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Comment on Convergence of macrostates under
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by

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Comment on "Convergence of macrostates under reproducible processes" [Phys. Lett. A 374: 3715-3717 (2010)]

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Abstract

In this paper, two counterexamples show that the superadditivity inequality of relative entropy is not true even for the full-ranked quantum states. Thus, an inequality of quantum channels and complementary channels is not also true. Finally, a conjecture of weak superadditivity inequality is presented.

1 Introduction

Let \mathcal{H} and \mathcal{K} be two finite dimensional Hilbert spaces, $L(\mathcal{H}, \mathcal{K})$ be the set of all linear operators from \mathcal{H} to \mathcal{K} , if $L(\mathcal{H}) = L(\mathcal{H})$, we denote $L(\mathcal{H}, \mathcal{K})$ for $L(\mathcal{H})$, $T(\mathcal{H}, \mathcal{K})$ the set of all linear super-operators from $L(\mathcal{H})$ to $L(\mathcal{K})$. A super-operator $\Lambda \in T(\mathcal{H}, \mathcal{K})$ is said to be a *completely positive linear map* if for each $k \in \mathbb{N}$,

$$\Lambda \otimes \mathbb{1}_{M_k(\mathbb{C})} : L(\mathcal{H}) \otimes M_k(\mathbb{C}) \rightarrow L(\mathcal{K}) \otimes M_k(\mathbb{C})$$

is positive, where $M_k(\mathbb{C})$ denotes the set of all $k \times k$ complex matrices. It follows from Choi's theorem [1] that every completely positive linear map Λ has a Kraus representation

$$\Lambda = \sum_{\mu} \text{Ad}_{M_{\mu}},$$

that is, for every $X \in L(\mathcal{H})$, $\Lambda(X) = \sum_{\mu} M_{\mu} X M_{\mu}^{\dagger}$, where $\{M_{\mu}\} \subseteq L(\mathcal{H}, \mathcal{K})$, $\sum_{\mu=1}^K M_{\mu}^{\dagger} M_{\mu} = \mathbb{1}_{\mathcal{H}}$, M_{μ}^{\dagger} is the adjoint operator of M_{μ} . A *quantum channel* is just a trace-preserving completely positive linear super-operator.

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Let $D(\mathcal{H})$ denote the set of all the density matrices ρ on \mathcal{H} . The *von Neumann entropy* $S(\rho)$ of ρ is defined by

$$S(\rho) \stackrel{\text{def}}{=} -\text{Tr}(\rho \log \rho).$$

The *relative entropy* of two mixed states ρ and σ is defined by

$$S(\rho||\sigma) \stackrel{\text{def}}{=} \begin{cases} \text{Tr}(\rho(\log \rho - \log \sigma)), & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma), \\ +\infty, & \text{otherwise.} \end{cases}$$

The relative entropy is an very important quantity in quantum information theory [2]. It satisfies many significant relations such as monotonicity property under quantum channels [3]. In [4, 5], Petz studied the strong superadditivity of relative entropy. In [6], Rau derived a monotonicity property of relative entropy under a reproducible process. From which he obtained the following *superadditivity inequality* of relative entropy:

$$S(\rho_{AB}||\sigma_{AB}) \geq S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B), \quad (1.1)$$

where ρ_{AB} and σ_{AB} are macrostates on tensor space $\mathcal{H}_A \otimes \mathcal{H}_B$, ρ_A, ρ_B, σ_A and σ_B are the reduced states of ρ_{AB} and σ_{AB} , respectively. Note that the inequality (1.1) holds if σ_{AB} is a product state.

In this paper, however, we show that the inequality (1.1) is not true even for the full-ranked quantum states. Thus, an inequality of quantum channels and complementary channels is not also true. Finally, we present a conjecture of weaker superadditivity inequality of relative entropy.

2 Counterexamples

Firstly, we show that the superadditivity inequality (1.1) is not true.

Example 2.1. Let $|\psi_X\rangle, |\phi_X\rangle \in \mathcal{H}_X$ such that $\langle\psi_X|\phi_X\rangle = 0$, where $X = A, B$. Set

$$\rho_{AB} = |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|$$

and

$$\sigma_{AB} = \lambda|\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B| + (1-\lambda)|\phi_A\rangle\langle\phi_A| \otimes |\phi_B\rangle\langle\phi_B|,$$

where $\lambda \in (0, 1)$. We have

$$S(\rho_{AB}||\sigma_{AB}) = S(\rho_A||\sigma_A) = S(\rho_B||\sigma_B) = -\log(\lambda) > 0,$$

which implies that

$$S(\rho_{AB}||\sigma_{AB}) < S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B).$$

Thus, the inequality (1.1) is violated.

The following numerical example of the diagonal and *full-ranked* states ρ_{AB} and σ_{AB} given by M. Mosonyi show that the inequality (1.1) is also not true.

Example 2.2 (Random research). Let

$$\rho_{AB} = 0.1568|00\rangle\langle 00| + 0.7270|10\rangle\langle 10| + 0.0804|01\rangle\langle 01| + 0.0358|11\rangle\langle 11|$$

and

$$\sigma_{AB} = 0.3061|00\rangle\langle 00| + 0.4243|10\rangle\langle 10| + 0.1713|01\rangle\langle 01| + 0.0983|11\rangle\langle 11|.$$

Thus we have

$$\begin{cases} \rho_A = 0.2372|0\rangle\langle 0| + 0.7628|1\rangle\langle 1| \\ \sigma_A = 0.4774|0\rangle\langle 0| + 0.5226|1\rangle\langle 1| \end{cases} \quad \text{and} \quad \begin{cases} \rho_B = 0.8838|0\rangle\langle 0| + 0.1162|1\rangle\langle 1| \\ \sigma_B = 0.7304|0\rangle\langle 0| + 0.2696|1\rangle\langle 1|. \end{cases} \quad (2.1)$$

Apparently, all states here are invertible and

$$S(\rho_{AB}||\sigma_{AB}) < S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B)$$

which contradicts with the superadditivity inequality again.

Remark 2.3. Now, we show that an inequality of quantum channels and complementary channels is not also true since the superadditivity inequality is not hold.

In fact, Let $\rho, \sigma \in \mathcal{D}(\mathcal{H})$. Let Φ be a quantum channel from \mathcal{H} to \mathcal{K} ,

$$\Phi = \sum_{\mu=1}^K \text{Ad}_{M_\mu},$$

where $M_\mu \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ are Kraus operators such that $\sum_{\mu=1}^K M_\mu^\dagger M_\mu = \mathbb{1}_{\mathcal{H}}$. Let $\mathcal{H}_E = \mathbb{C}^K$ be a complex Hilbert space with orthonormal basis $\{|\mu\rangle : \mu = 1, \dots, K\}$. Define

$$V|\psi\rangle \stackrel{\text{def}}{=} \sum_{\mu} M_\mu |\psi\rangle \otimes |\mu\rangle, \quad \forall |\psi\rangle \in \mathcal{H}.$$

According to the Stinespring representation of quantum channels, one has

$$\Phi(\rho) = \text{Tr}_E(V\rho V^\dagger).$$

The corresponding *complementary channel* is given by

$$\hat{\Phi}(\rho) = \text{Tr}_{\mathcal{K}}(V\rho V^\dagger) = \sum_{\mu, \nu=1}^K \text{Tr}(M_\mu \rho M_\nu^\dagger) |\mu\rangle\langle \nu|.$$

That $V \in \mathcal{L}(\mathcal{H}, \mathcal{K} \otimes \mathcal{H}_E)$ is a linear isometry, and for all $\tau \in \mathcal{D}(\mathcal{H})$, $V\tau V^\dagger$ has, up to multiplicities of zero, the same eigenvalues as τ are clear. Thus,

$$\begin{aligned} S(\rho||\sigma) &= S(V\rho V^\dagger||V\sigma V^\dagger) \\ &\geq S(\text{Tr}_E(V\rho V^\dagger)||\text{Tr}_E(V\sigma V^\dagger)) + S(\text{Tr}_{\mathcal{K}}(V\rho V^\dagger)||\text{Tr}_{\mathcal{K}}(V\sigma V^\dagger)) \\ &= S(\Phi(\rho)||\Phi(\sigma)) + S(\hat{\Phi}(\rho)||\hat{\Phi}(\sigma)). \end{aligned} \quad (2.2)$$

Taking $\rho = \rho_{AB}$ and $\Phi(\rho_{AB}) = \text{Tr}_B(\rho_{AB})$, we have $\widehat{\Phi}(\rho_{AB}) = W\rho_B W^\dagger$ for some linear isometry $W \in \mathcal{L}(\mathcal{H}_B, \mathcal{H}_E)$. It follows from inequality (2.2) that

$$\begin{aligned} S(\rho_{AB}||\sigma_{AB}) &\geq S(\Phi(\rho_{AB})||\Phi(\sigma_{AB})) + S(\widehat{\Phi}(\rho_{AB})||\widehat{\Phi}(\sigma_{AB})) \\ &= S(\rho_A||\sigma_A) + S(W\rho_B W^\dagger||W\sigma_B W^\dagger) \\ &= S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B). \end{aligned}$$

Therefore, the inequality (1.1) told us that the inequality (2.2) is also not valid, that is,

$$S(\rho||\sigma) \not\geq S(\Phi(\rho)||\Phi(\sigma)) + S(\widehat{\Phi}(\rho)||\widehat{\Phi}(\sigma)).$$

In [7], Li and Winter proposed the following question: For given quantum channel Φ from \mathcal{H}_A to \mathcal{H}_B and quantum states $\rho, \sigma \in \mathcal{D}(\mathcal{H}_A)$, does there exist a quantum channel Ψ from \mathcal{H}_B to \mathcal{H}_A with $\Psi \circ \Phi(\sigma) = \sigma$ and

$$S(\rho||\sigma) \geq S(\Phi(\rho)||\Phi(\sigma)) + S(\rho||\Psi \circ \Phi(\rho)) \quad (2.3)$$

They answered this question affirmatively in the classical case, however, the quantum case is still open. In view of this, we can ask the following questions:

- (i) Can we have $\widehat{\Phi}(\rho) = \widehat{\Phi}(\sigma)$ if $S(\rho||\sigma) = S(\Phi(\rho)||\Phi(\sigma))$?
- (ii) What can be derived from $\widehat{\Phi}(\rho) = \widehat{\Phi}(\sigma)$?
- (iii) What can be derived from $S(\rho||\sigma) = S(\widehat{\Phi}(\rho)||\widehat{\Phi}(\sigma))$?

For (i), M. Hayashi answered negatively in [9].

Let ρ, σ and Φ be as follows:

$$\rho = \sum_j \lambda_j(\rho) E_j, \quad \sigma = \sum_j \lambda_j(\sigma) E_j, \quad \Phi(X) = \sum_j E_j X E_j,$$

where E_j is a projector operator and $\sum_j E_j = \mathbb{1}$. Then $S(\rho||\sigma) = S(\Phi(\rho)||\Phi(\sigma))$ and

$$\widehat{\Phi}(\rho) = \sum_j \lambda_j(\rho) |j\rangle\langle j|, \quad \widehat{\Phi}(\sigma) = \sum_j \lambda_j(\sigma) |j\rangle\langle j|.$$

It is clear that if $\lambda(\rho) \neq \lambda(\sigma)$, then $\widehat{\Phi}(\rho) \neq \widehat{\Phi}(\sigma)$.

This showed that no matter how close together ρ and σ are, the inequality (2.2) does not hold. Therefore, it seems that the inequality (1.1) does not hold even if ρ_{AB} and σ_{AB} are closer in some sense.

3 Discussions

It is said in [6] that the second law of thermodynamics, i.e., any reproducible process increases entropy, similarly implies that under a reproducible process macrostates become less distinguishable from the uniform distribution

$$S(\mu_f || \mathbb{1} / \text{Tr}(\mathbb{1})) \leq S(\mu_g || \mathbb{1} / \text{Tr}(\mathbb{1})), \quad (3.1)$$

where μ_f, μ_g are the so-called *generalized canonical distribution*

$$\mu_g \propto \exp \left[\sum_a \lambda^a G_a \right],$$

$\{G_a\}$ are the observables whose expectation values characterize the system's macrostate. With properly adjusted Lagrange parameters $\{\lambda^a\}$ this canonical state encodes information about the relevant expectation values $\{g_a \equiv \langle G_a \rangle_\mu\}$, while discarding (by maximizing entropy) all other information. The initial macrostate μ_g evolves under the same reproducible process to the final macrostate μ_f .

It follows that full ranked states are some kind of macrostates and a reproducible process (coarse-graining) could be a process which maps a full-ranked state into another full-ranked state. As the second law of thermodynamics reflects the fact that the macrostates tend to be closer to equidistribution, Rau intuitively thought that not only the distinguishability between *any* macrostate and the uniform distribution diminishes, but also the mutual distinguishability (described by the relative entropy) between *arbitrary pairs* of macrostates decreases. Thus, he proposed the following monotonicity inequality: for *any* two initial macrostates μ_g and $\mu_{g'}$ evolving under the same reproducible process to final macrostates μ_f and $\mu_{f'}$, respectively, the relative entropy will decrease:

$$S(\mu_f || \mu_{f'}) \leq S(\mu_g || \mu_{g'}). \quad (3.2)$$

He concluded that the inequality (3.2) follows immediately if only one can show the monotonicity relation

$$S(\mu_{f(\rho)} || \mu_{f(\sigma)}) \leq S(\rho || \sigma), \quad (3.3)$$

where $\mu_{f(\rho)}, \mu_{f(\sigma)}$ are the final macrostates evolved from ρ, σ under the same reproducible process, respectively.

The inequality (3.3) is the main result of Rau in [6]. When the removing correlations are considered, Rau obtained the superadditivity inequality (1.1) from inequality (3.3). Since the inequality (1.1) is not true, so, the inequality (3.3) is not also true. Thus, ones need to reconsider some related results which based on inequalities (1.1) and (3.3), for instance, the Lemma B5 in [8], etc.

4 Weak superadditivity inequality: A conjecture

Although inequalities (1.1) and (2.2) are not valid in general, it seems that a modified version of the results is possible. If ρ equals to the coarse-grained $\mu_f(\sigma)$, e.g., for the case of removing correlations and $\rho_{AB} = \sigma_A \otimes \sigma_B$, then $\gamma = 0$. γ is bounded in the range $[0, 1]$ and varies continuously as a function of ρ . For ρ within some finite neighborhood of $\mu_f(\sigma)$ (for the case of removing correlations and ρ_{AB} within some finite neighborhood of $\sigma_A \otimes \sigma_B$), γ should still remain strictly smaller than one. Hence, while the strong monotonicity of the relative entropy may no longer hold globally for arbitrary pairs of states, it may still hold locally for nearby states within some finite region. Indeed, pursuing an alternative approach (within the framework of nonequilibrium thermodynamics) to prove exactly such local convergence of macrostates is deserved.

In [10, arXiv:1305.1472], we have obtained the following result:

For given two quantum states $\rho, \sigma \in \mathcal{D}(\mathcal{H}_d)$, one has

$$\begin{cases} \min_{U \in \mathcal{U}(\mathcal{H}_d)} S(U\rho U^\dagger || \sigma) = H(\lambda^\downarrow(\rho) || \lambda^\downarrow(\sigma)), \\ \max_{U \in \mathcal{U}(\mathcal{H}_d)} S(U\rho U^\dagger || \sigma) = H(\lambda^\downarrow(\rho) || \lambda^\uparrow(\sigma)), \end{cases} \quad (4.1)$$

where σ is full-ranked, $\lambda^\downarrow(\sigma)$ (resp. $\lambda^\uparrow(\sigma)$) stands for the vector with all eigenvalues of σ arranged in decreasing (resp. increasing) order, $\mathcal{U}(\mathcal{H}_d)$ denotes the set of all unitary operators on \mathcal{H}_d ; $H(p||q) := \sum_j p_j(\log p_j - \log q_j)$ is Shannon entropy between two probability distribution $p = \{p_j\}$ and $q = \{q_j\}$.

Based on the above result, we propose the following *conjecture*: There exist unitary operators $U_A \in \mathcal{U}(\mathcal{H}_A)$, $U_B \in \mathcal{U}(\mathcal{H}_B)$ and $U_{AB} \in \mathcal{U}(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that

$$S(U_{AB}\rho_{AB}U_{AB}^\dagger || \sigma_{AB}) \geq S(U_A\rho_AU_A^\dagger || \sigma_A) + S(U_B\rho_BU_B^\dagger || \sigma_B), \quad (4.2)$$

where the reference state σ_{AB} is required to be full-ranked state. Indeed, our numerical calculations show that inequality (4.2) is true. More specifically, the relative entropy is *weak superadditivity* in the following sense [10, arXiv:1305.2023]:

$$H(\lambda^\downarrow(\rho_{AB}) || \lambda^\uparrow(\sigma_{AB})) \geq H(\lambda^\downarrow(\rho_A) || \lambda^\downarrow(\sigma_A)) + H(\lambda^\downarrow(\rho_B) || \lambda^\downarrow(\sigma_B)), \quad (4.3)$$

where σ_{AB} is full-ranked states.

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References

- [1] M. Choi, Completely positive linear maps on complex matrices. *Lin. Alg. Appl.* **10**, 285 (1975).
- [2] M. Ohya and D. Petz, *Quantum Entropy and Its Use*. Springer-Verlag, Heidelberg, 1993. Second edition 2004.
- [3] G. Lindblad, Completely positive maps and entropy inequalities. *Commun. Math. Phys.* **40**, 147-151 (1975).
- [4] D. Petz, Properties of the relative entropy of states of von Neumann algebras. *Acta Math. Hung.* **47**(1-2), 65-72 (1986).
- [5] D. Petz, On certain properties of the relative entropy of states of operator algebras. *Math. Z.* **206**, 351-361 (1991).
- [6] J. Rau, Convergence of macrostates under reproducible processes. *Phys. Lett. A* **374**: 3715–3717 (2010).
- [7] A. Winter and K. Li, A stronger subadditivity relation? With applications to squashed entanglement, sharability and separability. See http://www.maths.bris.ac.uk/~csajw/stronger_subadditivity.pdf
- [8] M.G. Genoni, M.G.A. Paris, Quantifying non-Gaussianity for quantum information. *Phys. Rev. A* **82**, 052341 (2010).
- [9] The International Symposium on Quantum Information and Quantum Logic, held in Zhejiang Univeristy, 10-13, Aug. 2012. See www.math.zju.edu.cn/wjd
- [10] L. Zhang *et. al.*, arXiv:1305.1472; arXiv:1305.2023