limit solution on $\mathbb{R}^2_\lambda = \{(s,t) | (s,t) \in \mathbb{R}^2, t \ge -\lambda\}$ satisfying

$$\begin{cases} -\Delta \widetilde{u} &= 2e^{2\widetilde{u}} - e^{\widetilde{u}}|\psi|^2, & \text{in } \mathbb{R}^2_\lambda, \\ \not{D}\widetilde{\psi} &= -e^{\widetilde{u}}\widetilde{\psi}, & \text{in } \mathbb{R}^2_\lambda, \\ \frac{\partial \widetilde{u}}{\partial n} &= ce^{\widetilde{u}}, & \text{on } \partial\mathbb{R}^2_\lambda, \\ \mathbf{B}^{\pm}\widetilde{\psi} &= 0, & \text{on } \partial\mathbb{R}^2_\lambda, \end{cases} \tag{27}$$

with the energy condition $\int_{\mathbb{R}^2_\lambda} (e^{2\widetilde{u}} + |\widetilde{\psi}|^4) dx + \int_{\partial\mathbb{R}^2_\lambda} e^{\widetilde{u}} < \infty$. By translation invariance and the removability of a global boundary singularity (see Proposition 2.9 in Section 2.), there holds

$$\int_{\mathbb{R}^2_\lambda} (2e^{2\widetilde{u}} - e^{\widetilde{u}}|\widetilde{\psi}|^2) dx + \int_{\partial\mathbb{R}^2_\lambda} ce^{\widetilde{u}} = 2\pi$$

and we obtain a bubbling solution of (3) on $S^2_{c'}$.

Thus we get the first bubble at the blow-up point p.

In order to prove (24) or (25) we need to estimate the energy of ψ_n in the neck domain. Let

$$A^+_{\delta,R,n} = \{x \in \mathbb{R}^2_+ | \lambda_n R \le |x - x_n| \le \delta, t \ge 0\}.$$

We call $A^+_{\delta,R,n}$ the neck domain, and the image of (u_n, ψ_n) is called the neck. Then to prove (24) or (25), it suffices to prove the following

$$\lim_{\delta\to 0} \lim_{R\to+\infty} \lim_{n\to\infty} \int_{A^+_{\delta,R,n}} |\psi_n|^4 dv = 0. \tag{28}$$

Next we want to show two claims.

Claim 1: For any $\epsilon > 0$, there is an $N > 1$ such that for any $n \ge N$, we have

$$\int_{D^+_r(x_n)\setminus D^+_{e^{-1}r}(x_n)} (e^{2u_n} + |\psi_n|^4) + \int_{\partial(D^+_r(x_n)\setminus D^+_{e^{-1}r}(x_n))\cap\partial\mathbb{R}^2_+} e^{u_n} < \epsilon; \quad \forall r \in [e\lambda_n R, \delta].$$

To prove this claim, we note the following two facts:

Fact 1: For any $\epsilon > 0$ and any $T > 0$, there exists some $N(T) > 0$ such that for any $n \ge N(T)$, we have

$$\int_{D^+_\delta(x_n)\setminus D^+_{\delta e^{-T}}(x_n)} (e^{2u_n} + |\psi_n|^4) + \int_{\partial(D^+_\delta(x_n)\setminus D^+_{\delta e^{-T}}(x_n))\cap\partial\mathbb{R}^2_+} e^{u_n} < \epsilon. \tag{29}$$

Actually, since (u_n, ψ_n) has no blow-up point in $\overline{D}^+_{2\delta}\setminus\{p\}$, we know that ψ_n converges strongly to ψ in $L^4_{loc}(\overline{D}^+_{2\delta}\setminus\{p\})$, and u_n will either be uniformly bounded on any compact subset of $\overline{D}^+_{2\delta}\setminus\{p\}$ or uniformly tend to $-\infty$ on any compact subset of $\overline{D}^+_{2\delta}\setminus\{p\}$.

If u_n uniformly tends to $-\infty$ on any compact subset of $\overline{D}^+_{2\delta}\setminus\{p\}$, it is clear that, for any given $T > 0$, there is an $N(T) > 0$ big enough such that when $n \ge N(T)$, we have

$$D^+_\delta(x_n)\setminus D^+_{\delta e^{-T}}(x_n) \subset D^+_{2\delta}\setminus D^+_{\frac{\delta}{2}e^{-T}}$$

and

$$\int_{D^+_\delta(x_n)\setminus D^+_{\delta e^{-T}}(x_n)} e^{2u_n} + \int_{\partial\{D^+_\delta(x_n)\setminus D^+_{\delta e^{-T}}(x_n)\}\cap\partial\mathbb{R}^2_+} e^{u_n} < \frac{\epsilon}{2}.$$

Moreover, since ψ_n converges to ψ in $L^4_{loc}(M \setminus \Sigma_1)$ and hence

$$\int_{D^+_\delta(x_n)\setminus D^+_{\delta e^{-T}}(x_n)} |\psi_n|^4 \to \int_{D^+_\delta\setminus D^+_{\delta e^{-T}}} |\varphi|^4.$$

For any given $\epsilon > 0$ small, we can choose $\delta > 0$ small enough such that $\int_{D^+_\delta} |\psi|^4 < \frac{\epsilon}{4}$, then for any given $T > 0$, there is an $N(T) > 0$ big enough such that when $n \ge N(T)$

$$\int_{D^+_\delta(x_n)\setminus D^+_{\delta e^{-T}}(x_n)} |\varphi_n|^4 < \frac{\epsilon}{2}.$$

Consequently, we get (29).

If (u_n, ψ_n) is uniformly bounded on any compact subset of $\overline{D}^+_{2\delta}\setminus\{p\}$, then we know that (u_n, ψ_n) converges to a solution (u, ψ) strongly on any compact subset of $\overline{D}^+_{2\delta}\setminus\{p\}$ and hence

$$\begin{aligned} &\int_{D^+_\delta(x_n)\setminus D^+_{\delta e^{-T}}(x_n)} (e^{2u_n} + |\psi_n|^4) + \int_{\partial\{D^+_\delta(x_n)\setminus D^+_{\delta e^{-T}}(x_n)\}\cap\partial\mathbb{R}^2_+} e^{u_n} \\ \to \ &\int_{D^+_\delta\setminus D^+_{\delta e^{-T}}} (e^{2u} + |\psi|^4) + \int_{\partial\{D^+_\delta\setminus D^+_{\delta e^{-T}}\}\cap\partial\mathbb{R}^2_+} e^{u} \end{aligned}$$

Therefore, we can choose $\delta > 0$ small enough such that, for any given $\epsilon > 0$ and any given $T > 0$, there exists an $N(T) > 0$ big enough, when $n \geq N(T)$, (29) holds.

Fact 2: For any small $\epsilon > 0$, and $T > 0$, we may choose an $N(T) > 0$ such that when $n \geq N(T)$

$$\begin{aligned} & \int_{D^+_{\lambda_n R e^T}(x_n)\setminus D^+_{\lambda_n R}(x_n)} (e^{2u_n} + |\psi_n|^4) + \int_{\partial(D^+_{\lambda_n R e^T}(x_n)\setminus D^+_{\lambda_n R}(x_n))\cap\partial\mathbb{R}^2_+} e^{u_n} \\ = & \int_{\{D_{Re^T}\setminus D_R\}\cap\{t>-\frac{t_n}{\lambda_n}\}} (e^{2\widetilde{u}_n} + |\widetilde{\psi}_n|^4) + \int_{\partial(D_{Re^T}\setminus D_R)\cap\{t=-\frac{t_n}{\lambda_n}\}} e^{\widetilde{u}_n} \\ \to & \int_{\{D_{Re^T}\setminus D_R\}\cap\{t>-\lambda\}} (e^{2\widetilde{u}} + |\widetilde{\psi}|^4) + \int_{\partial(D_{Re^T}\setminus D_R)\cap\{t=-\lambda\}} e^{\widetilde{u}} \\ < & \ \epsilon, \end{aligned}$$

if R is big enough.

Now we can prove Claim 1. We argue by contradiction by using the above two facts. Thus we assume that there exists $\epsilon_0 > 0$ and a sequence $r_n \in [e\lambda_n R, \delta]$ such that

$$\int_{D^+_{r_n}(x_n)\setminus D^+_{e^{-1}r_n}(x_n)} (e^{2u_n} + |\psi_n|^4) + \int_{\partial(D^+_{r_n}(x_n)\setminus D^+_{e^{-1}r_n}(x_n))\cap\mathbb{R}^2_+} e^{u_n} \geq \epsilon_0.$$

Then, by the above two facts, we know that $\frac{\delta}{r_n} \to +\infty$ and $\frac{\lambda_n R}{r_n} \to 0$, in particular, $r_n \to 0$ as $n \to +\infty$.

Scaling again, we set

$$\begin{cases} v_n(x) = u_n(r_n x + x_n) + \ln r_n, \\ \varphi_n(x) = r_n^{\frac{1}{2}} \psi_n(r_n x + x_n) \end{cases}$$

for any x such that $r_n x + x_n \in D^+_{r_n}(x_n) \setminus D^+_{e^{-1}r_n}(x_n)$.

It is clear that

$$\int_{(D_1\setminus D_{e^{-1}})\cap\{t>-\frac{t_n}{r_n}\}} (e^{2v_n} + |\varphi_n|^4) + \int_{\partial(D_1\setminus D_{e^{-1}})\cap\{t=-\frac{t_n}{r_n}\}} e^{v_n} \geq \epsilon_0, \tag{30}$$

and (v_n, φ_n) satisfies

$$\begin{cases} -\Delta v_n(x) = 2e^{2v_n(x)} - e^{v_n(x)}|\varphi_n(x)|^2, & \text{in } (D_{\frac{\delta}{r_n}} \setminus D_{\frac{\lambda_n R}{r_n}}) \cap \{t > -\frac{t_n}{r_n}\}, \\ \not{D}\varphi_n(x) = -e^{v_n(x)}\varphi_n(x), & \text{in } (D_{\frac{\delta}{r_n}} \setminus D_{\frac{\lambda_n R}{r_n}}) \cap \{t > -\frac{t_n}{r_n}\}, \\ \frac{\partial v_n(x)}{\partial n} = ce^{v_n(x)}, & \text{on } \partial(D_{\frac{\delta}{r_n}} \setminus D_{\frac{\lambda_n R}{r_n}}) \cap \{t = -\frac{t_n}{r_n}\}, \\ \mathbf{B}^{\pm}\varphi_n(x) = 0, & \text{on } \partial(D_{\frac{\delta}{r_n}} \setminus D_{\frac{\lambda_n R}{r_n}}) \cap \{t = -\frac{t_n}{r_n}\}. \end{cases}$$

By passing to a subsequence, we assume w.l.o.g. that

$$\lim_{n\to\infty} \frac{t_n}{r_n} = \mu \in [0, +\infty]$$

If $0 \leq \mu < +\infty$, by Theorem 1.1, there are three possible cases:

(1). There exists some $R > 0$, some point $q \in (D_R \setminus D_{\frac{1}{R}}) \cap \{t \geq -\mu\}$ and energy concentration occurs near q, namely, setting $Q_n = (D_{\frac{\delta}{r_n}} \setminus D_{\frac{\lambda_n R}{r_n}}) \cap \{t \geq -\frac{t_n}{r_n}\}$, then along some subsequence

$$\lim_{n\to\infty} \int_{D_r(q_n)\cap Q_n} (e^{2v_n} + |\varphi_n|^4) + \int_{D_r(q_n)\cap\partial Q_n\cap\{t=-\frac{t_n}{r_n}\}} e^{v_n} \geq \epsilon_0 > 0$$

for any small $r > 0$. In such a case, we still obtain a second bubble on S^2 or $S^2_{c'}$ by the rescaling argument. Thus we get a contradiction to the assumption that there is only one bubble at the blow-up point p.

(2). For any $R>0$, there is no blow-up point in $(D_R \setminus D_{\frac{1}{R}}) \cap \{t \geq -\mu\}$ and

$$\lim_{n\to\infty} ||v_n||_{L^\infty(Q_{R,n})} = -\infty,$$

where $Q_{R,n} = (D_R \setminus D_{\frac{1}{R}}) \cap \{t \geq -\frac{t_n}{\lambda_n}\}$. Then, there is a solution φ satisfying

$$\begin{cases} \not{D}\varphi &= 0, \quad \text{in } (\mathbb{R}^2 \setminus \{0\}) \cap \{t > -\mu\}, \\ \mathbf{B}^{\pm}\bar{\varphi} &= 0, \quad \text{on } \partial(\mathbb{R}^2 \setminus \{0\}) \cap \{t = -\mu\} \end{cases}$$

with bounded energy $||\varphi||_{L^4(\mathbb{R}^2 \cap \{t>-\mu\})} < \infty$, such that

$$\lim_{n\to\infty} ||\varphi_n - \varphi||_{L^4(Q_{R,n} \cap \{t \geq -\mu\})} = 0, \quad \text{for any } R > 0.$$

We translate φ to get a harmonic spinor on $\mathbb{R}^2_+ \setminus \{0\}$ satisfying the corresponding chirality boundary condition and then extend it via reflection as in (17) to a harmonic spinor $\overline{\varphi}$ on $\mathbb{R}^2 \setminus \{0\}$ with bounded energy, i.e., $||\overline{\varphi}||_{L^4(\mathbb{R}^2)} < \infty$. Since harmonic spinors on surfaces can be considered as trivial Dirac-harmonic maps (with constant map component) studied in [CJLW], by removability of singularity for Dirac-haromic maps with bounded energy, we know that $\overline{\varphi}$ can be conformally extended to a harmonic spinor on S^2. Then, by the well known fact that there is no nontrivial harmonic spinor on S^2, we conclude that $\overline{\varphi} \equiv 0$ and hence $\lim_{n\to\infty} ||\varphi_n - \varphi||_{L^4(Q_{R,n})} = 0$ for any $R > 0$. This will contradict (30).

(3). For any $R > 0$, there is no blow-up point in $(D_R \setminus D_{\frac{1}{R}}) \cap \{t \geq -\mu\}$ and

$$\sup_n \left(||v_n||_{L^\infty(Q_{R,n})} + ||\varphi_n||_{L^\infty(Q_{R,n})}\right) < +\infty.$$

Then, there is a solution (v, φ) satisfying

$$\begin{cases} -\Delta v &= 2e^{2v} - e^v|\varphi|^2, & \text{in } (D_R \setminus D_{\frac{1}{R}}) \cap \{t > -\mu\} \\ \not{D}\varphi &= -e^v\varphi, & \text{in } (D_R \setminus D_{\frac{1}{R}}) \cap \{t > -\mu\} \\ \frac{\partial v}{\partial n} &= ce^v, & \text{on } \partial(D_R \setminus D_{\frac{1}{R}}) \cap \{t = -\mu\} \\ \mathbf{B}^{\pm}\varphi &= 0, & \text{on } \partial(D_R \setminus D_{\frac{1}{R}}) \cap \{t = -\mu\} \end{cases}$$

with finite energy $\int_{(D_R \setminus D_{\frac{1}{R}}) \cap \{t>-\mu\}} (e^{2v} + |\varphi|^4)dx + \int_{\partial(D_R \setminus D_{\frac{1}{R}}) \cap \{t=-\mu\}} e^v < \infty$, such that

$$\lim_{n\to\infty} \left(||v_n - v||_{C^k(Q_{R,n} \cap \{t \geq -\mu\})} + ||\varphi_n - \varphi||_{C^k(Q_{R,n} \cap \{t \geq -\mu\})}\right) = 0, \quad \text{for any } k \in \mathbb{N}_0 \text{ and any } R > 0.$$

Furthermore, we can show that (v, φ) satisfies

$$\int_{(\mathbb{R}^2 \setminus \{0\}) \cap \{t \geq -\mu\}} |T(z)|dz \leq C, \tag{31}$$

where

$$T(z)dz^2 = \left\{(\partial_z v)^2 - \partial_z^2 v + \frac{1}{4}\langle \varphi, dz \cdot \partial_{\bar{z}}\varphi \rangle + \frac{1}{4}\langle d\bar{z} \cdot \partial_z \varphi, \varphi \rangle\right\} dz^2$$

is the holomophic quadratic differential associated to (v, φ) and $C > 0$ is independent of μ. Indeed, this property is inherited from (u_n, ψ_n). To see this, let

$$T_n(z)dz^2 = \left\{(\partial_z u_n)^2 - \partial_z^2 u_n + \frac{1}{4}\langle \psi_n, dz \cdot \partial_{\bar{z}}\psi_n \rangle + \frac{1}{4}\langle d\bar{z} \cdot \partial_z \psi_n, \psi_n \rangle\right\} dz^2$$

be the holomophic quadratic differential associated to (u_n, ψ_n). Then by Proposition 2.10, $T_n(z)$ is holomorphic in $D^+_{2\delta}$ and is real on $\partial D^+_{2\delta} \cap \partial\mathbb{R}^2_+$ and hence it can be extended to a holomorphic function in $D_{2\delta}$. On the other hand, by Cauchy's integral formula, we have

$$T_n(z) = \frac{1}{2\pi i} \int_{\partial D_\rho} \frac{T_n(\xi)}{\xi - z} d\xi,$$

where ρ can be any number in $(0, 2\delta)$. It follows that

$$\int_{D_\delta^+} |T_n(z)| dz \le C.$$

Since the L^1-norm of the quadratic differential is conformally invariant and (v_n, φ_n) converges to (v, φ) strongly on $(D_R \setminus D_{\frac{1}{R}}) \cap \{t \ge -\mu\}$ for any $R > 0$, we conclude that (31) holds.

Note that $(\mathbb{R}^2 \setminus \{0\}) \cap \{t \ge -\mu\}$ is conformal to $S^2_{c'} \setminus \{p_1, p_2\}$, where $p_1 \in \partial S^2_{c'}$, $p_2 \in S^2_{c'}$ in case of $0 < \mu < +\infty$ and $p_1, p_2 \in \partial S^2_{c'}$ in case of $\mu = 0$. By the removability of local interior singularities (Proposition 2.6 in [JWZZ1]) and local boundary singularities (Proposition 2.11 in Section 2), we get another bubble on $S^2_{c'}$. Thus we get a contradiction.

If $\mu = +\infty$, we can apply a similar argument (which is the same as the case of interior blow-up developed in [JWZZ1]) to obtain a contradiction.

This completes the proof of Claim 1.

Claim 2: We can separate $A^+_{\delta,R,n}$ into finitely many parts

$$A^+_{\delta,R,n} = \bigcup_{k=1}^{N_k} A^+_k$$

such that on each part

$$\int_{A_k^+} e^{2u_n} \le \frac{1}{4\Lambda^2}, \quad k = 1, 2, \cdots, N_k. \tag{32}$$

where $N_k \le N_0$ for N_0 is a uniform integer for all n large enough, $A^+_k = D^+_{r^{k-1}}(x_n) \setminus D^+_{r^k}(x_n)$, $r^0 = \delta, r^{N_k} = \lambda_n R$, $r^k < r^{k-1}$ for $k = 1, 2, \cdots, N_k$, and Λ is the constant as in Lemma 3.1.

The proof of Claim 2 is very similar to the interior blow-up case done in [JWZZ1] (see also [Zh, Z1]). For the sake of completeness, we provide the details as follows.

W.l.o.g., we may assume that $m_n := -\log \frac{\lambda_n R}{\delta}$ is an integer and $\lim_{n\to\infty} m_n = +\infty$.

By Claim 1, for any $0 < \epsilon \le \frac{1}{8\Lambda^2}$, we can find $N > 0$ such that when $n \ge N$ we have

$$\int_{D^+_r(x_n) \setminus D^+_{e^{-1}r}(x_n)} e^{2u_n} < \epsilon \le \frac{1}{8\Lambda^2}, \quad \forall r \in [e\lambda_n R, r^0].$$

Then for any $n \ge N$, if

$$\int_{D^+_{r^0}(x_n) \setminus D^+_{e\lambda_n R}(x_n)} e^{2u_n} \le \frac{1}{4\Lambda^2},$$

we take $r^1 = e\lambda_n R$ and denote $A^+_1 = D^+_{r^0}(x_n) \setminus D^+_{r^1}(x_n) = D^+_{r^0}(x_n) \setminus D^+_{e\lambda_n R}(x_n)$. Otherwise, if

$$\int_{D^+_{r^0}(x_n) \setminus D^+_{e\lambda_n R}(x_n)} e^{2u_n} > \frac{1}{4\Lambda^2},$$

we can choose an integer m^1_n such that

$$\frac{1}{8\Lambda^2} < \int_{A^+_1} e^{2u_n} \le \frac{1}{4\Lambda^2} \quad \text{and} \quad \int_{D^+_{r^0}(x_n) \setminus D^+_{e^{-1}r^1}(x_n)} e^{2u_n} > \frac{1}{4\Lambda^2},$$

where $r^1 = r^0 e^{-m^1_n}$, $A^+_1 = D^+_{r^0}(x_n) \setminus D^+_{r^1}(x_n)$ and $1 \le m^1_n \le m_n - 1$. This is the first step of the division.

Inductively, we suppose that $A_l^+ = D_{r^{l-1}}^+(x_n) \setminus D_{r^l}^+(x_n)$ is chosen such that $\int_{A_l^+} e^{2u_n} \le \frac{1}{4\Lambda^2}$. If

$$\int_{D_{r^l}^+(x_n)\setminus D_{e\lambda_n R}^+(x_n)} e^{2u_n} \le \frac{1}{4\Lambda^2},$$

we take $r^{l+1} = \lambda_n R$ and set $A_{l+1}^+ = D_{r^l}^+(x_n) \setminus D_{r^{l+1}}^+(x_n)$. On the other hand, if

$$\int_{D_{r^l}^+(x_n)\setminus D_{e\lambda_n R}^+(x_n)} e^{2u_n} > \frac{1}{4\Lambda^2},$$

then similarly to the first step, we can find $r^{l+1} = r^l \cdot e^{-m_n^{l+1}}$, $A_{l+1}^+ = D_{r^l}^+(x_n) \setminus D_{r^{l+1}}^+(x_n)$ such that

$$\frac{1}{8\Lambda^2} < \int_{A_{l+1}^+} e^{2u_n} \le \frac{1}{4\Lambda^2} \quad \text{and} \quad \int_{D_{r^l}^+(x_n)\setminus D_{e^{-1}r^l}^+(x_n)} e^{2u_n} > \frac{1}{4\Lambda^2},$$

where $m_n^l + 1 \le m_n^{l+1} \le m_n - 1$. Thus we obtain one more part A_{l+1}^+ satisfying $\int_{A_{l+1}^+} e^{2u_n} \le \frac{1}{4\Lambda^2}$. Since $\int_{A_{\delta,R,n}^+} e^{2u_n} \le C$ for some constant $C > 0$, we can finish our division after at most $N_0 = [8\Lambda^2 C]$ steps. This completes the proof of Claim 2.

Now using Claim 1 and Claim 2, we can show (28). Let $0 < \epsilon < 1$ be small, δ be small enough, and let R and n be large enough. We apply Lemma 3.1 to each part A_l^+ and use (32) to calculate

$$\begin{aligned}
(\int_{A_l^+} |\psi_n|^4)^{\frac14} &\le \Lambda(\int_{D_{er^{l-1}}^+(x_n)\setminus D_{e^{-1}r^l}^+(x_n)} e^{2u_n})^{\frac12}(\int_{D_{er^{l-1}}^+(x_n)\setminus D_{e^{-1}r^l}^+(x_n)} |\psi_n|^4)^{\frac14} \\
&\quad + C(\int_{D_{er^{l-1}}^+(x_n)\setminus D_{r^{l-1}}^+(x_n)} |\psi_n|^4)^{\frac14} + C(\int_{D_{r^l}^+(x_n)\setminus D_{e^{-1}r^l}^+(x_n)} |\psi_n|^4)^{\frac14} \\
&\le \Lambda((\int_{A_l^+} e^{2u_n})^{\frac12} + \epsilon^{\frac12} + \epsilon^{\frac12})((\int_{A_l^+} |\psi_n|^4)^{\frac14} + \epsilon^{\frac14} + \epsilon^{\frac14}) + C\epsilon^{\frac14} \\
&\le \Lambda(\int_{A_l^+} e^{2u_n})^{\frac12}(\int_{A_l^+} |\psi_n|^4)^{\frac14} + C(\epsilon^{\frac14} + \epsilon^{\frac12} + \epsilon^{\frac34}) \\
&\le \frac12(\int_{A_l^+} |\psi_n|^4)^{\frac14} + C\epsilon^{\frac14}.
\end{aligned}$$

It follows that

$$(\int_{A_l^+} |\psi_n|^4)^{\frac14} \le C\epsilon^{\frac14}. \tag{33}$$

Using Lemma 3.1, (32), (33) and applying similar arguments, we get

$$(\int_{A_l^+} |\nabla\psi_n|^{\frac43})^{\frac34} \le C\epsilon^{\frac14}. \tag{34}$$

Summing up (33) and (34) on A_l^+, we conclude that

$$\int_{A_{\delta,R,n}^+} |\psi_n|^4 + \int_{A_{\delta,R,n}^+} |\nabla\psi_n|^{\frac43} = \sum_{l=1}^{N_0} \int_{A_l^+} |\psi_n|^4 + |\nabla\psi_n|^{\frac43} \le C\epsilon^{\frac13}. \tag{35}$$

Thus we have proved (28). This completes the proof of the theorem. □

4. Blow-up behavior

In this section, we will rule out the possibility that u_n is uniformly bounded in $L^\infty_{loc}(M\backslash\Sigma_1)$ in Theorem 1.1, in order to prove Theorem 1.3, which is an application of the energy identity of spinors, the removability for a local singularity at the boundary and the energy of a bubble.

Proof of Theorem 1.3: We shall prove this result again by contradiction. So, we assume that the conclusion of the theorem is false. Then u_n is uniformly bounded in L^∞ on any compact subset of $M\backslash\Sigma_1$ by Theorem 1.1. Since (u_n, ψ_n) is a sequence of solutions to (3) with uniformly bounded energy, by classical elliptic estimates for both the Laplacian Δ and the Dirac operator $\not{D}$ under appropate boundary constraints (see Theorem 2.3 and Theorem 2.4 in Section 2.), we know that (u_n, ψ_n) converges strongly on any compact subset of $M \setminus \Sigma_1$ to some limit solution (u, ψ) of (3) with bounded energy $\int_M (e^{2u} + |\psi|^4) + \int_{\partial M} e^u < C$.

Since the blow-up set Σ_1 is non empty, we can take a point $x_0 \in \Sigma_1$. Choose a small $\delta_0 > 0$ such that x_0 is the only point of Σ_1 in $\overline{B}^M_{2\delta_0}(x_0)$. Here $B^M_{2\delta_0}(x_0)$ is a geodesic ball at x_0 on M.

We start by considering the boundary case $x_0 \in \partial M$. By conformal invariance, we may assume that $x_0 = 0$ and $B^M_{2\delta_0}(x_0)$ is the Euclidean upper-half disc $D^+_{2\delta_0} = \{z = (s,t) \in \mathbb{R}^2_+ | |z| < \delta_0, t \geq 0\}$.

We will first show that the limit (u, ψ) is smooth at the isolated singularity x_0. To see this, let

$$T(z)dz^2 = \left\{(\partial_z u)^2 - \partial_z^2 u + \frac{1}{4}\langle \psi, dz \cdot \partial_{\bar{z}}\psi\rangle + \frac{1}{4}\langle d\bar{z} \cdot \partial_z \psi, \psi\rangle\right\} dz^2.$$

be the quadratic differential associated to (u, ψ). Then, by Proposition 2.10, $T(z)$ is holomorphic in $D^+_{2\delta_0} \setminus \{0\}$ and is real on $(\partial D^+_{2\delta_0} \setminus \{0\}) \cap \partial\mathbb{R}^2$. Hence, we can extend $T(z)dz^2$ to a holomorphic quadratic differential in $D_{2\delta_0} \setminus \{0\}$. On the other hand, for each n,

$$T_n(z)dz^2 = \left\{(\partial_z u_n)^2 - \partial_z^2 u_n + \frac{1}{4}\langle \psi_n, dz \cdot \partial_{\bar{z}}\psi_n\rangle + \frac{1}{4}\langle d\bar{z} \cdot \partial_z \psi_n, \psi_n\rangle\right\} dz^2$$

is a holomorphic quadratic differential in $D^+_{2\delta_0}$ and $T_n(z)$ is real on $\partial D^+_{2\delta_0} \cap \partial\mathbb{R}^2_+$. Again, we extend $T_n(z)dz^2$ to a holomorphic quadratic differential in $D_{2\delta_0}$. By Cauchy's integral formula, we write

$$T_n(z) = \frac{1}{2\pi i}\int_{\partial D_{\delta_0}} \frac{T_n(\xi)}{\xi - z} d\xi.$$

It follows that

$$\int_{D_{\delta_0/2}} |T_n(z)| dz \leq C.$$

Since (u_n, ψ_n) converges to (u, ψ) strongly on any compact subset of $\overline{D}^+_{2\delta_0} \setminus \{0\}$, we have

$$\int_{D^+_{\delta_0/2}} |T(z)| dz \leq C.$$

Applying the removability of a local boundary singularity (see Proposition 2.11), we conclude that (u, ψ) is smooth on $\overline{D}^+_{\delta_0/2}$ and hence it is a smooth solution of

$$\begin{cases} -\Delta u = 2e^{2u} - e^u|\psi|^2, & \text{in } D^+_{2\delta_0}, \\ \not{D}\psi = -e^u\psi, & \text{in } D^+_{2\delta_0}, \\ \frac{\partial u}{\partial n} = ce^u, & \text{on } \partial D^+_{2\delta_0} \cap \{t = 0\}, \\ \mathbf{B}^{\pm}\psi = 0, & \text{on } \partial D^+_{2\delta_0} \cap \{t = 0\}. \end{cases} \tag{36}$$

with bounded energy

$$\int_{D^+_{2\delta_0}} (e^{2u} + |\psi|^4) + \int_{\partial D^+_{2\delta_0} \cap \{t=0\}} e^u < C.$$

Now we choose some small $\delta_1 \in (0, \delta_0)$ such that for any $\delta \in (0, \delta_1)$,

$$\int_{D_\delta^+} (2e^{2u} - e^u |\psi|^2) + \int_{\partial D_\delta^+ \cap \{t=0\}} ce^u < \frac{1}{10} \tag{37}$$

Next, as in the proof of Theorem 1.2 (see Section 3), we rescale (u_n, ψ_n) near $x_0 = 0$. Choose $x_n = (s_n, t_n) \in D_{\delta_1}^+$ with $u_n(x_n) = \max_{\bar{D}_{\delta_1}^+} u_n(x)$. Then we have $x_n \to x_0$ and $u_n(x_n) \to +\infty$. Let $\lambda_n = e^{-u_n(x_n)} \to 0$. Denote

$$\begin{cases} \widetilde{u}_n(x) &= u_n(\lambda_n x + x_n) + \ln \lambda_n \\ \widetilde{\psi}_n(x) &= \lambda_n^{\frac{1}{2}} \psi_n(\lambda_n x + x_n) \end{cases}$$

for any x such that $\lambda_n x + x_n \in \bar{D}_{\frac{\delta_1}{2}}^+(x_n)$. Then, we can pass to a subsequence such that $\lim_{n\to\infty} \lambda_n^{-1} t_n = \lambda$ for some $0 \le \lambda \le +\infty$. We distinguish the following two cases:

Case I: $\lambda = \infty$. $(\widetilde{u}_n, \widetilde{\psi_n})$ converges in $C_{loc}^\infty(\mathbb{R}^2)$ to some limit solution $(\widetilde{u}, \widetilde{\psi})$ of (1) on $\mathbb{R}^2$ with

$$\int_{\mathbb{R}^2} (2e^{2\widetilde{u}} - e^{\widetilde{u}} |\widetilde{\psi}|^2) dx = 4\pi. \tag{38}$$

Case II: $0 \le \lambda < \infty$. There exists a limit solution $(\widetilde{u}, \widetilde{\psi})$ of (3) on $\mathbb{R}_\lambda^2 = \{(s,t) | (s,t) \in \mathbb{R}^2, t \ge -\lambda\}$ with

$$\int_{\mathbb{R}_\lambda^2} (2e^{2\widetilde{u}} - e^{\widetilde{u}} |\widetilde{\psi}|^2) dx + \int_{\partial \mathbb{R}_\lambda^2} ce^{\widetilde{u}} = 2\pi, \tag{39}$$

such that

$$\lim_{n\to\infty} \left(||\widetilde{u}_n - \widetilde{u}||_{C^k(\Omega_{R,n} \cap \mathbb{R}_\lambda^2)} + ||\widetilde{\psi}_n - \widetilde{\psi}||_{C^k(\Omega_{R,n} \cap \mathbb{R}_\lambda^2)} \right) = 0, \quad \text{for any } k \in \mathbb{N}_0 \text{ and any } R > 0,$$

where $\Omega_{R,n} = \overline{D}_R \cap \{t \ge -\frac{t_n}{\lambda_n}\}$.

Then for $\delta \in (0, \delta_1)$ small enough, $R > 0$ large enough and n large enough, we have

$$\begin{aligned} & \int_{D_\delta^+} (2e^{2u_n} - e^{u_n} |\psi_n|^2) + \int_{\partial D_\delta^+ \cap \{t=0\}} ce^{u_n} \\ = & \int_{D_{\lambda_n R}^+(x_n)} (2e^{2u_n} - e^{u_n} |\psi_n|^2) + \int_{\partial D_{\lambda_n R}^+(x_n) \cap \{t=0\}} ce^{u_n} \\ & + \int_{D_\delta^+ \setminus D_{\lambda_n R}^+(x_n)} (2e^{2u_n} - e^{u_n} |\psi_n|^2) + \int_{\partial \{D_\delta^+(x_n) \setminus D_{\lambda_n R}^+(x_n)\} \cap \{t=0\}} ce^{u_n} \\ \ge & \int_{D_R \cap \{t > -\frac{t_n}{\lambda_n}\}} (2e^{2\widetilde{u}_n} - e^{\widetilde{u}_n} |\widetilde{\psi}_n|^2) + \int_{D_R \cap \{t = -\frac{t_n}{\lambda_n}\}} ce^{\widetilde{u}_n} - \int_{D_\delta^+(x_n) \setminus D_{\lambda_n R}^+(x_n)} e^{u_n} |\psi_n|^2 \\ \ge & 2\pi - \frac{1}{10}. \end{aligned} \tag{40}$$

Here in the last step, we have used (38) or (39) and the fact from Theorem 1.2 that the neck energy of the spinor field ψ_n is converging to zero, On the other hand, we fix some $\delta \in (0, \delta_1)$ small such that (40) holds and then let $n \to \infty$ to conclude that

$$\begin{aligned}
2\pi - \frac{1}{10} &\le \int_{D_\delta^+} (2e^{2u_n} - e^{u_n}|\psi_n|^2) + \int_{\partial D_\delta^+ \cap \{t=0\}} ce^{u_n} \\
&= -\int_{D_\delta^+} \Delta u_n + \int_{\partial D_\delta^+ \cap \{t=0\}} \frac{\partial u_n}{\partial n} \\
&= -\int_{\partial D_\delta^+ \cap \{t>0\}} \frac{\partial u_n}{\partial n} \\
&\to -\int_{\partial D_\delta^+ \cap \{t>0\}} \frac{\partial u}{\partial n} \\
&= -\int_{D_\delta^+} \Delta u + \int_{\partial D_\delta^+ \cap \{t=0\}} \frac{\partial u}{\partial n} \\
&= \int_{D_\delta^+} (2e^{2u} - e^{u}|\psi|^2) + \int_{\partial D_\delta^+ \cap \{t=0\}} ce^{u} < \frac{1}{10}
\end{aligned}$$

Here in the last step, we have used (37). Thus we get a contradiction.

It is easy to see that the above argument can also be applied to the case that $x_0 \in M^o$ to obtain a contradiction.

Therefore, we have that $u_n \to -\infty$ uniformly on any compact subset of $M \backslash \Sigma_1$. Consequently, by applying a rescaling argument and using (38), (39), we can conclude (8). This completes the proof of Theorem 1.3. □

Remark 4.1. When $\psi \equiv 0$, the super-Liouville equation reduces to the classical Liouville equation and hence the method used in the proof of Theorem 1.3 provides a new proof of the corresponding result for the classical Liouville equation.

5. The blow-up value

In this section, we shall determine the blow-up value at blow-up points in Σ_1. For $p \in \Sigma_1$, define

$$m(p) = \lim_{r\to 0}\lim_{n\to\infty} \left\{ \int_{B_r^M(p)} (2e^{2u_n} - e^{u_n}|\psi_n|^2) + c\int_{B_r^M(p)\cap\partial M} e^{u_n} \right\}.$$

It is easy to see that $m(p) \neq 0$ if and only if $p \in \Sigma_1$. Actually, it is clear from Theorem 1.5 of [JWZZ1] that $m(p) = 4\pi$ when $p \in \Sigma_1 \cap M^o$. From the proof of Theorem 1.3 in Section 4., we know that $m(p) \geq 2\pi$ when $p \in \Sigma_1 \cap \partial M$. In this section, we shall show that $m(p) = 2\pi$ when $p \in \Sigma_1 \cap \partial M$.

Lemma 5.1. *There exists $G \in W^{1,q}(M) \cap C^2_{loc}(M\backslash\Sigma_1)$ with $\int_M G = 0$ for $1 < q < 2$ such that*

$$u_n - \frac{1}{|M|}\int_M u_n \to G$$

in $C^2_{loc}(M\backslash\Sigma_1)$ and weakly in $W^{1,q}(M)$. Denote $\Sigma_1 = \{p_1, p_2, \cdots, p_l\}$. Then for $p_k \in \Sigma_1 \cap M^o$, there exists $R > 0$ small enough such that $B_R^M(p_k) \cap \Sigma_1 = \{p_k\}$ and

$$G = \frac{1}{2\pi} m(p_k) \log\frac{1}{|x - p_k|} + g(x)$$

for $x \in B_R^M(p_k)\backslash\{p_k\}$ with $g \in C^2(B_R^M(p_k))$; For $p_l \in \Sigma_1 \cap \partial M$, there exists $R > 0$ small enough such that $B_R^M(p_l) \cap \Sigma_1 = \{p_l\}$ and

$$G = \frac{1}{\pi} m(p_l) \log\frac{1}{|x - p_l|} + g(x)$$

for $x \in B_R^M(p_l)\backslash\{p_l\}$ with $g \in C^2(B_R^M(p_l))$.

Proof: For any $p > 2$, let $q = \frac{p}{p-1} \in (1, 2)$. Then we get

$$||\nabla u_n||_{L^q(M)} = \sup\left\{ |\int_M \nabla u_n \nabla \varphi dv| | \forall \varphi \in W^{1,p}(M), \int_M \varphi dv = 0, ||\varphi||_{W^{1,p}(M)} = 1 \right\}.$$

By Sobolev embedding theorem, we have

$$||\varphi||_{L^\infty(M) \leq C}.$$

It follows that

$$\begin{aligned} |\int_M \nabla u_n \nabla \varphi dv| &= |\int_M -\Delta u_n \varphi dv + \int_{\partial M} \frac{\partial u_n}{\partial n} \varphi d\sigma| \\ &= |\int_M (2e^{2u_n} - e^{u_n}|\psi_n|^2 - K_g)\varphi dv + \int_{\partial M} (ce^{u_n} - h_g)\varphi d\sigma| \\ &\leq C. \end{aligned}$$

Therefore, $u_n - \overline{u}_n$ is uniformly bounded in $W^{1,q}(M)$.

Next, we define the Green function G by

$$\begin{cases} -\Delta G &= \sum_{p \in M^o \cap \Sigma_1} m(p)\delta_p - K_g, & \text{in } M^\circ \\ \frac{\partial G}{\partial n} &= \sum_{p \in \partial M \cap \Sigma_1} m(p)\delta_p - h_g, & \text{on } \partial M \\ \int_M G &= 0. \end{cases}$$

Then, for any $\varphi \in C^\infty(M)$, we have

$$\begin{aligned} &\int_M \nabla(u_n - G)\nabla\varphi dv \\ = & -\int_M \Delta(u_n - G)\varphi dv + \int_{\partial M} \frac{\partial(u_n - G)}{\partial n} \varphi d\sigma \\ = & \int_M (2e^{2u_n} - e^{u_n}|\psi_n|^2 - \sum_{p \in M^o \cap \Sigma_1} m(p)\delta_p)\varphi + \int_{\partial M} (ce^{u_n} - \sum_{p \in \partial M \cap \Sigma_1} m(p)\delta_p)\varphi dv \\ \to & \ 0, \quad \text{as } n \to \infty. \end{aligned}$$

Since $u_n - \overline{u}_n$ are uniformly bounded in $W^{1,q}(M)$, we obtain the conclusion of the lemma. □

Now we shall compute the blow-up value by using the Pohozaev identity in Propsition 1.4 and applying Lemma 5.1.

Proof of Theorem 1.5: W.l.o.g., we assume that $p = 0$ and 0 is the only blow-up point in $\overline{B}^M_{2R}(0) = \overline{B}^+_{2R} \subset M$ for some small $R > 0$. By Proposition 1.4, the Pohozaev identity for solution (u_n, ψ_n) is

$$\begin{aligned} & R\int_{\partial B^+_R \cap \mathbb{R}^2_+} (|\frac{\partial u_n}{\partial \nu}|^2 - \frac{1}{2}|\nabla u_n|^2) d\sigma \\ = & \int_{B^+_R} 2e^{2u_n} dv - \int_{B^+_R} e^{u_n}|\psi_n|^2 dv + \int_{\partial B^+_R \cap \partial \mathbb{R}^2_+} ce^{u_n} ds \\ & -R\int_{\partial B^+_R \cap \mathbb{R}^2_+} e^{2u_n} d\sigma - cse^{u_n(s,0)}|^{s=R}_{s=-R} \\ & +\frac{1}{4}\int_{\partial B^+_R \cap \mathbb{R}^2_+} \langle \frac{\partial \psi_n}{\partial \nu}, (x + \bar{x}) \cdot \psi_n \rangle d\sigma + \frac{1}{4}\int_{\partial B^+_R \cap \mathbb{R}^2_+} \langle (x + \bar{x}) \cdot \psi_n, \frac{\partial \psi_n}{\partial \nu} \rangle d\sigma \end{aligned} \tag{41}$$

By Lemma 5.1, we have

$$\lim_{R \to 0} \lim_{n \to \infty} R\int_{\partial B^+_R \cap \mathbb{R}^2_+} (|\frac{\partial u}{\partial \nu}|^2 - \frac{1}{2}|\nabla u|^2) d\sigma = \lim_{R \to 0} R\int_{\partial B^+_R \cap \mathbb{R}^2_+} (|\frac{\partial G}{\partial \nu}|^2 - \frac{1}{2}|\nabla G|^2) d\sigma = \frac{1}{2\pi} m^2(0).$$

Since $u_n \to -\infty$ uniformly on $\partial B_R^+ \cap \mathbb{R}_+^2$, we have

$$\lim_{R\to 0}\lim_{n\to\infty}(-R\int_{\partial B_R^+\cap\mathbb{R}_+^2} e^{2u_n}d\sigma - cse^{u_n(s,0)}|_{s=-R}^{s=R}) = 0.$$

Furthermore, by using the Schrödinger-Lichnerowicz formula $\not{D}^2 = -\Delta + \frac{1}{2}K_g$ and noticing that $K_g = 0$ in our case of a flat domain $B_{2R}^+(0)$, we have

$$\Delta\psi_n = e^{u_n}du_n\cdot\psi_n - e^{2u_n}\psi_n, \quad \text{ in } B_{2R}^+(0)\backslash B_{\frac{R}{4}}^+(0).$$

Since $u_n \to -\infty$ uniformly in $B_{2R}^+(0)\backslash B_{\frac{R}{4}}^+(0)$, $u_n - \overline{u}_n$ is uniformly bounded in $W^{1,q}(M)$ for $1 < q < 2$ and $|\psi_n|$ is uniformly bounded in $B_{2R}^+(0)\backslash B_{\frac{R}{4}}^+(0)$, by the standard elliptic estimates, we know that $|\psi_n|$ is uniformly bounded in $W^{2,q}(B_{\frac{3}{2}R}^+(0)\backslash B_{\frac{R}{2}}^+(0))$ for $1 < q < 2$. Then by the trace imbedding theorem, we get

$$\lim_{R\to 0}\lim_{n\to\infty}\int_{\partial B_R^+\cap\mathbb{R}_+^2} |\psi_n||(x+\bar{x})\cdot\nabla\psi_n|d\sigma = 0.$$

Let $R \to 0$ and $n \to \infty$ in (41), we get

$$\frac{1}{2\pi}m^2(0) = m(0).$$

It follows that $m(0) = 2\pi$. This completes the proof of Theorem 1.5. □

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