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Systems of reaction-diffusion equations with spatially distributed hysteresis

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SYSTEMS OF REACTION-DIFFUSION EQUATIONS WITH SPATIALLY DISTRIBUTED HYSTERESIS

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ABSTRACT. We study systems of reaction-diffusion equations with discontinuous spatially distributed hysteresis in the right-hand side. The input of hysteresis is given by a vector-valued function of space and time. Such systems describe hysteretic interaction of non-diffusive (bacteria, cells, etc.) and diffusive (nutrient, proteins, etc.) substances leading to formation of spatial patterns. We provide sufficient conditions under which the problem is well posed in spite of the discontinuity of hysteresis. These conditions are formulated in terms of geometry of manifolds defining hysteresis thresholds and the graph of initial data.

1. SETTING OF THE PROBLEM

1.1. **Introduction and setting.** Reaction-diffusion equations with spatially distributed hysteresis were first introduced in [6] to describe the growth of a colony of bacteria (Salmonella typhimurium) and explain emerging spatial patterns of the bacteria density. In [6, 7], numerical analysis of the problem was carried out, however without rigorous justification. First analytical results were obtained in [2, 17] (see also [1, 11]), where existence of solutions for multi-valued hysteresis was proved. Formal asymptotic expansions of solutions were recently obtained in a special case in [8]. Questions about the uniqueness of solutions and their continuous dependence on initial data as well as a thorough analysis of pattern formation remained open. In this paper, we formulate sufficient conditions that guarantee existence, uniqueness, and continuous dependence of solutions on initial data for systems of reaction-diffusion equations with discontinuous spatially distributed hysteresis. Analogous conditions for scalar equations have been considered by the authors in [4, 5].

Denote $Q_T = (0,1) \times (0,T)$, where $T > 0$. Let $\mathcal{U} \subset \mathbb{R}^k$ and $\mathcal{V} \subset \mathbb{R}^l$ $(k, l \in \mathbb{N})$ be closed sets. We assume throughout that $(x,t) \in \overline{Q}_T$, $u(x,t) \in \mathcal{U}$, $v(x,t) \in \mathcal{V}$.

We consider the system of reaction-diffusion equations

$$\begin{cases} u_t = Du_{xx} + f(u, v, W(\xi_0, u)), \\ v_t = g(u, v, W(\xi_0, u)) \end{cases} \tag{1.1}$$

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with the initial and boundary conditions

$$u|_{t=0} = \varphi(x), \quad v|_{t=0} = \psi(x), \quad u_x|_{x=0} = u_x|_{x=1} = 0. \tag{1.2}$$

Here D is a positive-definite diagonal matrix; W is a hysteresis operator which maps an initial configuration function $\xi_0(x)$ ($\in \{1, -1\}$) and an input function $u(x, \cdot)$ to an output function $W(\xi_0(x), u(x, \cdot))(t)$. As a function of (x, t), $W(\xi_0, u)$ takes values in a set $\mathcal{W} \subset \mathbb{R}^m$ ($m \in \mathbb{N}$). Now we shall define this operator in detail.

Let $\Gamma_\alpha, \Gamma_\beta \subset \mathcal{U}$ be two disjoint smooth manifolds of codimension one without boundary (hysteresis "thresholds"). For simplicity, we assume that they are given by $\gamma_\alpha(u) = 0$ and $\gamma_\beta(u) = 0$ with $\nabla\gamma_\alpha(u) \neq 0$ and $\nabla\gamma_\beta(u) \neq 0$, respectively, where γ_α and γ_β are C^∞-smooth functions (in the general situation, atlases can be used).

Denote $M_\alpha = \{u \in \mathcal{U} : \gamma_\alpha(u) \leq 0\}$, $M_\beta = \{u \in \mathcal{U} : \gamma_\beta(u) \leq 0\}$, $M_{\alpha\beta} = \{u \in \mathcal{U} : \gamma_\alpha(u) > 0,\ \gamma_\beta(u) > 0\}$. Assume that $M_\alpha \cap \Gamma_\beta = \varnothing$ and $M_\beta \cap \Gamma_\alpha = \varnothing$ (Fig. 1).

Next, we introduce locally Hölder continuous functions (*hysteresis "branches"*)

$$W_1 : D(W_1) = M_\alpha \cup \overline{M}_{\alpha\beta} \to \mathcal{W}, \qquad W_{-1} : D(W_{-1}) = M_\beta \cup \overline{M}_{\alpha\beta} \to \mathcal{W}.$$

We fix $T > 0$ and denote by $C_r[0, T)$ the space of functions continuous on the right in $[0, T)$. For any $\zeta_0 \in \{1, -1\}$ (*initial configuration*) and $u_0 \in C([0, T]; \mathcal{U})$ (*input*), we introduce the *configuration* function

$$\zeta : \{1, -1\} \times C([0, T]; \mathcal{U}) \to C_r[0, T), \quad \zeta(t) = \zeta(\zeta_0, u_0)(t)$$

as follows. Let $X_t = \{s \in (0, t] : u_0(s) \in \Gamma_\alpha \cup \Gamma_\beta\}$. Then $\zeta(0) = 1$ if $u_0(0) \in M_\alpha$, $\zeta(0) = -1$ if $u_0(0) \in M_\beta$, $\zeta(0) = \zeta_0$ if $u_0(0) \in M_{\alpha\beta}$; for $t \in (0, T]$, $\zeta(t) = \zeta(0)$ if $X_t = \varnothing$, $\zeta(t) = 1$ if $X_t \neq \varnothing$ and $u_0(\max X_t) \in \Gamma_\alpha$, $\zeta(t) = -1$ if $X_t \neq \varnothing$ and $u_0(\max X_t) \in \Gamma_\beta$ (Fig. 1).

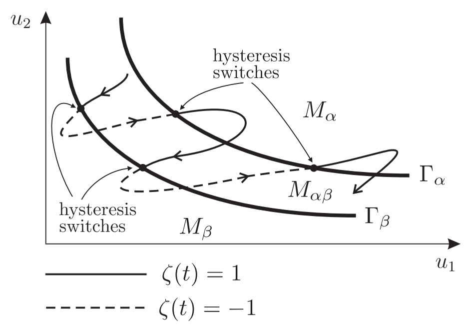

FIGURE 1. Regions of different behavior of hysteresis W

Now we introduce the *hysteresis operator* $W : \{1, -1\} \times C([0, T]; \mathcal{U}) \to C_r[0, T)$ by the following rule (cf. [12, 18, 10]). For any initial configuration $\zeta_0 \in \{1, -1\}$ and input $u_0 \in C([0, T]; \mathcal{U})$, the function $W(\zeta_0, u_0) : [0, T] \to \mathcal{W}$ (*output*) is given by

$$W(\zeta_0, u_0)(t) = W_{\zeta(t)}(u_0(t)), \tag{1.3}$$

where $\zeta(t)$ is the configuration function defined above.

Assume that the initial configuration and the input function depend on spatial variable x. Denote them by $\xi_0(x)$ and $u(x, t)$, where $u(x, \cdot) \in C([0, T]; \mathcal{U})$.

Using (1.3) and treating x as a parameter, we define the *spatially distributed hysteresis*

$$W(\xi_0(x), u(x,\cdot))(t) = W_{\xi(x,t)}(u(x,t)), \tag{1.4}$$

where $\xi(x,t) = \zeta(\xi_0(x), u(x,\cdot))(t)$ is the *spatial configuration.*

1.2. **Functional spaces.** Denote by $L_q(0,1)$, $q > 1$, the standard Lebesgue space and by $W_q^l(0,1)$ with natural l the standard Sobolev space. For a noninteger $l > 0$, denote by $W_q^l(0,1)$ the Sobolev space with the norm

$$\|\varphi\|_{W_q^{[l]}(0,1)} + \left(\int_0^1 dx \int_0^1 \frac{|\varphi^{([l])}(x) - \varphi^{([l])}(y)|^q}{|x-y|^{1+q(l-[l])}} dy\right)^{1/q},$$

where $[l]$ is the integer part of l. Introduce the anisotropic Sobolev spaces $W_q^{2,1}(Q_T)$ with the norm $\left(\int_0^T \|u(\cdot,t)\|_{W_q^2(0,1)}^q \, dt + \int_0^T \|u_t(\cdot,t)\|_{L_q(0,1)}^q \, dt\right)^{1/q}$ and the space $W_\infty^{0,1}(Q_T)$ of $L_\infty(0,1)$-valued functions continuously differentiable on $[0,T]$ with the norm $\|v\|_{L_\infty(Q_T)} + \|v_t\|_{L_\infty(Q_T)}$. Denote by $C^\gamma(\overline{Q}_T)$, $\gamma \in (0,1)$, the Hölder space.

For the vector-valued functions, we use the following notation. If, e.g., $u(x,t) \in \mathcal{U}$ and each component of u belongs to $W_q^{2,1}(Q_T)$, then we write $u \in W_q^{2,1}(Q_T;\mathcal{U})$.

Throughout, we fix q and γ such that $q \in (3,\infty)$ and $\gamma \in (0, 1-3/q)$. This implies that $u, u_x \in C^\gamma(\overline{Q}_T;\mathcal{U})$ for $u \in W_q^{2,1}(Q_T;\mathcal{U})$ (see Lemma 3.3 in [13, Chap. 2]).

To define the space of initial data, we use the fact that if $u \in W_q^{2,1}(Q_T;\mathcal{U})$, then the trace $u|_{t=0}$ is well defined and belongs to $W_q^{2-2/q}((0,1);\mathcal{U})$ (see Lemma 2.4 in [13, Chap. 2]). We denote the norm in the latter space by $\|\cdot\|_q$. Moreover, one can define the space $W_{q,N}^{2-2/q}((0,1);\mathcal{U})$ as the subspace of functions from $W_q^{2-2/q}((0,1);\mathcal{U})$ with the zero Neumann boundary conditions.

We assume that $\varphi \in W_{q,N}^{2-2/q}((0,1);\mathcal{U})$ and $\psi \in L_\infty((0,1);\mathcal{V})$ in (1.2).

Definition 1.1. A pair $(u,v) \in W_q^{2,1}(Q_T;\mathcal{U}) \times W_\infty^{0,1}(Q_T;\mathcal{V})$ is a *solution of problem* (1.1), (1.2) if $W(\xi_0, u)$ is measurable with respect to (x,t) and (1.1), (1.2) hold.

1.3. **Spatial transversality.** We will deal with the case where $\xi_0(x)$ has one discontinuity point. Generalization to finitely many discontinuity points is straightforward.

Condition 1.1. (1) *For some* $\overline{b} \in (0,1)$, *we have*

$$\xi_0(x) = 1 \ (x \le \overline{b}), \quad \xi_0(x) = -1 \ (x > \overline{b}). \tag{1.5}$$

(2) *For* $x \in [0,\overline{b}]$, *we have* $\varphi(x) \in M_\alpha \cup M_{\alpha\beta}$ *or, equivalently,* $\gamma_\beta(\varphi(x)) > 0$.
(3) *For* $x \in (\overline{b},1]$, *we have* $\varphi(x) \in M_\beta \cup M_{\alpha\beta}$ *or, equivalently,* $\gamma_\alpha(\varphi(x)) > 0$.
(4) *If* $\gamma_\alpha(\varphi(\overline{b})) = 0$, *then* $\left.\dfrac{d}{dx}\gamma_\alpha(\varphi(x))\right|_{x=\overline{b}} > 0$.

It follows from Condition 1.1 that the hysteresis in (1.4) at the initial moment equals $W_1(\varphi(x))$ for $x \le \overline{b}$ and $W_2(\varphi(x))$ for $x > \overline{b}$. Items 2 and 3 in Condition 1.1 are necessary for the hysteresis to be well-defined at the initial moment, while item 4 is an essential assumption. We refer to item 4 as the *spatial transversality* and say that $\varphi(x)$ *is transverse with respect to the spatial configuration* $\xi_0(x)$. This means that if $\varphi(\bar{b}) \in \Gamma_\alpha$, then the vector $\varphi'(\bar{b})$ is transverse to the manifold Γ_α at this point.

Consider time-dependent functions $u(x,t)$ such that $u, u_x \in C(\overline{Q}_T; \mathcal{U})$.

Definition 1.2. We say that a function u is *transverse on* $[0,T]$ (with respect to a spatial configuration $\xi(x,t)$) if, for every fixed $t \in [0,T]$, either $\xi(\cdot,t)$ has no discontinuity points for $x \in (0,1)$, or it has one discontinuity point and the function $u(\cdot,t)$ is transverse with respect to the spatial configuration $\xi(\cdot,t)$.

Definition 1.3. A function u *preserves spatial topology* (*of a spatial configuration* $\xi(x,t)$) *on* $[0,T]$ if there is $M > 0$ such that, for $t \in [0,T]$, there is a continuous function $b(t) \in (0,1)$ such that $\xi(x,t) = 1$ for $x \le b(t)$ and $\xi(x,t) = -1$ for $x > b(t)$.

The solution from Definition 1.1 is called *transverse* (*preserving spatial topology*) if the function $u(x,t)$ is transverse (preserving spatial topology).

Remark 1.1. The function $b(t)$ defining discontinuity of $\xi(x,t)$ plays a role of free boundary, which has much in common with free boundary in parabolic obstacle problems (see, e.g., [3, 15] and the references therein). However, in our case, the behavior of $b(t)$ is defined differently.

1.4. **Assumptions on the right-hand side.** First, we assume the following.

Condition 1.2. *The functions $f(u,v,w)$ and $g(u,v,w)$ are locally Lipschitz continuous in $\mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m$.*

Next, we formulate dissipativity conditions for f and g.

In the following condition, we denote by $\mathcal{U}_\mu$, $\mu \ge 0$, closed parallelepipeds in $\mathcal{U}$ with the edges parallel to respective coordinate axes such that $\varphi(x) \in \mathcal{U}_\mu$ for all $x \in [0,1]$.

Condition 1.3. *There is a parallelepiped $\mathcal{U}_0$ and, for each sufficiently small $\mu > 0$, there is a parallelepiped $\mathcal{U}_\mu$ and a locally Lipschitz continuous function $f_\mu(u,v)$ such that*

1. *$|f_\mu(u,v)|$ converges to 0 uniformly on compact sets in $\mathcal{U} \times \mathcal{V}$ as $\mu \to 0$,*
2. *At each point $u \in \partial\mathcal{U}_0 \cap D(W_{\pm 1})$, $v \in \mathcal{V}$, the vector $f(u,v,W_{\pm 1}(u)) + f_\mu(u,v)$ points strictly inside $\mathcal{U}_0$.*
3. *At each point $u \in \partial\mathcal{U}_\mu \cap D(W_{\pm 1})$, $v \in \mathcal{V}$, the vector $f(u,v,W_{\pm 1}(u_\mu)) + f_\mu(u,v)$ points strictly inside $\mathcal{U}_\mu$ for all $u_\mu \in \mathcal{U}_\mu$.*

To formulate the assumption on g, we fix $\mathcal{U}_0$ satisfying Condition 1.3 and set

$$\mathcal{W}_0 = \bigcup_{j=\pm 1} \{W_j(u) : u \in \mathcal{U}_0\}. \tag{1.6}$$

Condition 1.4. *For any $T_0 > 0$, there is a compact $\mathcal{V}_0 = \mathcal{V}_0(T_0, \mathcal{U}_0) \subset \mathcal{V}$ such that $\psi(x) \in \mathcal{V}_0$ (($x \in (0,1)$)) and the Cauchy problem*

$$v_t = g(u_0(x,t), v, w_0(x,t)), \quad v|_{t=0} = \psi(x) \tag{1.7}$$

has a solution $v \in W^{0,1}_\infty(Q_{T_0}; \mathbb{R}^l)$ satisfying $v(x,t) \in \mathcal{V}_0$ whenever

$$u_0 \in L_\infty(Q_{T_0}; \mathcal{U}), \quad w_0 \in L_\infty(Q_{T_0}; \mathcal{W}),$$
$$u_0(x,t) \in \mathcal{U}_0, \quad w_0(x,t) \in \mathcal{W}_0 \quad ((x,t) \in Q_{T_0}).$$

Remark 1.2. It follows from [14, Theorem 1, p. 111] that system (1.7) has a unique solution $v \in W^{0,1}_\infty(Q_{T_0};\mathbb{R}^l)$ for a sufficiently small $T_0 > 0$. Condition 1.4 additionally guarantees the absence of blow-up.

In particular, the uniform boundedness of v holds if $|g(u,v,w)| \le A(u,w)|v| + B(u,w)$, where $A(u,w)$ and $B(u,w)$ are bounded on compact sets (see Example 1.1). However, if $\mathcal{V} \ne \mathbb{R}^l$, one must additionally check that v never leaves $\mathcal{V}$. To fulfill Condition 1.4, one could alternatively assume the existence of invariant parallelepiped for g (similarly to Condition 1.3).

Example 1.1. The hysteresis operator and the right-hand side in the present paper apply to a model describing a growth of a colony of bacteria (Salmonella typhimurium) on a petri plate (see [6, 7]). Let $u_1(x,t)$ and $u_2(x,t)$ denote the concentrations of diffusing buffer (pH level) and histidine (nutrient), respectively, while $v(x,t)$ denote the density of nondiffusing bacteria. These three unknown functions satisfy the following equations:

$$\begin{cases} u_{1t} = D_1\Delta u_1 - a_1 W(\xi_0,u)v, \\ u_{2t} = D_2\Delta u_2 - a_2 W(\xi_0,u)v, \\ \ v_t = aW(\xi_0,u)v, \end{cases} \tag{1.8}$$

supplemented by initial and no-flux (Neumann) boundary conditions. In (1.8), $D_1, D_2, a, a_1, a_2 > 0$ are given constants and $W(\xi_0,u)$ is the hysteresis operator. In this example, we have $\mathcal{U} = \{u \in \mathbb{R}^2 : u_1,u_2 \ge 0\}$, $\mathcal{V} = [0,\infty)$, $\mathcal{W} = [0,\infty)$. The hysteresis thresholds Γ_α and Γ_β are the curves on the plane given by $\gamma_\alpha(u) := -u_1 + a_\alpha/u_2 + b_\alpha = 0$ and $\gamma_\beta(u) := u_1 - a_\beta/u_2 - b_\beta = 0$, respectively, where $a_\alpha, a_\beta, b_\alpha, b_\beta > 0$ are some constants (Fig. 1); the hysteresis "branches" are given by functions $W_1(u)$ (> 0) and $W_{-1}(u)$ $(\equiv 0)$.

2. Main results.

In what follows, we assume that Conditions 1.1–1.4 hold.

Theorem 2.1 (local existence)**.** *There is a number $T > 0$ such that*

(1) *There is at least one solution of problem* (1.1), (1.2) *in Q_T;*
(2) *Any solution in Q_T is transverse and preserves spatial topology.*

Theorem 2.2 (continuation)**.** *Let (u,v) be a transverse topology preserving solution of problem* (1.1), (1.2) *in Q_T for some $T > 0$. Then it can be continued to an interval $[0,T_{max})$, where $T_{max} > T$ has the following properties. 1. For any $t_0 < T_{max}$, the pair (u,v) is a transverse solution of problem* (1.1), (1.2) *in Q_{t_0}. 2. Either $T_{max} = \infty$, or $T_{max} < \infty$ and (u,v) is a solution in $Q_{T_{max}}$, but $u(\cdot,T_{max})$ is not transverse with respect to $\xi(\cdot,T_{max})$.*

Theorem 2.3 (continuous dependence on initial data)**.** *Assume the following.*

(1) *There is a number $T > 0$ such that problem* (1.1), (1.2) *with initial functions φ, ψ and initial configuration $\xi_0(x)$ defined by its discontinuity point $\overline{b}$ admits a unique transverse topology preserving solution (u,v) in Q_s for any $s \le T$.*
(2) *Let $\varphi_n \in W^{2-2/q}_{q,N}((0,1);\mathcal{U})$, $\psi_n \in L_\infty((0,1);\mathcal{V})$, $n = 1,2,\dots$, be a sequence of other initial functions such that $\|\varphi - \varphi_n\|_q \to 0$, $\|\psi - \psi_n\|_{L_q((0,1);\mathcal{V})} \to 0$ as $n \to \infty$.*

(3) *Let $\xi_{0n}(x)$, $n = 1, 2, \dots$, be a sequence of other initial configurations defined by their discontinuity points $\overline{b}_n$ similarly to* (1.5) *and $\overline{b}_n \to \overline{b}$ as $n \to \infty$.*

Then, for all sufficiently large n, problem (1.1), (1.2) *with initial data $(\varphi_n, \psi_n, \xi_{0n})$ has at least one transverse topology preserving solution (u_n, v_n). Each sequence of such solutions satisfies*

$$\|u_n - u\|_{W_q^{2,1}(Q_T;\mathcal{U})} \to 0, \quad \|b_{jn} - b_j\|_{C[0,T]} \to 0,$$
$$\sup_{t\in[0,T]} \left(\|v_n(\cdot,t) - v(\cdot,t)\|_{L_q((0,1);\mathcal{V})} + \|v_{nt}(\cdot,t) - v_t(\cdot,t)\|_{L_q((0,1);\mathcal{V})}\right) \to 0$$

as $n \to \infty$, where $b(t)$ and $b_n(t)$ are the respective discontinuity points of the configuration functions $\xi(x,t)$ and $\xi_n(x,t)$.

Remark 2.1. If one a priori knows that all u_n are transverse on some interval $[0,T] \subset [0, T_{max})$, then one can prove that u_n approximate u on $[0,T]$ even if u is not topology preserving on $[0,T]$.

Now we discuss the uniqueness of solutions. We strengthen the assumption about local Hölder continuity of $W_{\pm 1}$. Let $\mathcal{U}_0$ be the set from Condition 1.3.

Condition 2.1. *There are numbers $K > 0$ and $\sigma \in [0,1)$ such that*

$$|W_1(u) - W_1(\hat{u})| \le \frac{K}{(\gamma_\beta(u))^\sigma + (\gamma_\beta(\hat{u}))^\sigma}|u - \hat{u}| \qquad \forall u, \hat{u} \in M_\alpha \cup \overline{M}_{\alpha\beta},$$
$$|W_{-1}(u) - W_{-1}(\hat{u})| \le \frac{K}{(\gamma_\alpha(u))^\sigma + (\gamma_\alpha(\hat{u}))^\sigma}|u - \hat{u}| \qquad \forall u, \hat{u} \in M_\beta \cup \overline{M}_{\alpha\beta}.$$

We refer readers to [5] for the discussion about functions satisfying this condition.

Theorem 2.4 (uniqueness)**.** *Assume additionally that Condition* 2.1 *holds. Let (u, v) and $(\hat{u}, \hat{v})$ be two transverse solutions of problem* (1.1), (1.2) *in Q_T for some $T > 0$. Then $(u, v) = (\hat{u}, \hat{v})$.*

3. Local existence, continuation and continuous dependence of solutions on initial data

In this section, we prove Theorems 2.1–2.3. Throughout the section, we fix $\mathcal{U}_0$ satisfying Condition 1.3 and $\mathcal{W}_0$ given by (1.6). Next, we fix some $T_0 \in (0, 1]$ and then $\mathcal{V}_0$ satisfying Condition 1.4.

3.1. **Preliminaries.** The following result is straightforward.

Lemma 3.1. (1) *Let $\lambda \in [0,1)$, $a \in C^\lambda[0,T]$, and $b(t) = \max_{s\in[0,t]} a(s)$. Then $b \in C^\lambda[0,T]$ and $\|b\|_{C^\lambda[0,T]} \le \|a\|_{C^\lambda[0,T]}$.*

(2) *If $a_j \in C[0,T]$ and $b_j(t) = \max_{s\in[0,t]} a_j(s)$, $j = 1, 2$, then $\|b_1 - b_2\|_{C[0,T]} \le \|a_1 - a_2\|_{C[0,T]}$.*

For some $T \le T_0$, $u_1 \in L_\infty(Q_T;\mathcal{U})$, and $b_1 \in C[0,T]$ such that $u_1(x,t) \in \mathcal{U}_0$ ($(x,t) \in Q_T$) and $b_1(t) \in [0,1)$ ($t \in [0,T]$), we define the function $w_1(x,t)$ by

$$w_1(x,t) = \begin{cases} W_1(u_1(x,t)), & 0 \le x \le b_1(t), \\ W_{-1}(u_1(x,t)), & b_1(t) < x \le 1; \end{cases} \tag{3.1}$$

here we assume $W_{\pm 1}(u_1)$ to be extended to $\mathcal{U}_0$ without loss of regularity.

Lemma 3.2. *Let u_1, b_1 be functions with the above properties, and let $\hat{u}_1, \hat{b}_1$ be functions with the same properties. Let w_1 be defined by (3.1) and $\hat{w}_1$ by (3.1) with $\hat{u}_1$ and $\hat{b}_1$ instead of u_1 and b_1, respectively. Then, for any $p \in [1, \infty)$,*

$$\|w_1 - \hat{w}_1\|_{L_p(Q_T;\mathcal{W})} \le c_0 \left(T^{1/p} \|u_1 - \hat{u}_1\|^{\sigma_0}_{L_\infty(Q_T;\mathcal{U})} + \|b_1 - \hat{b}_1\|^{1/p}_{L_1(0,T)} \right),$$

where σ_0 is a Hölder exponent for $W_{\pm 1}(u_1)$ and $c_0 > 0$ depends on $\mathcal{U}_0$ and p, but does not depend on u_1, b_1, T.

Proof. We fix some $t \in [0, T]$ and assume that $b_1(t) \le \hat{b}_1(t)$ for this t. Then, using (3.1) and omitting the arguments of the integrands, we have

$$\int_0^1 |w_1 - \hat{w}_1|^p \, dx = \int_0^{b_1(t)} |W_1(u_1) - W_1(\hat{u}_1)|^p \, dx + \int_{\hat{b}_1(t)}^1 |W_{-1}(u_1) - W_{-1}(\hat{u}_1)|^p \, dx$$
$$+ \int_{b_1(t)}^{\hat{b}_1(t)} |W_{-1}(u_1) - W_1(\hat{u}_1)|^p \, dx.$$

Using the Hölder continuity and the boundedness of $W_{\pm 1}(u_1)$ for $u_1 \in \mathcal{U}_0$ and integrating with respect to t from 0 to T, we complete the proof. □

Now we introduce sets that "measure" the spatial transversality. Denote by E_m, $m \in \mathbb{N}$, the set of triples (φ, ψ, ξ_0) such that $\varphi \in W^{2-2/q}_{q,N}((0,1);\mathcal{U})$, $\psi \in L_\infty((0,1);\mathcal{V})$, $\xi_0(x)$ is of the form (1.5), and the following hold:

(1) $\overline{b} \in [1/m, 1 - 1/m]$,
(2) $\gamma_\beta(\varphi(x)) \ge 1/m^2$ for $x \in [0, \overline{b}]$,
(3) $\gamma_\alpha(\varphi(x)) \ge 1/m^2$ for $x \in [\overline{b} + 1/m, 1]$,
(4) if $x \in [\overline{b}, \overline{b} + 1/m]$ and $\gamma_\alpha(\varphi(x)) \in [0, 1/m^2]$, then $\dfrac{d}{dx}\gamma_\alpha(\varphi(x)) \ge 1/m$,
(5) $\|\varphi\|_q \le m$ and $\|\psi\|_{L_\infty((0,1);\mathcal{V})} \le m$.

It is easy to check that $E_m \subset E_{m+1}$. Moreover, one can show (Lemma 2.25 in [4]) that the union of all sets E_m coincides with the set of all data satisfying Condition 1.1. From now on, we fix $m \in \mathbb{N}$ such that $(\varphi, \psi, \xi_0) \in E_m$.

The next lemma follows from the implicit function theorem and Lemma 3.1.

Lemma 3.3. *Let $\lambda \in (0, 1)$, $u_1, u_{1x} \in C^\lambda(\overline{Q}_{T_0}; \mathcal{U})$,*

$$\|u_1\|_{C^\lambda(\overline{Q}_{T_0};\mathcal{U})} + \|u_{1x}\|_{C^\lambda(\overline{Q}_{T_0};\mathcal{U})} \le c$$

for some $c > 0$, $u_1|_{t=0} = \varphi(x)$, and $(\varphi, \psi, \xi_0) \in E_m$. Then there is $T_1 = T_1(m, \lambda, c) \le T_0$ and a natural number $N_1 = N_1(m, \lambda, c) \ge m$ which do not depend on u, φ, ξ_0 such that the following is true for any $t \in [0, T_1]$.

(1) *The equation $\gamma_\alpha(u_1(x, t)) = 0$ for $x \in [\overline{b}, 1]$ has no more than one root. If this root exists, we denote it by $a_1(t)$; otherwise, we set $a_1(t) = \overline{b}$. One has $a_1(t) \in [\overline{b}, \overline{b} + 1/N_1]$, $a_1 \in C^\lambda[0, T_1]$.*
(2) *The hysteresis $\mathcal{H}(\xi_0, u_1)$ and its configuration function $\xi_1(x, t)$ have exactly one discontinuity point $b_1(t)$; moreover, $b_1(t) = \max_{s \in [0,t]} a_1(s)$, $b_1 \in C^\lambda[0, T_1]$.*

3.2. **Auxiliary problem.** Consider functions $u_1 \in L_\infty(Q_T;\mathcal{U})$ and $w_1 \in L_\infty(Q_T;\mathcal{W})$ such that

$$u_1(x,t) \in \mathcal{U}_0, \quad w_1(x,t) \in \mathcal{W}_0 \quad ((x,t) \in Q_T)$$

for some $T > 0$. Define the functions

$$f_1(u,v,x,t) = f(u,v,w_1(x,t)), \quad g_1(v,x,t) = g(u_1(x,t),v,w_1(x,t)). \tag{3.2}$$

Consider the auxiliary problem

$$\begin{cases} u_t = Du_{xx} + f_1(u,v,x,t), \\ v_t = g_1(v,x,t) \\ u|_{t=0} = \varphi(x), \quad v|_{t=0} = \psi(x), \quad u_x|_{x=0} = u_x|_{x=1} = 0. \end{cases} \tag{3.3}$$

Set $f_U = \sup f(u,v,w)$ and $g_U = \sup g(u,v,w)$, where $(u,v,w) \in \mathcal{U}_0 \times \mathcal{V}_0 \times \mathcal{W}_0$.

The next result follows from the standard estimates for solutions of linear parabolic equations [13], from Conditions 1.2–1.4 combined with the principle of invariant rectangles [16], and from Lemma 3.3.

Lemma 3.4. (1) *For any $T \le T_0$, problem* (3.3) *has a unique solution $(u,v) \in W_q^{2,1}(Q_T;\mathcal{U}) \times W_\infty^{0,1}(Q_T;\mathcal{V})$ and*

$$u(x,t) \in \mathcal{U}_0, \quad v(x,t) \in \mathcal{V}_0 \quad ((x,t) \in Q_T),$$

$$\|u\|_{W_q^{2,1}(Q_T;\mathcal{U})} + \max_{t\in[0,T]} \|u(\cdot,t)\|_q \le c_1(\|\varphi\|_q + f_U),$$

$$\|v\|_{W_\infty^{0,1}(Q_T;\mathcal{V})} \le \|\psi\|_{L_\infty((0,1);\mathcal{V})} + 2g_U,$$

$$\|u\|_{C^\gamma(\overline{Q}_T;\mathcal{U})} + \|u_x\|_{C^\gamma(\overline{Q}_T;\mathcal{U})} \le c_2(\|\varphi\|_q + f_U), \tag{3.4}$$

where $c_1, c_2 > 0$ depend only on T_0.

(2) *If u_n, v_n, $n = 1,2,\dots$, are solutions of problem* (3.2), (3.3) *with u_1, w_1 replaced by u_{1n}, w_{1n}* (*with the same properties*) *and*

$$\|u_{1n} - u_1\|_{L_\infty(Q_T;\mathcal{U})} + \|w_{1n} - w_1\|_{L_q(Q_T;\mathcal{W})} \to 0, \quad n \to \infty,$$

then

$$\|u_n - u\|_{W_q^{2,1}(Q_T;\mathcal{U})} + \|v_n - v\|_{W_q^{0,1}(Q_T;\mathcal{V})} \to 0, \quad n \to \infty.$$

(3) *There is $T_2 = T_2(m) \le T_0$ and a natural number $N_2 = N_2(m) \ge m$ such that, for any $t \in [0,T_2]$, conclusions (1) and (2) from Lemma* 3.3 *hold for $u(x,t)$, for the corresponding "root" function $a(t)$, for the configuration function $\xi(x,t)$ of the hysteresis $\mathcal{H}(\xi_0,u)$, for its discontinuity point $b(t)$, and for T_2, N_2 instead of T_1, N_1. Furthermore, $(u(\cdot,t), v(\cdot,t), \xi(\cdot,t)) \in E_{N_2}$.*

3.3. **Local existence: proof of Theorem** 2.1. 1. Let us prove the first assertion.

1.1. Fix λ in Lemma 3.3 such that $\lambda \in (0,\gamma)$. Fix c_2 from Lemma 3.4. Set $c = c_{\lambda,\gamma} c_2(m + f_U)$, where $c_{\lambda,\gamma} > 0$ is the embedding constant such that $\|u\|_{C^\lambda(\overline{Q}_T;\mathcal{U})} \le c_{\lambda,\gamma}\|u\|_{C^\gamma(\overline{Q}_T;\mathcal{U})}$. Set $T = \min(T_1,T_2)$, where T_1, T_2 are defined in Lemmas 3.3, 3.4.

Let $R^\lambda(\overline{Q}_T)$ be the set of functions $u(x,t)$ such that $u|_{t=0} = \varphi(x)$,

$$u, u_x \in C^\lambda(\overline{Q}_T;\mathcal{U}), \quad u(x,t) \in \mathcal{U}_0 \quad ((x,t) \in Q_T),$$

$$\|u\|_{C^\lambda(\overline{Q}_T;\mathcal{U})} + \|u_x\|_{C^\lambda(\overline{Q}_T;\mathcal{U})} \le c. \tag{3.5}$$

The set $R^\lambda(\overline{Q}_T)$ is a closed convex subset of the Banach space endowed with the norm given by the left-hand side in (3.5). Similarly, we define $R^\gamma(\overline{Q}_T)$.

1.2. We construct a map $\mathcal{R} : R^\lambda(\overline{Q}_T) \to R^\gamma(\overline{Q}_T)$. Take any $u_1 \in R^\lambda(\overline{Q}_T)$ and define $a_1(t)$ and $b_1(t)$ according to Lemma 3.3. Then define $w_1(x,t)$ by (3.1) and, using this w_1, define f_1, g_1 by (3.2). Finally apply Lemma 3.4 and obtain a solution (u,v) of auxiliary problem (3.3). We now define $\mathcal{R} : u_1 \mapsto u$.

The operator $\mathcal{R}$ is continuous. Indeed, it is not difficult to check that the mapping $u_1 \mapsto a_1$ is continuous from $R^\lambda(\overline{Q}_T)$ to $C[0,T]$. Thus, the continuity of $\mathcal{R}$ follows by consecutively applying Lemmas 3.1 (part 2), 3.2, 3.4 (part 2), and the continuity of the embedding $W_q^{2,1}(Q_T;\mathcal{V}) \subset R^\gamma(\overline{Q}_T)$.

Furthermore, due to (3.4) and the choice of c, the operator $\mathcal{R}$ maps $R^\lambda(\overline{Q}_T)$ into itself. As an operator acting from $R^\lambda(\overline{Q}_T)$ into itself, it is compact due to (3.4) and the compactness of the embedding $R^\gamma(\overline{Q}_T) \subset R^\lambda(\overline{Q}_T)$. Therefore, applying the Schauder fixed-point, we conclude the proof of the first assertion of the theorem. Note that $\mathcal{R}$ is not Lipschitz continuous (mind the exponent $1/p$ in (3.2)). Hence, the contraction principle does not apply. We prove uniqueness separately in Sec. 4.

2. The second assertion follows by applying the principle of invariant rectangles (see [16]) and Lemma 3.4.

3.4. **Continuation: proof of Theorem** 2.2**.** Theorem 2.2 follows from part 3 of Lemma 3.4 and from the following fact (see Lemma 2.25 in [4]). Assume (1) $(\varphi_m, \psi_m, \xi_m) \in E_m \setminus E_{m-1}$, $m = 2,3,\dots$; (2) $\|\varphi_m - \varphi\|_q \to 0$ and $\|\psi_m - \psi\|_{L_\infty((0,1);\mathcal{V})} \to 0$ as $m \to \infty$ for some $\varphi \in W_q^{2-2/q}((0,1);\mathcal{U})$ and $\psi \in L_\infty((0,1);\mathcal{V})$; (3) $\overline{b}_m - \overline{b} \to 0$ as $m \to \infty$ for some $\overline{b} \in [0,1]$. Then $\overline{b} \in \{0,1\}$ or $\varphi(x)$ is not transverse with respect to $\xi_0(x)$, where $\xi_0(x)$ is given by (1.5).

3.5. **Continuous dependence on initial data: proof of Theorem** 2.3**.** 1. It suffices to prove the theorem for a sufficiently small time interval. Since $(\varphi, \psi, \xi_0) \in E_m$, it is easy to show that there is $n_1 = n_1(m) > 0$ such that $(\varphi, \psi, \xi_0), (\varphi_n, \psi_n, \xi_{0n}) \in E_{m+1}$ for all $n \ge n_1(m)$. Hence, by Theorem 2.1, there is $T \in (0,1]$ for which problem (1.1), (1.2) has transverse topology preserving solutions (u,v) and (u_n, v_n) with the corresponding initial data. Moreover, any solution of problem (1.1), (1.2) in Q_T is transverse and preserves topology.

We introduce the functions $a(t)$ and $a_n(t)$ corresponding to u and u_n as described in part 3 of Lemma 3.4. Then the discontinuity points of the corresponding configuration functions $\xi(x,t)$, $\xi_n(x,t)$ are given by $b(t) = \max\limits_{s\in[0,t]} a(s)$ and $b_n(t) = \max\limits_{s\in[0,t]} a_n(s)$.

2. Assume that there is $\varepsilon > 0$ such that

$$\|u_n - u\|_{W_q^{2,1}(Q_T;\mathcal{U})} \ge \varepsilon, \quad n = 1,2,\dots, \tag{3.6}$$

for some subsequence of u_n, which we denote u_n again. Theorem 2.1 implies that u_n and a_n are uniformly bounded in $W_q^{2,1}(Q_T;\mathcal{U})$ and $C^\gamma[0,T]$, respectively. Hence, we can choose subsequences of u_n and a_n (which we denote u_n and a_n again) such that

$$\|u_n - \hat{u}\|_{C^\gamma(\overline{Q}_T;\mathcal{U})} \to 0, \quad \|(u_n)_x - \hat{u}_x\|_{C^\gamma(\overline{Q}_T;\mathcal{U})} \to 0, \quad n \to \infty, \tag{3.7}$$

$$\|a_n - \hat{a}\|_{C[0,T]} \to 0, \quad n \to \infty \tag{3.8}$$

for some function $\hat{u} \in C^\gamma(\overline{Q}_T;\mathcal{U})$ with $\hat{u}_x \in C^\gamma(\overline{Q}_T;\mathcal{U})$ and some $\hat{a} \in C[0,T]$.

Set $\hat b(t) = \max\limits_{s\in[0,t]} \hat a(s)$. Due to (3.7), (3.8), and Lemma 3.1, we have

$$\|b_n - \hat b\|_{C[0,T]} \to 0, \quad n\to\infty, \tag{3.9}$$

$$W(\xi_0(x), \hat u(x,\cdot))(t) = \begin{cases} W_1(\hat u(x,t)), & 0\le x\le \hat b(t),\\ W_2(\hat u(x,t)), & \hat b(t) < x \le 1.\end{cases} \tag{3.10}$$

3. Now we show that

$$\sup_{t\in[0,T]} \left(\|v_n(\cdot,t) - \hat v(\cdot,t)\|_{L_q((0,1);\mathcal V)} + \|v_{nt}(\cdot,t) - \hat v_t(\cdot,t)\|_{L_q((0,1);\mathcal V)}\right) \to 0, \quad n\to\infty, \tag{3.11}$$

for some $\hat v$. Take an arbitrary $\delta > 0$. It follows from the assumptions of the theorem, from (3.7), (3.9), (3.10), and from Lemma 3.2 that

$$\begin{gathered} \|\psi_n - \psi_k\|_{L_q((0,1);\mathcal V)} \le \delta, \quad \|u_n(\cdot,t) - u_k(\cdot,t)\|_{L_q((0,1);\mathcal U)} \le \delta,\\ \|W(\xi_n,u_n)(t) - W(\xi_k,u_k)(t)\|_{L_q((0,1);\mathcal W)} \le \delta, \end{gathered} \tag{3.12}$$

provided n,k are large enough. Estimates (3.12), the second equation in (1.1), and the local Lipschitz continuity of g yield

$$\|v_n(\cdot,t) - v_k(\cdot,t)\|_{L_q((0,1);\mathcal V)} \le (1+2L)\delta + L\int_0^t \|v_n(\cdot,s) - v_k(\cdot,s)\|_{L_q((0,1);\mathcal V)}\,ds,$$

where $L>0$ does not depend on n,k. Hence, by Gronwall's inequality,

$$\|v_n(\cdot,t) - v_k(\cdot,t)\|_{L_q((0,1);\mathcal V)} \le k_1\delta, \tag{3.13}$$

where $k_1 > 0$ does not depend δ, n, k, and $t\in[0,T]$. A similar inequality for the time derivative of v_n follows from (3.12), (3.13), and from the second equation in (1.1). Since $\delta>0$ is arbitrary, (3.11) does hold.

Now consider (1.1), (1.2) with the subsequence $\varphi_n, \psi_n, \xi_{0n}, u_n, v_n$ and pass to the limit as $n\to\infty$. Due to the uniqueness assumption, $(u,v) = (\hat u, \hat v)$. Therefore, (3.6) is not true and we have the convergence for the whole sequence (u_n, v_n).

4. Uniqueness of solutions

In this section, we prove Theorem 2.4. For the clarity of exposition, we restrict ourselves to the case where initial data satisfy the equality $\gamma_\alpha(\varphi(\overline b)) = 0$ in addition to Condition 1.1. (The case $\gamma_\alpha(\varphi(\overline b)) \ne 0$ can be treated easily because then the hysteresis $W(\xi_0, u)$ remains constant on some time interval.)

Set

$$\overline\varphi := \frac12 \frac{d}{dx}\gamma_\alpha(\varphi(x))|_{x=\overline b}\ (>0).$$

We fix T_1 such that the conclusions of Lemma 3.3 are true for the u, $\hat u$ on $(0,T_1)$. Let $a(t)$, $b(t)$, $\hat a(t)$, $\hat b(t)$ be the functions defined in Lemma 3.3 for u and $\hat u$, respectively. We fix $T\in(0,T_1)$ and $\delta>0$ such that the following hold for $t\in[0,T]$:

$$\frac{d}{dx}\gamma_\alpha(u(x,t)) \ge \overline\varphi, \quad x\in[\overline b - \delta, \overline b + \delta], \tag{4.1}$$

$$\gamma_\beta(u(x,t)) < 0, \quad x\in[0, b(t)], \tag{4.2}$$

and the analogous inequalities hold for $\hat u$.

Due to (4.1) and (4.2), we have

$$W(\xi_0(x), u(x,\cdot)(t) = \begin{cases} W_1(u(x,t)), & 0 \le x \le b(t), \\ W_{-1}(u(x,t)), & b(t) < x \le 1, \end{cases}$$
$$W(\xi_0(x), \hat{u}(x,\cdot)(t) = \begin{cases} W_1(\hat{u}(x,t)), & 0 \le x \le \hat{b}(t), \\ W_{-1}(\hat{u}(x,t)), & \hat{b}(t) < x \le 1. \end{cases} \tag{4.3}$$

Let us now prove Theorem 2.4.

1. Denote $w = u - \hat{u}$, $z = v - \hat{v}$. The functions w, z satisfy the equations

$$\begin{cases} w_t = w_{xx} + h_w(x,t), \\ z_t = h_z(x,t), \end{cases} \tag{4.4}$$

and the zero boundary and initial conditions, where

$$h_w(x,t) = f(u, v, W(u)) - f(\hat{u}, \hat{v}, W(\hat{u})),$$
$$h_z(x,t) = g(u, v, W(u)) - g(\hat{u}, \hat{v}, W(\hat{u})).$$

Obviously, $h_{w,z} \in L_\infty(Q_T)$. The function w can be represented via the Green function $G(x,y,t,s)$ of the heat equation with the Neumann boundary conditions:

$$w(x,t) = \int_0^t \int_0^1 G(x,y,t,s) h_w(y,s)\, dyds.$$

Therefore, using the estimate $|G(x,y,t,s)| \le \dfrac{k_1}{\sqrt{t-s}}$, $0 < s < t$, with $k_1 > 0$ not depending on $(x,t) \in Q_T$ (see, e.g., [9]), we obtain

$$|w(x,t)| \le k_1 \int_0^t \frac{ds}{\sqrt{t-s}} \int_0^1 |h_w(y,s)|\, dy. \tag{4.5}$$

Set $Z(t) = \int_0^1 |h_z(y,t)| dy$. Due to the second equation in (4.4),

$$Z(t) \le \int_0^t \int_0^1 |h_z(y,s)| dyds. \tag{4.6}$$

2. Now we prove that, for some $k_2 > 0$,

$$\int_0^1 |h_{w,z}(y,s)| dy \le k_2 (\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)}), \quad s \in (0,T). \tag{4.7}$$

Let us prove this inequality for the function h_w, assuming that $b(s) < \hat{b}(s)$. (The cases of h_z and $b(s) \ge \hat{b}(s)$ are treated analogously.) Since f is locally Lipschitz,

$$\int_0^1 |h_w(y,s)| dy \le k_3 \int_0^1 (|w(y,s)| + |z(y,s)| + |W(u(y,s)) - W(\hat{u}(y,s))|) \le \tag{4.8}$$
$$\le k_3 \left(\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)} + \int_0^1 |W(u(y,s)) - W(\hat{u}(y,s))| dy \right),$$

where $k_3 > 0$ and the constants $k_4, k_5, \ldots > 0$ below do not depend on $s \in [0, T]$.

Denote $\theta(y,s) = W(u(y,s)) - W(\hat{u}(y,s))$. Due to (4.3), we have

$$\theta(y,s) = \begin{cases} W_1(u) - W_1(\hat{u}), & 0 < y < b(s), \\ W_{-1}(u) - W_1(\hat{u}), & b(s) < y < \hat{b}(s), \\ W_{-1}(u) - W_{-1}(\hat{u}), & \hat{b}(s) < y < 1. \end{cases}$$

2.1. Inequality (4.2) implies that $\gamma_\beta(u(y,s)) < 0, \gamma_\beta(\hat{u}(y,s)) < 0$ on the closed set $\{(y,s) : y \in [0, b(s)],\ s \in [0,T]\}$. Hence, the values $\gamma_\beta(u(y,s))$ and $\gamma_\beta(\hat{u}(y,s))$ are separated from 0. Therefore, using Condition 2.1, we obtain

$$\int\limits_0^{b(s)} |\theta(y,s)|\,dy \le k_4 \int\limits_0^{b(s)} |u(y,s) - \hat{u}(y,s)|\,dy \le k_4 \|w\|_{C(\overline{Q}_T)}. \tag{4.9}$$

2.2. Boundedness of $W_1(\hat{u})$ and $W_{-1}(u)$ for $(y,s) \in \overline{Q}_T$ and Lemma 3.1 imply

$$\int\limits_{b(s)}^{\hat{b}(s)} |\theta(y,s)|\,dy \le k_5 \int\limits_{b(s)}^{\hat{b}(s)} dy \le k_5 \|b - \hat{b}\|_{C[0,T]} \le k_5 \|a - \hat{a}\|_{C[0,T]}. \tag{4.10}$$

Using (4.1), we obtain for any $t \in [0,T]$ the inequalities

$$\begin{aligned} |a(t) - \hat{a}(t)| &\le \frac{1}{\overline{\varphi}} |\gamma_\alpha(u(a(t),t)) - \gamma_\alpha(\hat{u}(a(t),t))| \le \\ &\le \frac{L_\alpha}{\overline{\varphi}} |u(a(t),t) - \hat{u}(a(t),t)| \le \frac{L_\alpha}{\overline{\varphi}} \|u - \hat{u}\|_{C(\overline{Q}_T)}, \end{aligned} \tag{4.11}$$

where $L_\alpha > 0$ is a respective Lipschitz constant for $\gamma_\alpha(u)$ and hence does not depend on $T \in (0, T_1)$. Inequalities (4.10) and (4.11) yield

$$\int\limits_{b(s)}^{\hat{b}(s)} |\theta(y,s)|\,dy \le k_6 \|w\|_{C(\overline{Q}_T)}. \tag{4.12}$$

2.3. Let $y \in [\hat{b}(s), \overline{b} + \delta]$. Inequality (4.1) and the mean-value theorem imply

$$\gamma_\alpha(\hat{u}(y,s)) = \gamma_\alpha(\hat{u}(y,s)) - \gamma_\alpha(\hat{u}(\hat{a}(s),s)) \ge (y - \hat{a}(s))\overline{\varphi} \ge (y - \hat{b}(s))\overline{\varphi},$$
$$|\gamma_\alpha(u(y,s))| \ge (y - b(s))\overline{\varphi}.$$

Taking into account these two inequalities and using Condition 2.1, we obtain

$$\int\limits_{\hat{b}(s)}^{\overline{b}+\delta} |\theta(y,s)|\,dy \le k_7 \int\limits_{\hat{b}(s)}^{\overline{b}+\delta} \frac{|u(y,s) - \hat{u}(y,s)|}{(y - \hat{b}(s))^\sigma}\,dy \le k_8 \|w\|_{C(\overline{Q}_T)}. \tag{4.13}$$

2.4. Similarly to item 2.1, we conclude that

$$\int\limits_{\overline{b}+\delta}^{1} |\theta(y,s)|\,dy \le k_9 \|w\|_{C(\overline{Q}_T)}. \tag{4.14}$$

Finally, (4.8)–(4.14) imply (4.7).

3. Combining estimates (4.5)–(4.7), we obtain

$$
\begin{aligned}
|w(x,t)| &\le k_{10}(\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)}) \int\limits_0^t \frac{ds}{\sqrt{t-s}} \\
&= 2k_{10}T^{1/2}(\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)}), \\
Z(t) &\le k_2 T(\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)}).
\end{aligned}
$$

Taking the supremum with respect to $t \in (0,T)$, we see that

$$
\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)} \le (2k_{10}T^{1/2} + k_2T)(\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)}).
$$

Thus, $w = 0$ and $z = 0$, provided that $T > 0$ is small enough. □

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