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in Yang-Mills gauge theories

by

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# Applications of hypercomplex automorphic forms in Yang-Mills gauge theories

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## Abstract

In this paper we show how hypercomplex function theoretical objects can be used to construct explicitly self-dual  $SU(2)$ -Yang-Mills instanton solutions on certain classes of conformally flat 4-manifolds. We use a hypercomplex argument principle to establish a natural link between the fundamental solutions of  $D\Delta f = 0$  and the second Chern class of the  $SU(2)$  principal bundles over these manifolds. The considered base manifolds of the bundles are not simply-connected, in general. Actually, this paper summarizes an extension of the corresponding results of Gürsey and Tze on a hyper-complex analytical description of  $SU(2)$  instantons. Furthermore, it provides an application of the recently introduced new classes of hypercomplex-analytic automorphic forms.

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# 1 Basic concepts of quaternionic analyticity in Euclidean Space

In this section we summarize some basics on quaternions and their function theory. For more details we refer the reader, for instance, to [3, 7]. Basic applications of this framework to  $SU(2)$ -Yang Mills gauge theories may be found in the seminal works by Gürsey and Tze [12, 13].

Let  $\mathbb{H} \cong \mathbb{R} \oplus \mathbb{R}^3$  be the set of real quaternions. Each quaternion  $a \in \mathbb{H}$  is written as  $a = \sum_{\mu=0}^3 a_\mu e_\mu = a_0 e_0 + \sum_{i=1}^3 a_i e_i \equiv Sc(a) + Vec(a)$ ,  $a_\mu \in \mathbb{R}$ . Here,  $Sc(a) = a_0$  is called the scalar or real part of the quaternion  $a$ ;  $Vec(a) = \sum_{i=1}^3 a_i e_i$  is called the vector or imaginary part of  $a$ .

The quaternionic multiplication is defined by  $e_i^2 = -1$  for  $i = 1, 2, 3$ ,  $e_0^2 = 1$  and  $e_1 e_2 = e_3 = -e_2 e_1$ ,  $e_2 e_3 = e_1 = -e_3 e_2$ ,  $e_3 e_1 = e_2 = -e_1 e_3$ , as well as  $e_0 e_i = e_i e_0$  for  $i = 1, 2, 3$ .

The quaternionic conjugate is given by  $\bar{a} := Sc(a) - Vec(a)$ . Accordingly, the norm is defined as  $N(a) \equiv |a|^2 := a\bar{a} = \bar{a}a = \sum_{\mu=0}^3 a_\mu^2$ . It coincides with the Euclidean norm of  $\mathbb{R}^4$ , such that as Euclidean spaces  $\mathbb{H} \simeq \mathbb{R}^{4,0}$ .

Next, we call in mind two different concepts of quaternionic analyticity. For more details on quaternionic analysis we refer, for instance, to [3, 7, 10, 11]. For new connections between the different concepts of analyticity in hypercomplex spaces we also refer to the recent works [6, 5].

Let  $U \subset \mathbb{H} \simeq \mathbb{R}^{4,0}$  be an open subset. A quaternion-valued function  $f : U \rightarrow \mathbb{H}$ , which is real differentiable in each real component, is called *left monogenic* (resp. *right monogenic*) on  $U$  if  $Df = 0$  (resp.  $fD = 0$ ). Here, the first order differential operator

$$D := \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}$$

is the quaternionic analogue of the Cauchy-Riemann operator  $\mathcal{D} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$  of complex analysis.

The operator  $D$  often abbreviated as  $\partial_0 + e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3$ . In what follows we focus on the class of left monogenic functions and will call them simply monogenic for short.

An important property of the quaternionic Cauchy-Riemann operator is, that

is factorizes the Euclidean Laplacian

$$\Delta = \sum_{i=0}^3 \frac{\partial^2}{\partial x_i^2}$$

$$\text{viz } \Delta = D\bar{D} = \left( \partial_0 + e_1\partial_1 + e_2\partial_2 + e_3\partial_3 \right) \overline{\left( \partial_0 + e_1\partial_1 + e_2\partial_2 + e_3\partial_3 \right)}.$$

Each quaternion  $a \in \mathbb{H}$  is known to have a polar decomposition:  $a = a_0 + \omega r$ , with  $r > 0$  and  $\omega$  being a unit vector in  $\mathbb{R}^3 \subset \mathbb{H}$ . Since  $\omega^2 = -1$ , the vector  $\omega$  defines a complex structure on  $\mathbb{R}^4$ .

Let  $F : \mathbb{C} \rightarrow \mathbb{C}$ ,  $F(x + iy) = u(x, y) + iv(x, y)$  be a complex analytic function. It follows that

$$G : \mathbb{H} \rightarrow \mathbb{H}, \quad G(a_0 + \omega r) = u(a_0, r) + \omega v(a_0, r)$$

satisfies the linear third order equation  $D\Delta G = 0$ , see for instance [8]. The function  $G$  is called (left) *Fueter-holomorphic*, cf. for instance [8, 26, 7, 20, 18]. Note that the function  $f := \Delta G$  is left monogenic. This was proved already in the papers cited above.

**Remark:** Since  $\Delta$  is a scalar operator, one has  $\Delta D = D\Delta$  and therefore  $\Delta Df = 0$  if and only if  $D\Delta f = 0$ , whenever  $f$  is at least three times continuously differentiable.

All integer powers of  $z = x_\mu e_\mu = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 \in \mathbb{H}$  have the property that they are Fueter-holomorphic, cf. for instance [20]. However, none of these are left or right monogenic. As suggested in [3] and elsewhere, in the monogenic context the negative power function  $\frac{1}{z}$  is replaced by the fundamental solution of  $D$  having the form

$$q_0(z) = \frac{\bar{z}}{|z|^4} = \frac{x_0 - x_1 e_1 + x_2 e_2 + x_3 e_3}{(x_0^2 + x_1^2 + x_2^2 + x_3^2)^2} = -\frac{1}{4} \Delta z^{-1}. \quad (1)$$

The other negative powers  $z^{-m}$  with  $m \in \mathbb{N}^{\geq 2}$  are replaced by the partial derivatives of  $q_0(z)$ . The partial derivatives of  $q_0(z)$  will be denoted by

$$q_{\mathbf{m}}(z) := q_{m_1, m_2, m_3}(z) := \frac{\partial^{m_1+m_2+m_3}}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} q_0(z),$$

where  $\mathbf{m} := (m_1, m_2, m_3)$  is a multi index from  $\mathbb{N}_0^3$ .

Consequently, all quaternionic Möbius transformations are Fueter-holomorphic, but none of them is left monogenic. For convenience we recall that Möbius transformations can be expressed in the quaternionic language in the form

$$M(z) = (az + b)(cz + d)^{-1},$$

where  $a, b, c, d$  are arbitrary quaternions satisfying the determinant condition (see for instance [1, 27]):

$$|b - ac^{-1}d||c| \neq 0, \text{ if } c \neq 0 \text{ and } |ad| \neq 0 \text{ if } c = 0.$$

Due to Liouville's theorem (see for example [2]), these are exactly the conformal maps in  $\mathbb{H}$ .

Both operators  $D$  and  $D\Delta$  are conformally invariant up to an automorphy factor. According to [9, 15, 24, 25], suppose that  $D_y f(y) = 0$  resp.  $D_y \Delta_y f(y) = 0$  and  $y = (az + b)(cz + d)^{-1}$  is a Möbius transformation. Then

$$q_0(cz + d)f((az + b)(cz + d)^{-1}),$$

resp.

$$(cz + d)^{-1}f((az + b)(cz + d)^{-1})$$

lies again in an appropriate domain of  $\mathbb{H}$  in  $\text{Ker } D_z$  resp.  $\text{Ker } D_z \Delta_z$ .

Similar to complex function theory, as proved in [3] and elsewhere, quaternion-valued functions in  $\text{Ker}(D)$  satisfy a Cauchy integral formula of the following form

$$f(z) = \frac{1}{8\pi^2} \int_{\partial K} q_0(z - w) d\sigma(w) f(w),$$

where  $K$  is some compact subset with strongly Lipschitz boundary. Furthermore,  $K$  is contained in an open set  $U$ , such that  $f$  satisfies  $Df = 0$ . Here, and in all that follows

$$d\sigma(w) = dw_1 \wedge dw_2 \wedge dw_3 - dw_0 \wedge dw_2 \wedge dw_3 e_1 + dw_0 \wedge dw_1 \wedge dw_3 e_2 - dw_1 \wedge dw_2 \wedge dw_3 e_3$$

denotes the oriented surface 3-form. Notice that the expression  $q_0(z - w)$  replaces the complex function  $\frac{1}{z-w}$  in the ordinary Cauchy integral formula. In the quaternionic version the factor  $8\pi^2$  replaces the factor  $2\pi i$  from complex analysis.

From this generalized Cauchy integral formula one may establish a Green's integral formula for functions in  $\text{Ker}(D\Delta)$ , see [15]. We also need to consider the more general topological version of Cauchy's integral formula, cf. [15]

**Theorem 1** *Let  $U \subset \mathbb{H}$  be open. Let  $\Gamma$  be a 3-dimensional null-homologous cycle in  $U$ . Suppose that  $z \in U \setminus \Gamma$  and that  $f : U \rightarrow \mathbb{H}$  is left monogenic. Then*

$$\frac{1}{8\pi^2} \int_{\Gamma} q_0(z - \zeta) d\sigma(\zeta) f(\zeta) = w_{\Gamma}(z) f(z),$$

where  $w_{\Gamma}(z)$  stands for the winding number of  $\Gamma$  with respect to  $z$ .

The winding number counts how often  $\Gamma$  wraps around the point  $z$ . This generalized topological Cauchy formula provides us with a quaternionic analogue of the usual residue theorem in the form

$$\int_{\Gamma} d\sigma(z) f(z) = 8\pi^2 \sum_{i=1}^s \text{res}(f; \beta_i), \quad (2)$$

where  $f$  is supposed to be left monogenic in  $\text{int}\Gamma$  except of in a finite number of isolated points denoted by  $\beta_i$ .

**Remark:** In complete analogy to complex function theory, the residue of a function  $f$  at a point  $\beta_i$  is nothing else than the first Laurent coefficient of the singular part of the Laurent series expansion, i.e. the coefficient that is associated with  $q_0$  in the expansion given below in (3). Following for instance [3, 15], the Laurent series of a function that is left monogenic in the open pointed ball  $B(\beta_i, \varepsilon) \setminus \{\beta_i\}$  of radius  $\varepsilon$  and center  $\beta_i$  is of the form

$$f(z) = \sum_{\mathbf{m} \in \mathbb{N}_0^3} q_{\mathbf{m}}(z - \beta_i) b_{\mathbf{m}} + \sum_{\mathbf{m} \in \mathbb{N}_0^3} V_{\mathbf{m}}(z - \beta_i) a_{\mathbf{m}} \quad (3)$$

where  $V_{\mathbf{m}}$  are the so-called Fueter polynomials,  $q_{\mathbf{m}}$  the partial derivatives of the Cauchy kernel and  $a_{\mathbf{m}}$  and  $b_{\mathbf{m}}$  quaternionic coefficients. Hence, the residue of  $f$  at  $\beta_i$  equals  $\text{res}(f; \beta_i) = b_0$ .

In the special case where the function  $f$  is left monogenic at each point of the open ball  $B(\beta_i, \varepsilon)$ , this series expansion simplifies to the Taylor type series expansion of the form

$$f(z) = \sum_{\mathbf{m} \in \mathbb{N}_0^3} V_{\mathbf{m}}(z - \beta_i) a_{\mathbf{m}}.$$

Consequently, this residue formula then simplifies to

$$\int_{\Gamma} d\sigma(z) f(z) = 0,$$

and we have a generalization of Cauchy's theorem in a ball, which can be extended to open star-like domains by applying standard arguments from the literature.

As a further consequence of the generalized Cauchy integral formula combined with the generalized Cauchy integral theorem, the following generalization of the complex argument principle was proved a few years ago to hold for left monogenic functions with isolated  $a$ -points, cf. [14, 15].

For convenience we also recall that  $c$  is an isolated  $a$ -point of  $f$  if  $f(c) = a$  and if additionally there exists a sufficiently small neighborhood  $V$  around  $c$  where  $f(x) \neq a$  for all  $x \in V \setminus \{a\}$ .

**Theorem 2** (cf. [14, 15]).

Let  $G \subset \mathbb{H}$  be a domain. Suppose that  $f : G \rightarrow \mathbb{H}$  is left monogenic in  $G$  and that  $c \in G$  is an isolated zero point of  $f$ .

Next, let  $\varepsilon > 0$  so that  $\overline{B}(c, \varepsilon) \subseteq G$  and  $f|_{\overline{B}(c, \varepsilon) \setminus \{c\}} \neq 0$ . Then

$$\text{ord}(f; c) = \frac{1}{8\pi^2} \int_{\partial B(c, \varepsilon)} q_0(f(z)) \cdot \left[ (Jf)^{ad}(z) \right] * \left[ d\sigma(z) \right]$$

Here “ $*$ ” denotes the matrix multiplication and “ $\cdot$ ” the quaternionic multiplication. The vector  $[(Jf)^{ad}(z')] * [d\sigma'(z')] \in \mathbb{R}^4$  then is re-interpreted as a quaternion.  $(Jf)^{ad}$  stands for the adjunct matrix of the Jacobian of  $f$ .

These tools will now be used to study self-dual  $SU(2)$  Yang-Mills instantons on specific classes of conformally flat 4-manifolds. These new tools together with the new class of automorphic forms developed in [15, 4, 6] allow to round off some classical studies by Gürsey and Tze on self-duality in  $SU(2)$ -Yang-Mills gauge theories and to review their results from the viewpoint of a new mathematical fundamental theory.

## 2 Self-duality of Yang-Mills instantons and quaternionic analyticity

### 2.1 Self-duality in $SU(2)$ -Yang-Mills gauge theory

In [12, 13], F. Gürsey and Tze constructed an explicit relation between the self-duality condition for  $SU(2)$ -Yang-Mills instantons on  $\mathbb{R}^4$  and Fueter holomorphic functions. To get started, we briefly summarize some of their results.

For this we consider the quaternionic Hopf-bundle  $Sp(1) \hookrightarrow S^7 \rightarrow S^4$  with typical fiber  $S^3 \simeq SU(2) \subset \mathbb{H}$ . The base manifold  $S^4 \simeq \mathbb{H}P_1$  is regarded as being the one-point compactification of  $\mathbb{R}^{4,0} \simeq \mathbb{H}$ . Moreover, the total space of the quaternionic Hopf-bundle  $S^7$  is considered as being a sub-manifold of  $\mathbb{H} \times \mathbb{H}$ . In other words, the quaternionic Hopf-bundle is considered as being a natural sub-bundle of the trivial quaternionic line bundle  $\mathbb{R}^4 \times \mathbb{H} \rightarrow \mathbb{R}^4$ . On the latter, respectively, the gauge group  $\mathcal{G}$  and the affine set of connections  $\mathcal{A}$  can be identified with  $C^\infty(\mathbb{R}^4, SU(2))$  and  $C^\infty(\mathbb{R}^4, \Lambda^1 \mathbb{R}^4 \otimes su(2))$ . Here,  $su(2)$  denotes the Lie algebra of  $SU(2)$ .

The field strength  $F_A \in C^\infty(\mathbb{R}^4, \Lambda^2 \mathbb{R}^4 \otimes su(2))$  of  $A \in C^\infty(\mathbb{R}^4, \Lambda^1 \mathbb{R}^4 \otimes su(2))$  is defined by  $F_A := dA + A \wedge A$ . As a consequence, the field strength satisfies the Bianchi-Identity:  $dF_A = -[A, F_A]$ . The gauge group  $\mathcal{G}$  naturally acts on  $\mathcal{A}$  from the right via

$$\mathcal{A} \times \mathcal{G} \rightarrow \mathcal{A}, (A, g) \mapsto A^g := g^{-1}Ag + g^{-1}dg.$$

It follows that  $F_A^g \equiv F_{A^g} = g^{-1}F_A g$ . The Yang-Mills functional

$$S_{\text{YM}} : \mathcal{A} \rightarrow \mathbb{R}, A \mapsto \int_{S^4} tr(F_A \wedge *F_A)$$

is thus invariant with respect to the right-action of  $\mathcal{G}$  on  $\mathcal{A}$ . Here, “ $tr$ ” refers to the ordinary matrix trace and “ $*$ ” denotes the Hodge map with respect to the orientation and the Fubini-Study metric on  $\mathbb{H}P_1$ .

The critical points of the Yang-Mills functional fulfill the *Yang-Mills equation*  $d * F_A = -[A, *F_A]$ . In the definition of the Yang-Mills functional the Yang-Mills field strength  $F_A$  is identified with its pull-back with respect to the stereographic projection  $S^4 \rightarrow \mathbb{R}^4$ . Of particular interest are the solutions  $F_A$  satisfying  $F_A \pm *F_A = 0$ . These are the anti-self dual (resp. self-dual) *instanton solutions*. Geometrically  $A \in \mathcal{A}$  is interpreted as a gauge potential of an  $SU(2)$ -connection on the Hopf bundle.

In [12, 13] it has been shown that

$$a_\mu = \frac{1}{2} \frac{t \overset{\leftrightarrow}{\partial}_\mu t}{1 + t^\sharp t}, \quad t \in \mathbb{H},$$

transforms under the group  $C^\infty(\mathbb{R}^4, Sp(1))$  as:  $a_\mu \rightarrow \bar{r}a_\mu r + \bar{r}\partial_\mu r$ . This means that  $A := dx^\mu \otimes a_\mu$  transforms like a gauge potential under  $Sp(1) = SU(2)$ .

Hence, one may define an  $SU(2)$  Yang-Mills field  $F_A := \frac{1}{2}dx^\mu \wedge dx^\nu \otimes f_{\mu\nu}$  with  $f_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu]$ . Because of the Bianchi identity, the Yang-Mills equation are automatically satisfied if  $F_A$  is self-dual or anti-self-dual.

Next, one may introduce the Fueter holomorphic function

$$\mathcal{F}(z) = (z + a)^{-1} + \bar{\alpha}(z + \beta)^{-1}\alpha,$$

where  $a, \beta, \alpha$  are some constant quaternions. From [12, 13] one infers that

$$a_\mu = \frac{1}{2} \text{Vec}[e_\mu(\Delta\mathcal{F})(D\mathcal{F})^{-1}].$$

Let  $\Phi_{\mu\nu} = f_{\mu\nu} + \tilde{f}_{\mu\nu}$  denote the self-dual part of  $f_{\mu\nu}$  and  $\Phi'_{\mu\nu} = f_{\mu\nu} - \tilde{f}_{\mu\nu}$  the anti-self-dual part of  $f_{\mu\nu}$ . Here,  $\tilde{f}_{\mu\nu}$  are the components of the Hodge dual  $*F_A$  of  $F_A$ . It follows that

$$\Phi'_{\mu\nu} = -\frac{1}{2}e'_{\mu\nu}(D\mathcal{F})^{-1}(D\Delta\mathcal{F}) \quad \text{and} \quad \Phi_{\mu\nu} = -\frac{1}{2}(D\mathcal{F})(De_{\mu\nu}\bar{D})(D\mathcal{F})^{-1}.$$

The self-duality condition  $\Phi'_{\mu\nu} \equiv 0$  thus is satisfied if  $D\Delta\mathcal{F} = 0$ . The self-duality condition may thus be rephrased in terms of Fueter holomorphy for the special ansatz proposed by Gürsey and Tze. Note that  $(z + a)^{-1}$  actually is the Green's kernel of  $D\Delta$  up to a constant.

## 2.2 The relation between the Chern-Pontryagin index and the quaternionic winding number

In [12, 13] it has been shown that the second Chern class is proportional to

$$\Pi = -Sc(f_{\mu\nu}\tilde{f}_{\mu\nu}) = \frac{1}{4}\Delta[\Delta \ln(D\mathcal{F})].$$

The density  $*\Pi$  coincides with the Yang-Mills Lagrangian density in the case of self-dual instantons.

Next let  $V \subset \mathbb{H}$  be a four-dimensional domain having a three dimensional Lipschitz boundary  $\partial V$  that is homeomorphic to the three-dimensional unit sphere  $S^3 = \{z \in \mathbb{H} \mid |z| = 1\}$ .

Following [12, 13], the second Chern number of the  $SU(2)$  principal bundle reads

$$\begin{aligned} c_2 &= \frac{1}{8\pi^2} \int_V \Pi dx_0 dx_1 dx_2 dx_3 \\ &= \frac{1}{8\pi^2} \int_V \left( \frac{1}{4} \Delta[\Delta \ln(D\mathcal{F})] \right) dx_0 dx_1 dx_2 dx_3 \\ &= \frac{1}{32\pi^2} \int_{\partial V} d\sigma(z) [\Delta \bar{D} \ln(D\mathcal{F})]. \end{aligned}$$

In the previous line we have applied the classical Gauss-Ostrogradskii-Green formula.

The second Chern number is a topological invariant. As a consequence, the topological deformation of the three-dimensional contour  $\partial V$  reduces to the sum of small hyper-spheres (or cycles)  $\partial B(\beta_i, \varepsilon)$  surrounding each pole (denoted by  $\beta_i$ ) of the integrand  $[\Delta \bar{D} \ln(D\mathcal{F})]$ . In the function theoretic language this topological invariance is rephrased as the application of the Cauchy integral theorem. This allows one to reduce the integration over sufficiently small spheres of radius  $\varepsilon$  surrounding the poles, instead of extending the integration over the full boundary contour  $\partial V$ . In general, the latter is very difficult to parametrize. To be more precise, we have

$$\begin{aligned} c_2 &= \frac{1}{32\pi^2} \int_{\partial V} d\sigma(z) [\Delta \bar{D} \ln(D\mathcal{F})] \\ &= \frac{1}{32\pi^2} \int_{\cup_{i=0}^n \partial B(\beta_i, \varepsilon)} d\sigma(z) \Delta \left( \sum_{i=0}^n (\beta_i - z)^{-1} \right) \\ &= \frac{1}{8\pi^2} \int_{\cup_{i=0}^n \partial B(\beta_i, \varepsilon)} d\sigma(z) \sum_{i=0}^n q_0(z - \beta_i). \end{aligned}$$

In the previous equality we exploited the relation (1), namely that

$$q_0(z) = -\frac{1}{4} \Delta(z^{-1}).$$

Applying in the next step a suitable Möbius transformation, one can remove one of these  $n+1$  poles from  $\bar{D} \ln D\mathcal{F}$ , namely by sending for instance  $\beta_0$  to  $\infty$ , where again  $\infty$  is interpreted in the sense of the one-point compactification of  $\mathbb{H}$ . The other quaternions  $\beta_1, \dots, \beta_n$  are then sent to other quaternions, say  $\gamma_1, \dots, \gamma_n$ .

By the application of the quaternionic residue theorem, see (2), one finally obtains

$$c_2 = \frac{1}{8\pi^2} \int_{\cup_{i=1}^n \partial B(\gamma_i, \varepsilon)} d\sigma(z) \sum_{i=1}^n q_0(z - \gamma_i) = \sum_{i=1}^n \text{res}(q_0; \gamma_i) = n.$$

### 3 Quaternionic-analytic automorphic forms and instanton solutions on conformally flat manifolds

On the 4-sphere and the projective space  $\mathbb{C}P_2$  self-dual solutions of a  $SU(2)$ -Yang-Mills instantons are well studied. In this section we discuss how hypercomplex analysis may be used to explicitly construct self-dual instanton solutions on various other classes of conformally flat spin manifolds. In recent works [16, 4] we have been able to construct monogenic spinor bundles over large classes of examples of such manifolds in terms of hypercomplex-analytic Eisenstein- and Poincaré series that have been introduced in [15] in a different context. Our main goal in this paper is to develop an explicit link between the fundamental solutions of  $D\Delta$  on some manifolds and the Chern number of  $SU(2)$  principal bundles over these manifolds.

Conformally flat manifolds are Riemannian manifolds with vanishing Weyl tensor. Equivalently, conformally flat manifolds are manifolds with an atlas whose transition functions are Möbius transformations. Remember that Möbius transformations coincide exactly with the conformal maps in  $\mathbb{H}$ . From this viewpoint (cf. [16]) conformally flat manifolds can be regarded as higher dimensional analogues of holomorphic Riemann surfaces. Following for example the classical paper [19], a large class of examples can be constructed by factoring out a domain  $U \subseteq \mathbb{H}$  by a discrete group of the Möbius group. Then  $\mathcal{M} = U/\Gamma$  is a conformally flat manifold. The simplest examples are of course  $\mathbb{H} \cong \mathbb{R}^4$  and  $S^4$ . Further important examples are given by

- $C_p = \mathbb{H}/\mathcal{T}(\mathbb{Z}^p)$ , where  $\mathcal{T}(\mathbb{Z}^p) = \left\langle \begin{pmatrix} 1 & e_0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & e_{p-1} \\ 0 & 1 \end{pmatrix} \right\rangle$   $1 \leq p \leq 4$  is the discrete translation group associated to the lattice  $\mathbb{Z}^p$ . If  $p \leq 3$ , then  $C_p$  are  $p$ -cylinders and for  $p = 4$ :  $T_4 = \mathbb{H}/\mathcal{T}(\mathbb{Z}^4)$  is the flat 4-torus.

- Let  $U = \mathbb{H} \setminus \{0\}$  and  $\Gamma = \{m^k, k \in \mathbb{Z}\}$  which is a discrete dilatation group. Then,  $U/\Gamma \cong S^3 \times S^1$  is the Hopf manifold
- $U = H^+(\mathbb{H}) = \{z \in \mathbb{H}, \text{Sc}(z) > 0\}$ .

Let  $\Gamma_p = \left\langle \begin{pmatrix} 1 & e_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & e_p \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$  ( $p < 4$ ) denote the special hypercomplex modular group as discussed in [15]. For  $N \in \mathbb{N}$  consider

$$\Gamma_p[N] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_p, \mid a-1, b, c, d-1 \in N\mathbb{Z}^4 \right\}.$$

The associated class of conformally flat manifolds contains  $k$ -handled cylinders ( $p < 3$ ) and  $k$ -handled tori ( $p = 3$ ).

Instead of the standard orthonormal lattice we can also consider some other choices of lattices, for instance the set of reduced Hurwitz quaternions where the basis  $\{e_1, e_2, e_3\}$  is replaced by  $\{e_1, e_2, \frac{e_1+e_2+e_3}{2}\}$ . This lattice plays a crucial role in analytic number theory, see for instance [17]. This lattice is more dense than the standard one, and the fundamental domain of the associated hypercomplex modular group hence has a smaller volume.

In all these cases  $U \subset \mathbb{H}$  is the universal covering space of a manifold  $\mathcal{M} = U/\Gamma$ . Hence, there exists a unique projection map  $p : U \rightarrow \mathcal{M}, x \mapsto x \pmod{\Gamma}$ . Let  $x' := p(x)$ . If  $A \subseteq \mathbb{H}$  is open, then  $A' = p(A)$  is open on  $\mathcal{M}$ . Provided  $f$  is  $\Gamma$ -invariant on  $U$ , then  $f' = p(f)$  represents a well-defined spinor-valued section on  $\mathcal{M}$ . Notice that different decompositions of the period lattice may give rise to different choices of spinor bundles over  $\mathcal{M}$ , but for details about this topic we refer the reader to [16, 4] where this issue has been adressed extensively.

Furthermore,  $D' = p(D)$  and  $\Delta' = p(\Delta)$  yields the hyper-complex analogue of the Cauchy-Riemann operator and the Laplace operator on  $\mathcal{M}$ .

In the spirit of [12, 13], self-dual  $SU(2)$  instantons on these manifolds are induced by  $\Gamma$ -“periodizations” of the self-dual instantons that we have on Euclidean space. Consider the Yang-Mills connection

$$a_\mu = \frac{1}{2} \text{Vec}\{e_\mu(\Delta\mathcal{F})(D\mathcal{F})^{-1}\}$$

associated with superpositions of  $\Gamma$ -“periodic” actions on the basic Fueter holomorphic function  $z^{-1}$ . Their partial derivatives then provide us in particular with further self-dual instanton solutions on the associated manifolds. The key question that arises in this context immediately is to ask for the fundamental solutions in terms of explicit formulas of  $D$  (resp. of  $D\Delta$ ) on these manifolds. In the mainly mathematical oriented works [16, 4] an explicit answer to this question has been given. These results will now be interpreted in terms of self-dual  $SU(2)$ -instantons on these manifolds and Chern numbers.

### 3.1 Conformally flat cylinders and tori

Following [16], the fundamental solution of  $D$  (resp  $D\Delta$ ) on  $\mathbb{H}/\mathbb{Z}^p$  ( $1 \leq p \leq 3$ ) are the generalized monogenic (resp. Fueter holomorphic) cotangent functions. The simplest example  $p = 1$  is

$$\cot_{D\Delta}^{(1)}(z) = (\beta z)^{-1} + \sum_{n=1}^{+\infty} \left[ \left( \frac{\beta}{2\pi} z - n \right)^{-1} + \left( \frac{\beta}{2\pi} z + n \right)^{-1} \right].$$

In the spirit of [13], this function is physically interpreted as infinite superpositions of 't Hooft instantons along the  $x_0$ -axis placed at periodic intervals. Notice that in the mathematical context, 't Hooft instantons are hypermonogenic functions [22, 23]. Further special instanton solutions on this cylinder with values in the trivial spinor bundle are obtained by considering derivatives:

$$\partial_{x_0}^{m_0} \partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \partial_{x_3}^{m_3} \cot_{D\Delta}^{(1)}(z), \dots,$$

where  $(m_0, m_1, m_2, m_3) \in \mathbb{N}_0^4$ , or finite superpositions for example. Rectangular multi-periodic instanton solutions for the cases  $p = 2, 3, 4$  are obtained by considering the generalized  $p$ -fold periodic cotangent functions (see also [21])

$$\cot_{D\Delta}^{(p)}(z) = z^{-1} + \sum_{\mathbb{Z}^p \setminus \{0\}} \left[ (z - w)^{-1} + \sum_{\mu=0}^3 (w^{-1}z)^\mu w \right] \text{ for } p = 2, 3, 4.$$

The application of the quaternionic residue theorem leads to

$$\begin{aligned} \frac{1}{8\pi^2} \int_{\partial P} d\sigma(z) \cot_D^{(4)}(z) &= \frac{1}{8\pi^2} \int_{\partial P} d\sigma(z) \Delta \cot_{D\Delta}^{(4)}(z) = \sum_{a \in P} \text{res}(\cot_D^{(4)}; a) \\ &= \text{res}(\cot^{(4)}; 0) = 1, \end{aligned}$$

where  $\cot_D^{(4)}$  denotes the four-fold periodic monogenic cotangent function discussed in [15]. The choice of the ansatz

$$a_\mu = \frac{1}{2} \text{Vec}(e_\mu \Delta \cot_{D\Delta}^{(4)}(z) (D \cot_{D\Delta}(z))^{-1})$$

corresponds to a Yang-Mills solution with unit instanton number per period parallelepiped. The same result can be established for  $p < 4$  with a similar argument. Since  $\cot_{D\Delta}^{(p)}$  is the fundamental solution of  $D\Delta$ , it provides us with all solutions to  $D\Delta$  on  $\mathbb{H}/\mathbb{Z}^p$ . This can be seen by applying the following Green's integral formula taken from [16]. In the case  $p = 1$  we have

$$\begin{aligned} w_{\Gamma'}(y')g(y') &= \frac{1}{8\pi^2} \int_{\Gamma'} \cot_D^{(1)}(z' - y') d\sigma(z') g'(z') \\ &\quad - \frac{1}{4\pi^2} \int_{\Gamma'} \cot_\Delta^{(1)}(z' - y') \overline{d\sigma(z')} D' g'(z') \\ &\quad + \frac{1}{2\pi^2} \int_{\Gamma'} \cot_{D\Delta}^{(1)}(z' - y') \Delta' g'(z'). \end{aligned}$$

In the cases  $p = 2, 3, 4$  this formula can be adapted directly using properly chosen sums of generalized cotangent functions. By means of the generalized monogenic cotangent functions we also obtain an argument principle on  $C_p$  for isolated zero points. Due to what has been shown in [15] Chapter 2.11,

$$\text{ord}(f'; c') = \frac{1}{8\pi^2} \int_{\partial B'(c', \varepsilon)} \cot_D^{(p)}(f'(z')) \cdot [(Jf)^{ad}(z')] * [d\sigma'(z')].$$

For  $p = 4$  the function  $\cot^{(4)}$  needs to be slightly modified like for the Green's integral formula. Here again “ $*$ ” denotes the matrix multiplication and “ $\cdot$ ” the quaternionic multiplication. The vector  $[(Jf)^{ad}(z')] * [d\sigma'(z')] \in \mathbb{R}^4$  is again re-interpreted as a quaternion. This formula provides a direct relation between the fundamental solution and the Chern number of the  $SU(2)$  principle bundles with base manifolds  $C_p$ .

## 3.2 The Hopf manifold

According to [16] the fundamental solution of  $D'$  and  $D\Delta'$  on  $S^3 \times S^1$  with values in the trivial spinor bundle are given by

$$G_D^H(z', y') = \sum_{k=-\infty}^0 m^{3k/2} q_0(m^k z' - y') + q_0(z') \left[ \sum_{k=1}^{+\infty} m^{-3k/2} q_0(m^{-k} z'^{-1} - y'^{-1}) \right] q_0(y'),$$

$$G_{D\Delta}^H(z', y') = \sum_{k=-\infty}^0 m^{k/2} (m^k z' - y'^{-1}) + z'^{-1} \left[ \sum_{k=1}^{+\infty} m^{-k/2} (m^{-k} z'^{-1} - y'^{-1})^{-1} \right] y'^{-1}.$$

These induce self-dual  $SU(2)$ -instanton solutions on  $S^3 \times S^1$ . By applying the Green's formula we obtain all solutions of  $D\Delta$  on  $S^3 \times S^1$ . For this we have to replace the generalized cotangent functions from the previous formula by the projection of  $G^H$ . This yields an argument principle for isolated zeroes for monogenic functions on  $S^3 \times S^1$ . More precisely, we can establish

**Theorem 3** *Let  $G' \subseteq S^3 \times S^1 =: H$  be a domain. Let  $f' : G' \rightarrow H$  be left monogenic in  $G'$  and  $c' \in G'$  be an isolated zero point. Let  $\varepsilon > 0$  be a real such that  $f'|_{B'(c', \varepsilon)} \neq 0$ , and that  $B'(c', \varepsilon) \subset G'$ , where  $B'(c', \varepsilon)$  denotes the projection of the ball  $B(c, \varepsilon)$  onto the Hopf manifold  $H$ . Then we have*

$$\text{ord}(f', c') = \frac{1}{8\pi^2} \int_{\partial B(c', \varepsilon)} G_D^H(f'(z')) \cdot [(Jf)^{ad}(z')] * [d\sigma'(z')].$$

In order to show that  $\text{ord}(f'; c')$  actually is an integer, we need to use the generalized Cauchy integral formula on the Hopf manifold. The rest of the argumentation is then similar as in the Euclidean case treated in [14]. In view of the transformation rule for differential forms from [27] reading  $d\sigma'(f'(z')) = [(Jf')^{ad}(z')] * [d\sigma'(z')]$  the integral above can be written in the form

$$\text{ord}(f', c') = \frac{1}{8\pi^2} \int_{\partial f(B(c', \varepsilon))} G_D^H(z') d\sigma'(z').$$

The generalized Cauchy integral formula for left monogenic functions on the Hopf-manifold tells us that

$$\frac{1}{8\pi^2} \int_{\Gamma} G_D^H(z', y') d\sigma(z') g(z') = w_{\Gamma'}(y') g(y')$$

Now we replace  $g' \equiv 1$  and  $y'$  by  $f'(c')$  and  $\Gamma'$  by  $f'(\partial B'(c', \varepsilon))$ . This leads consequently to

$$\frac{1}{8\pi^2} \int_{f'(\partial B'(c', \varepsilon))} G_D^H(z', \underbrace{f'(c')}_{=0}) d\sigma'(z') = w_{f'(\partial B'(c', \varepsilon))}(\underbrace{f'(c')}_{=0}).$$

The value  $\text{ord}(f', c') = w_{f'(\partial B'(c', \varepsilon))}(0)$  is thus the integer counting how often the image of  $B'$  under  $f'$  around the isolated zero wraps around zero.

Similarly to the Euclidean case, one can replace the projection of the ball by a null-homologous 3-dimensional cycle parameterizing an 3-dimensional surface of a 4-dimensional simply connected domain inside of  $G' \subset H$  which contains the isolated zero  $c'$  in its interior and no further zeroes neither in its interior nor on its boundary.

Again, this allows to relate the fundamental solution on  $S^3 \times S^1$  and the Chern numbers of  $SU(2)$  principle bundles.

### 3.3 k-handled cylinders and tori

According to [4], the fundamental solution of  $D\Delta$  on  $\mathcal{M} = H^+/\Gamma_p[N]$  ( $N \geq 3$ ) is induced by the Fueter holomorphic Poincaré series on  $\Gamma_p[N]$ . Adapting the representation formulas of [4] to the particular context considered here, the corresponding Poincaré series then reads

$$G_{\mathcal{M}}(z, y) = \frac{1}{8\pi^2} \sum_{M \in \Gamma_p[N]} x_n^2 \frac{\overline{cz + d}}{|cz + d|^6} (M \langle z \rangle - y)^{-1}.$$

Similar to the previously discussed example, this series provides us with the Green's kernel functions. Likewise, a residue and argument formula on this class of conformally flat manifolds can be obtained which can be used to express the Chern number of  $SU(2)$  principal bundles over k-handled cylinders and tori in terms of the above given fundamental solution. Notice that we exploited in the proof to show that the topological winding number (which gives the order of an  $a$  point) is an integer that the constant function  $g \equiv 1$  is a solution to the Cauchy-Riemann operator on the manifold. This is true since we are dealing with conformally flat manifolds where the scalar curvature is zero. In the case of dealing with other manifolds where we have a non-zero scalar curvature we have to apply a more sophisticated argument to relate the second Chern number with the fundamental solution of  $D\Delta$ .

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