A note on zero sets of fractional sobolev functions with negative power of integrability

by

Armin Schikorra

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A NOTE ON ZERO SETS OF FRACTIONAL SOBOLEV FUNCTIONS WITH NEGATIVE POWER OF INTEGRABILITY

ARMIN SCHIKORRA

ABSTRACT. We extend a Poincaré-type inequality for functions with large zero-sets by Jiang and Lin to fractional Sobolev spaces. As a consequence, we obtain a Hausdorff dimension estimate on the size of zero sets for fractional Sobolev functions whose inverse is integrable. Also, for a suboptimal Hausdorff dimension estimate, we give a completely elementary proof based on a pointwise Poincaré-style inequality.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. For functions $u : \Omega \to \mathbb{R}^n$ we are interested in the size of the zero set $\Sigma$,

$$\Sigma := \{ x \in \Omega : \lim_{r \to 0} \int_{B_r(x)} |f| = 0 \},$$

under the condition that for some $\alpha > 0$,

$$\int_{\Omega} |f|^{-\alpha} < \infty. \quad (1.1)$$

Here and henceforth, for a measurable set $A \subset \mathbb{R}^n$ we denote the mean value integral

$$\int_A f \equiv (f)_A := |A|^{-1} \int_A f.$$

In [7] Jiang and Lin showed that if $f \in W^{1,p}(\Omega)$, then

$$\mathcal{H}^s(\Sigma) = 0 \quad \text{where} \quad s = \max\{0, n - \frac{p\alpha}{p+\alpha} \}.$$

They were motivated by the analysis of rupture sets of thin films, which is described by a singular elliptic equation. We do not go into the details of this and instead, for applications we refer to, e.g., [3, 6, 2, 8].

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In this note, we extend Jiang and Lin’s result to fractional Sobolev spaces and obtain

**Theorem 1.1.** For \( \sigma \in (0, 1) \) and for any \( f \in W^{\sigma,p}(\Omega) \) satisfying (1.1), \( \mathcal{H}^s(\Sigma) = 0 \), where \( s = \max\{0, n - \sigma \frac{p\alpha}{p+\alpha}\} \).

Here, we use the following definitions for the (fractional) Sobolev space. For more on these we refer to, e.g., [4, 1, 10].

**Definition 1.2.** The homogeneous \( W^{\sigma,p} \)-norms are defined as follows:

\[
[f]_{W^{1,p}(\Omega)} := \|\nabla f\|_{L^p(\Omega)}.
\]

For \( \sigma \in (0, 1) \) we define the Slobodeckij-norm,

\[
[f]_{W^{\sigma,p}(\Omega)} := \begin{cases}
\left( \int_{\Omega} \int_{\Omega} \left( \frac{|f(x) - f(y)|}{|x-y|^p} \right)^p \frac{dx}{|x-y|^p} \right)^{1/p} & \text{if } p \in [1, \infty), \\
\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^p} & \text{if } p = \infty.
\end{cases}
\]

The respective Sobolev space \( W^{\sigma,p}, \sigma \in (0, 1], p \in [1, \infty] \) is then the collection of functions \( f : \Omega \to \mathbb{R} \) with finite Sobolev norms \( \|f\|_{W^{\sigma,p}(\Omega)} \),

\[
\|f\|_{W^{\sigma,p}(\Omega)} := \|f\|_{L^p(\Omega)} + [f]_{W^{\sigma,p}(\Omega)}.
\]

To prove Theorem 1.1, the case \( p \leq n/\sigma \) is the relevant one, since for the other cases we can use the embedding into the Hölder spaces, see [7]. We have the following extension to fractional Sobolev spaces of a Poincaré-type inequality from [7].

**Theorem 1.3.** For any \( \theta > 0, \sigma \in (0, 1], p \in (1, n/\sigma], s \in (n - \sigma p, n] \), there is a constant \( C > 0 \) such that the following holds for any \( R > 0 \):

Let \( B_R \) be any ball in \( \mathbb{R}^n \) with radius \( R \), \( f \in W^{\sigma,p}(B_R) \) and assume that there is a closed set \( T \subset B_R \) such that

\[
T \subset \{x \in B_R : \limsup_{r \to 0} \int_{B_r} |f| = 0\},
\]

(1.2)

\[
\mathcal{H}^s(T) > \frac{1}{\theta} R^s,
\]

and for any ball \( B_r \) with some radius \( r > 0 \),

(1.3)

\[
\mathcal{H}^s(T \cap B_r) \leq \theta r^s.
\]

Then,

\[
\|f\|_{L^p(B_R)} \leq C R^s [f]_{W^{\sigma,p}(B_R)}.
\]
In [7] this was proven for the classical Sobolev space $W^{1,p}$, using an argument based on the $p$-Laplace equation with measures and the Wolff potential. Our argument, on the other hand, is completely elementary and adapts the classical blow-up proof of the Poincaré inequality, see Section 2.

Once Theorem 1.3 is established, one can follow the arguments in [7] to obtain Theorem 1.1. These rely heavily on the theory of Sousslin sets, [9], to find the closed set $T \subset \Sigma$ with the condition (1.2) and (1.3) satisfied. Those arguments are by no means elementary, but we were unable to remove them in order to show that $\mathcal{H}^s(\Sigma) = 0$. However, if one is satisfied in showing that $\mathcal{H}^t(\Sigma) = 0$ for any $t > s$, then there is a completely elementary argument, the details of which we will present in Section 3. There, we prove the following “pointwise” Poincaré-style inequality, from which the suboptimal Hausdorff dimension estimate easily follows, see Corollary 3.1.

**Lemma 1.4.** For any $\varepsilon > 0$, $p \in [1, \infty)$, there exists $C > 0$, such that the following holds. Let $f \in L^p_{\text{loc}}$, and assume $x \in \mathbb{R}^n$, such that

$$\lim_{r \to 0} \int_{B_r(x)} |f| = 0$$

then for any $R > 0$, there exists $\rho \in (0, R)$ such that

$$\int_{B_\rho(x)} |f|^p \leq C \left( \frac{R}{\rho} \right)^\varepsilon \int_{B_\rho(x)} \|f| - (|f|)_{B_\rho} |^p.$$

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## 2. Poincaré Inequality: Proof of Theorem 1.3

By a scaling argument, Theorem 1.3 follows from the following

**Lemma 2.1.** For any $\theta > 0$, $\sigma \in (0, 1]$, $p \in (1, n/\sigma]$, $s \in (n - \sigma p, n]$, there is a constant $C > 0$ such that the following holds:

Let $f \in W^{\sigma,p}(B_1, [0, \infty))$ and assume that there is a closed set $T \subset B_1$ such that

$$T \subset \{ x \in B_1 : \limsup_{r \to 0} \int_{B_r} f = 0 \},$$

and

$$\mathcal{H}^s(T) > \frac{1}{\theta},$$

as well as

$$\mathcal{H}^s(T \cap B_r) \leq \theta r^s$$

for any ball $B_r$ with radius $r > 0$. 

Then,
\[ \|f\|_{L^p(B_1)} \leq C \|f\|_{\dot{W}^{\sigma,p}(B_1)}. \]

**Proof.** We proceed by the usual blow-up proof of the Poincaré inequality: Assume the claim is false, and that for fixed \( \theta, p, s, \sigma \) for any \( k \in \mathbb{N} \) there are \( f_k \in W^{\sigma,p}(B_1, [0, \infty)) \) such that
\[ T_k \subset \{ x \in B_1 : \limsup_{r \to 0} \int_{B_r} f_k = 0 \}, \]
\[ \mathcal{H}^s(T_k) > \frac{1}{\theta}, \quad \mathcal{H}^s(T_k \cap B_r) \leq \theta r^s \forall B_r, \]
and
\[ \|f_k\|_{L^p(B_1)} > k \|f_k\|_{\dot{W}^{\sigma,p}(B_1)}. \]
Replacing \( f_k \) by \( f_k \|f_k\|_{L^p} \) (note that this does not change the definition and size of \( T_k \)), we can assume w.l.o.g.
\[ \|f_k\|_{L^p} \equiv 1, \]
and
\[ \|f_k\|_{L^p(B_1)} \xrightarrow{k \to \infty} 0. \]
In particular, \( f_k \) is uniformly bounded in \( W^{\sigma,p} \), and by the Rellich-Kondrachov theorem, up to taking a subsequence, \( f_k \) converges strongly in \( L^p \), and weakly in \( W^{\sigma,p} \) to some \( f \in W^{\sigma,p} \), with \( \|f\|_{\dot{W}^{\sigma,p}(B_1)} \equiv 0, \|f\|_{L^p} = 1. \) Thus,
\[ f \equiv |B_1|^{-\frac{1}{p}}, \]
and setting \( g_k := |B_1|^\frac{1}{p} f_k \), we have found a sequence such that
\[ g_k \to 1 \quad \text{in} \quad W^{\sigma,p}(B_1), \]
\[ \mathcal{H}^s(T_k) > \frac{1}{\theta}, \]
and
\[ \mathcal{H}^s(T_k \cap B_r) \leq \theta r^s \quad \text{for any ball} \ B_r. \]
This is a contradiction to Lemma 2.2. \( \square \)

We used the following lemma, which essentially quantifies the intuition, that a function approximating 1 in \( W^{\sigma,p} \) cannot be zero on a large set.

**Lemma 2.2.** Let \( \sigma \in (0,1], \ s \in (n-\sigma p, n], \ f_k \in W^{\sigma,p}(B_1, [0, \infty)) \), and assume that
\[ \|f_k - 1\|_{W^{\sigma,p}(B_1)} \xrightarrow{k \to \infty} 0. \]
Then, for any \( T_k \subset B_1 \) closed and
\[ T_k \subset \{ x \in B_1 : \limsup_{r \to 0} \int_{B_r} f_k = 0 \}, \]
as well as for some $\theta > 0$,

\begin{equation}
\mathcal{H}^s(T_k \cap B_r) \leq \theta r^s \quad \text{for any } B_r, \text{ for all } k
\end{equation}

we have

$$\lim_{k \to \infty} \mathcal{H}^s(T_k) = 0.$$ 

Proof. By the subsequence principle, it suffices to show

$$\liminf_{k \to \infty} \mathcal{H}^s(T_k) = 0.$$ 

By extension, we also can assume that $f_k - 1 \to 0$ in $W^{\sigma,p}(\mathbb{R}^n)$, and $f_k \equiv 1$ on $\mathbb{R}^n \setminus B_2$.

On the one hand, we have

$$[f_k]_{W^{\sigma,p}(\mathbb{R}^n)} \xrightarrow{k \to \infty} 0.$$ 

On the other hand, up to picking a subsequence, we can assume the existence of $R_k \in (0,1)$, for $k \in \mathbb{N}$, and $\lim_{k \to \infty} R_k = 0$, such that

$$\inf_{r>R_k, x \in B_1} \int_{B_r(x)} f_k \geq \frac{9}{10}.$$ 

Since for any point $x \in T_k$ we have that $\lim_{l \to 0} \int_{B_2-l} f_k(x) = 0$, we expect the average (fractional) gradient around $x$ to be fairly large. More precisely, we have the following

Claim. There is a uniform constant $c_{s,\sigma,p} > 0$, such that the following holds: For any $x \in T_k$, there exists $\rho = \rho_{k,x} \in (0,R_k)$ such that

\begin{equation}
c_{s,\sigma,p} \rho^s \leq \rho^{-\sigma p} \int_{B_\rho} |f_k - (f_k)_{B_\rho}|^p \leq C [f_k]_{W^{\sigma,p}(B_\rho)}^p.
\end{equation}

Of course, we only have to show the first inequality, the second inequality is the classical Poincaré inequality.

For the proof let us write $f$ instead of $f_k$. Then, since for $x \in T$,

$$\lim_{l \to \infty} \int_{B_{2^{-l-1}R_k(x)}} f = 0,$$
we have that
\[
\frac{9}{10} \leq \sum_{l=0}^{\infty} \left( \int_{B_{2^{-l}R_k}(x)} f - \int_{B_{2^{-l-1}R_k}(x)} f \right) \leq C \sum_{l=0}^{\infty} \left( (2^{-l}R_k)^{-n} \int_{B_{2^{-l}R_k}} |f - (f)_{B_{2^{-l}R_k}}| \right).
\]
Consequently, for any \( \varepsilon > 0 \), there has to be some \( c_\varepsilon > 0 \) and some \( l \in \mathbb{N} \) such that
\[
\left( (2^{-l}R_k)^{-n} \int_{B_{2^{-l}R_k}} |f - (f)_{B_{2^{-l}R_k}}| \right) \geq c_\varepsilon \left( 2^{-l}R_k \right)^\varepsilon,
\]
because if the opposite inequality was true for all \( l \in \mathbb{N} \) we would have
\[
\frac{9}{10} \leq C c_\varepsilon R_k \sum_{l \in \mathbb{N}} 2^{-\varepsilon l} \leq C c_\varepsilon \sum_{l \in \mathbb{N}} 2^{-\varepsilon l},
\]
which is false for \( c_\varepsilon \) small enough.

Thus, for \( \rho := 2^{-l}R_k \in (0, R_k) \),
\[
\rho^{n-\sigma + \varepsilon} \leq C_\varepsilon \rho^{-\sigma} \int_{B_{\rho}} |f - (f)_{B_{\rho}}| \leq C_\varepsilon \left( \rho^{-\sigma p} \int_{B_{\rho}} |f - (f)_{B_{\rho}}|^p \right)^{1/p} \rho^{n-\frac{n}{p}},
\]
that is
\[
\rho^{n-\sigma p + \varepsilon p} \leq C_\varepsilon \rho^{-\sigma p} \int_{B_{\rho}} |f - (f)_{B_{\rho}}|^p,
\]
Setting \( \varepsilon = \frac{s-(n-\sigma p)}{p} > 0 \), we have shown for any \( x \in T \) the existence of some \( \rho \in (0, R_k) \) satisfying (2.2), and the claim is proven.

For any \( k \) we cover \( T_k \) by the family
\[
\mathcal{F}_k := \{ B_{\rho}(x), \ x \in T, \ B_{\rho}(x) \text{ satisfies (2.2)} \}.
\]
Since \( T \subset B_2 \) is closed and bounded, i.e. compact, we can find a finite subfamily still covering all of \( T_k \), and then using Vitali’s (finite) covering theorem, we find a subfamily \( \tilde{\mathcal{F}}_k \subset \mathcal{F}_k \) of disjoint balls \( B_{\rho}(x) \), so that the union of the \( B_{5\rho} \) covers all of \( T_k \). We use this \( \tilde{\mathcal{F}}_k \) as a cover for an estimate of the Hausdorff measure:
\[
\mathcal{H}^s(T_k) \leq \sum_{B_{\rho} \in \tilde{\mathcal{F}}_k} \mathcal{H}^s(B_{5\rho} \cap T_k) \overset{(2.1)}{=} \theta 5^s \sum_{B_{\rho} \in \tilde{\mathcal{F}}_k} \rho^s \leq C_{\theta,s} \sum_{B_{\rho} \in \tilde{\mathcal{F}}_k} [\mathcal{H}^p_{W^{\sigma,p}}(B_{\rho})] \leq C_{\theta,s} [\mathcal{H}^p_{W^{\sigma,p}(\mathbb{R}^n)}] \overset{k \to \infty}{\longrightarrow} 0.
\]
3. An elementary proof for the suboptimal case

We start with the proof of the pointwise inequality, Lemma 1.4.

Proof. First, let us show the claim for \( p = 1 \):

Fix \( R, \varepsilon > 0, f \in L^1_{\text{loc}} \) and assume \( x = 0 \). W.l.o.g., \( f \geq 0 \). Set

\[
\tau = 2^{-n-1} \left( \sum_{l=-\infty}^{0} 2^{\varepsilon l} \right)^{-1} R^{-\varepsilon},
\]

and \( C_\varepsilon := R^{-\varepsilon} \tau^{-1} \). Assume by contradiction that the claim was false, i.e. assume that for any \( \rho \in (0, R) \),

\[
\int_{B_\rho} |f - (f)_{B_\rho}| < \tau \rho^\varepsilon \int_{B_\rho} f.
\]

Then for any \( K \in \mathbb{N} \),

\[
\int_{B_\rho} |f - (f)_{B_\rho}| < \tau \rho^\varepsilon \sum_{k=-K}^{0} \int_{B_{2k+1,\rho}} f - \int_{B_{2k-1,\rho}} f + \tau \rho^\varepsilon \int_{B_{2-K-1,\rho}} f
\]

\[
\leq 2^n \tau \rho^\varepsilon \sum_{k=-K}^{0} \int_{B_{2k,\rho}} |f - (f)_{B_{2k,\rho}}| + \tau \rho^\varepsilon \int_{B_{2-K-1,\rho}} f
\]

Setting now for \( l \in \mathbb{Z} \),

\[
a_l := \int_{B_{2^l R}} |f - (f)_{B_{2^l R}}|,
\]

\[
b_l := \int_{B_{2^l R}} f,
\]

the above equation applied to \( \rho = 2^l R \) reads as

\[
a_l \leq 2^n R^\varepsilon \tau 2^{\varepsilon l} \sum_{k=-K}^{0} a_{k+l} + \tau (2^l R)^\varepsilon b_{-K+l-1} \quad \text{for any } K \in \mathbb{N}, l \in \mathbb{Z}.
\]
In particular for any $L \in \mathbb{N}$,
\[
\sum_{l=-L}^{0} a_l \leq 2^n R^\varepsilon \tau \sum_{l=-L}^{0} 2^{2l} \sum_{k=-K}^{0} a_{k+l} + \tau R^\varepsilon \sum_{l=-L}^{0} 2^{2l} b_{-K+l-1} \\
\leq 2^n R^\varepsilon \tau \sum_{l=-L}^{0} 2^{2l} \sum_{k=-K+l}^{0} a_k + \tau R^\varepsilon \left( \sup_{j \leq -K} b_j \right) \sum_{l=-\infty}^{0} 2^{2l} \\
\leq 2^n R^\varepsilon \tau \sum_{k=-L-K}^{0} a_k \sum_{l=-L}^{k+K} 2^{2l} + \tau R^\varepsilon \left( \sup_{j \leq -K} b_j \right) \sum_{l=-\infty}^{0} 2^{2l} \\
\overset{(3.1)}{\leq} \frac{1}{2} \sum_{k=-L-K}^{0} a_k + \frac{1}{2} \sup_{j \leq -K} b_j.
\]

Under the additional assumption that
\[
\sum_{l=-\infty}^{0} a_l < \infty,
\]
letting $L, K \to \infty$, using that by (1.4) we have $\lim_{l \to \infty} b_l = 0$, the above estimates implies that $a_k = 0$ for all $k \leq 0$. This means that $f$ is a constant on $B_R$, and in particular by (1.4), $f$ is constantly zero in $B_R$. This contradicts the strict inequality (3.2).

To see (3.3), fix $K \in \mathbb{N}$ such that $\sup_{j \leq -K} b_j \leq 2$. Then for
\[
c_L := \sum_{l=-L}^{0} a_l,
\]
the above estimate becomes
\[
c_L \leq \frac{1}{2} c_{L+K} + 1 \quad \text{for any } L \in \mathbb{N}.
\]
In particular, for any $i \in \mathbb{N}$,
\[
c_{L+iK} \leq 2^{-i} c_L + \sum_{j=0}^{i} 2^{-j}.
\]
Since $c_i$ is monotonically increasing,
\[
\sup_{i \geq L+K} c_i \leq c_L + \sum_{j=0}^{\infty} 2^{-j} < \infty.
\]
This proves Lemma 1.4 for $p = 1$.

If $p > 1$, we apply this to $f^p$, and obtain
\[
(3.4) \quad \int_{B_\rho(x)} f^p \leq C \left( \frac{R}{\rho} \right)^\varepsilon \int_{B_\rho(x)} |f^p - (f^p)_{B_\rho}|.
\]
We now need the following estimate, which holds for any $p \in [1, \infty)$, and $\delta \in (0, 1)$,

$$|a - b|^p - |a|^p - |b|^p \leq \delta |a|^p + \frac{C_p}{\delta^p} |b|^p.$$ 

Since $B_\rho$ is fixed, let us write $(f)$ for $(f)_{B_\rho}$. Firstly, for any $\delta \in (0, 1)$,

$$|f^p - (f^p)| \leq |f - (f)|^p + |(f)^p - (f^p)| + \frac{C_p}{\delta^p} |f - (f)|^p + \delta |f|^p.$$ 

Plugging this in (3.4), for $\delta = \tilde{\delta}(R/\rho)^{-\varepsilon}$ small enough, we arrive at (3.5)

$$\int_{B_\rho(x)} f^p \leq C \left( \frac{R}{\rho} \right)^{(1+p)\varepsilon} \int_{B_\rho(x)} |f - (f)|^p + C \rho^\alpha \left( \frac{R}{\rho} \right)^{(1+p)\varepsilon} |(f)^p - (f^p)|.$$ 

Next,

$$|(f)^p - (f^p)| \leq (|(f)^p - f^p|) \leq (|f - (f)|^p) + \delta f^p + \frac{C_p}{\delta^p} (|f - (f)|^p).$$ 

Plugging this now for $\delta = \tilde{\delta}(R/\rho)^{-\varepsilon}$ into (3.5), by absorbing we arrive at (3.5)

$$\int_{B_\rho(x)} f^p \leq C \left( \frac{R}{\rho} \right)^{\varepsilon \rho^a} \int_{B_\rho(x)} |f - (f)|^p.$$ 

Since this holds for $\varepsilon > 0$ is arbitrarily small, this proves the Lemma 1.4. $\square$

**Corollary 3.1.** For $\sigma \in (0, 1)$ and for any $f \in W^{\sigma,p}(\Omega)$ satisfying (1.1), $H^t(\Sigma) = 0$, whenever $t > s = \max\{0, n - \sigma \frac{p}{p+\alpha}\}$.

**Proof.** Let $\varepsilon > 0$, $R > 0$, and $x \in \Sigma$. Pick $\rho < R$ from Lemma 1.4, so that

$$\int_{B_\rho(x)} |f|^p \leq C R^\varepsilon \rho^{\sigma p - \varepsilon} \int_{W^{\sigma,p}(B_\rho)} f^p.$$ 

By Hölder and Young inequality, as in [7, Corollary 2.1],

$$\rho^{n + (2\varepsilon - \sigma p) \frac{\alpha}{p+\alpha}} \leq C \rho^{2\varepsilon - \sigma p} \int_{B_\rho(x)} |f|^p + C \rho^\varepsilon \int_{B_\rho(x)} |f|^{-\alpha} \leq C R^{2\varepsilon} \int_{W^{\sigma,p}(B_\rho)} f^p + C R^\varepsilon \int_{B_\rho(x)} |f|^{-\alpha}.$$ 

Let now $\varepsilon > 0$ such that $t > n + (2\varepsilon - \sigma p) \frac{\alpha}{p+\alpha}$, then what we have shown is that for any $R > 0$ and any $x \in \Sigma$ there exists $\rho \in (0, R)$ such that

$$\rho^t \leq C R^\varepsilon \int_{W^{\sigma,p}(B_\rho)} f^p + C \int_{B_\rho(x)} |f|^{-\alpha}. \quad (3.6)$$
Let now
\[ V_R := \{ B_\rho(x) : x \in \Sigma, \ \rho < R, (3.6) \text{ holds} \} . \]
Any countable disjoint subclass \( U_R \subset V_R \) satisfies
\[ \sum_{B_\rho \subset U_R} \rho^t \leq C R^t \left[ f \right]_{W^{s,p}(\Omega)}^{p} + C R^\varepsilon \int_\Omega |f|^{-\alpha}. \]
By the Besicovitch covering theorem, as in, e.g., [5, Theorem 18.1], we find for any \( R \) a countable subclass \( U_R \subset V_R \), such that any point of \( \Sigma \) is covered at least once, and at most a fixed number of times. Thus,
\[ \mathcal{H}^t(\Sigma) = \lim_{R \to 0} \mathcal{H}^t_R(\Sigma) \leq C \lim_{R \to 0} \sum_{B_\rho \subset U_R} \rho^t \leq C_f \lim_{R \to 0} R^\varepsilon = 0. \]
\[ \square \]

References


Armin Schikorra, Max-Planck Institut MiS Leipzig, Inselstr. 22, 04103 Leipzig, Germany, armin.schikorra@mis.mpg.de