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A FAMILY OF ROTATION NUMBERS FOR DISCRETE RANDOM DYNAMICS ON THE CIRCLE

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Abstract. We revisit the problem of well-defining rotation numbers for discrete random dynamical systems on S^1. We show that, contrasting with deterministic systems, the topological (i.e. based on Poincaré lifts) approach does depend on the choice of lifts (e.g. continuously for nonatomic randomness). Furthermore, the winding orbit rotation number does not agree with the topological rotation number. Existence and conversion formulae between these distinct numbers are presented. Finally, we prove a sampling in time theorem which recover the rotation number of continuous Stratonovich stochastic dynamical systems on S^1 out of its time discretisation of the flow.

1. Introduction

Rotation number is one of the most fundamental quantities characterising the behaviour of dynamics on the circle. Since its introduction by Poincaré more than a century ago, it has played essential role on the understanding of the iterated deterministic dynamics on S^1: from the existence of periodic orbits to proving linear conjugacies. Its importance stems from the fact that it is an invariant under conjugacy, allowing for classification of possible asymptotic dynamics; we refer to Katok and Hasselblatt [6], and references therein for the classical definitions. Among many other basic properties, the rotation number of deterministic dynamics equals the average speed of orbits winding around the circle (mod1). Moreover, it is intrinsic in the sense that it is independent: of the choice of the lift, of the initial condition for the lifts, and of the orbit chosen to count the winding. These well known properties are indispensable in order to obtain simultaneous linear conjugacy of commuting diffeomorphisms to pure rotations [2], to establish that the group of orientation preserving homeomorphisms of the circle is a simple group where each of its element is generated by a product of (at most three) involutions [3], or, as it has been recently accomplished, to investigate general rigidity conditions of critical maps [4], just to cite a few instances.

Contrasting with deterministic dynamics, for random dynamics on the circle many of these properties actually do not hold. Rational rotation number, for example, does not imply the existence of periodic orbit. Neither does

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irrationality of the rotation number (and regularity) imply dense orbit. One can think of simple examples of random systems with irrational rotation number and finite orbits. We are going to show that for random dynamics of homeomorphism preserving orientation on the circle, the topological concept based on Poincaré lifts for each random mapping and the orbit winding rotation concept even lead to different numbers. Therefore, they measure distinct features of the dynamics. Yet, there are several reasons why one is interested in such numbers. In this paper, we study these different concepts, proving existence, investigating ergodic properties, and the relation among them. We introduce a convenient parametrisation of the topological rotation number, denoted here by $\rho_{q,\alpha}$, and the orbit winding rotation number, starting at $s_0 \in S^1$, denoted here by OR_{s_0}. The topological rotation number depends precisely on a choice of such parameters (q, α), which encodes the dependence on the lifts for each random homeomorphism.

Amongst many natural motivations to deal with rotation number for discrete random dynamical systems on the circle, consider, for example, the random homeomorphisms on S^1 induced by the action of random 2×2-matrices. The rotation number for this random system on S^1 is the angular counterpart of the Lyapunov exponents for the product of these random matrices. In this sense, for a product of random (ergodically generated) matrices, the rotation number in invariant 2-subspaces represents the imaginary part of a generalised eigenvalues, whose real part is represented by the Lyapunov exponents, described in the multiplicative ergodic theorem (see e.g. Arnold [1] and references therein).

Another motivation comes from discretisation in time of continuous systems. In general, one would like to recover the original rotation number based on observations of the system at discrete time, sampled at intervals of length Δt. For deterministic systems, the original rotation number can be recovered after renormalisation, if the interval Δt is smaller than $\frac{1}{2\rho}$, where ρ is the rotation number (frequency). This is the classical Nyquist frequency sampling theorem, see e.g. Higgins [5], Oppenheim and Schafer [11]. We show here that for appropriate parameters (q, α), a sampling theorem for stochastic system holds, as a limit when Δt goes to zero, if one considers the topological rotation number $\rho_{q,\alpha}$.

Several authors, from different perspectives, have addressed random dynamics on the circle. Just to mention a few, Ruffino has proved a sampling theorem for linear random systems on S^1 [13]. In [14, 15] Zmarrou and Homburg studied the influence of bounded noise on bifurcation of diffeomorphism, while Li and Lu [9] have proved the continuity of the rotation number with respect to L^1-norm on the lifts. Numerical simulations have also been performed in McSharry and Ruffino [10]. Our contribution here on clarifying the distinction among different rotation numbers is not only to prevent ambiguities, but also to show how rich the angular asymptotic behaviour is on random systems on S^1. The paper is organised as follows.

In the next Section we introduce the random dynamics based on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Under *uniform conditions* on the random lifts we introduce the definition of the topological rotation number $\rho_{q,\alpha}$. We prove an ergodic result of existence $\mathbb{P}$-a.s. for $\rho_{q,\alpha}$. In Section 3, we introduce the orbital rotation number. An ergodic result on existence also holds in this case. Here, remarkably, it appears the dependence on the initial condition, *i.e.* the dependence on the ergodic invariant measure chosen in the domain of the skew product on $\Omega \times S^1$ (or just on S^1 if the system is *i.i.d.*). Formulae which compare each one of these distinct rotation numbers are presented. Finally, in Section 4, we prove a sampling in time theorem to recover the rotation number of continuous Stratonovich stochastic dynamical systems on S^1 out of its time discretisation of the flow.

2. Topological rotation numbers $\rho_{q,\alpha}$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $f : \Omega \times S^1 \to S^1$ be a random variable in $\mathcal{H}^+$, the space of homeomorphisms of S^1 which preserve the orientation. Let also $\theta : \Omega \to \Omega$ be an ergodic transformation with respect to $\mathbb{P}$. We consider the dynamics given by the cocycle generated by the composition of the sequence of random homeomorphisms in the following sense (see e.g. Arnold [1]),

$$f^n(\omega, s_0) = f(\theta^{n-1}\omega, \cdot) \circ \cdots \circ f(\theta\omega, \cdot) \circ f(\omega, s_0), \text{ for } n \in \mathbb{N}.$$

Lifts of random homeomorphims. We shall denote the covering map of S^1 by

$$\begin{aligned} p : \mathbb{R} &\longrightarrow S^1, \\ x &\longmapsto e^{i2\pi x}. \end{aligned}$$

A lift of an orientation preserving homeomorphism $f : S^1 \to S^1$ is an increasing continuous function on the covering space $F : \mathbb{R} \to \mathbb{R}$ which is semiconjugate to f by p, *i.e.*, $p \circ F = f \circ p$. Given a lift F of f, then $F + k$ is also a lift for any $k \in \mathbb{Z}$. For a fixed lift F, one can write $F(x) = Id(x) + \delta(x)$, where $\delta : \mathbb{R} \to \mathbb{R}$, is a periodic function. It measures the deviation or the distance of a point s to its image $f(s)$ along one of the many possible chosen geodesics (counting periodic geodesics). This choice of angle (geodesic) is related to the fact that the lifts are not unique. For any lift F, the associated deviation function δ has bounded amplitude:

$$\max_{x \in \mathbb{R}} \{\delta(x)\} - \min_{x \in \mathbb{R}} \{\delta(x)\} < 1.$$

Moreover, the deviation function $\delta \circ p^{-1} : S^1 \to \mathbb{R}$ is well defined by periodicity of δ. For a random homeomorphism $f(\omega, s)$ in S^1, given a random lift $F(\omega, x)$, the corresponding random deviation as a function on S^1 will be denoted, by abuse of notation, by $\delta \circ p^{-1}$, in the sense that, $\delta \circ p^{-1}(\omega, s) = \delta(\omega, p^{-1}(s))$.

Next, we establish a criterion of *uniformness* on the random lifts:

Definition 1. *Given an orientation preserving homeomorphism f and $q, \alpha \in \mathbb{R}$, we denote by $F_{q,\alpha} : \mathbb{R} \to \mathbb{R}$ the unique lift of f such that $F_{q,\alpha}(q) \in [\alpha, \alpha+1)$. We call this function the (q,α)-lift of f.*

We shall refer to such choices of parameters fulfilling Definition 1 as *uniform choice of lifts*. The following elementary properties of the deviation periodic function δ are direct consequences of Definition 1.

Proposition 2.1 (Basic properties)**.** *For all $q, \alpha \in \mathbb{R}$ and $f \in \mathcal{H}^+$, write the corresponding lift $F_{q,\alpha} = Id + \delta_{q,\alpha}$. Then the deviation function $\delta_{q,\alpha}$ satisfies:*

(1) Boundedness*: For every $x \in \mathbb{R}$,*

$$(\alpha - q) - 1 < \delta_{q,\alpha}(x) < (\alpha - q) + 2; \tag{1}$$

(2) Periodicity*: for all $k, l \in \mathbb{Z}$,*

$$\delta_{q+k,\alpha+l} = \delta_{q,\alpha} + l - k. \tag{2}$$

Proof. For item (1); note that, directly from the definition of $F_{q,\alpha}$, we have that $\alpha - q \leq \delta_{q,\alpha}(q) < (\alpha - q) + 1$. Adding this inequality to the bounded amplitude $-1 < \delta_{q,\alpha}(x) - \delta_{q,\alpha}(q) < 1$, the result follows. Item (2) is obvious. □

We are particularly interested in the dependence of the rotation number with respect to the parameters (q, α). In contrast to the deterministic dynamics, in random systems, the rotation number does depend on (q, α). In order to understand this dependency and to obtain comparison formulae relating the family of rotation numbers, it is convenient at this point to consider that the rotation numbers live in the real line $\mathbb{R}$ rather than in $\mathbb{R}/\mathbb{Z}$. We introduce the following definition:

Definition 2. *For a choice of parameters $q, \alpha \in \mathbb{R}$, the (q, α)-rotation number of the random dynamical system generated by a cocycle (f, θ) is defined by the asymptotic limit*

$$\rho_{q,\alpha} = \lim_{n\to\infty} \frac{F^n(\omega, x) - x}{n},$$

when the limit exists, where $F^n(\omega, x) = F_{q,\alpha}(\theta^{n-1}\omega, \cdot) \circ F_{q,\alpha}(\theta^{n-2}\omega, \cdot) \circ \cdots \circ F_{q,\alpha}(\omega, x)$.

The almost surely existence of $\rho_{q,\alpha}$ is proved in Theorem 2.3. Moreover, if $\rho_{q,\alpha}$ exists for a certain $\omega \in \Omega$, it is independent of the starting point $x \in \mathbb{R}$.

Proposition 2.2. *If the rotation number $\rho_{q,\alpha}$ of Definition 2 exists for a fixed $\omega \in \Omega$ and an initial $x \in \mathbb{R}$, then it is independent of $x \in \mathbb{R}$.*

Proof. For every positive integer n, we have that $F^n(\omega, x)$ defined as before is again a lift of $f^n \circ \cdots \circ f^1$ (although not necessarily with $F^n(\omega, q) \in [\alpha, \alpha+1)$). Hence, its deviation $(F^n - Id)$ has bounded amplitude

$$\max_{x\in\mathbb{R}}\{(F^n - Id)(x)\} - \min_{x\in\mathbb{R}}\{(F^n - Id)(x)\} < 1 \ .$$

Then, for all $x, y \in \mathbb{R}$,

$$|F^n(\omega,x)-F^n(\omega,y)| \leq |(F^n(\omega,x)-x)-(F^n(\omega,y)-y)|+|x-y| \leq |x-y|+1,$$

therefore

$$\lim_{n\to\infty}\left(\frac{F^n(\omega,x)}{n}-\frac{F^n(\omega,y)}{n}\right)=0.$$

□

Let $\Theta : \Omega\times S^1 \to \Omega\times S^1$ be the skew product of the dynamics (f, θ) given by $\Theta(\omega, s) = (\theta(\omega), f(\omega, s))$. An invariant probability measure μ on $\Omega \times S^1$ factorise uniquely (a.s.) as $\mu(d\omega, ds) = \nu_\omega(ds)\,\mathbb{P}(d\omega)$, where ν_ω are random probability measures on S^1, see *e.g.* Arnold [1, Sect. 1.4 and 1.5] and references therein. In particular, if the sequence of random homeomorphisms $(f(\theta^n, \cdot))_{n\geq 0}$ is *i.i.d* with respect to an appropriate probability space, then $\mu = \nu(ds)\mathbb{P}(d\omega)$, where $\nu(ds)$ is the stationary measure on S^1 for the Markov process induced by $(f(\theta^n, \cdot))_{n\geq 0}$ a. s., see [1, §1.4.7].

Theorem 2.3 (Existence of (q, α)-rotation numbers)**.** *Let $\mu = \nu_\omega(ds)\mathbb{P}(d\omega)$ be an invariant probability measure on $\Omega \times S^1$ for the skew product $\Theta(\omega, s)$ associated to the cocycle (f, θ). Then we have that the rotation number $\rho_{q,\alpha}$ exists and satisfies:*

$$\rho_{q,\alpha} = \mathbb{E}\int_{S^1} [\delta_{q,\alpha}(\omega, p^{-1}(s))]\ d\nu_\omega(s) \qquad \mathbb{P}\text{-}a.s.. \tag{3}$$

Proof. For a fixed pair of parameters (q, α), we write the deviation function $\delta(\omega, x) := \delta_{q,\alpha}(\omega, x)$ and $\delta\circ p^{-1}(\omega, s) := \delta_{q,\alpha}(\omega, p^{-1}(s))$, for sake of notation. From Definition 2 and induction on n we have that

$$\begin{aligned}
F^n(\omega,x) = x &+ \delta(\omega,x) + \delta\Big(\theta\omega, x+\delta(\omega,x)\Big)\\
&+ \delta\Big(\theta^2\omega, x+\delta(\omega,x)+\delta\Big(\theta\omega, x+\delta(\omega,x)\Big)\Big) + \cdots\\
&+ \delta\Big(\theta^{n-1}\omega, x+\delta(\omega,x)+\delta(\theta\omega, x+\delta(\omega,x)) + \cdots + \delta(\theta^{n-2}\omega,\cdots)\Big)\ .
\end{aligned}$$

For every positive integer i, one has that $F^i(\omega, \cdot)$ is a lift of $f(\theta^{i-1}(\omega), \cdot) \circ \cdots \circ f(\omega, \cdot)$, *i.e.*,

$$p\Big(x+\delta(\omega,x)+\cdots+\delta(\theta^{i-1}(\omega),\ldots)\Big) \;=\; f(\theta^{i-1}(\omega),\cdot)\circ\cdots\circ f(\omega,\cdot)\circ p(x).$$

Additionally, for every deviation function and every $z \in \mathbb{R}$, we have, by periodicity that $\delta(\cdot, z) = \delta \circ p^{-1}(\cdot, p(z))$. Thus, the terms in the expression of $F^n(\omega, x)$ above can be written as

$$\begin{aligned}
&\delta\Big(\theta^{i-1}(\omega), x+\delta(\omega,x)+\cdots+\delta(\theta^{i-2}(\omega),\cdots)\Big)\\
&\qquad = \delta\circ p^{-1}\Big(\theta^{i-1}(\omega), f(\theta^{i-1}(\omega),\cdot)\circ\cdots\circ f(\omega,\cdot)\circ p(x)\Big)\\
&\qquad = \delta\circ p^{-1}\Big(\Theta^{i-1}(\omega,p(x))\Big).
\end{aligned}$$

Therefore,

$$F^n(\omega, x) = x + \sum_{i=1}^{n} \delta \circ p^{-1}\Big(\Theta^{i-1}(\omega, p(x))\Big).$$

Suppose μ is an ergodic invariant probability measure for the skew product. We obviously have that $\delta \circ p^{-1} \in L^1(\mu)$. Then, by the Birkhoff's ergodic theorem, one has that

$$\rho_{q,\alpha} = \lim_{n\to\infty} \frac{F^n(\omega, x) - x}{n} = \mathbb{E}\left[\int_{S^1} \delta \circ p^{-1}(\omega, s)\ d\nu_\omega(s) \right] \qquad \mu\text{-a.s.}.$$

Since $(\theta, \mathbb{P})$ is ergodic, the last formula says that there exists a subset $\Omega' \subset \Omega$ of probability one such that for each $\omega' \in \Omega'$, there exists at least one point $y \in S^1$ such that the equality above holds for (ω', y). Moreover, Proposition 2.2 says that the rotation number $\rho_{q,\alpha}$ is independent of the initial point in S^1. Thus, this implies that the equality also holds a.s. for (ω', s) for almost every $s \in S^1$. Finally, if there exist more than one ergodic invariant probability measure for the skew product, again the fact that $\rho_{q,\alpha}$ is independent of the initial condition implies that the formula of the statement gives the same number, independently of the choice of the ergodic invariant probability measure μ.

□

Note that the boundedness of the deviations $\delta_{q,\alpha}$ for the (q,α)-lifts stated by inequality (1) implies that $(\alpha - q) - 1 < \rho_{q,\alpha} < (\alpha - q) + 2$, and inequality (2) implies that $\rho_{q+k,\alpha+l} = \rho_{q,\alpha} + (l - k)$.

In contrast to the uniform choice of lifts as established in Definition 1, consider a nonuniform choice of random lifts $F(\omega)$, as in [9]. Then, for fixed parameters (q,α), we have $F(\omega) = F_{q,\alpha} + N(\omega)$, where $N(\omega)$ is an integer random variable.

Corollary 2.4. *If we consider a nonuniform choice of random lift given by $F(\omega) = F_{q,\alpha} + N(\omega)$, where $N(\omega)$ is an integrable integer random variable, then the associated rotation number for this lift exists and it is given by*

$$\rho_{F(\omega)} = \lim_{n\to\infty} \frac{F^n(x) - x}{n} = \rho_{q,\alpha} + \mathbb{E}[N].$$

Proof. The proof of Theorem 2.3 holds for any random lift $F(\omega)$. Given $F(\omega) = F_{q,\alpha} + N(\omega)$, with $N(\omega)$ integrable, we have that the deviation function for this lift is $\delta = \delta_{q,\alpha}(\omega, x) + N(\omega)$. Hence the statement follows directly from the ergodic formula (3) of Theorem 2.3.

□

We can now obtain a probabilistic formula of comparison between distinct rotation numbers $\rho_{q,\alpha}$ and $\rho_{q',\alpha'}$ in terms of their parameters. Before doing that, it is convenient to set the following notation. For the sets of parameters (q, α) and (q', α'), and $i, j \in \mathbb{Z}$, consider

$$\begin{aligned} A_i :=& \{\omega \in \Omega : F_{q,\alpha'}(\omega, q') \in [\alpha' + i, \alpha' + i + 1)\}, \text{ and} \\ B_j :=& \{\omega \in \Omega : F_{q,\alpha}(\omega, q) \in [\alpha' - j, \alpha' - j + 1)\}. \end{aligned}$$

Note that there exist at most two parameters i, j such that the respective probabilities are positive. Let us define

$$\begin{aligned} k =& \min \{i \in \mathbb{Z}, \text{ such that } \mathbb{P}(A_i) > 0\}, \text{ and} \\ l =& \min \{j \in \mathbb{Z}, \text{ such that } \mathbb{P}(B_j) > 0\}. \end{aligned}$$

We have the following relation between the rotation numbers $\rho_{q,\alpha}$, and $\rho_{q',\alpha'}$.

Proposition 2.5. *Let (q, α) and (q', α') be two sets of parameters for the rotation number of Definition (2). Then we have that*

$$\rho_{q',\alpha'} = \rho_{q,\alpha} + \mathbb{P}(A_k) - \mathbb{P}(B_l) - k + l.$$

Proof. By the definition of the uniform random lift $F_{q,\alpha'}(\omega, \cdot)$ we have that $\mathbb{P}(A_k \dot\cup A_{k+1}) = 1$. Therefore,

$$\begin{aligned} F_{q',\alpha'}(\omega, x) &= F_{q,\alpha'}(\omega, x) - k\, 1_{A_k} - (k+1)\, 1_{A_{k+1}} \\ &= F_{q,\alpha'}(\omega, x) + 1_{A_k} - k - 1, \end{aligned}$$

and thus, the deviations $\delta_{q',\alpha'} = \delta_{q,\alpha'} + 1_{A_k} - k - 1$. This implies, by Theorem 2.3, in particular that

$$\rho_{q',\alpha'} = \rho_{q,\alpha'} + \mathbb{P}(A_k) - k - 1. \tag{4}$$

Again, by the uniformness of the lifts $F_{q,\alpha}$ we have that $\mathbb{P}(B_l \dot\cup B_{l+1}) = 1$, and,

$$\begin{aligned} F_{q,\alpha'}(\omega, x) &= F_{q,\alpha}(\omega, x) + l\, 1_{\{B_l\}} + (l+1)\, 1_{\{B_{l+1}\}} \\ &= F_{q,\alpha}(\omega, x) - 1_{B_l} + l + 1. \end{aligned}$$

Hence, the deviations $\delta_{q,\alpha'} = \delta_{q,\alpha} - 1_{B_l} + l + 1$. This implies, by Theorem 2.3, that

$$\rho_{q,\alpha'} = \rho_{q,\alpha} - \mathbb{P}(B_l) + l + 1. \tag{5}$$

The result now follows by Formulas (5) and (4) above.

□

In the statement of last proposition, we have that $k \in \{\lfloor q' - q \rfloor, \lfloor q' - q \rfloor + 1\}$ and k is non-decreasing as a function of q'. The probability $\mathbb{P}(A_{k(q')})$ is a periodic function depending on q', with the same discontinuities of $k(q')$. It is decreasing in the interval $[q, q+1)$, with $\mathbb{P}(A_0) = 1$ if $q' = q$ and decreases to zero when q' approaches $q + 1$ in this interval. For a fixed α, the rotation number $\rho_{q,\alpha}$ is decreasing with respect to q. Analogously, $l \in \{\lfloor \alpha' - \alpha \rfloor, \lfloor \alpha' - \alpha \rfloor + 1\}$ and l is non-decreasing as a function of α'. The probability $\mathbb{P}(B_{l(\alpha')})$ is a real periodic function depending on α'. If $\alpha' = \alpha \pmod 1$, then $\mathbb{P}(B_l) = 1$ and $k = \alpha' - \alpha$. Moreover $\mathbb{P}(B_{l(\alpha')})$ decreases to zero when q' approaches $q+m$ from below, for any $m \in \mathbb{Z}$. For a fixed q, the rotation number $\rho_{q,\alpha}$ is increasing with respect to α.

For a fixed q, from the previous dependence formula, we have that $\rho_{q,\alpha}$ is continuous with respect to α if the distribution of $F_{q,\alpha}(\omega, q)$ in $[\alpha, \alpha+1)$ has no atoms. Note that, if this distribution does have atoms, it generates discontinuities, which does not contradict the continuity result on $L^1(\Omega) \mapsto \rho$, [9, Thm. A.ii], since $q \mapsto L^1(\Omega)$ is not continuous.

3. Rotation number of orbits

In this section we explore the concept of rotation number based on the physical observation of the angular behaviour of single orbits. Consider a random trajectory $s_n = f(\theta^{n-1}(\omega), \cdot) \circ \cdots \circ f(\omega, s_0)$. We are going to lift this orbit to $\mathbb{R}$ as an increasing random angle with bounded jumps. The asymptotic average speed in $\mathbb{R}$ is the rotation number of its orbit. More precisely, let $\gamma_0 \in [0,1)$ be the initial normalised angle, *i.e.* $p(\gamma_0) = s_0$. Defining by induction $\gamma_n \in \mathbb{R}$ as the angle in $\mathbb{R}$, such that, $p(\gamma_n) = s_n$ and $\gamma_{n-1} \leq \gamma_n < \gamma_{n-1} + 1$, gives us the following.

Definition 3. *The rotation number of the random orbit s_n starting at s_0 is defined by the random variable*

$$OR_{(s_0)} = \lim_{n\to\infty} \frac{\gamma_n - \gamma_0}{n},$$

when the limit exists.

Next, we show that the rotation number of orbits exists almost surely with respect to ergodic invariant measures for the skew product on $\Omega \times S^1$. Differently from the topological rotation number $\rho_{q,\alpha}$ of the previous section, the rotation number of orbits can in fact depend on the initial condition.

Theorem 3.1. *Let μ be an ergodic invariant probability measure for the skew product Θ on $\Omega \times S^1$. Then the rotation number of an orbit (s_n) exists for μ-almost every (ω, s_0).*

Proof. Let $\bar{\delta}(x)$ be the unique (possibly discontinuous) deviation function with image in $[0,1)$, such that, $\bar{F}(x) = Id(x) + \bar{\delta}(x)$ satisfies $f(\omega,\cdot) \circ p = p \circ \bar{F}(\omega, x)$, *i.e.*, $\bar{F}$ is a discontinuous lift of the homeomorphism f. The dynamics of the increasing angles γ_n can be described in terms of this discontinuous lift $\bar{F}$ as follows:

$$\gamma_n = \bar{F}(\theta^{n-1}(\omega), \cdot) \circ \cdots \circ \bar{F}(\omega, \gamma_0).$$

The rest of the proof follows as in the proof of Theorem 2.3, since the same calculation there also holds for discontinuous lifts. We have obviously that $\bar{\delta} \in L^1(\mu)$, hence,

$$OR_{(s_0)} = \mathbb{E} \int_{S^1} \bar{\delta}(\omega, x)\ \nu_\omega(ds) \qquad \mu\text{- a.s.}$$

□

Next Corollary gives a comparison formula between the orbit rotation number OR_{s_0} of this section and the family of topological rotation numbers $\rho_{q,\alpha}$ of the last section.

Corollary 3.2. *Fix a pair of parameters (q, α). For each ergodic invariant probability measure $\mu(d\omega, ds) = \nu_\omega(ds)\mathbb{P}(d\omega)$ on $\Omega \times S^1$ we have that*

$$\begin{aligned} OR_{(s_0)} = \quad \rho_{q,\alpha} \quad & -k \;\; \mathbb{P}\,[\,\mu\{s \in S^1 : \delta_{q,\alpha}(p^{-1}(s)) \in (k, k+1)\}] \\ & -(k+1)\; \mathbb{P}\,[\,\mu\{s \in S^1 : \delta_{q,\alpha}(p^{-1}(s)) \in [k+1, k+2)\}] \\ & -(k+2)\; \mathbb{P}\,[\,\mu\{s \in S^1 : \delta_{q,\alpha}(p^{-1}(s)) \in [k+2, k+3)\}] \end{aligned}$$

μ-a.s., where $k = \lfloor(\alpha - q) - 1\rfloor$.

Proof. We write the discontinuous deviation $\bar{\delta}(\omega, x) \in [0, 1)$ associated to $OR_{(s_0)}$ in terms of the uniform deviation $\delta_{q,\alpha}$. In order to do that, we only have to add to $\delta_{q,\alpha}$ appropriate integers which depend on ω and on $x \in \mathbb{R}$. Inequality (1) says that $(\alpha - q) - 1 < \delta_{q,\alpha}(x) < (\alpha - q) + 2$, which implies that this correction random integer ranges only in the set $\{k, k+1, k+2\}$. In particular, one checks that

$$\begin{aligned} \bar{\delta}(\omega, x) = \delta_{q,\alpha}(x) \quad & - \quad k\; 1_{\{\delta_{q,\alpha}(\omega,x) \in (k,k+1)\}} \\ & - \quad (k+1)\; 1_{\{\delta_{q,\alpha}(\omega,x) \in [k+1,k+2)\}} \\ & - \quad (k+2)\; 1_{\{\delta_{q,\alpha}(\omega,x) \in [k+2,k+3)\}}. \end{aligned}$$

The result follows by the ergodic formula at the end of the proof of Theorem 3.1.

□

As an example, one verifies that with parameters $(q, \alpha) = (0, 0)$, we have that $OR_{(s_0)} = \rho_{0,0} + \mathbb{P}[\mu\{s \in S^1 : \delta_{0,0}(p^{-1}(s)) < 0\}] - \mathbb{P}[\mu\{s \in S^1 : \delta_{0,0}(p^{-1}(s)) \geq 1\}]$, μ-a.s..

Next example is a simple construction which shows the dependency of the orbit rotation number with the initial condition of the skew product Θ.

Example 1. Parametrise S^1 by $x \mapsto e^{2\pi x i}$ with $x \in (-1/2, 1/2]$. Let $\Omega = \{1, 2, 3, 4\}$, with $\mathbb{P}[i] = \frac{1}{4}$ for every $i \in \Omega$. Consider the ergodic transformation given by the permutation which in cycle notation is $\theta = (1, 2, 3, 4)$. Let $f(1), f(2), f(3), f(4) : S^1 \to S^1$ be homeomorphisms in $\mathcal{H}^+$ such that $f(\omega, 0) = 0$, for all $\omega \in \Omega$ and $f(1, 1/8) = 3/8$, $f(2, 3/8) = -3/8$, $f(3, -3/8) = -1/8$ and $f(4, -1/8) = 1/8$. For this random dynamics we have that $\rho_{0,\alpha} = \lfloor\alpha\rfloor$, since zero is a fixed point and $\delta_{0,\alpha}(\omega, 0) = \lfloor\alpha\rfloor$ for all $\omega \in \Omega$. The orbit rotation number starting at zero is given by $OR_0 = 0$, corresponding to a Dirac invariant measure at zero for all $\omega \in \Omega$. Nevertheless, for $x_0 = 1/8$, $\omega = 1$ we have that the evolution of increasing angles is given by

$$\gamma_n = \frac{2n+1}{8},$$

which yields $OR_{\frac{1}{8}} = 1/4$, corresponding to the ergodic invariant measure μ given by the normalised sum of Dirac measures in $(1, \frac{1}{8})$, $(2, \frac{3}{8})$, $(3, -\frac{3}{8})$ and $(4, -\frac{1}{8})$, *i.e.* a periodic orbit of Θ.

4. Sampling time Theorem

In [13] it has been studied rotation number for a sequence of random matrices acting on $\mathbb{R}^2$. Such quantity corresponds to a counter part of the Oseledet's theorem for Lyapunov exponents for a product of random matrices with invariant 2-subspaces, see *e.g.*, Arnold [1] and references therein. In the context of linear systems on S^1, it was proved in that paper a sampling theorem for discretisation of the flow at time interval Δt, such that, the rotation number of the discrete system, when rescaled by $\frac{1}{\Delta t}$, converges to the rotation number of the original continuous systems. The main result in this section generalises this sampling theorem, extending its scope from linear systems acting on S^1 to nonlinear equations intrinsic on S^1. We are going to show that, adequate choices of the parameters (q, α) will lead to a compatibility of the definitions of rotation numbers for discrete systems with continuous systems.

To get started, consider a classical stochastic flow of diffeomorphisms on S^1 generated by a Stratonovich differential equation:

$$ds_t = H^0(s_t)\ dt + \sum_{j=1}^{m} H^j(s_t)\ \circ dB_t^j \tag{6}$$

with initial condition $s_0 \in S^1$, where $H^0, H^1, \ldots, H^m$ are smooth vector fields on S^1 and $(B_t^1, \ldots, B_t^m)$ is a standard Brownian motion on $\mathbb{R}^m$ with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. This equation can be lifted to the covering space $\mathbb{R}$, and written as

$$dx_t = h^0(x_t)\ dt + \sum_{j=1}^{m} h^j(x_t)\ \circ dB_t^j \tag{7}$$

with initial condition x_0 such that $p(x_0) = s_0$, where the functions h^i, $i = 0, 1, \ldots, m$ are periodic with $h^i(x) = H^i(p(x))$ for all $x \in \mathbb{R}$. If x_t is a solution of equation (7), then $s_t = p(x_t)$ solves Equation (6). Let φ_t denote the stochastic flows of diffeomorphisms on S^1 generated by Equations (6), and let ψ_t denote the stochastic flows of diffeomorphisms on $\mathbb{R}$ generated by Equations (7). Then $p \circ \psi_t = \varphi_t \circ p$, *i.e.* ψ_t are Poincaré lifts of φ_t for all $t \geq 0$. The classical rotation number of the continuous stochastic flow $\varphi_t : S^1 \to S^1$ is given by the average winding of trajectories:

$$\mathrm{rot}(\varphi) = \lim_{t\to\infty} \frac{\psi_t(\omega, x_0)}{t}$$

whose existence $\mathbb{P}$-almost surely is guaranteed by standard stochastic analysis technique and the ergodic theorem for Markov process, see *e.g.* [12].

This rotation number is independent of x_0 and is given by

$$\mathrm{rot}(\varphi) = \int_{S^1} h^0(s) + \frac{1}{2}\sum_{j=1}^{m} \left(dh^i(s)h^i(s)\right)\ d\nu(s) \qquad \mathbb{P}\text{-a.s.},$$

where ν is an ergodic invariant measure on S^1.

Next theorem says that if the parameters q and α are in an appropriate range, when we sample the time periodically at an interval $\Delta t > 0$, then the normalised rotation number $\rho_{q,\alpha}$ of the discrete cocycle of diffeomorphisms $\varphi_{\Delta t}$ recover the original (continuous) rotation number $\mathrm{rot}(\varphi)$, when the frequency of sampling tends to infinite.

Theorem 4.1. *For $q-1<\alpha<q\in\mathbb{R}$, the rotation number of a stochastic continuous systems (6) can be recovered by the limit of the rescaled rotation:*

$$\mathrm{rot}(\varphi_t) = \lim_{\Delta t\to 0}\frac{\rho_{q,\alpha}(\varphi_{\Delta t})}{\Delta t},$$

where $\rho_{q,\alpha}(\varphi_{\Delta t})$ is the topological rotation number of the discrete random system generated by $\varphi_{\Delta t}(\omega,\cdot)$ with θ the canonical shift by Δt on the probability space Ω.

For proving 4.1, we exploit the fact that at each time $t \geq 0$, the flow $\psi_t(\omega,\cdot)$ for the system on $\mathbb{R}$, Equation (7), is a lift of the system $\varphi_t(\omega,\cdot)$ on S^1, Equation (6). We have to control, for $t>0$, those lifts $\psi_t = Id+\delta_t$ whose corresponding deviation δ_t are not in the prescribed interval $\delta_t(q)\in[\alpha,\alpha+1)$.

Proof of Theorem 4.1. Let $F_{\Delta t}(\omega):\mathbb{R}\to\mathbb{R}$ denote the (q,α)-lift of φ_t according to Definition 1. Then $F_{\Delta t} = \psi_{\Delta t} + N_{\Delta t}(\omega)$, where $N_{\Delta t}(\omega)$ is a random integer variable. By the cocycle property and Corollary 2.4, we have that for $n\in\mathbb{N}$,

$$\begin{aligned}
\mathrm{rot}(\varphi) &= \lim_{n\to\infty}\frac{\psi_{n\Delta t}(x_0)-x_0}{n\Delta t}\\
&= \lim_{n\to\infty}\frac{\psi_{\Delta t}(\theta^{n-1}\omega,\cdot)\circ\cdots\circ\psi_{\Delta t}(\theta\omega,x_0)\circ\psi_{\Delta t}(\omega,x_0)}{n\Delta t}\\
&= \frac{1}{\Delta t}\left(\rho_{q,\alpha}(\varphi_{\Delta t})+\mathbb{E}[N]\right).
\end{aligned}$$

Therefore, we only have to prove that

$$\lim_{\Delta t\searrow 0}\frac{\mathbb{E}[|N|]}{\Delta t} = 0.$$

Note that the random variable N counts how many times continuous trajectories of the original system on S^1 cross, in the anticlockwise direction, the point $p(\alpha)\in S^1$ up to time Δt. Thus, we have that

$$N(\omega) = \sum_{n\in\mathbb{Z}} n 1_{\Omega_n}\ ,$$

where

$$\Omega_n = \{\psi_{\Delta t}(\omega, q) \in [\alpha + n, \alpha + n + 1)\}.$$

We control the expectation of N using boundedness on the distribution of $\psi_{\Delta t}(\omega, q)$. Let $p(t, x, y)$ be the density of the transition probability measure associated to a non-degenerate diffusions given by Equation (7). Then, there exists a constant $M > 0$ such that,

$$\frac{1}{M\sqrt{t}} e^{-M\frac{(x-y)^2}{t}} \leq p(t, x, y) \leq \frac{M}{\sqrt{t}} e^{-\frac{(x-y)^2}{Mt}}.$$

See Kusuoka and Stroock [7, 8]. Let $N^+ = \max\{N, 0\}$ and $N^- = \max\{-N, 0\}$, such that $N = N^+ - N^-$. Hence, for the positive part N^+

$$\begin{aligned} \mathbb{E}[N^+] &\leq M \int_{\alpha+1}^{\infty} (\lfloor x - (\alpha+1) \rfloor + 1) \frac{1}{\sqrt{\Delta t}} \exp\left\{-\frac{(x-q)^2}{M\Delta t}\right\} dx \\ &\leq M \int_{\alpha+1}^{\infty} (x - \alpha) \frac{1}{\sqrt{\Delta t}} \exp\left\{-\frac{(x-q)^2}{M\Delta t}\right\} dx. \end{aligned}$$

And for the negative part:

$$\begin{aligned} \mathbb{E}[N^-] &\leq M \int_{-\infty}^{\alpha} (\lfloor \alpha - x \rfloor + 1) \frac{1}{\sqrt{\Delta t}} \exp\left\{-\frac{(x-q)^2}{M\Delta t}\right\} dx \\ &\leq M \int_{-\infty}^{\alpha} (\alpha - x + 1) \frac{1}{\sqrt{\Delta t}} \exp\left\{-\frac{(x-q)^2}{M\Delta t}\right\} dx. \end{aligned}$$

Changing variables, for $\Delta t \in (0, 1)$ we have that

$$\mathbb{E}[N^+] \leq M^{\frac{3}{2}} \int_{\frac{\alpha-q+1}{\sqrt{M\Delta t}}}^{\infty} \left(\sqrt{M}u + q - \alpha\right) \exp\left\{-u^2\right\} du$$

and

$$\mathbb{E}[N^-] \leq M^{\frac{3}{2}} \int_{\frac{q-\alpha}{\sqrt{M\Delta t}}}^{\infty} \left(\sqrt{M}u + \alpha - q + 1\right) \exp\left\{-u^2\right\} du$$

Hence $\mathbb{E}[N^+]$ and $\mathbb{E}[N^-]$ goes to zero when Δt goes to zero. Moreover, by standard calculus argument, using that $\lim_{z\to 0} \exp\left\{-\frac{1}{z}\right\} z^{\beta} = 0$ for any exponent $\beta \in \mathbb{R}$, then finally we get that

$$\lim_{\Delta t \searrow 0} \frac{\mathbb{E}[N^+]}{\Delta t} = \lim_{\Delta t \searrow 0} \frac{\mathbb{E}[N^-]}{\Delta t} = 0.$$

□

We remark that this sampling theorem does not hold either if $\alpha \geq q$ or if $\alpha \leq q-1$. In this case, note that either $\mathbb{E}[N^+]$ or $\mathbb{E}[N^-]$ does not converge to zero when Δt goes to zero. Also, the theorem does not hold if one consider the rotation number of orbits (Section 3). As a counterexample consider a deterministic system driven by a North-South vector fields in S^1 embedded in $\mathbb{R}^2$. The rotation number $\mathrm{rot}(\varphi)$ of the continuous system is obviously zero. But the orbit rotation number OR_{s_0} for any discretisation in time of the flow is zero for orbits with s_0 on the right hand side of S^1, and 1 for orbits with s_0 on the left hand side of S^1, independently of the time interval of discretisation Δt.

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