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by

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Well-Posedness for the Navier-Slip Thin-Film Equation in the Case of Complete Wetting

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Abstract

We are interested in the thin-film equation with zero-contact angle and quadratic mobility, modeling the spreading of a thin liquid film, driven by capillarity and limited by viscosity in conjunction with a Navier-slip condition at the substrate. This degenerate fourth-order parabolic equation has the contact line as a free boundary. From the analysis of the self-similar source-type solution, one expects that the solution is smooth only as a function of *two* variables (x, x^β) (where x denotes the distance from the contact line) with $\beta = \frac{\sqrt{13}-1}{4} \approx 0.6514$ irrational. Therefore, the well-posedness theory is more subtle than in case of *linear* mobility (coming from Darcy dynamics) or in case of the *second*-order counterpart (the porous medium equation).

Here, we prove global existence and uniqueness for one-dimensional initial data that are close to traveling waves. The main ingredients are maximal regularity estimates in weighted L^2-spaces for the linearized evolution, after suitable subtraction of $a(t) + b(t)x^\beta$-terms.

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Contents

1. Introduction

1.1. *The Free Boundary Problem to the Thin-Film Equation*

We study regularity and stability properties of the free boundary problem

$$\partial_t h + \partial_z \left(h^2 \partial_z^3 h\right) = 0 \quad \text{for } t > 0,\ z \in (Z_0(t), \infty), \tag{1.1a}$$

$$h = \partial_z h = 0 \quad \text{for } t > 0,\ z = Z_0(t), \tag{1.1b}$$

$$\dot{Z}_0(t) = \lim_{z \searrow Z_0(t)} h \partial_z^3 h \quad \text{for } t > 0. \tag{1.1c}$$

Equation (1.1a) is a thin-film equation that can be derived in a lubrication approximation from the underlying Navier-Stokes equations of a two-dimensional viscous thin film on a one-dimensional flat solid [1, 2]. The function $h(t,z)$ describes the height of the film, whereas $Z_0(t)$ denotes the position of the free boundary, which, in two space dimensions, corresponds to the contact line separating the three phases gas, liquid, and solid. Then $h = 0$ for $z = Z_0(t)$ merely defines the position of the contact line, while $\partial_z h = 0$ for $z = Z_0(t)$ enforces a zero contact angle between the liquid-gas and liquid-solid interfaces. In a quasi-static model, the contact angle is determined by a local equilibrium of surface tensions at the interfaces solid-gas, solid-liquid, and liquid-gas. The assumption $\partial_z h = 0$

at $z = Z_0(t)$ then implies that a global equilibrium is never attained, which is why the film will not stop spreading. This situation is commonly referred to as 'complete wetting regime' as opposed to the case of partial wetting, i.e. $\partial_z h \neq 0$.

Equation (1.1a) has the form of a continuity equation

$$\partial_t h + \partial_z (hV) = 0,$$

with $V = h\partial_z^3 h$ the velocity with which the film height h is transported. In fact, within the lubrication approximation, V is the vertically averaged horizontal velocity of the fluid. The third boundary condition (1.1c) then states that the boundary value V_0 of the velocity V at the contact line equals the velocity $\dot{Z}_0$ of the contact line. In particular, (1.1c) implies that the mass $\int_{Z_-(t)}^{Z_+(t)} h \,\mathrm{d}z$ of a droplet with compact support $[Z_-(t), Z_+(t)]$ is conserved in time. Equation (1.1a) is a special case of the larger class of thin-film equations

$$\partial_t h + \partial_z \left(h^n \partial_z^3 h\right) = 0, \tag{1.2}$$

with mobility exponents $n \in (0,3)$. Hence the velocity in general reads $V = h^{n-1}\partial_z^3 h$. The case $n = 2$ corresponds to the physically relevant situation of linear Navier-slip at the liquid-solid interface for the underlying Navier-Stokes equations and can be seen as the typical case for mobility exponents $n \in \left(\frac{3}{2}, 3\right)$, cf. [3, 4]. We remark that the case of complete wetting for mobility exponent $n = 1$ was investigated by three, respectively two, of the authors in earlier works within weighted L^2- and Hölder-spaces, respectively [5, 6]. Moreover, this problem in arbitrary space dimensions, i.e.

$$\partial_t h + \nabla \cdot (h \nabla \Delta h) = 0$$

with appropriate boundary conditions, was recently investigated by means of the theory of singular integral operators in weighted L^p-spaces by John [7]. The latter model, in the $1 + 1$-dimensional case, can be seen as the lubrication approximation of the two-dimensional Hele-Shaw cell. Indeed in [8], it has been rigorously proven that solutions of the two-dimensional Hele-Shaw cell converge to solutions of the thin-film equation with $n = 1$. A corresponding rigorous lubrication approximation for non-zero contact angle and for classical solutions has been derived in [9, 10]. The solution $h(t, z)$ turns out to be smooth up to the boundary in that case.

We mark that the case of partial wetting, i.e. without loss of generality $\partial_z h = 1$ at $z = Z_0(t)$, was treated by one of the authors in earlier works [9–12] covering the range of exponents $n \in \left(0, \frac{14}{5}\right)$. We also note that an extensive theory for global existence of weak solutions [13–17] and their qualitative properties [18–28] has been developed in the case of complete wetting, also in higher space dimensions. On the other hand, the corresponding theory in the partial wetting case is so far limited to [29, 30]. Referenced discussions may be found e.g. in [31, 32].

1.2. Stationary and Traveling Wave Solutions

Stationary solutions to (1.1) have the form

$$h_{\mathrm{ST}} = \mu_0 (z - z_0)^2, \tag{1.3}$$

where $\mu_0 > 0$ and $z_0 \in \mathbb{R}$ are two time-independent parameters. Indeed, these solutions play an important role in the analysis of the thin-film equation for $n \in \left(0, \frac{3}{2}\right)$. In particular the mentioned regularity results [5–7] linearize around the profile of the quadratic stationary solution (1.3). For the range of mobility exponents $n \in \left(\frac{3}{2}, 3\right)$, however, the expected generic solutions are traveling waves:

In the case of a traveling wave solution, the film moves with a constant velocity $\dot{Z}_0 = V_0 \in \mathbb{R}$. In fact we will linearize around a traveling wave profile in the following analysis. We pass to a moving coordinate system[1], by considering $h(t, z) := H_{\mathrm{TW}}(x)$ with $x = z - V_0 t$, and arrive at

$$-V_0 \frac{\mathrm{d} H_{\mathrm{TW}}}{\mathrm{d} x} + \frac{\mathrm{d}}{\mathrm{d} x} \left(H_{\mathrm{TW}}^2 \frac{\mathrm{d}^3 H_{\mathrm{TW}}}{\mathrm{d} x^3} \right) = 0.$$

We can integrate this equation, by using boundary conditions (1.1b) and (1.1c), additionally assuming without loss of generality that $x = 0$ defines the contact line, to the result

$$H_{\mathrm{TW}} \frac{\mathrm{d}^3 H_{\mathrm{TW}}}{\mathrm{d} x^3} = V_0 \ \text{ for } \ x > 0, \tag{1.4a}$$

$$H_{\mathrm{TW}} = \frac{\mathrm{d} H_{\mathrm{TW}}}{\mathrm{d} x} = 0 \ \text{ at } \ x = 0. \tag{1.4b}$$

We assume $V_0 < 0$. Then, by rescaling $x \mapsto x \left(-\frac{8V_0}{3}\right)^{\frac{1}{3}}$, we can assume without loss of generality that the speed of the wave is given by $-\frac{3}{8}$. Thus the solution of (1.4) reads

$$H_{\mathrm{TW}} = x^{\frac{3}{2}}. \tag{1.5}$$

Note that the leading-order asymptotics as $x \searrow 0$ of the traveling-wave solution (1.5) and the stationary solution (1.3) differ, unlike in the case of the thin-film equation with $n = 1$ [5].

1.3. Reformulation of the Problem

It is one of the major aims of this paper to show that the traveling wave solution (1.5) is stable under small perturbations. We therefore return to the full parabolic free boundary problem (1.1) and apply the hodograph transform by putting

$$h\left(t, Z(t, x)\right) := x^{\frac{3}{2}}. \tag{1.6}$$

[1] Coordinates x denote in general coordinates in the moving frame, while coordinates z (or Z) denote coordinates in the frame fixed to the solid, on which the film adheres.

This equality implicitly defines the function $Z(t,x)$ for $t \geq 0$ and $x \geq 0$. We have several reasons for choosing the transformation (1.6): First of all we note that in the new coordinates the free boundary is fixed and the position of the contact line is recovered by $Z_0(t) = Z(t,0)$. Using the analogy to fluid dynamics, coordinates x can be viewed as Lagrangian coordinates, while coordinates z are the Eulerian coordinates of the problem. In the new coordinates the traveling wave solution $h_{\mathrm{TW}} = x^{\frac{3}{2}} = \left(z + \frac{3}{8}t\right)^{\frac{3}{2}}$ is linear and given by $Z_{\mathrm{TW}}(t,x) = x - \frac{3}{8}t$. We note that the similar von Mises transformation was used to treat a related problem, the free boundary problem for the porous medium equation

$$\begin{aligned} h_t - \partial_z \left(h^n \partial_z h\right) &= 0 \quad \text{for } \ t > 0,\ z \in (Z_0(t), \infty), \\ h &= 0 \quad \text{for } \ t > 0,\ z = Z_0(t), \\ \dot{Z}_0(t) &= \lim_{z \searrow Z_0(t)} h^{n-1} \partial_z h \quad \text{for } \ t > 0, \end{aligned}$$

where $n > 0$ [33]. We refer to [5, 11] for further discussions on analog results for the porous medium equation and the choice of boundary conditions, such as [34–36].

Let us reformulate the free boundary problem (1.1) in our new coordinates. Therefore we set

$$F(t,x) := \frac{1}{\partial_x Z(t,x)} \tag{1.7a}$$

so that the traveling wave is now merely the constant solution $F = F_{\mathrm{TW}} \equiv 1$. Perturbations of the traveling wave are hence given by

$$u(t,x) := F(t,x) - 1. \tag{1.7b}$$

In Appendix A.1 we discuss in detail how we then arrive at the degenerate parabolic initial value problem

$$x \partial_t u + p(D) u = \mathcal{N}(u) \ \text{ for } \ t > 0,\ x > 0, \tag{1.8a}$$

$$u_{|t=0} = u_0 \ \text{ at } \ t = 0, \tag{1.8b}$$

where we used the following definitions:

- The scaling-invariant logarithmic derivative

$$D := x \partial_x = \frac{\partial}{\partial s}, \tag{1.9a}$$

  where $s := \ln x$.

- The 5-linear form[2]

$$\mathcal{M}\left(F_1, \cdots, F_5\right) := F_1 F_2 D \left(D + \frac{3}{2}\right) F_3 \left(D - \frac{1}{2}\right) F_4 \left(D + \frac{1}{2}\right) F_5. \tag{1.9b}$$

[2] In the sequel, differential operators act on everything following them: e.g. $DFDFDF = D(FD(FDF))$.

- The fourth-order linear operator

$$\mathcal{L}u = \mathcal{M}(u, 1, \cdots, 1) + \cdots + \mathcal{M}(1, \cdots, 1, u) = p(D)u, \tag{1.9c}$$

where $p(\zeta)$ is a fourth-order polynomial given by

$$p(\zeta) := \zeta^4 + 2\zeta^3 - \frac{9}{8}\zeta = \left(\zeta + \frac{3}{2}\right)\left(\zeta + \beta + \frac{1}{2}\right)\zeta(\zeta - \beta), \tag{1.9d}$$

with the irrational number

$$\beta = \frac{\sqrt{13} - 1}{4} \approx 0.6514. \tag{1.9e}$$

- The nonlinearity

$$\mathcal{N}(u) := p(D)u - \mathcal{M}(1 + u, \cdots, 1 + u). \tag{1.9f}$$

It is instructive to note that, despite the fact that we deal with a fourth-order PDE, we do not have to impose boundary conditions on the solution. The boundary conditions (1.1b) and (1.1c) are implicitly fulfilled by our coordinate transformation (1.6). Also note that the differential operator $p(D)$, a polynomial in the logarithmic derivative $D = x\partial_x$, remains invariant under the rescaling $x \mapsto \mu x$ for $\mu > 0$. Then we can further conclude that problem (1.8) remains invariant under the rescaling

$$(t, x) \mapsto (\mu t, \mu x) \quad \text{for } \mu > 0.$$

Hence the time-variable t and the space-variable x have the same scaling, $t \sim x$. Unlike in the case of *linear* fourth-order *non-degenerate* parabolic equations, where $t \sim x^4$, our fourth-order *degenerate* parabolic equation (1.8a) has the scale invariance of a *first order* PDE. We can apply the same reasoning to the linear problem associated to (1.8), i.e.

$$x\partial_t u + p(D)u = f \quad \text{for } t > 0,\, x > 0, \tag{1.10a}$$

$$u_{|t=0} = u_0 \quad \text{at } t = 0. \tag{1.10b}$$

Again, it is the degeneracy of the linear operator in (1.10a) that leads to the fact that no boundary conditions need to be imposed.

The rest of the paper will be concerned with the analysis of problem (1.8).

1.4. Notation

Throughout the paper we will write $f \lesssim_S g$, whenever a constant $C \geq 1$, depending on the set of parameters S, exists such that $f \leq Cg$. We write $f \sim_S g$, if $f \lesssim_S g$ and $g \lesssim_S f$. We say that a property is true for $x \gg_S 1$ ($x \ll_S 1$), whenever a constant $C > 0$, depending on parameters S, exists such that the property is true for all x with $x \geq C$ ($x \leq \frac{1}{C}$). If $S = \emptyset$ or if the dependence is specified within the text, we just write $f \lesssim g$ etc. The space $C_0^k(U)$, where $U \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}_0 \cup \{\infty\}$, denotes the space of test functions on U, i.e. the set of k-times differentiable functions with compact support contained in U.

2. An Overview of our Approach

In this section, we motivate the norms we express our main result in. In doing so, we also give a summary of our approach. Throughout the subsection, estimates may depend k, ℓ, α, or ϱ, but they are independent of the functions u or w.

At the basis of all estimates is the *coercivity* of $p(D)$: There is a range of weight exponents α for which $p(D)$ is coercive with respect to the weighted inner product

$$(u, w)_\alpha := (x^{-\alpha} u, x^{-\alpha} w)_0, \quad \text{where} \quad (u, w)_0 := \int_0^\infty u w \, \frac{\mathrm{d}x}{x}$$

is the L^2-inner product with respect to the logarithmic variable $s = \ln x$. This actually means that

$$(u, p(D)u)_\alpha \gtrsim |u|_\alpha^2 \,, \tag{2.1}$$

where $|u|_\alpha^2 := (u, u)_\alpha$. Coercivity will be worked out in Section 5.

We now look at the initial value problem of the linear degenerate parabolic equation (1.10). By (at least formally) elementary arguments (cf. Subsection 7.1) coercivity of $p(D)$ with respect to α implies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(|u|_{\alpha-\frac{1}{2}}^2 + C_1 \left| D^{\ell+2} u \right|_{\alpha-\frac{1}{2}}^2 \right) + |\partial_t u|_{\ell,\alpha-1}^2 + |u|_{\ell+4,\alpha}^2 \lesssim |f|_{\ell,\alpha}^2 \,, \tag{2.2}$$

where $|u|_{\ell,\alpha}^2 := \sum_{m=0}^{\ell} |D^m u|_\alpha^2$ and the constant $C_1 > 0$ only depends on ℓ and α. Estimate (2.2) is at the basis of all further a priori estimates, which we obtain from it by by interpolation and integrating in the time variable t. Since we need the flexibility to introduce a time weight, we stick to the *differential* version (2.2) for a while.

In fact, equation (2.2) immediately implies L^2-maximal regularity estimates for the unshifted equation, i.e.

$$\sup_{t \in I} |u(t)|_{\ell+2,\alpha-\frac{1}{2}}^2 + \int_I \left(|\partial_t u(t)|_{\ell,\alpha-1}^2 + |u(t)|_{\ell+4,\alpha}^2 \right) \mathrm{d}t \lesssim |u_0|_{\ell+2,\alpha-\frac{1}{2}}^2 + \int_I |f(t)|_{\ell,\alpha}^2 \, \mathrm{d}t. \tag{2.3}$$

Note that using the logarithmic transformation $s = \ln x$ and setting $\tilde{u}(s) := e^{-\alpha s} u(e^s)$, $\tilde{f}(s) := e^{-\alpha s} f(e^s)$, and $\tilde{p}(\zeta) := p(\zeta + \alpha)$, equation (1.10a) is equivalent to the problem

$$e^s \partial_t \tilde{u} + \tilde{p}\left(\partial_s\right) \tilde{u} = \tilde{f} \quad \text{for } t > 0 \text{ and } s \in \mathbb{R}. \tag{2.4}$$

The operator $\tilde{p}(\partial_s)$ is formally coercive with respect to the standard L^2-scalar product on $\mathbb{R}$. Replacing $\tilde{p}(\partial_s)$ more generally by a formally coercive pseudo-differential operator (with certain additional properties), the well-posedness of (2.4) in L^q with $q \in (1, \infty)$ was investigated in a work by Prüss and Simonett [37] using semigroup theory: For $q = 2$ the authors are able to establish (2.2) as well, whereas for $q \neq 2$ they are able to prove appropriate estimates for a shifted equation $e^s \partial_t \tilde{u} + \omega \tilde{u} + \tilde{p}\left(\partial_s\right) \tilde{u} = \tilde{f}$, where $\omega \geq \omega_0$ is arbitrary and $\omega_0 \geq 0$ is fixed and unknown.

We observe, however, that (2.2) or (2.3) are *not* higher regularity results: since $D = x\partial_x$, all estimates have the same scaling with respect to x. Since furthermore α is in the coercivity range only if $\alpha < 0$ (cf. Corollary 5.4), none of the norms in (2.2) does control $u_{|x=0}$ — no matter how large ℓ is. Since the treatment of the nonlinearity by a fixed point argument, which we are aiming for, requires at least control of the supremum norm $\sup_{t\geq 0}\|u\| = \sup_{t\geq 0,\, x\geq 0}|u|$ (where $\|\cdot\|$ denotes the supremum norm in the space variable x), (2.2) or (2.3) are not enough. A first attempt is to apply (2.2) with u replaced by $\partial_x u$. However, because of the commutation relation $\partial_x D = (D+1)\partial_x$, this means that the elliptic operator in (1.10a) changes to $p(D+1)$. From the commutation relation $x(D+1) = Dx$ we learn that if α is in the coercivity range of $p(D)$, then $\alpha - 1$ is in the coercivity range of $p(D+1)$. Hence this approach does not yield better regularity near 0 in terms of scaling. A better idea is to apply (2.2) with u replaced by $v := p(D)u$. From the commutation relation $Dx = x(D+1)$ we learn that v satisfies (1.10) with the elliptic operator $p(D)$ replaced by $p(D-1)$:

$$x\partial_t v + p(D-1)v = p(D-1)f. \tag{2.5}$$

Now if α is in the coercivity range of $p(D)$, then $\alpha+1$ is in the coercivity range of $p(D-1)$. Hence we obtain from (2.2) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|v|^2_{\alpha+\frac{1}{2}} + C_1\left|D^{\ell+2}v\right|^2_{\alpha+\frac{1}{2}}\right) + |\partial_t v|^2_{\ell,\alpha} + |v|^2_{\ell+4,\alpha+1} \lesssim |f|^2_{\ell+4,\alpha+1}\,. \tag{2.6}$$

In formal terms of level of scaling, $|v|_{\ell+2,\alpha+\frac{1}{2}}$ is indeed stronger than $\|u\|$ near $x=0$ for $\alpha > -\frac{1}{2}$, and there are such α in the coercivity range; we will argue below that this is not just formal (cf. (2.9)–(2.11)). This procedure can be iterated and gives control of $|p(D-1)v|_{\ell+2,\alpha+\frac{3}{2}}$ in the second step. However, the use of the second and further steps in the iteration is limited, as we shall explain below (cf. (2.11)).

In order to leverage (2.6), we have to study another property of the elliptic operator $p(D)$ next to coercivity, namely *maximal regularity*: For instance, does $|p(D)u|_{\ell+4,\alpha+1} = |v|_{\ell+4,\alpha+1}$ control $|u|_{\ell+8,\alpha+1}$? Here we give a short summary of Section 7 (cf. Lemma 7.2): We have quite generally for a polynomial $q(\zeta)$, a weight exponent ϱ, a D-regularity level ℓ, and any $w : (0,\infty) \to \mathbb{R}$ that

$$|w|_{\ell+\deg,\varrho} \sim |q(D)w|_{\ell,\varrho} \quad \text{provided } w = o(x^\varrho) \text{ as } x \searrow 0 \text{ and } x \nearrow \infty, \tag{2.7}$$

where deg denotes the degree of $q(\zeta)$, and provided ϱ does not coincide with the real part of one of the roots of $q(\zeta)$ (cf. Lemma 7.4). Hence the expected regularity as $x \searrow 0$ (and $x \nearrow \infty$) of the solution to the *nonlinear* problem (1.8) is crucial. We know from [3] that the source-type solution (expressed in our coordinates) is an analytic function in the *two* spatial variables (x, y), where $y := x^\beta$. We expect this to be the generic behavior of solutions of the nonlinear parabolic equation (1.8), i.e. we expect that $u(t,x) = \bar{u}\left(t, x, x^\beta\right)$, where $\bar{u}(t,x,y)$ is a smooth function in (x,y) up to the contact line. This behavior is natural: Since β and 0 are the two largest roots of the quartic polynomial $p(\zeta)$, we expect that a generic solution of $x\partial_t u + p(D)u = f$ with compactly supported f contains an $y = x^\beta$-term near $x = 0$. Because of the $x\partial_t u$-term, it also generically contains an x-term. Since for the *nonlinear* equation, f is a nonlinear (analytic) function of u (and of Du, D^2u, D^3u, and D^4u), u will generically contain *all* monomials $x^k y^\ell$. Our study of the resolvent equation in Section 6 shows that as expected, we also do not need more than these monomials, and that u can be assumed to decay like a Schwartz function as $x \nearrow \infty$ if f does.

In view of this regularity and since $\beta \in \left(\frac{1}{2}, 1\right)$, the expected ordered expansion of u near $x = 0$ is given by

$$u = a + bx^\beta + cx + dx^{2\beta} + O\left(x^{1+\beta}\right) \tag{2.8}$$

with t-dependent coefficients a, b, c, and d. Since $-\beta - \frac{1}{2}$ is the first negative root of $p(\zeta)$, we thus obtain from (2.7) applied to $q(\zeta) = p(\zeta)$ that

$$|u|_{\ell+4,\varrho} \sim |v|_{\ell,\varrho} \quad \text{for } -\beta - \tfrac{1}{2} < \varrho < 0. \tag{2.9}$$

Applying (2.7) to $w = u - a \overset{(2.8)}{=} O(x^\beta)$, we obtain thanks to $p(D)a = 0$

$$|u - a|_{\ell+4,\varrho} \sim |v|_{\ell,\varrho} \quad \text{for } 0 < \varrho < \beta. \tag{2.10}$$

Likewise, applying (2.7) to $w = u-a-bx^\beta \overset{(2.8)}{=} O(x)$, we obtain thanks to $p(D)(a+bx^\beta) = 0$

$$\left|u - a - bx^\beta\right|_{\ell+4,\varrho} \sim |v|_{\ell,\varrho} \quad \text{for } \beta < \varrho < 1. \tag{2.11}$$

We can even go to the second step and apply (2.7) to $q(\zeta) = p(\zeta - 1)p(\zeta)$ and to

$$v = u - a - bx^\beta - cx \overset{(2.8)}{=} O(x^{2\beta})$$

and obtain because of $p(D-1)p(D)(a + bx^\beta + cx) = 0$ that

$$\left|u - a - bx^\beta - cx\right|_{\ell+8,\varrho} \sim |p(D-1)v|_{\ell,\varrho} \quad \text{for } 1 < \varrho < 2\beta. \tag{2.12}$$

However, the procedure stops here! Indeed, despite the fact that

$$u - a - bx^\beta - cx - dx^{2\beta} \overset{(2.8)}{=} O\left(x^{1+\beta}\right),$$

it is *not* true that

$$\left|u - a - bx^\beta - cx - dx^{2\beta}\right|_{\ell+8,\varrho} \lesssim |p(D-1)v|_{\ell,\varrho} \quad \text{for } 2\beta < \varrho < 1 + \beta,$$

since the monomial $y^2 = x^{2\beta}$, which is generated by the nonlinearity $\mathcal{N}(u)$, is not annihilated by $p(D-1)p(D)$. Hence besides (2.12), the second and all further steps in the iteration cannot be leveraged. We thus limit our attention to (2.9)–(2.11). Hence our entire approach is limited to the construction of solutions of moderate regularity.

Our goal is to capture at least the x^β-term, which means that in view of (2.11) (which in view of (2.6) we apply for $\varrho = \alpha+1$) we want the weight exponent α to satisfy $\beta-1 < \alpha < 0$, say $\alpha = \beta - 1 + \delta$ for some sufficiently small $\delta > 0$, which is compatible with the coercivity range (cf. Section 5, Corollary 5.4). From integrating (2.6) in t and applying (2.11), we thus get the following a priori estimate

$$\begin{aligned}&\sup_{t\geq 0}|u-a|^2_{\ell+6,\beta-\frac{1}{2}+\delta}+\int_0^\infty\left(|\partial_t u|^2_{\ell+4,\beta-1+\delta}+\left|u-a-bx^\beta\right|^2_{\ell+8,\beta+\delta}\right)\mathrm{d}t\\&\lesssim|u_0-a_0|_{\ell+6,\beta-\frac{1}{2}+\delta}+\int_0^\infty|f|^2_{\ell+4,\beta+\delta}\,\mathrm{d}t.\end{aligned}\tag{2.13}$$

If we based our existence result on (2.13), we would require control (and smallness for the nonlinear equation) of the initial data u_0 in $|u_0 - a_0|_{\ell+6,\beta-\frac{1}{2}+\delta}$, which is supercritical by the amount $\beta - \frac{1}{2} + \delta$. We are a bit more ambitious and want to control the initial data in a norm that in terms of scaling is closer to the critical norm $\|u_0\|$, say, the only slightly supercritical norm $|u_0|_{\ell+6,\delta}$. In order to do so, we employ the following strategy: We use (2.6) for $\alpha = \beta - 1 + \delta$ (and ℓ replaced by k) after integration with respect to the *time weight* $t^{2\beta-1}$ and combine it with the integrated version of (2.6) for $\alpha = -\frac{1}{2} + \delta$ (which also is in the coercivity range for $\delta \ll 1$, cf. Section 5, Corollary 5.4). As we argue in Section 7, for $\ell \geq k + 2$, this implies with help of the interpolation estimate Lemma 7.5 that

$$\begin{aligned}&\sup_{t\geq 0}\left(t^{2\beta-1}|u-a|^2_{k+6,\beta-\frac{1}{2}+\delta}+|u-a|^2_{\ell+6,\delta}\right)+\int_0^\infty\left(t^{2\beta-1}|\partial_t u|^2_{k+4,\beta-1+\delta}+|\partial_t u|^2_{\ell+4,-\frac{1}{2}+\delta}\right)\mathrm{d}t\\&+\int_0^\infty\left(t^{2\beta-1}\left|u-a-bx^\beta\right|^2_{k+8,\beta+\delta}+|u-a|^2_{\ell+8,\frac{1}{2}+\delta}\right)\mathrm{d}t\\&\lesssim|u_0-a_0|_{\ell+6,\delta}+\int_0^\infty\left(t^{2\beta-1}|f|^2_{k+4,\beta+\delta}+|f|_{\ell+4,\frac{1}{2}+\delta}\right)\mathrm{d}t,\end{aligned}\tag{2.14}$$

see the beginning of Section 7 for more details. However, these slightly supercritical norms by themselves do not yield control of $\sup_{t\geq 0}\|u\|^2$. Hence the estimate needs to be combined with a similar estimate that involves slightly subcritical norms, namely

$$\begin{aligned}&\sup_{t\geq 0}\left(t^{2\beta-1}|u-a|^2_{k+6,\beta-\frac{1}{2}-\delta}+|u|^2_{\ell+6,-\delta}\right)+\int_0^\infty\left(t^{2\beta-1}|\partial_t u|^2_{k+4,\beta-1-\delta}+|\partial_t u|^2_{\ell+4,-\frac{1}{2}-\delta}\right)\mathrm{d}t\\&+\int_0^\infty\left(t^{2\beta-1}|u-a|^2_{k+8,\beta-\delta}+|u-a|^2_{\ell+8,\frac{1}{2}-\delta}\right)\mathrm{d}t\\&\lesssim|u_0|_{\ell+6,-\delta}+\int_0^\infty\left(t^{2\beta-1}|f|^2_{k+4,\beta-\delta}+|f|_{\ell+4,\frac{1}{2}-\delta}\right)\mathrm{d}t.\end{aligned}\tag{2.15}$$

As we will see in Section 8, because of the nonlinear term, we will not just need $\ell \geq k + 2$ but $\ell \geq k + 3$ which we specify to $\ell = k + 3$. Hence we will work with the following compound a priori estimate for the linear problem (1.10):

$$|||u||| \lesssim |||u_0|||_0 + |||f|||_1,\tag{2.16}$$

where we have used the following abbreviations: The norm for the solution is given via

$$\begin{aligned}
|||u|||^2 := &\sup_{t\geq 0}\left(t^{2\beta-1}\left|u-a\right|^2_{k+6,\beta-\frac{1}{2}+\delta}+\left|u-a\right|^2_{k+9,\delta}\right)\\
&+\sup_{t\geq 0}\left(t^{2\beta-1}\left|u-a\right|^2_{k+6,\beta-\frac{1}{2}-\delta}+\left|u\right|^2_{k+9,-\delta}\right)\\
&+\int_0^\infty\left(t^{2\beta-1}\left|\partial_t u\right|^2_{k+4,\beta-1+\delta}+\left|\partial_t u\right|^2_{k+7,-\frac{1}{2}+\delta}\right)\mathrm{d}t\\
&+\int_0^\infty\left(t^{2\beta-1}\left|\partial_t u\right|^2_{k+4,\beta-1-\delta}+\left|\partial_t u\right|^2_{k+7,-\frac{1}{2}-\delta}\right)\mathrm{d}t\\
&+\int_0^\infty\left(t^{2\beta-1}\left|u-a-bx^\beta\right|^2_{k+8,\beta+\delta}+\left|u-a\right|^2_{k+11,\frac{1}{2}+\delta}\right)\mathrm{d}t\\
&+\int_0^\infty\left(t^{2\beta-1}\left|u-a\right|^2_{k+8,\beta-\delta}+\left|u-a\right|^2_{k+11,\frac{1}{2}-\delta}\right)\mathrm{d}t.
\end{aligned} \tag{2.17}$$

The norm for the initial data is given via

$$|||u_0|||_0^2 := \left|u_0-a_0\right|^2_{k+9,\delta}+\left|u_0\right|^2_{k+9,-\delta} \tag{2.18}$$

(and is just δ away from a critical norm). Finally the norm for the right hand side is given by

$$|||f|||_1^2 := \int_0^\infty\left(t^{2\beta-1}\left|f\right|^2_{k+4,\beta+\delta}+\left|f\right|^2_{k+7,\frac{1}{2}+\delta}+t^{2\beta-1}\left|f\right|^2_{k+4,\beta-\delta}+\left|f\right|^2_{k+7,\frac{1}{2}-\delta}\right)\mathrm{d}t. \tag{2.19}$$

3. Main Result and Discussion

3.1. Well-Posedness in a Quasi-Minimal Setting

We are now ready to state our main result:

Theorem 3.1. *Suppose $k\in\mathbb{N}_0$ and $0<\delta<\min\left\{1-\beta,\beta-\frac{1}{2}\right\}$. Then there exists $\varepsilon>0$ such that for all locally integrable*[3] *$u_0:(0,\infty)\to\mathbb{R}$ with $|||u_0|||_0<\varepsilon$, problem (1.8) has a unique solution $u:(0,\infty)^2\to\mathbb{R}$ that is locally integrable with $|||u|||<\infty$. Furthermore, the a priori estimate $|||u|||\lesssim_{k,\delta}|||u_0|||_0$ holds.*

For a precise discussion of the norms and underlying function spaces, we refer to Section 4.

In view of (2.17), we have

$$\int_0^\infty t^{2\beta-1}\left|u(t)-a(t)-b(t)x^\beta\right|^2_{k+6,\beta+\delta}\mathrm{d}t\leq |||u|||^2<\infty.$$

[3]Note that for all locally integrable functions $u:(0,\infty)\to\mathbb{R}$, the derivatives $D^\ell u$, $\ell\in\mathbb{N}$, exist in a distributional sense. Hence expressions like $|u|_{k,\alpha}$ make sense if allowed to assume the value ∞.

Since $\beta + \delta > 0$ and since sufficiently many spatial derivatives of u are locally square integrable, this implies in particular that $u(t,x)$ has the following expansion close to the contact line:

$$u(t,x) = a(t) + b(t)x^\beta + o(x^\beta) \quad \text{as } x \searrow 0 \tag{3.1}$$

almost everywhere in $t > 0$. We emphasize that we expect the power series expansion (3.1) to hold in an even more general fashion, that is, we expect that $u(t,x) = \bar{u}\left(t, x, x^\beta\right)$, where $\bar{u}(t,x,y)$ is a smooth function up to the contact line in (x,y). As we explained in Section 2, our method to prove Theorem 3.1 does, however, not generalize in an obvious way to such higher order regularity results.

Besides that, let us further study the behavior at the contact line: In Appendix A.2 we show that

$$V(t, Z(t,x)) = -\frac{3}{8}(1 + a(t))^3 + o(x^\beta) \quad \text{as } x \searrow 0. \tag{3.2}$$

Hence the vertically averaged speed of the droplet, expressed in Lagrangian coordinates, does not contain the leading order x^β-term. Thinking more generally of an expansion of $V(t, Z(t,x))$ as

$$V(t, Z(t,x)) = \sum_{0 \le k + \beta\ell \le N} a_{k\ell}(t) x^{k+\beta\ell} + o\left(x^N\right) \quad \text{as } x \searrow 0,$$

we are not able to conclude that $a_{k\ell}(t) \equiv 0$ for $\ell \ge 2$ or $k \ge 1$ and $\ell \ge 1$. Furthermore, to our knowledge it is not clear whether the underlying Stokes or Navier-Stokes equations show an analogous expansion in fractional powers at the contact line.

Remark 3.2. *From (3.2) we recover the speed of the moving contact line as*

$$V_0(t) = -\frac{3}{8}(1 + a(t))^3. \tag{3.3}$$

In addition, in Lemma B.4 we prove that $a(t)$ is continuous in time t with the limit $a(t) \to 0$ as $t \to \infty$, i.e. the speed of the contact line converges to the constant $-\frac{3}{8}$.

Remark 3.3. *Undoing the transformations (1.6) and (1.7) (cf. Appendix A.3), we obtain the expansion*

$$h(t,z) = \tilde{x}^{\frac{3}{2}}\left(1 + \tilde{a}(t) + \tilde{b}(t)\tilde{x}^\beta + o\left(\tilde{x}^\beta\right)\right) \quad \textit{as } \tilde{x} \searrow 0 \tag{3.4a}$$

almost everywhere in $t > 0$, where

$$\tilde{x} := z - Z_0(t), \quad \tilde{a}(t) := (1 + a(t))^{\frac{3}{2}} - 1, \quad \tilde{b}(t) := \frac{3b(t)(1 + a(t))^{\beta + \frac{1}{2}}}{2(1+\beta)}, \tag{3.4b}$$

and $Z_0(t) = z_0 - \frac{3}{8}\int_0^t \left(1 + a\left(t'\right)\right)^3 \mathrm{d}t'$ identifies the position of the contact line at time t. Hence, after factoring off the traveling wave $\sim \tilde{x}^{\frac{3}{2}}$, the function h exhibits an analogous expansion as u, F, or V.

Remark 3.4. *It is interesting to consider the case of the general thin-film equation* (1.2) *with* $n \in \left(\frac{3}{2}, 3\right)$*, since in this range of exponents the traveling wave solution* (1.5) *generalizes to* $H_{\mathrm{TW}} = x^{\frac{3}{n}}$*. The hodograph transform* (1.6) *with right hand side* $x^{\frac{3}{n}}$ *can be applied as well and yields structurally the same equation as* (1.8a)*, where* $p(\zeta)$ *is a fourth-order polynomial and the nonlinearity* $\mathcal{N}(u)$ *is build up of summands consisting of two to five factors* u*,* Du*,* D^2u*,* D^3u*, or* D^4u*. More precisely the polynomial* $p(\zeta)$ *is given by*

$$p(\zeta) = (\zeta + \nu)(\zeta + \beta + 3\nu - 4)\zeta(\zeta - \beta),$$

where

$$\nu = \frac{3}{n} \in (1, 2) \quad \textit{and} \quad \beta = \frac{\sqrt{-3\nu^2 + 12\nu - 8} + 4 - 3\nu}{2} \in (0, 1).$$

Then the discussion of the resolvent equation (Section 6) can be carried out only changing the numerical constants of the equation. However, the argumentation for the linear equation (Section 7) needed the restriction $\beta > \frac{1}{2}$*, which implies the lower bound*

$$n > \frac{3}{17}\left(15 - \sqrt{21}\right) \approx 1.84. \tag{3.5a}$$

On the other hand, both, $\beta - 1$ *and* $-\frac{1}{2}$ *need to be in the coercivity range of* $p(D)$*. Whereas the first condition does not lead to any additional constraint on* n*, the second leads to the restriction* $-\frac{1}{2} > -\beta - 3\nu + 4$ *or equivalently*

$$n < \frac{3}{11}(7 + \sqrt{5}) \approx 2.52. \tag{3.5b}$$

Then, provided the restrictions (3.5) *are fulfilled, the analog of Theorem 3.1 is valid. It seems that these restrictions are of technical origin and related to the fact that we work in an* L^2*-setting (rather than e.g. in* L^q *with* $q \in (1, \infty)$*). Whether a well-posedness result for the TFE* (1.2) *for the full range* $n \in \left(\frac{3}{2}, 3\right)$ *(or other mobility exponents) can be achieved, remains unanswered.*

3.2. Outline of the Paper

We briefly outline the content of the paper: In Section 4 we list the norms that are used in the following sections and summarize some of their properties. In Section 5 we discuss the coercivity properties of the differential operator $p(D)$ with respect to the inner product $(\cdot, \cdot)_\alpha$. We characterize in particular the coercivity range of $p(D)$, i.e., the set of α's for which $p(D)$ is coercive (cf. Proposition 5.3 and Corollary 5.4). The coercivity of $p(D)$ is crucial for proving existence and uniqueness of solutions to the corresponding resolvent equation

$$xu + p(D)u = f \tag{3.6}$$

(cf. Proposition 5.5). A standard way to obtain smooth solutions to the resolvent problem for smooth right hand side was carried out in [5] by starting with the Lax-Milgram solution and proving regularity afterwards. Instead, in this work we opted for ODE-based arguments. These have the advantage that they can be carried out independently of a special choice of norms. For that we construct families of solutions to (3.6) for $x \ll 1$ and $x \gg 1$ and characterize their asymptotics explicitly (cf. Section 6). Using coercivity, we are able to match these families of solutions and to prove well-posedness for the resolvent problem (3.6) (cf. Proposition 6.3). This was already done similarly by Angenent [35] for the PME.

In Section 7 we then treat the linear parabolic problem (1.10). Solutions can be obtained by a time discretization procedure leading to (3.6) as the time-discretized version of (1.10a). Appropriate maximal regularity estimates of the solution u in terms of the initial data u_0 and the right hand side f, however, are then formulated in our L^2-setting, as we do not expect them to hold in sup-norms (cf. Proposition 7.6).

Finally in Section 8 we prove appropriate estimates for the nonlinearity (cf. Lemma 8.1), leading to well-posedness of (1.8) by a fixed point argument.

Appendix A contains details of the coordinate transformations outlined in the introduction. Appendix B provides interpolation inequalities and approximation results that are used in Sections 7 and 8.

4. The Norms and their Properties

Let us list the norms introduced in Section 2. For any integrability exponent $\alpha \in \mathbb{R}$ and any interval $I \subseteq (0,\infty)$, our basic inner product is given by

$$(u,w)_{\alpha,I} := \int_I x^{-2\alpha} u(x) w(x) \, \frac{\mathrm{d}x}{x}. \tag{4.1a}$$

This inner product induces a norm

$$|u|_{\alpha,I} := \left(\int_I x^{-2\alpha} (u(x))^2 \, \frac{\mathrm{d}x}{x} \right)^{\frac{1}{2}}. \tag{4.1b}$$

We also use the following higher order norms and inner products (cf. (1.9a)):

$$(u,w)_{k,\alpha,I} := \sum_{\ell=0}^{k} (D^\ell u, D^\ell w)_{\alpha,I} = \sum_{\ell=0}^{k} \int_I x^{-2\alpha} D^\ell u(x) D^\ell w(x) \, \frac{\mathrm{d}x}{x}, \tag{4.2a}$$

$$|u|_{k,\alpha,I} := \sqrt{(u,u)_{k,\alpha,I}} = \left(\sum_{\ell=0}^{k} \int_I x^{-2\alpha} (D^\ell u(x))^2 \, \frac{\mathrm{d}x}{x} \right)^{\frac{1}{2}}. \tag{4.2b}$$

In the case of $I = (0,\infty)$ we omit to specify I, e.g., we write $|u|_{k,\alpha} := |u|_{k,\alpha,(0,\infty)}$.

We further note that D is skew-symmetric with respect to $(\cdot,\cdot)_0$, i.e. $(Du,w)_0 = -(u,Dw)_0$ for all $u, w : (0,\infty) \to \mathbb{R}$ smooth with $|u|_1, |w|_1 < \infty$. By the commutation relation between multiplication and differentiation operators,

$$x^\mu D = (D - \mu) x^\mu \quad \text{for } \mu \in \mathbb{R}, \tag{4.3}$$

we have more generally

$$(Du, w)_\alpha = -(u, (D - 2\alpha)w)_\alpha \tag{4.4}$$

for all $u, w : (0, \infty) \to \mathbb{R}$ smooth with $|u|_{1,\alpha}, |w|_{1,\alpha} < \infty$.

In fact, we can relate our norms to standard Sobolev norms. To this aim, we recall the definition of the Fourier transform

$$\mathcal{F}F(\xi) := \hat{F}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi s} F(s) \mathrm{d}s \tag{4.5}$$

for a Schwartz function $F : \mathbb{R} \to \mathbb{R}$, and we note that

$$\widehat{\partial_s F}(\xi) = i\xi \hat{F}(\xi). \tag{4.6}$$

Lemma 4.1. *Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. For $\tilde{u}(s) := e^{-\alpha s} u(e^s)$ ($s \in \mathbb{R}$) we have*

$$|u|_{k,\alpha} \sim_{k,\alpha} \|\tilde{u}\|_{W^{k,2}(\mathbb{R})}. \tag{4.7}$$

Furthermore if $\alpha \neq 0$, then $\left|D^k u\right|_\alpha \sim_{k,\alpha} \|\tilde{u}\|_{W^{k,2}(\mathbb{R})}$.

Proof. Passing to the logarithmic variable, i.e. setting $s := \ln x$, we have $D = x\partial_x = \partial_s$ and $\tilde{u}(s) := e^{-\alpha s} u(e^s)$ so that we arrive at

$$\left|D^\ell u\right|_\alpha^2 = \int_{-\infty}^{\infty} e^{-2\alpha s} \left(\partial_s^\ell u\left(e^s\right)\right)^2 \mathrm{d}s \overset{(4.3)}{=} \int_{-\infty}^{\infty} \Big(\left(\partial_s + \alpha\right)^\ell \tilde{u}(s)\Big)^2 \mathrm{d}s.$$

If $\alpha = 0$, then

$$|u|_{k,0}^2 = \sum_{\ell=0}^{k} \int_{-\infty}^{\infty} (\partial_s^\ell \tilde{u}(s))^2 \mathrm{d}s = \|\tilde{u}\|_{W^{k,2}(\mathbb{R})}^2,$$

hence (4.7). If $\alpha \neq 0$, in view of (4.5) and (4.6), we know that

$$\begin{aligned}\int_{-\infty}^{\infty} \Big(\left(\partial_s + \alpha\right)^\ell \tilde{u}(s)\Big)^2 \mathrm{d}s &= \int_{-\infty}^{\infty} (\xi^2 + \alpha^2)^\ell \left|\mathcal{F}\tilde{u}(\xi)\right|^2 \mathrm{d}\xi \sim_{\alpha,\ell} \int_{-\infty}^{\infty} (1 + \xi^{2\ell}) \left|\mathcal{F}\tilde{u}(\xi)\right|^2 \mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} \left(\tilde{u}^2(s) + (\partial_s^\ell \tilde{u}(s))^2\right) \mathrm{d}s \sim_\ell \|\tilde{u}\|_{W^{\ell,2}(\mathbb{R})},\end{aligned}$$

which implies (4.7). ◻

Remark 4.2. *We note that the space of test functions $C_0^\infty(\mathbb{R})$ is dense in $W^{k,2}(\mathbb{R})$ for $k \in \mathbb{N}_0$ and the transformation $u \mapsto \tilde{u}$, with $\tilde{u}(s) := e^{-\alpha s} u(e^s)$ ($s \in \mathbb{R}$), is a bijection $C_0^\infty((0,\infty)) \to C_0^\infty(\mathbb{R})$. Thus the equivalence (4.7) shows that we can approximate every locally integrable $u : (0,\infty) \to \mathbb{R}$ with $|u|_{k,\alpha} < \infty$ by a sequence of functions in $C_0^\infty((0,\infty))$ in the norm $|\cdot|_{k,\alpha}$.*

For later purposes (cf. Section 8) we also introduce the corresponding supremum norms, i.e. for any $k \in \mathbb{N}_0$ and any interval $I \subseteq (0,\infty)$, we set

$$\|u\|_{k,I} := \max_{\ell=0,\cdots,k} \left\|D^\ell u\right\|_I \quad \text{where} \quad \|w\|_I := \sup_{x\in I} |w(x)| \,. \tag{4.8}$$

If $I = (0,\infty)$, we just write $\|\cdot\|_k := \|\cdot\|_{k,I}$ and $\|\cdot\| := \|\cdot\|_I$.

We now discuss the norms which we will use for our solutions: For the initial data we use the following norm:

$$|||u_0|||_0 := \inf_{a_0\in\mathbb{R}} \left(|u_0|^2_{k+9,-\delta} + |u_0 - a_0|^2_{k+9,\delta}\right)^{\frac{1}{2}} \tag{4.9}$$

for a locally integrable $u_0 : (0,\infty) \to \mathbb{R}$. One easily sees that if $|||u_0|||_0 < \infty$ then the minimizer a_0 is uniquely determined and $a_0 = \lim_{x\searrow 0} u_0(x)$. This proves the equivalence to the definition in (2.18).

For any interval $I = (0,\tau) \subseteq (0,\infty)$, $k \in \mathbb{N}_0$, and $\delta > 0$, we define $|||\cdot|||_I$ as

$$\begin{aligned} |||u|||_I^2 := \inf_{a,b:I\to\mathbb{R}} \Bigg(& \sup_{t\in I} t^{2\beta-1} \left(|u-a|^2_{k+6,\beta-\frac{1}{2}+\delta} + |u-a|^2_{k+6,\beta-\frac{1}{2}-\delta}\right) \\ & + \sup_{t\in I} \left(|u-a|^2_{k+9,\delta} + |u|^2_{k+9,-\delta}\right) \\ & + \int_I \left(t^{2\beta-1} |\partial_t u|^2_{k+4,\beta-1+\delta} + |\partial_t u|^2_{k+7,-\frac{1}{2}+\delta}\right) \mathrm{d}t \\ & + \int_I \left(t^{2\beta-1} |\partial_t u|^2_{k+4,\beta-1-\delta} + |\partial_t u|^2_{k+7,-\frac{1}{2}-\delta}\right) \mathrm{d}t \\ & + \int_I \left(t^{2\beta-1} \left|u-a-bx^\beta\right|^2_{k+8,\beta+\delta} + |u-a|^2_{k+11,\frac{1}{2}+\delta}\right) \mathrm{d}t \\ & + \int_I \left(t^{2\beta-1} |u-a|^2_{k+8,\beta-\delta} + |u-a|^2_{k+11,\frac{1}{2}-\delta}\right) \mathrm{d}t \Bigg) \end{aligned} \tag{4.10}$$

for a locally integrable $u : I \times (0,\infty) \to \mathbb{R}$, where the infimum is taken among all locally integrable $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$. As before, if $|||u|||_I < \infty$, then a and b are uniquely defined almost everywhere in time t and given by the boundary values

$$a(t) := \lim_{x\searrow 0} u(t,x) \quad \text{and} \quad b(t) := \lim_{x\searrow 0} (u(t,x) - a(t))x^{-\beta}. \tag{4.11}$$

This also demonstrates that the definitions in (2.17) and (4.10) are equivalent. For $I = (0,\infty)$ we just write $|||\cdot||| := |||\cdot|||_{(0,\infty)}$ (cf. (2.17)).

Furthermore, the norm for the right hand side of (1.8a) and (1.10a), i.e. a locally integrable $f : I \times (0,\infty) \to \mathbb{R}$, is defined as in (2.19) where the time integrals are restricted to I instead of $(0,\infty)$.

We have the following linear estimates for the boundary values:

Lemma 4.3. *Suppose $k \in \mathbb{N}$ with $k \geq 2$, $\delta > 0$, and let $I = (0,\tau) \subseteq (0,\infty)$. Then the following estimates hold true:*

(a) For every locally integrable $u_0 : (0,\infty) \to \mathbb{R}$ such that $|||u_0|||_0 < \infty$, we have

$$|a_0| \lesssim_\delta |||u_0|||_0. \tag{4.12}$$

(b) For every locally integrable $u : I \times (0,\infty) \to \mathbb{R}$ such that $|||u|||_I < \infty$, we have

$$\sup_{t\in I} |a(t)|^2 + \int_I t^{2\beta-1} |b(t)|^2 \, \mathrm{d}t \lesssim_\delta |||u|||_I^2. \tag{4.13}$$

Proof. For part *(a)* we observe that

$$\begin{aligned}
|a_0|^2 &= \int_1^2 |a_0|^2 \, \mathrm{d}x \lesssim \int_1^2 |u_0(x) - a_0|^2 \, \mathrm{d}x + \int_1^2 |u_0(x)|^2 \, \mathrm{d}x \\
&\lesssim_\delta \int_0^\infty x^{-2\delta} |u_0(x) - a_0|^2 \, \frac{\mathrm{d}x}{x} + \int_0^\infty x^{2\delta} |u_0(x)|^2 \, \frac{\mathrm{d}x}{x} \\
&\leq |u_0|_{k+9,-\delta}^2 + |u_0 - a_0|_{k+9,\delta}^2 = |||u_0|||_0^2.
\end{aligned}$$

The same calculation also shows that

$$|a(t)|^2 \lesssim_\delta |u(t)|_{k+9,-\delta}^2 + |u(t) - a(t)|_{k+9,\delta}^2 \quad \text{for } t \in I.$$

Taking the supremum yields the desired estimate for the first summand on the left hand side of (4.13). For the second one observe that

$$\begin{aligned}
|b(t)|^2 &\lesssim \int_1^2 \left|b(t)x^\beta\right|^2 \mathrm{d}x \\
&\lesssim \int_1^2 |u(t,x) - a(t)|^2 \, \mathrm{d}x + \int_1^2 \left|u(t,x) - a(t) - b(t)x^\beta\right|^2 \mathrm{d}x \\
&\lesssim_\delta \int_0^\infty x^{-2(\beta-\delta)} |u(t,x) - a(t)|^2 \, \frac{\mathrm{d}x}{x} + \int_0^\infty x^{-2(\beta+\delta)} \left|u(t,x) - a(t) - b(t)x^\beta\right|^2 \frac{\mathrm{d}x}{x} \\
&\leq |u(t) - a(t)|_{k+8,\beta-\delta}^2 + \left|u(t) - a(t) - b(t)x^\beta\right|_{k+8,\beta+\delta}^2.
\end{aligned}$$

We can multiply this expression with $t^{2\beta-1}$ and integrate over t to obtain

$$\begin{aligned}
\int_I t^{2\beta-1} |b(t)|^2 \, \mathrm{d}t \lesssim_\delta &\int_I t^{2\beta-1} |u(t) - a(t)|_{k+8,\beta-\delta}^2 \, \mathrm{d}t \\
&+ \int_I t^{2\beta-1} \left|u(t) - a(t) - b(t)x^\beta\right|_{k+8,\beta+\delta}^2 \mathrm{d}t \leq |||u|||_I^2.
\end{aligned}$$

□

In Subsection B.2, we provide approximation results by smooth functions with specific properties for the norms $|||\cdot|||_0$ and $|||\cdot|||_I$.

5. The Coercivity Range

In this section we consider a general differential operator $P(D)$, where

$$P(\zeta) = (\zeta - \gamma_1)(\zeta - \gamma_2)(\zeta - \gamma_3)(\zeta - \gamma_4) \tag{5.1}$$

is a polynomial of fourth-order with ordered zeros $\gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \gamma_4$. Recalling the definition (4.5) of the Fourier transform and (4.6), we see that the differential operator $P(D)$ transforms into a multiplication with $P(i\xi)$ for the Fourier transform.

Let $\alpha \in \mathbb{R}$. Further recalling the symmetry property of D (cf. (4.4)), the polynomial $P(D)$ induces a bilinear form

$$B_\alpha(\varphi, \psi) := \left((D - \gamma_1)(D - \gamma_2)\varphi, (D + \gamma_3 - 2\alpha)(D + \gamma_4 - 2\alpha)\psi\right)_\alpha,$$

with $\varphi, \psi \in C_0^\infty((0, \infty))$.

Definition 5.1. *We call $P(D)$ formally coercive with respect to $|\cdot|_\alpha$, if and only if there exists a $\lambda > 0$ such that for all $\varphi \in C_0^\infty((0, \infty))$*

$$B_\alpha(\varphi, \varphi) = (P(D)\varphi, \varphi)_\alpha \geq \lambda \, |\varphi|_\alpha^2$$

holds. The set of $\alpha \in \mathbb{R}$ for which $P(D)$ is formally coercive with respect to $|\cdot|_\alpha$, is called the coercivity range of $P(D)$.

We can characterize the coercivity range by its symbol:

Lemma 5.2. *For the operator $P(D)$ defined in (5.1) and $\alpha \in \mathbb{R}$ the following statements are equivalent:*

(a) $P(D)$ is formally coercive with respect to $|\cdot|_\alpha$.

(b) There exists $\lambda > 0$ such that $\mathcal{R}e\, P(i\xi + \alpha) \geq \lambda$ for all $\xi \in \mathbb{R}$.

(c) There exists $\lambda' > 0$ such that for all $\varphi \in C_0^\infty((0, \infty))$

$$B_\alpha(\varphi, \varphi) = (P(D)\varphi, \varphi)_\alpha \geq \lambda' \, |\varphi|_{2,\alpha}^2 .$$

(d) There exists $\lambda' > 0$ such that for all locally integrable $\varphi : (0, \infty) \to \mathbb{R}$ with $|\varphi|_{2,\alpha} < \infty$

$$B_\alpha(\varphi, \varphi) \geq \lambda' \, |\varphi|_{2,\alpha}^2 .$$

Proof. Obviously *(d)* implies *(c)* and *(d)* follows from *(c)* by approximation (cf. Remark 4.2). By Definition 5.1 and since $|\varphi|_{2,\alpha} \geq |\varphi|_\alpha$, *(c)* implies *(a)*.

In order to show the other implications we note that, letting $\tilde{\varphi}(s) = e^{-\alpha s}\varphi(e^s)$, we have

$$(P(D)\varphi, \varphi)_\alpha = \int_{-\infty}^{\infty} e^{-2\alpha s}\varphi(e^s) P(\partial_s)\varphi(e^s) \, \mathrm{d}s = \int_{-\infty}^{\infty} \tilde{\varphi}(s) P(\partial_s + \alpha)\tilde{\varphi}(s) \, \mathrm{d}s,$$

where we used the commutator relation $[\partial_s, e^{-\alpha s}] = -\alpha e^{-\alpha s}$. We can now apply Plancherel's Theorem and arrive at

$$(P(D)\varphi,\varphi)_\alpha \overset{(4.6)}{=} \int_{-\infty}^{\infty} |\mathcal{F}\tilde{\varphi}(\xi)|^2 P(i\xi+\alpha)\,\mathrm{d}\xi = \int_{-\infty}^{\infty} |\mathcal{F}\tilde{\varphi}(\xi)|^2 \mathcal{R}e\, P(i\xi+\alpha)\,\mathrm{d}\xi. \qquad (5.2)$$

In the last identity we used that the inner product $(P(D)\varphi,\varphi)_\alpha$ is real.

We now show that *(a)* implies *(b)*: Indeed, suppose *(b)* does not hold, i.e. $\xi_0 \in \mathbb{R}$ exists such that $\mathcal{R}e\, P(i\xi_0+\alpha) = 0$. Then we can choose a sequence $(\varphi_n)_{n\in\mathbb{N}}$ such that $\tilde{\varphi}_n(s) := e^{-\alpha s}\varphi_n(e^s)$ are Schwartz functions obeying $|\mathcal{F}\tilde{\varphi}_n|^2 \to \delta_{\xi_0}$ as distributions. Then by (5.2) $(P(D)\varphi_n,\varphi_n)_\alpha \to 0$, which implies that *(a)* does not hold.

It remains to show that *(b)* implies *(c)*: If *(b)* holds, then $\mathcal{R}e\, P(i\xi+\alpha) \geq \lambda'(1+\xi^2+\xi^4)$ for some $\lambda' \in (0,\lambda)$ (since the polynomial $P(\zeta)$ is of fourth-order), and *(c)* follows. □

We can derive a partial but explicit characterization of the coercivity range:

Proposition 5.3. *The operator $P(D)$ of the form defined in (5.1) is formally coercive with respect to $|\cdot|_\alpha$, if the following conditions hold:*

$$\alpha \in (-\infty,\gamma_1) \cup (\gamma_2,\gamma_3) \cup (\gamma_4,\infty), \qquad (5.3a)$$

$$|\alpha - m(\gamma)| \leq \frac{1}{\sqrt{3}}\sigma(\gamma). \qquad (5.3b)$$

Here $m(\gamma)$ denotes the algebraic mean of the zeros γ_ℓ, i.e.

$$m(\gamma) := \frac{1}{4}\sum_{\ell=1}^{4}\gamma_\ell,$$

and $\sigma(\gamma)$ the nonnegative root of the variance

$$\sigma^2(\gamma) = \frac{1}{4}\sum_{\ell=1}^{4}\gamma_\ell^2 - m^2(\gamma) = \frac{1}{4}\sum_{\ell=1}^{4}\left(\gamma_\ell - m(\gamma)\right)^2.$$

Evaluating equations (5.3) for the polynomial (1.9d) of interest shows:

Corollary 5.4. *The operator $p(D)$ defined through equation (1.9d) is formally coercive with respect to $|\cdot|_\alpha$ if $\alpha \in (-1,0)$.*

Proof of Proposition 5.3. Because of $P(-i\xi+\alpha) = \prod_{\ell=1}^4(-i\xi-(\gamma_\ell-\alpha))$ and the fact that odd powers of ξ in the symbol $P(-i\xi+\alpha)$ have imaginary pre-factors, we obtain

$$\mathcal{R}e\, P(-i\xi+\alpha) = \kappa^2 - 2a\kappa + b, \quad \text{where } \kappa := \xi^2 \geq 0,$$

and

$$a := \frac{1}{2}\sum_{1\leq j<\ell\leq 4}(\gamma_j-\alpha)(\gamma_\ell-\alpha), \quad b := (\gamma_1-\alpha)(\gamma_2-\alpha)(\gamma_3-\alpha)(\gamma_4-\alpha).$$

Hence in view of Lemma 5.2 *(b)*, $P(D)$ is formally coercive with respect to $|\cdot|_\alpha$ if and only if

$$(a \leq 0 \text{ and } b > 0) \quad \text{or} \quad (a > 0 \text{ and } b > a^2). \tag{5.4}$$

Since it can be characterized explicitly in terms of α, we focus on the first part of (5.4), thus deriving a sufficient criterion for coercivity. Clearly, the condition $b > 0$ is equivalent to (5.3a). In order to see that $a \leq 0$ is equivalent to (5.3b), we note that by definition of a and γ_j:

$$2a = 6\alpha^2 - 3\left(\sum_{j=1}^4 \gamma_j\right)\alpha + \sum_{1\leq j<\ell\leq 4}\gamma_j\gamma_\ell = 6\left(\alpha - \frac{1}{4}\sum_{j=1}^4\gamma_j\right)^2 - \frac{3}{8}\left(\sum_{j=1}^4\gamma_j\right)^2 + \sum_{1\leq j<\ell\leq 4}\gamma_j\gamma_\ell.$$

It remains to notice that

$$\frac{3}{8}\left(\sum_{j=1}^4\gamma_j\right)^2 - \sum_{1\leq j<\ell\leq 4}\gamma_j\gamma_\ell = \frac{1}{2}\sum_{j=1}^4\gamma_j^2 - \frac{1}{8}\sum_{1\leq j,\ell\leq 4}\gamma_j\gamma_\ell = 2\left(\frac{1}{4}\sum_{j=1}^4\gamma_j^2 - \left(\frac{1}{4}\sum_{j=1}^4\gamma_j\right)^2\right),$$

showing

$$2a = 6(\alpha - m(\gamma))^2 - 2\sigma^2(\gamma).$$

□

The coercivity of $P(D)$ is essential for characterizing the coercivity of its associated resolvent operator:

Proposition 5.5. *Suppose that $u \in C^\infty((0,\infty))$ solves*

$$xu + P(D)u = 0 \quad \textit{in} \quad (0,\infty). \tag{5.5}$$

If $|u|_{2,\alpha} < \infty$ for some $\alpha \in \mathbb{R}$ in the coercivity range of $P(D)$, then $u(x) \equiv 0$.

Proof. We set $\tilde{u}(s) := u(e^s)$ so that (5.5) turns into

$$e^s\tilde{u} + P(\partial_s)\tilde{u} = 0. \tag{5.6}$$

Next we take $\eta \in C_0^\infty(\mathbb{R})$ with $0 \leq \eta \leq 1$, $\eta(s) = 1$ for $|s| \leq 1$, and $\eta(s) = 0$ for $|s| \geq 2$, and set $\eta_n(s) := \eta\left(\frac{s}{n}\right)$. Then we test equation (5.6) with $e^{-2\alpha s}\eta_n\tilde{u}$ and perform an integration by parts so that

$$(\eta_n\tilde{u},\tilde{u})_{\alpha-\frac{1}{2}} + \tilde{B}_\alpha(\tilde{u},\eta_n\tilde{u}) = 0,$$

with

$$\tilde{B}_\alpha(\tilde{u}_1,\tilde{u}_2) := \int_{-\infty}^\infty e^{-2\alpha s}((D-\gamma_1)(D-\gamma_2)\tilde{u}_1)((D+\gamma_3-2\alpha)(D+\gamma_4-2\alpha)\tilde{u}_2)\,\mathrm{d}s.$$

Obviously $(\eta_n\tilde{u},\tilde{u})_{\alpha-\frac{1}{2}} \geq 0$ so that $\tilde{B}_\alpha(\tilde{u},\eta_n\tilde{u}) \leq 0$. By Lebesgue's theorem on dominated convergence it is elementary to see that $|\tilde{u}-\eta_n\tilde{u}|_{2,\alpha} \to 0$ as $n \to \infty$. Then also $\tilde{B}_\alpha(\tilde{u},\eta_n\tilde{u}) \to \tilde{B}_\alpha(\tilde{u},\tilde{u})$ as $n \to \infty$, i.e. $\tilde{B}_\alpha(\tilde{u},\tilde{u}) \leq 0$. Since $\tilde{B}_\alpha(\tilde{u},\tilde{u}) = B_\alpha(u,u) \geq \lambda'|u|_{2,\alpha}^2$ with $\lambda' > 0$ by Lemma 5.2 *(d)*, we need to have $|u|_{2,\alpha} = 0$, viz. $u(x) \equiv 0$. □

6. The Resolvent Equation

6.1. The Main Result

In this section we construct a unique solution to the resolvent equation

$$xu + p(D)u = f \quad \text{for } x > 0. \tag{6.1}$$

The understanding of solutions to (6.1) will be crucial in order to understand the linear parabolic equation associated to (1.8a), (6.1) being its time-discrete analog. Here well-posedness of (6.1) holds for functions $u(x)$ and data $f(x)$ that are regular and bounded as $x \searrow 0$ and decay as $x \nearrow \infty$. More precisely we define:

Definition 6.1. *A function $f \in C^\infty((0,\infty))$ is said to satisfy (G_0), if there exist $\varepsilon > 0$ and a function $\bar{f}(x,y)$, analytic in $[0,\varepsilon] \times [0,\varepsilon^\beta]$, with $(\bar{f}, \partial_y \bar{f})(0,0) = (0,0)$ such that $f(x) = \bar{f}(x, x^\beta)$ for $0 \le x \le \varepsilon$.*

Definition 6.2. *A function $f \in C^\infty((0,\infty))$ is said to satisfy (G_∞), if*

$$\limsup_{x\to\infty} \left| e^{4\nu \sqrt[4]{x}} \partial_x^j f(x) \right| < \infty \quad \textit{for all } j \in \mathbb{N}_0 \textit{ and all } \nu \in \left[0, \tfrac{1}{\sqrt{2}}\right).$$

The main result of this section then reads:

Proposition 6.3 (Resolvent equation)**.** *Suppose $f \in C^\infty((0,\infty))$ satisfies (G_0) and (G_∞). Then there exists exactly one solution $u(x)$ to the resolvent equation (6.1) such that $|u|_{2,\alpha} < \infty$ for some α in the coercivity range of $p(D)$. In addition, u satisfies (G_0) and (G_∞).*

We obtain solutions to (6.1) in three steps:

1. We construct a two-parameter family of solutions to (6.1) for $x \ll 1$ satisfying (G_0).
2. We construct a two-parameter family of solutions to (6.1) for $x > 0$ fulfilling (G_∞).
3. Using the coercivity result, Proposition 5.5, we find exactly one solution of (6.1) obeying both (G_0) and (G_∞).

6.2. The Resolvent Equation for $x \ll 1$

For the first step we notice that the term xu can be treated as a perturbation of the fourth-order equation

$$p(D)u = f. \tag{6.2}$$

The corresponding homogeneous equation has two linearly independent bounded solutions, namely x^0 and x^β. Hence one expects that a generic bounded solution $u(x)$ to (6.2) behaves as

$$u = a_1 + a_2 x^\beta + o(x^\beta) \quad \text{as } x \searrow 0, \quad \text{with } a_1, a_2 \in \mathbb{R} \tag{6.3}$$

and that $f = o(x^\beta)$ as $x \searrow 0$. Therefore it is convenient to unfold the singular behavior at $x \searrow 0$ by introducing a second spatial variable according to

$$u(x) = \bar{u}(x, x^\beta), \quad f(x) = \bar{f}(x, x^\beta) \tag{6.4}$$

and to identify the derivative D with

$$\bar{D} := x\partial_x + \beta y \partial_y, \tag{6.5}$$

i.e. $Dv(x) = \bar{D}v(x, x^\beta)$ whenever $\bar{v}(x,y)$ and $v(x) = \bar{v}(x, x^\beta)$ are smooth in (x,y) and x, respectively (cf. [3]). Then, instead of solving equation (6.2), we look for solutions to the equation

$$p(\bar{D})\bar{u} = \bar{f} \quad \text{for } (x,y) \in Q := [0,\ell_x] \times [0,\ell_y] \tag{6.6a}$$

for some $\ell_x, \ell_y > 0$. Instead of (6.3), we impose the boundary conditions

$$(\bar{u}, \partial_y \bar{u})(0,0) = (a_1, a_2). \tag{6.6b}$$

Then, solving problem (6.6) and using the identification (6.4) as defining equations for $u(x)$ and $f(x)$, we also obtain a solution of (6.2) obeying the boundary behavior (6.3). Exploiting the linearity of (6.6a), we can write $\bar{u} = a_1 + a_2 y + \bar{u}_0$, where $\bar{u}_0$ satisfies (6.6) with homogeneous boundary conditions. Hence it is sufficient to consider the case $a_1 = a_2 = 0$.

Lemma 6.4. *Let $Q = [0,\ell_x] \times [0,\ell_y]$ with $\ell_x, \ell_y > 0$. For all $\bar{f}(x,y)$ smooth in Q such that $\left(\bar{f}, \partial_y \bar{f}\right)(0,0) = (0,0)$, there exists $\bar{u}(x,y) =: \left(T\bar{f}\right)(x,y)$ smooth satisfying (6.6a) and $(\bar{u}, \partial_y \bar{u})(0,0) = (0,0)$. Furthermore $\bar{u}(x,y)$ obeys the estimates*

$$\sum_{m=0}^{4} \left\| \bar{D}^m \partial_x^k \partial_y^\ell \bar{u} \right\|_{C^0(Q)} \lesssim \left\| \partial_x^k \partial_y^\ell \bar{f} \right\|_{C^0(Q)} \tag{6.7}$$

for all $(k,\ell) \in \mathbb{N}_0^2 \setminus \{(0,0),(0,1)\}$.

Proof. We will not go into the details of the proof of Lemma 6.4, as it is mainly contained in a previous work by three of the authors [3]. The crucial point is to notice that the product form of $p(\bar{D})$

$$p(\bar{D}) = \left(\bar{D} + \frac{3}{2}\right)\left(\bar{D} + \beta + \frac{1}{2}\right)\bar{D}\left(\bar{D} - \beta\right)$$

induces a product form of the solution operator T as

$$T = T_\beta T_0 T_{-\beta-\frac{1}{2}} T_{-\frac{3}{2}}.$$

Here T_γ for $\gamma \in \left\{-\frac{3}{2}, -\beta - \frac{1}{2}, 0, \beta\right\}$ is the solution operator associated to the problem

$$\begin{cases} (\bar{D} - \gamma)\bar{v} = \bar{g} \quad \text{for } (x,y) \in Q,, \\ (\bar{v}, \partial_y \bar{v})(0,0) = (0,0), \end{cases}$$

where $\bar{g}(x,y)$ is smooth with $\left(\bar{g}, \partial_y \bar{g}\right)(0,0) = (0,0)$. The solution has an explicit representation

$$\bar{v}(x,y) = (T_\gamma \bar{g})(x,y) = \int_0^1 r^{-\gamma} \bar{g}(rx, r^\beta y)\, \frac{\mathrm{d}r}{r}, \tag{6.8}$$

which is finite for all $\gamma \in \{-\frac{3}{2}, -\beta - \frac{1}{2}, 0, \beta\}$ only if $(\bar{g}, \partial_y \bar{g})(0,0) = (0,0)$. Differentiating (6.8), we obtain

$$\partial_x^k \partial_y^\ell \bar{v}(x,y) = \int_0^1 r^{-\gamma+k+\beta\ell} \bar{\partial}_x^k \partial_y^\ell \bar{g}(rx, r^\beta y) \, \frac{\mathrm{d}r}{r}.$$

For $(k,\ell) \in \mathbb{N}_0^2 \setminus \{(0,0),(0,1)\}$, the power of r in the integrand is larger than -1 and estimates (6.7) follow directly. □

We now return to the resolvent equation (6.1) and apply the unfolding described above, leading to

$$\begin{cases} x\bar{u} + p(\bar{D})\bar{u} = \bar{f} \quad \text{for } (x,y) \in Q, \\ (\bar{u}, \partial_y \bar{u})(0,0) = (a_1, a_2). \end{cases} \tag{6.9}$$

We can accomplish the first step, outlined at the beginning of this section, by employing Lemma 6.4 and using a fixed point argument:

Lemma 6.5 (The resolvent equation for $x \ll 1$)**.** *There exists $\varepsilon > 0$ such that for any $L > 0$, any $a_1, a_2 \in \mathbb{R}$, and any $\bar{f}(x,y)$ analytic in $Q = [0,\varepsilon] \times [0,L]$ with $\left(\bar{f}, \partial_y \bar{f}\right)(0,0) = (0,0)$ and*

$$\sum_{k,\ell=0}^{\infty} \frac{\varepsilon^k L^\ell}{k!\ell!} \left\| \partial_x^k \partial_y^\ell (\bar{f} - a_1 x - a_2 xy) \right\|_{C^0(Q)} =: K < \infty, \tag{6.10}$$

problem (6.9) has an analytic solution $\bar{u}(x,y)$. Furthermore

$$\sum_{k,\ell=0}^{\infty} \sum_{m=0}^{4} \frac{\varepsilon^k L^\ell}{k!\ell!} \left\| \partial_x^k \partial_y^\ell \bar{D}^m (\bar{u} - a_1 - a_2 y) \right\|_{C^0(Q)} \lesssim K. \tag{6.11}$$

In particular, $u(x) := \bar{u}(x, x^\beta)$ is a solution to (6.1) for $x \le \varepsilon$ with

$$u = a_1 + a_2 x^\beta + o(x^\beta) \quad \text{as } x \searrow 0.$$

Proof. We write

$$\bar{u}_0 := \bar{u} - a_1 - a_2 y \quad \text{and} \quad \bar{f}_0 := \bar{f} - a_1 x - a_2 xy,$$

so that problem (6.9) is equivalent to

$$\begin{cases} x\bar{u}_0 + p(\bar{D})\bar{u}_0 = \bar{f}_0 \quad \text{for } (x,y) \in Q, \\ (\bar{u}_0, \partial_y \bar{u}_0) = (0,0). \end{cases}$$

Note that $\left(\bar{f}_0, \partial_y \bar{f}_0\right)(0,0) = \left(\bar{f}, \partial_y \bar{f}\right)(0,0) = (0,0)$ holds, too. Hence we can apply the solution operator T constructed in Lemma 6.4, which leaves us with the fixed point equation

$$\bar{u}_0 = \mathcal{T}[\bar{u}_0] \; := \; T\left[\bar{f}_0 - x\bar{u}_0\right] = T\bar{f}_0 - T\left[x\bar{u}_0\right]. \tag{6.12}$$

For applying the contraction mapping theorem, we introduce the norms

$$\|\bar{u}_0\|_1 := \sum_{k,\ell=0}^{\infty} \sum_{m=0}^{4} \frac{\varepsilon^k L^\ell}{k!\ell!} \left\|\partial_x^k \partial_y^\ell \bar{D}^m \bar{u}_0\right\|_{C^0(Q)} \tag{6.13}$$

and

$$\left\|\bar{f}_0\right\|_0 := \sum_{k,\ell=0}^{\infty} \frac{\varepsilon^k L^\ell}{k!\ell!} \left\|\partial_x^k \partial_y^\ell \bar{f}_0\right\|_{C^0(Q)}, \tag{6.14}$$

mimicking the Taylor series of $\bar{u}_0$ and $\bar{f}_0$.

The maximal regularity estimates (6.7) of Lemma 6.4 cease to hold for the cases $(k,\ell) \in \{(0,0),(0,1)\}$. Therefore, we need to show that the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ are equivalent to the ones not containing the indices $(k,\ell) \in \{(0,0),(0,1)\}$. This follows from the basic estimate

$$\|\bar{g}\| + L\|\partial_y \bar{g}\| \lesssim \varepsilon \|\partial_x \bar{g}\| + L^2 \left\|\partial_y^2 \bar{g}\right\| \tag{6.15}$$

for all smooth $\bar{g}(x,y)$ with $(\bar{g}, \partial_y \bar{g})(0,0) = (0,0)$. In order to show (6.15), by scaling $x = \varepsilon \hat{x}$ and $y = L\hat{y}$, it is enough to treat the case of $\varepsilon = L = 1$. Then estimate (6.15) is an immediate consequence of the representations

$$\bar{g}(x,y) = \int_0^x \partial_x \bar{g}\left(x', y\right) \mathrm{d}x' + \int_0^y \int_0^{y'} \partial_y^2 \bar{g}\left(0, y''\right) \mathrm{d}y'' \,\mathrm{d}y',$$

which yields $\|\bar{g}\| \le \|\partial_x \bar{g}\| + \left\|\partial_y^2 \bar{g}\right\|$, and

$$\partial_y \bar{g}(x,y) = \bar{g}(x,1) - \bar{g}(x,0) + \int_0^y y' \partial_y^2 \bar{g}\left(x, y'\right) \mathrm{d}y' - \int_y^1 \left(1 - y'\right) \partial_y^2 \bar{g}\left(x, y'\right) \mathrm{d}y',$$

demonstrating $\|\partial_y \bar{g}\| \lesssim \|\bar{g}\| + \left\|\partial_y^2 \bar{g}\right\|$.

It follows from (6.15) and (6.7) that there exists a universal constant $M \in (0,\infty)$ such that

$$\|T\bar{g}\|_1 \le M \|\bar{g}\|_0 \tag{6.16}$$

for all smooth $\bar{g}(x,y)$ with $\left(\bar{g}, \partial_y \bar{g}\right)(0,0) = (0,0)$.

For $\varepsilon > 0$ we introduce the complete metric space

$$X_C := \{\bar{u}_0 \,:\, \|\bar{u}_0\|_1 \le CK,\, (\bar{u}_0, \partial_y \bar{u}_0)(0,0) = (0,0)\}, \tag{6.17a}$$

with K as in (6.10), endowed with the distance function

$$d_X\left(\bar{u}_0^{(1)}, \bar{u}_0^{(2)}\right) := \left\|\bar{u}_0^{(1)} - \bar{u}_0^{(2)}\right\|_1. \tag{6.17b}$$

We will specify the constant $C > 0$ further below. Using Leibniz' rule, we obtain

$$\begin{aligned}
\|T[x\bar{u}_0]\|_1 &\overset{(6.16)}{\leq} M\|x\bar{u}_0\|_0 \overset{(6.14)}{=} M\sum_{k,\ell=0}^{\infty}\frac{\varepsilon^k L^\ell}{k!\ell!}\left\|\partial_x^k\partial_y^\ell(x\bar{u}_0)\right\|_{C^0(Q)} \\
&\leq M\sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty}\frac{\varepsilon^k L^\ell}{k!\ell!}k\left\|\partial_x^{k-1}\partial_y^\ell\bar{u}_0\right\|_{C^0(Q)} + M\sum_{k,\ell=0}^{\infty}\frac{\varepsilon^k L^\ell}{k!\ell!}\|x\|_{C^0(Q)}\left\|\partial_x^k\partial_y^\ell\bar{u}_0\right\|_{C^0(Q)} \\
&\leq M\varepsilon\sum_{k,\ell=0}^{\infty}\frac{\varepsilon^k L^\ell}{k!\ell!}\left\|\partial_x^k\partial_y^\ell\bar{u}_0\right\|_{C^0(Q)} + M\varepsilon\sum_{k,\ell=0}^{\infty}\frac{\varepsilon^k L^\ell}{k!\ell!}\left\|\partial_x^k\partial_y^\ell\bar{u}_0\right\|_{C^0(Q)} \\
&\leq 2M\varepsilon\|\bar{u}_0\|_0, \qquad (6.18)
\end{aligned}$$

where in the third inequality we have used the gain in scaling encoded in the two summands by the index k and by the $\|x\|_{C^0(Q)}$ term, respectively. Therefore

$$\begin{aligned}
\left\|\mathcal{T}\left[\bar{u}_0^{(1)}\right] - \mathcal{T}\left[\bar{u}_0^{(2)}\right]\right\|_1 &\overset{(6.12)}{=} \left\|T\left[x\left(\bar{u}_0^{(1)} - \bar{u}_0^{(2)}\right)\right]\right\|_1 \overset{(6.18)}{\leq} 2M\varepsilon\left\|\bar{u}_0^{(1)} - \bar{u}_0^{(2)}\right\|_0 \\
&\leq 2M\varepsilon\left\|\bar{u}_0^{(1)} - \bar{u}_0^{(2)}\right\|_1 \qquad (6.19)
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{T}[\bar{u}_0]\|_1 &\overset{(6.12)}{\leq} \left\|T\bar{f}_0\right\|_1 + \|T[x\bar{u}_0]\|_1 \overset{(6.16),(6.18)}{\leq} M\left\|\bar{f}_0\right\|_0 + 2M\varepsilon\|\bar{u}_0\|_0 \\
&\overset{(6.10),(6.14),(6.17a)}{\leq} M(2C\varepsilon + 1)K. \qquad (6.20)
\end{aligned}$$

Choosing $C := 2M$ and $\varepsilon \leq \frac{1}{4M}$, we infer from (6.20) that $\|\mathcal{T}[\bar{u}_0]\|_1 \leq CK$. Together with (6.19), this shows that the mapping $\mathcal{T}$ maps X_C into itself and is a contraction provided $\varepsilon \leq \frac{1}{4M}$. Hence we obtain a unique fixed point $\bar{u}(x,y)$, which by (6.17a) obeys (6.11). □

6.3. The Resolvent Equation for $x \gg 1$

We now derive the two-parameter family of solutions, which are well-behaved for $x \gg 1$. The solutions are characterized in the following way:

Lemma 6.6 (The resolvent equation for $x \gg 1$)**.** *Assume $f \in C^\infty((0,\infty))$ satisfies (G_∞). Then:*

(a) there exists a smooth solution $u_\infty(x)$ of the resolvent equation (6.1) that satisfies (G_∞);

(b) there exist two linearly independent solutions u_3, u_4 of the resolvent equation (6.1) with $f = 0$ that satisfy (G_∞);

(c) there exists a two-parameter family of solutions of (6.1) that satisfy (G_∞) and that is of the form

$$u(x) = u_\infty(x) + a_3u_3(x) + a_4u_4(x) \quad \textit{for } x > 0, \quad a_1, a_2 \in \mathbb{R}.$$

Proof. We split the proof into three parts:

Proof of (a): Construction of a particular solution. Using the transformations

$$r := 4\sqrt[4]{x}, \quad \tilde{u}(r) := u(x), \quad \text{and} \quad \tilde{f}(r) := x^{-1}f(x), \tag{6.21}$$

and noting that $x\partial_x = \frac{1}{4}r\partial_r$, equation (6.1) may be rewritten as

$$\left(1 + \partial_r^4\right)\tilde{u} + r^{-1}q\left(r^{-1}, \partial_r\right)\tilde{u} = \tilde{f} \quad \text{for } r > 0, \tag{6.22}$$

where

$$q\left(\frac{1}{r}, \partial_r\right) = 14\partial_r^3 + \alpha_2 r^{-1}\partial_r^2 + \alpha_3 r^{-2}\partial_r, \tag{6.23}$$

is a homogeneous polynomial of degree 3 (mind the scaling) with $\alpha_2, \alpha_3 \in \mathbb{R}$.

Take a smooth cut-off function $\eta_0(r)$ with $\eta_0(r) = 1$ for $r \geq 1$, $\eta_0(r) = 0$ for $r \leq \frac{1}{2}$, and let $\eta(r) = \eta_0\left(\frac{r}{R}\right)$ for some $R \geq 1$ (to be specified later), so that $|\partial_r^j \eta(r)| \lesssim_j R^{-j}$ for $j \geq 0$. We seek a solution $\tilde{v}$ of

$$\left(1 + \partial_r^4\right)\tilde{v} + Q\left(r^{-1}, \partial_r\right)\tilde{v} = \eta\tilde{f} \quad \text{for } r \in \mathbb{R}, \tag{6.24}$$

where

$$Q\left(r^{-1}, \partial_r\right) := \eta(r)r^{-1}q\left(r^{-1}, \partial_r\right) =: \beta_1(r)\partial_r^3 + \beta_2(r)\partial_r^2 + \beta_3(r)\partial_r.$$

Then, for $r \geq R$, $\tilde{v}(r)$ is a solution of (6.22) that, by standard ODE theory, can be extended to a solution $\tilde{u}(r)$ of (6.22) for all $r > 0$. Undoing our transformation, we can define $u_\infty(x)$ by $u_\infty(x) := \tilde{u}(4\sqrt[4]{x})$ for $x > 0$.

By construction, the coefficients $\beta_j(r)$ vanish for $r \leq \frac{R}{2}$ and obey $\left|\partial_r^k \beta_j(r)\right| \lesssim_k R^{-k-j}$ for $k \geq 0$. The fundamental solution $g(r)$ of the operator $(1 + \partial_r^4)$ is defined by

$$(1 + \partial_r^4)g = \delta_0, \quad \text{and} \quad \lim_{r \to \pm\infty} g(r) = 0. \tag{6.25}$$

It is explicitly given by

$$g(r) = \begin{cases} \frac{1}{\sqrt{2}}\sin\left(\frac{r}{\sqrt{2}}\right)e^{\frac{r}{\sqrt{2}}} - \frac{1}{\sqrt{2}}\cos\left(\frac{r}{\sqrt{2}}\right)e^{\frac{r}{\sqrt{2}}} & \text{for } r < 0, \\ -\frac{1}{\sqrt{2}}\sin\left(\frac{r}{\sqrt{2}}\right)e^{-\frac{r}{\sqrt{2}}} - \frac{1}{\sqrt{2}}\cos\left(\frac{r}{\sqrt{2}}\right)e^{-\frac{r}{\sqrt{2}}} & \text{for } r > 0. \end{cases} \tag{6.26}$$

Convolving (6.24) with g, we arrive at the equivalent fixed point problem

$$\tilde{v} = g * \left(\eta\tilde{f}\right) - g * \left(Q\left(r^{-1}, \partial_r\right)\tilde{v}\right) \quad \text{for } r \in \mathbb{R}. \tag{6.27}$$

It is our aim to apply the contraction mapping theorem to equation (6.27). Therefore, in view of (6.27) and the decay properties of g (cf. (6.26)), we define the norms

$$\|\tilde{v}\|_{N,\nu} := \sup_{r \in \mathbb{R}} \max_{j=0,\cdots,N} \left|e^{\nu r}\partial_r^j \tilde{v}(r)\right|$$

with $\nu \in \left[0, \frac{1}{\sqrt{2}}\right)$. Using (6.26), we notice that for $j = 0, 1, 2, 3$, $e^{\nu r}\partial_r^j g(r)$ exists and decays exponentially as $r \to \pm\infty$. In particular $e^{\nu r}\partial_r^j g(r)$ is absolutely integrable for $j = 0, 1, 2, 3$. We further use $\partial_r^4 g = \delta_0 - g$ (cf. (6.25)) and the general fact that $e^{\nu \cdot}(f * g) = (e^{\nu \cdot} f) * (e^{\nu \cdot} g)$, so that for any $j \geq 0$ and any $\tilde{v} \in C^{\max\{j-1,3\}}(\mathbb{R})$ we can estimate

$$\begin{aligned}
\sup_{r\in\mathbb{R}} \left| e^{\nu r}\partial_r^j \left(g * \left(Q\left(r^{-1}, \partial_r\right) \tilde{v} \right) \right) \right| &= \sup_{r\in\mathbb{R}} \left| \left(e^{\nu r}\partial_r^{\min\{4,j\}} g \right) * \left(e^{\nu r}\partial_r^{\max\{0,j-4\}} Q\left(r^{-1}, \partial_r\right) \tilde{v} \right) \right| \\
&\lesssim \sup_{r\in\mathbb{R}} \left| e^{\nu r}\partial_r^{\max\{0,j-4\}} \left(Q\left(r^{-1}, \partial_r\right) \tilde{v} \right) \right| \\
&\lesssim_j \frac{1}{R} \sup_{r\in\mathbb{R}} \max_{k=1,\cdots,\max\{j-1,3\}} \left| e^{\nu r}\partial_r^k \tilde{v} \right|. \qquad (6.28)
\end{aligned}$$

In the same way we obtain for any $j \geq 0$

$$\sup_{r\in\mathbb{R}} \left| e^{\nu r}\partial_r^j \left(g * (\eta\tilde{f}) \right) \right| \lesssim_j \sup_{r\in\mathbb{R}} \left| e^{\nu r}\partial_r^{\max\{0,j-4\}} (\eta\tilde{f}) \right|. \qquad (6.29)$$

Recalling (6.21) and since f satisfies (G_∞), the right hand side of (6.29) is finite for all $j \geq 0$ and all $\nu \in \left[0, \frac{1}{\sqrt{2}}\right)$.

We seek a solution to (6.27) as a fixed point in the Banach space

$$X_\nu := \left\{ \tilde{v} \in C^4(\mathbb{R}) : \|\tilde{v}\|_{4,\nu} < \infty \right\}$$

for the mapping

$$\mathcal{T}[\tilde{v}] := g * (\eta\tilde{f}) - g * \left(Q\left(r^{-1}, \partial_r\right) \tilde{v} \right) \quad \text{for } r \in \mathbb{R}.$$

Inequalities (6.28) and (6.29) imply that

$$\|\mathcal{T}[\tilde{v}]\|_{4,\nu} \lesssim \left\| \eta\tilde{f} \right\|_{0,\nu} + \|\tilde{v}\|_{4,\nu} < \infty.$$

Hence $\mathcal{T}$ maps X_ν into itself. Inequality (6.28) together with the linearity of $\mathcal{T}$ implies that

$$\|\mathcal{T}[\tilde{v}_1] - \mathcal{T}[\tilde{v}_2]\|_{4,\nu} \lesssim \frac{1}{R} \|\tilde{v}_1 - \tilde{v}_2\|_{4,\nu}.$$

Therefore $\mathcal{T}$ is a contraction in X_ν for $R \gg 1$.

Hence we have shown the existence and uniqueness of a solution to (6.27) in X_ν for any $\nu \in \left[0, \frac{1}{\sqrt{2}}\right)$. Since the spaces X_ν are nested, the solution is the same for all $\nu \in \left[0, \frac{1}{\sqrt{2}}\right)$. Finally, we have

$$\|\tilde{v}\|_{N,\nu} \overset{(6.27)}{\leq} \left\| g * (\eta\tilde{f}) \right\|_{N,\nu} + \left\| g * \left(Q\left(r^{-1}, \partial_r\right) \tilde{v} \right) \right\|_{N,\nu} \overset{(6.29),(6.28)}{\lesssim_N} \left\| \eta\tilde{f} \right\|_{N-4,\nu} + \frac{1}{R} \|\tilde{v}\|_{N-1,\nu},$$

hence a repeated application of this formula starting from $N = 5$ yields $\|\tilde{v}\|_{N,\nu} < \infty$ for any $N \in \mathbb{N}$ and $\nu \in \left[0, \frac{1}{\sqrt{2}}\right)$. Undoing transformation (6.21) completes the proof of *(a)*.

Proof of (b): Construction of u_3 and u_4. Let $u(x)$ solve the resolvent equation (6.1) with right hand side $f(x) \equiv 0$. Again we carry out the change of variables (6.21). Then $\tilde{u}(r)$ solves the homogeneous analog of (6.22), i.e.

$$(1+\partial_r^4)\tilde{u} + r^{-1} q\left(r^{-1}, \partial_r\right) \tilde{u} = 0 \quad \text{for } r > 0, \tag{6.30}$$

with $q\left(r^{-1}, \partial_r\right)$ given by (6.23). It is convenient to apply the following transformation:

$$\tilde{u}(r) =: r^{\alpha} e^{\mu r} v(r), \quad \alpha := -\frac{7}{2}, \tag{6.31}$$

where μ is one of the two fourth roots of -1 with $\mathcal{R}e\, \mu < 0$. Without loss of generality we focus on $\mu := \frac{i-1}{\sqrt{2}}$. The reason for choosing transformation (6.31) is two-fold:

- Because of $\partial_r^4 e^{\mu r} = e^{\mu r}(\mu + \partial_r)^4$, the factor $e^{\mu r}$ splits off the behavior of solutions $(1+\partial_r^4)\tilde{u} = 0$, which we expect to play the dominant role as $r \nearrow \infty$.
- The perturbation $r^{-1} q\left(r^{-1}, \partial_r\right)$ of the operator $(1+\partial_r^4)$ is of order $O\left(r^{-1}\right)$ as $r \nearrow \infty$, the leading order perturbation being given by $14\, r^{-1} \partial_r^3$ (cf. (6.23)). Since r^{-1} is not integrable around $r = \infty$, one cannot use Duhamel's principle to treat the perturbation by a contraction argument. Because of $\partial_r^4 r^{\alpha} = r^{\alpha}\left(\partial_r + \alpha r^{-1}\right)^4$, the factor r^{α} annihilates the term $14 r^{-1}\partial_r^3$ for $\alpha = -\frac{14}{4} = -\frac{7}{2}$, so that the remainder is of order $O\left(r^{-2}\right)$ and thus behaves well (in the sense sketched above) as $r \nearrow \infty$.

We expect the function $v(r)$ to converge to a constant as $r \nearrow \infty$, without loss of generality $\lim_{r \nearrow \infty} v(r) = 1$. This means that solutions to the perturbed equation (6.30) do not stay close to the solutions of $(1+\partial_r^4)\tilde{u} = 0$, but in fact differ by an algebraic pre-factor r^{α}.

Applying the transformation (6.31) to (6.30) by using Leibniz' rule, we arrive at the problem

$$\left[(\mu + \partial_r)^4 + 1\right] v + r^{-2} \tilde{q}\left(r^{-1}, \partial_r\right) v = 0 \quad \text{for } r > 0, \tag{6.32a}$$

$$v \to 1 \quad \text{as } r \nearrow \infty, \tag{6.32b}$$

where

$$\tilde{q}\left(r^{-1}, \partial_r\right) = \gamma_1 \partial_r^2 + \underbrace{\left(a_{2,1} + a_{2,2} r^{-1}\right)}_{=:\gamma_2(r)} \partial_r + \underbrace{\left(a_{3,1} + a_{3,2} r^{-1} + a_{3,3} r^{-1}\right)}_{=:\gamma_3(r)}, \tag{6.32c}$$

with numerical constants $\gamma_1, a_{2,1}, a_{2,2}, a_{3,1}, a_{3,2}, a_{3,3} \in \mathbb{C}$. Writing $v =: 1 + w$, we can reformulate system (6.32) as

$$\left[(\mu + \partial_r)^4 + 1\right] w + r^{-2} \tilde{q}\left(r^{-1}, \partial_r\right) w = -\gamma_3(r) r^{-2} \quad \text{for } r > 0 \tag{6.33a}$$

$$w \to 0 \quad \text{as } r \nearrow \infty. \tag{6.33b}$$

Our next step is to construct a linear solution operator $\mathcal{T}$ to the leading order equation

$$\left[(\mu + \partial_r)^4 + 1\right] w = f \quad \text{for } r > 0, \tag{6.34}$$

with the condition $w \to 0$ as $r \nearrow \infty$, for a smooth and sufficiently decaying $f(r)$. By construction, the zeros of the characteristic polynomial associated to the linear differential operator $[(\mu + \partial_r)^4 + 1]$ are those of $(1 + \partial_r^4)$ shifted by $-\mu$. Hence the zeros are explicitly given by

$$\zeta_1 = 0, \quad \zeta_2 = -i\sqrt{2}, \quad \zeta_3 = \sqrt{2}(1-i), \quad \text{and} \quad \zeta_4 = \sqrt{2}, \tag{6.35}$$

where we ordered them so that $\mathcal{R}e\,\zeta_j = 0$ for $j = 1, 2$ and $\mathcal{R}e\,\zeta_j > 0$ for $j = 3, 4$. We can factorize the solution operator $\mathcal{T}$ into $\mathcal{T} = \mathcal{T}_{\zeta_4}\mathcal{T}_{\zeta_3}\mathcal{T}_0$, where $\mathcal{T}_0$ and $\mathcal{T}_\zeta$ are defined through

$$\partial_r(\partial_r - \zeta_2)\mathcal{T}_0 f(r) = f(r), \quad (\partial_r - \zeta)\mathcal{T}_\zeta f(r) = f(r) \quad \text{for } \zeta \in \{\zeta_3, \zeta_4\}, \tag{6.36}$$

and explicitly given by

$$(\mathcal{T}_0 f)(r) = \frac{1}{\zeta_2}\int_0^\infty \left(1 - e^{-\zeta_2 r'}\right) f(r + r')\,\mathrm{d}r', \tag{6.37a}$$

$$(\mathcal{T}_\zeta f)(r) = \int_0^\infty e^{-\zeta r'} f(r + r')\,\mathrm{d}r' \quad \text{for } \zeta \in \{\zeta_3, \zeta_4\}. \tag{6.37b}$$

We now show that the explicit representations (6.37), together with the fact that $\mathcal{R}e\,\zeta_2 = 0$ and $\mathcal{R}e\,\zeta_j > 0$ for $j = 3, 4$, imply the following maximal regularity estimates for the solution operator:

$$\sup_{\substack{r \geq R \\ k=0,1,2}} \left| r\partial_r^{j+k}(\mathcal{T}_0 f)(r) \right| \lesssim \sup_{r \geq R} \left| r^2 \partial_r^j f(r) \right|, \tag{6.38a}$$

$$\sup_{\substack{r \geq R \\ k=0,1}} \left| r\partial_r^{j+k}(\mathcal{T}_\zeta f)(r) \right| \lesssim \sup_{r \geq R} \left| r \partial_r^j f(r) \right| \quad \text{for } \zeta \in \{\zeta_3, \zeta_4\}, \tag{6.38b}$$

for $j \in \mathbb{N}_0$ and $R > 0$ arbitrary. First of all, since ∂_r commutes with $\mathcal{T}_\zeta$ and $\mathcal{T}_0$, it is enough to consider the case of $j = 0$. Furthermore, an integration by parts of (6.37a) yields

$$\partial_r(\mathcal{T}_0 f)(r) = -\int_0^\infty e^{-\zeta_2 r'} f(r + r')\,\mathrm{d}r'. \tag{6.39}$$

Hence we can estimate

$$\begin{aligned}
\sup_{r \geq R} |r(\mathcal{T}_0 f)(r)| &\overset{(6.37a)}{\leq} \sup_{r \geq R} r\frac{2}{|\zeta_2|}\int_0^\infty |f(r + r')|\,\mathrm{d}r' \\
&\leq \sup_{r \geq R} r\frac{2}{|\zeta_2|}\int_0^\infty \frac{1}{(r + r')^2}\,\mathrm{d}r' \sup_{r \geq R}\left|r^2 f(r)\right| \\
&= \frac{2}{|\zeta_2|}\sup_{r \geq R}\left|r^2 f(r)\right|, \qquad (6.40a)
\end{aligned}$$

$$\sup_{r \geq R} |r(\mathcal{T}_\zeta f)(r)| \overset{(6.37b)}{\leq} \left|\int_0^\infty e^{-\zeta r'}\,\mathrm{d}r'\right| \sup_{r \geq R}|rf(r)| \leq \frac{1}{|\mathcal{R}e\,\zeta|}\sup_{r \geq R}|rf(r)| \tag{6.40b}$$

for $\zeta \in \{\zeta_3, \zeta_4\}$, and

$$\sup_{r\geq R} |r\partial_r(\mathcal{T}_0 f)(r)| \overset{(6.39)}{\leq} \sup_{r\geq R} r \int_0^\infty |f(r+r')| \, \mathrm{d}r' \leq \sup_{r\geq R} r \int_0^\infty \frac{1}{(r+r')^2} \, \mathrm{d}r' \sup_{r\geq R} \left|r^2 f(r)\right| = \sup_{r\geq R} \left|r^2 f(r)\right| . \tag{6.40c}$$

By using equations (6.36), we can further upgrade estimates (6.40) to one derivative higher and obtain the maximal regularity estimates (6.38) for $j = 0$.

Estimates (6.38) for $\mathcal{T}_0$, $\mathcal{T}_{\zeta_3}$, and $\mathcal{T}_{\zeta_4}$ can be combined to obtain the following maximal regularity estimate for the linear solution operator $\mathcal{T}$:

$$\sup_{\substack{r\geq R\\ k=0,1,2,3,4}} \left|r\partial_r^{j+k}(\mathcal{T}f)(r)\right| \lesssim \sup_{r\geq R} \left|r^2 \partial_r^j f(r)\right| \tag{6.41}$$

for $j \in \mathbb{N}_0$ and $R > 0$ arbitrary.

In order to solve the fixed point equation

$$w = \mathcal{S}[w] := -\mathcal{T}\left[r^{-2}\tilde{q}\left(r^{-1}, \partial_r\right) w + \gamma_3(r) r^{-2}\right] \tag{6.42}$$

that we obtain by applying $\mathcal{T}$ to (6.33), we introduce the Banach spaces

$$X_{N,R} := \left\{ w : \sup_{r\geq R} \left|r\partial_r^j w\right| < \infty \text{ for all } j = 0, \cdots, N+4 \right\}, \tag{6.43}$$

where $N \in \mathbb{N}_0$ is arbitrary and $R > 0$ will be chosen later. We then note that

$$\sup_{\substack{r\geq R\\ j=0,\cdots,N+4}} \left|r\partial_r^j \mathcal{T}\left[\gamma_3(r) r^{-2}\right]\right| \overset{(6.41)}{\lesssim} \sup_{\substack{r\geq R\\ j=0,\cdots,N}} \left|r^2 \partial_r^j \gamma_3(r) r^{-2}\right| \overset{(6.32c)}{<} \infty, \tag{6.44a}$$

as well as

$$\begin{aligned} \sup_{\substack{r\geq R\\ j=0,\cdots,N+4}} \left|r\partial_r^j \mathcal{T}\left[r^{-2}\tilde{q}\left(r^{-1}, \partial_r\right) w\right]\right| &\overset{(6.41)}{\lesssim} \sup_{\substack{r\geq R\\ j=0,\cdots,N}} \left|r^2 \partial_r^j r^{-2}\tilde{q}\left(r^{-1}, \partial_r\right) w\right| \\ &\overset{(6.32c)}{\lesssim_N} \sup_{\substack{r\geq R\\ j=0,\cdots,N+2}} \left|\partial_r^j w\right| \\ &\leq \frac{1}{R} \sup_{\substack{r\geq R\\ j=0,\cdots,N+4}} \left|r\partial_r^j w\right| . \end{aligned} \tag{6.44b}$$

Then we see by the triangle inequality

$$\begin{aligned} \sup_{\substack{r\geq R\\ j=0,\cdots,N+4}} \left|r\partial_r^j \mathcal{S}[w]\right| &\overset{(6.42)}{\leq} \sup_{\substack{r\geq R\\ j=0,\cdots,N+4}} \left|r\partial_r^j \mathcal{T}\left[r^{-2}\tilde{q}\left(r^{-1}, \partial_r\right) w\right]\right| \\ &\quad + \sup_{\substack{r\geq R\\ j=0,\cdots,N+4}} \left|r\partial_r^j \mathcal{T}\left[\gamma_3(r) r^{-2}\right]\right| \overset{(6.43),(6.44)}{<} \infty. \end{aligned}$$

Thus we recognize that $\mathcal{S}$ maps $X_{N,R}$ into itself. Furthermore, from (6.44b) we obtain

$$\begin{aligned} \sup_{\substack{r\geq R\\ j=0,\cdots,N+4}} \left|r\partial_r^j \mathcal{S}[w_1-w_2]\right| &= \sup_{\substack{r\geq R\\ j=0,\cdots,N+4}} \left|r\partial_r^j \mathcal{T}\left[r^{-2}\tilde{q}\left(r^{-1},\partial_r\right)(w_1-w_2)\right]\right| \\ &\lesssim_N \frac{1}{R} \sup_{\substack{r\geq R\\ j=0,\cdots,N+4}} \left|r\partial_r^j (w_1-w_2)\right| \end{aligned}$$

for $w_j \in X_{N,R}$. Hence $\mathcal{S}$ is a contraction for any $R \geq R_N \gg_N 1$. The contraction mapping theorem yields for every N and every $R \geq R_N$ a unique fixed point $w_{N,R}$ solving (6.33). Obviously $w_{N,R_N}|_{[R,\infty)}$ is a fixed point in $X_{N,R}$, hence $w_{N,R_N}|_{[R,\infty)} = w_{N,R}$ by the uniqueness part of the contraction mapping theorem. Let $w_N := w_{N,R_N}$. If we now pick two integers $N_1, N_2 \in \mathbb{N}_0$ with $N_1 < N_2$, then we can take $R := \max\{R_{N_1}, R_{N_2}\}$. In view of the above, $w_{N_i}|_{[R,\infty)}$ are the unique fixed points in $X_{N_i,R}$. Since $X_{N_1,R} \supset X_{N_2,R}$, we have $w_{N_1} = w_{N_2}$ in $[R,\infty)$, again by the uniqueness part of the contraction mapping theorem. As any solution of (6.33) for $r \gg 1$ can be uniquely extended to a solution of (6.33) for $r > 0$ by standard ODE theory, this defines a unique function $w : (0,\infty) \to \mathbb{C}$ such that $w|_{[R,\infty)}$ is a fixed point in $X_{N,R}$ for all $R \geq R_N$. Hence w is a smooth solution of (6.33) such that

$$\limsup_{r\to\infty} \left|r\partial_r^j w\right| < \infty \quad \text{for all } j \in \mathbb{N}_0.$$

Undoing the transformation $v = 1 + w$, (6.31), and (6.21), we obtain a solution of the resolvent equation (6.1) for all $x > 0$ with the asymptotic behavior

$$u(x) = x^{-\frac{7}{8}} \exp\left[-\frac{4}{\sqrt{2}}\sqrt[4]{x}\right]\left[\cos\left(\frac{4}{\sqrt{2}}\sqrt[4]{x}\right) + i\sin\left(\frac{4}{\sqrt{2}}\sqrt[4]{x}\right)\right]\cdot(1+o(1))$$

as $x \nearrow \infty$. We can define $u_3(x)$ and $u_4(x)$ as real and imaginary parts of $u(x)$ (note that the coefficients of (6.1) are all real) and obtain two smooth and linearly independent solutions of (6.1) that obey the decay properties of the lemma.

Proof of (c): Construction of a two-parameter family of solutions. This follows directly from assertions *(a)* and *(b)* of the lemma. □

6.4. Matching of the Regimes $x \ll 1$ and $x \gg 1$

Lemmas 6.5 and 6.6 can be combined to the desired result on solutions to equation (6.1):

Proof of Proposition 6.3. Let $\bar{u}_1(x,y)$ and $\bar{u}_2(x,y)$ denote the solutions of (6.9) for $(a_1,a_2) = (1,0)$ and $(a_1,a_2) = (0,1)$ and right hand side $f(x) \equiv 0$ that we constructed in Lemma 6.5. These are analytic in a sufficiently small rectangle $[0,\varepsilon]\times[0,\varepsilon^\beta]$. Hence $u_1(x) := \bar{u}_1(x,x^\beta)$ and $u_2(x) := \bar{u}_2(x,x^\beta)$, $x \in [0,\varepsilon]$, are solutions of (6.1) that satisfy (G_0). These functions can, by standard ODE theory, be extended to solutions of the resolvent equation for $0 < x < \infty$: we denote these extensions again by $u_1(x)$ and $u_2(x)$. The solutions (u_1,u_2) to (6.1) are linearly independent (as their asymptotic behavior as $x \searrow 0$ is different, cf. (6.11)).

By construction, the solutions (u_3, u_4) of the homogeneous resolvent equation, defined in Lemma 6.6 *(b)*, are linearly independent. We argue that also $u_1, \cdots, u_4$ are linearly independent and thus span the four-dimensional solution space of the homogeneous resolvent equation (6.1). Let $(a_1, \cdots, a_4) \in \mathbb{R}^4$ be arbitrarily such that $a_1 u_1 + \cdots + a_4 u_4 = 0$. Consider

$$u := a_1 u_1 + a_2 u_2 = -a_3 u_3 - a_4 u_4.$$

From the first representation we learn that u satisfies (G_0) and from the second one we learn that it also satisfies (G_∞). In particular $|u|_{2,\alpha} < \infty$ for some α in the coercivity range of $p(D)$. Hence, by Proposition 5.5 $u(x) \equiv 0$, which by the definition of u amounts to $a_1 u_1 + a_2 u_2 = a_3 u_3 + a_4 u_4 = 0$. Since (u_1, u_2) as well as (u_3, u_4) are both linearly independent, we necessarily have $a_1 = a_2 = 0$ as well as $a_3 = a_4 = 0$.

By Lemma 6.5 there exists a particular solution u_0 of (6.1) that satisfies (G_0) and by Lemma 6.6 there exists a particular solution u_∞ of (6.1) that satisfies (G_∞). By the above, we learn that $u_\infty - u_0 = a_1 u_1 + \cdots + a_4 u_4$ for certain $a_1, \cdots, a_4 \in \mathbb{R}$. Now consider

$$u := u_0 + a_1 u_1 + a_2 u_2 = u_\infty - a_3 u_3 - a_4 u_4.$$

Then u solves (6.1). From the first representation we learn that u satisfies (G_0) and from the second we learn that u satisfies (G_∞). In particular $|u|_{2,\alpha} < \infty$ for every α in the coercivity range of $p(D)$. This establishes the existence part of the proposition.

It remains to prove the uniqueness of u. Let $\tilde{u}(x)$ be another solution of (6.1) with $|\tilde{u}|_{2,\alpha} < \infty$ for some α in the coercivity range of $p(D)$. We set $w(x) := u(x) - \tilde{u}(x)$. Then $|w|_{2,\alpha} < \infty$ and $w(x)$ solves (6.1) with $f(x) \equiv 0$. By Proposition 5.5 we need to have $w(x) \equiv 0$, which completes the proof of Proposition 6.3. □

7. The Linear Degenerate Parabolic Equation

In this section we consider the linear degenerate parabolic problem (1.10), i.e.

$$x\partial_t u + p(D)u = f \quad \text{for } t \in I,\ x > 0, \tag{7.1}$$

with $u_{|t=0} = u_0$ for some given initial data u_0 and where $I = (0, \tau) \subseteq (0, \infty)$ is an interval. Note that if u is a distributional solution of (7.1) such that $|||u|||_I < \infty$, then both (7.1) and $u_{|t=0} = u_0$ hold classically (cf. Remark 7.7). The aim is to derive the maximal regularity estimate (2.16), i.e.

$$|||u|||_I \lesssim_{k,\delta} |||u_0|||_0 + |||f|||_{1,I}. \tag{7.2}$$

In parallel, we derive a corresponding existence and uniqueness result. The technical derivation of both will be based on a time discretization and the existence and regularity result for the resolvent equation, Proposition 6.3. Since deriving the a priori estimates on the level of the time discretized equation is somewhat cumbersome, we start by a formal derivation of (7.2).

7.1. Heuristics

Throughout the subsection estimates may depend on k, ℓ, and α. The main ingredients for the formal derivation of (7.2) (which we will work out without loss of generality for $I = (0, \infty)$, since we may otherwise extend f by 0 in time t) are the coercivity and the maximal regularity of $p(D)$, cf. Corollary 5.4 and Lemma 7.2, respectively.

We first formally argue how to get the parabolic estimate (2.2), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|_{\alpha-\frac{1}{2}}^2 + C_1\left|D^{\ell+2}u\right|_{\alpha-\frac{1}{2}}^2\right) + |\partial_t u|_{\ell,\alpha}^2 + |u|_{\ell+4,\alpha}^2 \lesssim |f|_{\alpha}^2 \tag{7.3}$$

for $\ell \in \mathbb{N}_0$ and some constant $C_1 > 0$ only depending on ℓ and α, from the coercivity estimate (2.1), i.e.

$$(u, p(D)u)_{\alpha} \gtrsim |u|_{\alpha}^2 . \tag{7.4}$$

Testing equation (7.1) with u in the inner product $(\cdot,\cdot)_{\alpha}$ and using Young's inequality for the right hand side, we obtain from (7.4)

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|_{\alpha-\frac{1}{2}}^2 + |u|_{\alpha}^2 \lesssim |g|_{\alpha}^2 . \tag{7.5}$$

This basic estimate serves as an anchoring for estimates containing higher powers of D. For the latter we test equation (7.1) with $(D+2\alpha-1)^{\ell+2}D^{\ell+2}u$. Due to the symmetry properties of D (cf. (4.4)) integrating by parts yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left|D^{\ell+2}u\right|_{\alpha-\frac{1}{2}}^2 + \left((D-1)^{\ell}p(D-1)u, (D+2\alpha-1)^2D^{\ell+2}u\right)_{\alpha}\\ &= \left((D-1)^{\ell}f, (D+2\alpha-1)^2D^{\ell+2}u\right)_{\alpha} .\end{aligned} \tag{7.6}$$

By interpolation (cf. Lemma B.1) we know that

$$\left((D-1)^{\ell}p(D-1)u, (D+2\alpha-1)^2D^{\ell+2}u\right)_{\alpha} \gtrsim \left|D^{\ell+4}u\right|_{\alpha}^2 - C\,|u|_{\alpha}^2 \tag{7.7}$$

for a sufficiently large $C < \infty$. Again by interpolation and by the Cauchy-Schwarz inequality we have

$$\left((D-1)^{\ell}f, (D+2\alpha-1)^2D^{\ell+2}u\right)_{\alpha} \lesssim |f|_{\ell,\alpha}\sqrt{|u|_{\alpha}^2 + |D^{\ell+4}u|_{\alpha}^2}. \tag{7.8}$$

The combination of (7.7) and (7.8) in (7.6) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|D^{\ell+2}u\right|_{\alpha-\frac{1}{2}}^2 + \left|D^{\ell+4}u\right|_{\alpha}^2 - C\,|u|_{\alpha}^2 \lesssim |f|_{\ell,\alpha}^2$$

for sufficiently large $C < \infty$. Adding a large multiple of (7.5) and using once more equation (7.1) to obtain control over the time derivative $\partial_t u$, we arrive at (7.3).

In Section 2, we have already argued that (7.3) yields (2.6), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|v|_{\alpha+\frac{1}{2}}^2 + C_1\left|D^{\ell+2}v\right|_{\alpha+\frac{1}{2}}^2\right) + |\partial_t v|_{\ell,\alpha}^2 + |v|_{\ell+4,\alpha+1}^2 \lesssim |f|_{\ell+4,\alpha+1}^2 , \tag{7.9}$$

where $v := p(D)u$. We now assume that

$$\beta - 1 \pm \delta, -\frac{1}{2} \pm \delta \text{ are in the coercivity range,} \tag{7.10}$$

which is the case for $0 < \delta \ll 1$ (cf. Section 5, Corollary 5.4). Under the assumption (7.10) we shall argue that (7.9) implies for $\ell \geq k+2$ the following preliminary version of the maximal regularity estimates (2.14) and (2.15):

$$\begin{aligned}
&\sup_{t\geq 0} \left(t^{2\beta-1} |v|^2_{k+2,\beta-\frac{1}{2}\pm\delta} + |v|^2_{\ell+2,\pm\delta} \right) + \int_0^\infty \left(t^{2\beta-1} |\partial_t v|^2_{k,\beta-1\pm\delta} + |\partial_t v|^2_{\ell,-\frac{1}{2}\pm\delta} \right) \mathrm{d}t \\
&+ \int_0^\infty \left(t^{2\beta-1} |v|^2_{k+4,\beta\pm\delta} + |v|^2_{\ell+4,\frac{1}{2}\pm\delta} \right) \mathrm{d}t \\
&\lesssim |v_0|_{\ell+2,\pm\delta} + \int_0^\infty \left(t^{2\beta-1} |f|^2_{k+4,\beta\pm\delta} + |f|_{\ell+4,\frac{1}{2}\pm\delta} \right) \mathrm{d}t.
\end{aligned} \tag{7.11}$$

Integrating (7.9) in time, we obtain

$$\sup_{t\geq 0} |v|^2_{\ell+2,\alpha+\frac{1}{2}} + \int_0^\infty \left(|\partial_t v|^2_{\ell,\alpha} + |v|^2_{\ell+4,\alpha+1} \right) \mathrm{d}t \lesssim |v_0|^2_{\ell+2,\alpha+\frac{1}{2}} + \int_0^\infty |f|^2_{\ell+4,\alpha+1} \, \mathrm{d}t. \tag{7.12}$$

Multiplying (7.9) (with ℓ replaced by k) with the time weight $t^{2\sigma}$ ($\sigma > 0$) and integrating, we likewise obtain

$$\begin{aligned}
&\sup_{t\geq 0} t^{2\sigma} |v|^2_{k+2,\alpha+\frac{1}{2}} + \int_0^\infty t^{2\sigma} \left(|\partial_t v|^2_{k,\alpha} + |v|^2_{k+4,\alpha+1} \right) \mathrm{d}t \\
&\lesssim \int_0^\infty t^{2\sigma-1} |v|^2_{k+2,\alpha+\frac{1}{2}} \, \mathrm{d}t + \int_0^\infty t^{2\sigma} |f|^2_{k+4,\alpha+1} \, \mathrm{d}t.
\end{aligned} \tag{7.13}$$

In view of (7.10), we may use (7.12) for $\alpha = -\frac{1}{2} \pm \delta$ and (7.13) for $\alpha = \beta - 1 \pm \delta$ and $\sigma = \beta - \frac{1}{2} > 0$:

$$\sup_{t\geq 0} |v|^2_{\ell+2,\pm\delta} + \int_0^\infty \left(|\partial_t v|^2_{\ell,-\frac{1}{2}\pm\delta} + |v|^2_{\ell+4,\frac{1}{2}\pm\delta} \right) \mathrm{d}t \lesssim |v_0|^2_{\ell+2,\pm\delta} + \int_0^\infty |f|^2_{\ell+4,\frac{1}{2}\pm\delta} \, \mathrm{d}t \tag{7.14}$$

and

$$\begin{aligned}
&\sup_{t\geq 0} t^{2\beta-1} |v|^2_{k+2,\beta-\frac{1}{2}\pm\delta} + \int_0^\infty t^{2\beta-1} \left(|\partial_t v|^2_{k,\beta-1\pm\delta} + |v|^2_{k+4,\beta\pm\delta} \right) \mathrm{d}t \\
&\lesssim \int_0^\infty t^{2\beta-2} |v|^2_{k+2,\beta-\frac{1}{2}\pm\delta} \, \mathrm{d}t + \int_0^\infty t^{2\beta-1} |f|^2_{k+4,\beta\pm\delta} \, \mathrm{d}t.
\end{aligned} \tag{7.15}$$

From Lemma 7.5 (see below) applied to $v = p(D)u$, we learn that the first right hand side term of (7.15) is estimated by the left hand side of (7.14) for $\ell \geq k+2$. For later purposes (cf. Lemma 8.1), we choose $\ell = k+3$. Then adding a large multiple of (7.14) to (7.15) yields (7.11).

We recall from Section 4 that, in order to derive control of u itself from control of $v = p(D)u$, we need to apply elliptic maximal regularity for $p(D)$. To this aim, we need to control the behavior of u as $x \searrow 0$ and $x \nearrow \infty$: In this formal discussion on the time continuum level we shall assume that we have

$$u = a + bx^\beta + O(x) \quad \text{as } x \searrow 0$$

and

$$u = O(x^{-\varrho}) \quad \text{as } x \nearrow \infty \text{ for all } \varrho < \infty.$$

Below we shall make this rigorous on the time discrete level based on the precise regularity result for the resolvent equation, Proposition 6.3. Furthermore, we note that for $0 < \delta \ll 1$ the weights stay away from the zeros of $p(\zeta)$:

$$-\beta - \frac{1}{2} < -\delta < 0, \quad 0 < \delta, \beta - \frac{1}{2} \pm \delta, \frac{1}{2} \pm \delta, \beta - \delta < \beta, \quad \beta < \beta + \delta < 1.$$

This is all we need to apply elliptic maximal regularity for $p(D)$ in form of (2.9)–(2.11) (cf. Lemma 7.2 for the precise statement). Hence (7.11) with the $+$ sign turns into

$$\begin{aligned}
&\sup_{t\geq 0}\left(t^{2\beta-1}\left|u-a\right|_{k+6,\beta-\frac{1}{2}+\delta}^2+\left|u-a\right|_{\ell+6,\delta}^2\right)+\int_0^\infty\left(t^{2\beta-1}\left|\partial_t u\right|_{k+4,\beta-1+\delta}^2+\left|\partial_t u\right|_{\ell+4,-\frac{1}{2}+\delta}^2\right)\mathrm{d}t\\
&+\int_0^\infty\left(t^{2\beta-1}\left|u-a-bx^\beta\right|_{k+8,\beta+\delta}^2+\left|u-a\right|_{\ell+8,\frac{1}{2}+\delta}^2\right)\mathrm{d}t\\
&\lesssim\left|u_0-a_0\right|_{\ell+6,\delta}+\int_0^\infty\left(t^{2\beta-1}\left|f\right|_{k+4,\beta+\delta}^2+\left|f\right|_{\ell+4,\frac{1}{2}+\delta}\right)\mathrm{d}t
\end{aligned} \tag{7.16}$$

and (7.11) with the $-$ sign turns into

$$\begin{aligned}
&\sup_{t\geq 0}\left(t^{2\beta-1}\left|u-a\right|_{k+6,\beta-\frac{1}{2}-\delta}^2+\left|u\right|_{\ell+6,-\delta}^2\right)+\int_0^\infty\left(t^{2\beta-1}\left|\partial_t u\right|_{k+4,\beta-1-\delta}^2+\left|\partial_t u\right|_{\ell+4,-\frac{1}{2}-\delta}^2\right)\mathrm{d}t\\
&+\int_0^\infty\left(t^{2\beta-1}\left|u-a\right|_{k+8,\beta-\delta}^2+\left|u-a\right|_{\ell+8,\frac{1}{2}-\delta}^2\right)\mathrm{d}t\\
&\lesssim\left|u_0\right|_{\ell+6,-\delta}+\int_0^\infty\left(t^{2\beta-1}\left|f\right|_{k+4,\beta-\delta}^2+\left|f\right|_{\ell+4,\frac{1}{2}-\delta}\right)\mathrm{d}t.
\end{aligned} \tag{7.17}$$

After taking the sum of (7.16) and (7.17), we obtain (7.2), the desired estimate.

7.2. Statement of Results

Here we provide the rigorous arguments for the construction of solutions to the linear degenerate parabolic equation (7.1) with initial condition $u_{|t=0} = u_0$.

In order control the boundary values of u, we will employ the mentioned time discretization procedure and derive maximal regularity estimates for $v := p(D)u$. We first consider a single discrete time step where the initial data for $t = 0$ is given by $u_0(x)$. Then, for $t = h \ll 1$, $u(t,x)$ approximately solves

$$x\frac{u-u_0}{h} + p(D)u = f. \tag{7.18}$$

The solution to this equation is obtained by Proposition 6.3. We additionally prove appropriate L^2-based estimates:

Lemma 7.1. *Assume that $f, u_0 \in C^\infty((0,\infty))$ satisfy (G_0) and (G_∞). Then there exists a solution $u \in C^\infty((0,\infty))$ to (7.18) which satisfies (G_0) and (G_∞). Furthermore for all $\alpha \in \mathbb{R}$ in the coercivity range of $p(D)$ and all $\ell \in \mathbb{N}_0$ there exist $C_1, C_2 \in (0,\infty)$, which only depend on ℓ and α, such that*

$$\left|D^{\ell+2}v\right|^2_{\alpha+\frac{1}{2}} + C_1\left|v\right|^2_{\alpha+\frac{1}{2}} + \frac{h}{C_2}\left|v\right|^2_{\ell+4,\alpha+1} \le \left|D^{\ell+2}v_0\right|^2_{\alpha+\frac{1}{2}} + C_1\left|v_0\right|^2_{\alpha+\frac{1}{2}} + C_2 h\left|f\right|^2_{\ell+4,\alpha+1}, \tag{7.19}$$

where $v = p(D)u$ and $v_0 = p(D)u_0$.

The following elliptic estimates then permit to control u itself and its traces in terms of $p(D)u$:

Lemma 7.2. *Let $k \in \mathbb{N}_0$ and $0 < \delta < \min\left\{1-\beta, \beta - \frac{1}{2}\right\}$. Furthermore suppose that $u \in C^\infty((0,\infty))$ satisfies (G_0), $|u|_{k+9,-\delta} < \infty$, and*

$$D^\ell u(x) = o(x^{-\delta}) \quad \text{as } x \nearrow \infty \text{ for all } 0 \le \ell \le k+11. \tag{7.20}$$

We set $a = \bar{u}(0,0)$, $b := \partial_y \bar{u}(0,0)$ (where $\bar{u}(x,y)$ is defined as in Definition 6.1), and $v := p(D)u$. Then the following elliptic estimates, only depending on k and δ, hold:

$$\begin{aligned}
|u|_{k+9,-\delta} &\sim |v|_{k+5,-\delta}, & |u-a|_{k+9,\delta} &\sim |v|_{k+5,\delta}, \\
|u-a|_{k+6,\beta-\frac{1}{2}-\delta} &\sim |v|_{k+2,\beta-\frac{1}{2}-\delta}, & |u-a|_{k+6,\beta-\frac{1}{2}+\delta} &\sim |v|_{k+2,\beta-\frac{1}{2}+\delta}, \\
|u-a|_{k+11,\frac{1}{2}-\delta} &\sim |v|_{k+7,\frac{1}{2}-\delta}, & |u-a|_{k+11,\frac{1}{2}+\delta} &\sim |v|_{k+7,\frac{1}{2}+\delta}, \\
|u-a|_{k+8,\beta-\delta} &\sim |v|_{k+4,\beta-\delta}, & \left|u-a-bx^\beta\right|_{k+8,\beta+\delta} &\sim |v|_{k+4,\beta+\delta}.
\end{aligned}$$

Remark 7.3. *We note that all the norms of u in the statement are finite: the first one, $|u|_{k+9,-\delta}$, is explicitly assumed to be; the other ones are since u satisfies (G_0) and decays as in (7.20).*

It will turn out that Lemma 7.2 is in fact a corollary of the following basic estimate:

Lemma 7.4. *Suppose that $w \in C^\infty((0,\infty))$, $\gamma, \varrho \in \mathbb{R}$ with $\gamma \ne \varrho$, $|w|_{1,\varrho} < \infty$, and*

$$w(x) = o(x^\varrho) \quad \text{as} \quad \begin{cases} x \searrow 0 & \text{if } \gamma < \varrho. \\ x \nearrow \infty & \text{if } \gamma > \varrho. \end{cases}$$

Then

$$|w|_{1,\varrho} \lesssim_{\gamma,\varrho} |(D-\gamma)w|_\varrho \tag{7.21}$$

As explained in Subsection 7.1 (see (7.15) and the sentence below), we need the following interpolation inequality in order to derive the maximal regularity estimate (7.2):

Lemma 7.5. *For any weight exponents $\alpha \in \mathbb{R}$, $\beta \in (\frac{1}{2}, 1)$, and any order of derivative $k \in \mathbb{N}_0$, we have the interpolation estimate*

$$\int_0^\infty t^{2\beta-2} |v|_{k,\alpha+\beta}^2 \, \mathrm{d}t \lesssim \int_0^\infty |\partial_t v|_{k,\alpha}^2 \, \mathrm{d}t + \int_0^\infty |v|_{k+1,\alpha+1}^2 \, \mathrm{d}t$$

for all locally integrable $v = v(t,x) : (0,\infty)^2 \to \mathbb{R}$.

Note that, though $\beta = \frac{\sqrt{13}-1}{4}$ is a fixed constant throughout the paper (fulfilling $\frac{1}{2} < \beta < 1$), within Lemma 7.5 we treat it as a parameter for the sake of generality. Note also that the constant in the estimate only depends on α, β, and k, and blows up as $\beta \searrow \frac{1}{2}$.

Finally we can prove maximal regularity with respect to the norm $|||\cdot|||$ (cf. equation (4.10)):

Proposition 7.6. *Suppose $I = (0,\tau) \subseteq (0,\infty)$, $k \in \mathbb{N}_0$, and δ as in Lemma 7.2. Then for every locally integrable $u_0 : (0,\infty) \to \mathbb{R}$ such that $|||u_0|||_0 < \infty$ and every locally integrable $f : I \times (0,\infty) \to \mathbb{R}$ such that $|||f|||_{1,I} < \infty$, there exists exactly one solution $u : I \times (0,\infty) \to \mathbb{R}$ of (7.1) that is locally integrable with $|||u|||_I < \infty$ and $u_{|t=0} = u_0$. This solution obeys the maximal regularity estimate (7.2).*

Remark 7.7. *Since $u : I \times (0,\infty) \to \mathbb{R}$ with $|||u|||_I < \infty$ implies that sufficiently many spatial derivatives of u and $\partial_t u$ are locally in space square integrable, we infer that:*

- *u is continuous,*
- *$p(D)u$ is continuous and defined classically,*
- *the initial condition $u_{|t=0} = u_0$ holds in the classical sense.*

For quantitative estimates we refer to Lemma 4.3 and the proof of Lemma 8.1.

Note that since we do not control $\partial_t^2 u$, the derivative $\partial_t u$ is only defined distributionally, i.e. the linear equation (7.1) is only fulfilled in the sense of distributions. However, we can differentiate the nonlinear equation (1.8a) in time t and thus also obtain control on $\partial_t^2 u$, that is, $\partial_t^2 u$ is locally square integrable. Hence the nonlinear equation (1.8a) is fulfilled classically.

7.3. Proofs

We now turn to the proofs of Lemmas 7.1, 7.2, 7.4, and Proposition 7.6:

Proof of Lemma 7.1. By rescaling $x \mapsto \frac{x}{h}$, equation (7.18) is equivalent to

$$xu + p(D)u = xu_0 + f. \tag{7.22}$$

Since the assumptions of Proposition 6.3 are fulfilled (in particular, xu_0 satisfies (G_0) and (G_∞) since u_0 does), there exists a solution to (7.18) which satisfies (G_0) and (G_∞). It hence remains to show estimate (7.19). Rescaling $x \mapsto \frac{x}{h}$ it is enough to show

$$\left|D^{\ell+2}v\right|^2_{\alpha+\frac{1}{2}} + C_1 \left|v\right|^2_{\alpha+\frac{1}{2}} + \frac{1}{C_2}\left|v\right|^2_{\ell+4,\alpha+1} \leq \left|D^{\ell+2}v_0\right|^2_{\alpha+\frac{1}{2}} + C_1\left|v_0\right|^2_{\alpha+\frac{1}{2}} + C_2\left|g\right|^2_{\ell,\alpha+2} \tag{7.23}$$

for all $\ell \geq 0$, where we recall that $v = p(D)u$, $v_0 = p(D)u_0$, and put $g = p(D-1)f$.

By applying $p(D-1)$ to (7.22) and using the commutator relation $[D, x] = x$, we obtain:

$$xv + p(D-1)v = xv_0 + g. \tag{7.24}$$

We test equation (7.24) with v in the inner product $(\cdot,\cdot)_{\alpha+1}$ and get

$$(xv, v)_{\alpha+1} + (p(D-1)v, v)_{\alpha+1} = (xv_0, v)_{\alpha+1} + (g, v)_{\alpha+1}. \tag{7.25}$$

Since α is in the coercivity range of $p(D)$, $p(D-1)$ is formally coercive with respect to $|\cdot|_{\alpha+1}$ (cf. Proposition 5.3). Hence by Lemma 5.2 *(d)*, there exists a constant $\lambda > 0$ such that $(p(D-1)v, v)_{\alpha+1} \geq \lambda\left|v\right|^2_{\alpha+1}$. Trivially $(xv, v)_{\alpha+1} = \left|v\right|^2_{\alpha+\frac{1}{2}}$ and by Young's inequality the other terms in (7.25) simplify to

$$(xv_0, v)_{\alpha+1} \leq \frac{1}{2}\left|v_0\right|^2_{\alpha+\frac{1}{2}} + \frac{1}{2}\left|v\right|^2_{\alpha+\frac{1}{2}} \quad \text{and} \quad (g, v)_{\alpha+1} \leq \frac{1}{2\lambda}\left|g\right|^2_{\alpha+1} + \frac{\lambda}{2}\left|v\right|^2_{\alpha+1}.$$

Then (7.25) yields the estimate

$$\left|v\right|^2_{\alpha+\frac{1}{2}} + \lambda\left|v\right|^2_{\alpha+1} \leq \left|v_0\right|^2_{\alpha+\frac{1}{2}} + \frac{1}{\lambda}\left|g\right|^2_{\alpha+1}. \tag{7.26}$$

We now want to upgrade the basic estimate (7.26) to obtain (7.23). Therefore, we test equation (7.24) with $(D+2\alpha+1)^{\ell+2}D^{\ell+2}v$ with respect to $(\cdot,\cdot)_{\alpha+1}$, so that by the symmetry properties (4.4) of D we get

$$\begin{aligned} &\left|D^{\ell+2}v\right|^2_{\alpha+\frac{1}{2}} + \left((D-1)^\ell p(D-1)v, (D+2\alpha+1)^2 D^{\ell+2}v\right)_{\alpha+1} \\ &= \left(D^{\ell+2}v_0, D^{\ell+2}v\right)_{\alpha+\frac{1}{2}} + ((D-1)^\ell g, (D+2\alpha+1)^2 D^{\ell+2}v)_{\alpha+1}. \end{aligned} \tag{7.27}$$

The first term on the right hand side of (7.27) is estimated by Young's inequality:

$$\left(D^{\ell+2}v_0, D^{\ell+2}v\right)_{\alpha+\frac{1}{2}} \leq \frac{1}{2}\left|D^{\ell+2}v_0\right|^2_{\alpha+\frac{1}{2}} + \frac{1}{2}\left|D^{\ell+2}v\right|^2_{\alpha+\frac{1}{2}}.$$

For the last term on the right hand side of (7.27), by applying the Cauchy-Schwarz and the Young inequality and the interpolation estimate (B.1b) (cf. Lemma B.1), we obtain

$$\begin{aligned} \left((D-1)^\ell g, (D+2\alpha+1)^2 D^{\ell+2}v\right)_{\alpha+1} &\leq \sqrt{C}\left|g\right|_{\ell,\alpha+1}\sqrt{\left|v\right|^2_{\alpha+1} + \left|D^{\ell+4}v\right|^2_{\alpha+1}} \\ &\leq C\left|g\right|^2_{\ell,\alpha+1} + \frac{1}{4}\left(\left|v\right|^2_{\alpha+1} + \left|D^{\ell+4}v\right|^2_{\alpha+1}\right) \end{aligned}$$

for some $C \in (0,\infty)$ (depending only on ℓ and γ). Making once more use of the interpolation estimates (B.1b) (cf. Lemma B.1), we get for the second term on the left hand side of (7.27)

$$\big((D-1)^\ell p(D-1)v, (D+2\alpha+1)^2 D^{\ell+2} v\big)_{\alpha+2} \geq \frac{3}{4} \left|D^{\ell+4} v\right|_{\alpha+1}^2 - C \left|v\right|_{\alpha+1}^2,$$

upon possibly enlarging C. We insert these inequalities into (7.27) which yields

$$\left|D^{\ell+2} v\right|_{\alpha+\frac{1}{2}}^2 + \left|D^{\ell+4} v\right|_{\alpha+1}^2 - \left(2C + \frac{1}{2}\right) \left|v\right|_{\alpha+1}^2 \leq \left|D^{\ell+2} v_0\right|_{\alpha+\frac{1}{2}}^2 + 2C \left|g\right|_{\ell,\alpha+1}^2.$$

We add C_1 times inequality (7.26) (where C_1 will be chosen below), so that

$$\begin{aligned} &\left(\left|D^{\ell+2} v\right|_{\alpha+\frac{1}{2}}^2 + C_1 \left|v\right|_{\alpha+\frac{1}{2}}^2\right) + \left|D^{\ell+4} v\right|_{\alpha+1}^2 + \left(C_1 \lambda - 2C - \frac{1}{2}\right) \left|v\right|_{\alpha+1}^2 \\ &\leq \left(\left|D^{\ell+2} v_0\right|_{\alpha+\frac{1}{2}}^2 + C_1 \left|v_0\right|_{\alpha+\frac{1}{2}}^2\right) + 2C \left|g\right|_{\ell,\alpha+1}^2 + \frac{C_1}{\lambda} \left|g\right|_{\alpha+1}^2. \end{aligned} \tag{7.28}$$

Then we choose C_1 so large that $C_1\lambda - 2C - \frac{1}{2} = 1$. Using the interpolation inequality (B.1b) (cf. Lemma B.1) one last time for the second and third term in (7.28), we obtain estimate (7.23) for sufficiently large $C_2 \in (0,\infty)$. □

Proof of Lemma 7.4. Throughout the proof, estimates only depend on γ and ϱ.

We first note that

$$\left|(D-\gamma)w\right|_\varrho^2 = \left|Dw\right|_\varrho^2 - 2\gamma (Dw, w)_\varrho + \gamma^2 \left|w\right|_\varrho^2. \tag{7.29}$$

For $\gamma \neq 0$ by Young's inequality we have

$$\left|(Dw, w)_\varrho\right| \leq \frac{1}{4\left|\gamma\right|} \left|Dw\right|_\varrho^2 + \left|\gamma\right| \left|w\right|_\varrho^2. \tag{7.30}$$

The combination of (7.29) and (7.30) shows

$$\left|(D-\gamma)w\right|_\varrho^2 \geq \frac{1}{2} \left|Dw\right|_\varrho^2 - \gamma^2 \left|w\right|_\varrho^2, \tag{7.31}$$

which is trivially true for $\gamma = 0$. Next we observe that

$$\left|(D-\gamma)w\right|_\varrho \overset{(4.1b)}{=} \left|x^{-\gamma}(D-\gamma)w\right|_{\varrho-\gamma} = \left|D(x^{-\gamma} w)\right|_{\varrho-\gamma}.$$

By assumption, $x^{-\varrho} w(x) = o(1)$ as $x \searrow 0$ if $\gamma < \varrho$, or $x \nearrow \infty$ if $\gamma > \varrho$. Hence we may apply Hardy's inequality (cf. [5, Lemma A.1] for a proof):

$$\left|D(x^{-\gamma} w)\right|_{\varrho-\gamma} \gtrsim \left|x^{-\gamma} w\right|_{\varrho-\gamma}.$$

Using also the identification $\left|w\right|_\varrho = \left|x^{-\gamma} w\right|_{\varrho-\gamma}$ (cf. (4.1b)), we obtain that

$$\left|(D-\gamma)w\right|_\varrho \gtrsim \left|w\right|_\varrho. \tag{7.32}$$

The combination of (7.31) and (7.32) yields estimate (7.21). □

Proof of Lemma 7.2. Throughout the proof, estimates only depend on k and ϱ and all norms of u are finite in view of Remark 7.3.

First of all, we note that estimating $|v|_{\ell,\varrho}$ in terms of $|u|_{\ell+4,\varrho}$ if $\varrho < 0$, $|u-a|_{\ell+4,\varrho}$ if $0 < \varrho < \beta$, or $\left|u - a - bx^\beta\right|_{\ell+4,\varrho}$ if $\beta < \varrho < 1$, directly follows from applying the triangle inequality.

For the nontrivial estimates we start with the last inequality of the lemma. In fact, we will more generally show that

$$\left|u - a - bx^\beta\right|_{k+4,\varrho} \lesssim |p(D)u|_{k,\varrho} \quad \text{for all } \varrho \in (\gamma_4, 1) = (\beta, 1), \tag{7.33}$$

from which the last estimate follows:

$$\varrho = \beta + \delta \in (\beta, 1) \iff 0 < \delta < 1 - \beta.$$

By the definition of the norms (cf. (4.2b)) and using the factorization

$$p(\zeta) = (\zeta - \gamma_4)(\zeta - \gamma_3)(\zeta - \gamma_2)(\zeta - \gamma_1),$$

where the γ_j are given by (1.9d), (7.33) follows from

$$\left|D^k\left(u - a - bx^\beta\right)\right|_\varrho \lesssim \left|(D-\gamma_4)(D-\gamma_3)(D-\gamma_2)(D-\gamma_1)D^k u\right|_\varrho$$

for all $k \in \mathbb{N}_0$. Because of $\gamma_4 = \beta$ and $\gamma_3 = 0$, this can be reformulated in terms of $u - a - bx^\beta$ solely:

$$\left|D^k\left(u - a - bx^\beta\right)\right|_\varrho \lesssim \left|(D-\gamma_4)(D-\gamma_3)(D-\gamma_2)(D-\gamma_1)D^k\left(u - a - bx^\beta\right)\right|_\varrho.$$

In order to establish this estimate, we will iteratively apply Lemma 7.4 with weight exponent ϱ and roots $\gamma = \gamma_j$, $j = 1, \cdots, 4$, on functions w that are linear combinations of $D^k\left(u - a - bx^\beta\right)$. Because of the assumption $\varrho > \beta$ and since $\gamma_j \le \beta$, we are in the case of $\varrho > \gamma$ in Lemma 7.4. In this case, we have to check that $D^k\left(u - a - bx^\beta\right) = o\left(x^\varrho\right)$ as $x \searrow 0$. For this, we use the lemma's assumption that u satisfies (G_0), which means that its extension $\bar{u}$ is smooth in (x, y) with $a = \bar{u}(0,0)$ and $b = \partial_y \bar{u}(0,0)$, yielding $\bar{D}^k(\bar{u} - a - by) = O(x) + O(y^2) + O(xy)$ as $x, y \searrow 0$, which because of $\beta \ge \frac{1}{2}$ translates back into $D^k\left(u - a - bx^\beta\right) = O(x)$ as $x \searrow 0$. This yields as desired $D^k\left(u - a - bx^\beta\right) = o\left(x^\varrho\right)$ if $\varrho < 1$.

Let us now prove the estimates which involve $u - a$. In fact, we will more generally show that

$$|u - a|_{k+4,\varrho} \lesssim |p(D)u|_{k,\varrho} \quad \text{for all } \varrho \in (\gamma_3, \gamma_4) = (0, \beta). \tag{7.34}$$

from which the estimates follow: indeed,

$$\begin{array}{rclcl}
\varrho = \beta - \delta \in (0,\beta) & \iff & 0 < \delta < \beta & \Leftarrow & 0 < \delta < \beta - \frac{1}{2} \\
\varrho = \frac{1}{2} + \delta \in (0,\beta) & \iff & -\frac{1}{2} < \delta < \beta - \frac{1}{2} & \Leftarrow & 0 < \delta < \beta - \frac{1}{2} \\
\varrho = \frac{1}{2} - \delta \in (0,\beta) & \iff & \frac{1}{2} - \beta < \delta < \frac{1}{2} & \Leftarrow & 0 < \delta < 1 - \beta \quad (\text{since } \beta \ge \frac{1}{2}) \\
\varrho = \beta - \frac{1}{2} + \delta \in (0,\beta) & \iff & \frac{1}{2} - \beta < \delta < \frac{1}{2} & \Leftarrow & 0 < \delta < 1 - \beta \quad (\text{since } \beta \ge \frac{1}{2}) \\
\varrho = \beta - \frac{1}{2} - \delta \in (0,\beta) & \iff & -\frac{1}{2} < \delta < \beta - \frac{1}{2} & \Leftarrow & 0 < \delta < \beta - \frac{1}{2} \\
\varrho = \delta \in (0,\beta) & \iff & 0 < \delta < \beta & \Leftarrow & 0 < \delta < 1 - \beta \quad (\text{since } \beta \ge \frac{1}{2}).
\end{array}$$

As above, (7.34) follows from

$$\left|D^k(u-a)\right|_\varrho \lesssim \left|(D-\gamma_4)(D-\gamma_3)(D-\gamma_2)(D-\gamma_1)D^k u\right|_\varrho .$$

Because of $\gamma_3 = 0$, this can be reformulated in terms of $u-a$ solely:

$$\left|D^k(u-a)\right|_\varrho \lesssim \left|(D-\gamma_4)(D-\gamma_3)(D-\gamma_2)(D-\gamma_1)D^k(u-a)\right|_\varrho .$$

As above, we apply Lemma 7.4 four times:

- For the root $\gamma_4 = \beta$, we use Lemma 7.4 with $\varrho < \gamma = \beta$ on $w = (D-\gamma_3)(D-\gamma_2)(D-\gamma_1)D^k(u-a)$.
- For the other roots, we have $\gamma_3, \gamma_2, \gamma_1 \leq 0$ and therefore we use it with $\varrho > \gamma$ and on $w = (D-\gamma_2)(D-\gamma_1)D^k(u-a)$, $w = (D-\gamma_1)D^k(u-a)$, and $w = D^k(u-a)$.

In order to be able to apply Lemma 7.4, we need to check the assumptions as $x \nearrow \infty$ and $x \searrow 0$, respectively:

- In case of γ_4, we need that $(D-\gamma_3)D^k(u-a) = D^{k+1}(u-a) = D^{k+1}u = o\left(x^\varrho\right)$ as $x \nearrow \infty$, which is implied by the lemma's assumption (7.20).
- In case of γ_3, γ_2, and γ_1, we need $D^k(u-a) = o\left(x^\varrho\right)$ as $x \searrow 0$. For this we use the lemma's assumption that u satisfies (G_0), which yields $\bar{D}^k(\bar{u}-a) = O(x) + O(y)$ as $x, y \searrow 0$. Because of $\beta \leq 1$, this translates back into $D^k(u-a) = O\left(x^\beta\right) \overset{\varrho \leq \beta}{=} o\left(x^\varrho\right)$.

Let us finally prove the estimate which involves u alone. In fact, we will more generally show that

$$|u|_{k+4,\varrho} \lesssim |p(D)u|_{k,\varrho} \quad \text{for all} \quad \varrho \in (\gamma_2, \gamma_3) = \left(-\beta - \frac{1}{2}, 0\right). \tag{7.35}$$

from which the estimate follows: indeed,

$$\varrho = -\delta \in (\gamma_2, 0) \iff 0 < \delta < -\gamma_2 = \beta + \frac{1}{2} \Leftarrow 0 < \delta < \beta - \frac{1}{2}.$$

As above, (7.34) follows from

$$\left|D^k u\right|_\varrho \lesssim \left|(D-\gamma_4)(D-\gamma_3)(D-\gamma_2)(D-\gamma_1)D^k u\right|_\varrho .$$

This time, we apply Lemma 7.4 four times:

- For the roots $\gamma_3 = 0$ and $\gamma_4 = \beta$, we use Lemma 7.4 with $\varrho < \gamma$ on $w = (D-\gamma_2)(D-\gamma_1)D^k u$.
- For the other roots we have $\gamma_1 \leq \gamma_2 < \varrho$ and therefore we use it with $\varrho > \gamma$ on $w = (D-\gamma_1)D^k(u-a)$, and $w = D^k(u-a)$.

In order to be able to apply Lemma 7.4, we need to check the assumptions as $x \nearrow \infty$ and $x \searrow 0$, respectively:

- In case of γ_3 and γ_4, we need that $D^k u = o\,(x^{\varrho})$ as $x \nearrow \infty$, which holds by the lemma's assumption (7.20).
- In case of γ_2 and γ_1, we need $D^k u = o\,(x^{\varrho})$ as $x \searrow 0$. For this, we use the lemma's assumption that u satisfies (G_0), which yields $\bar{D}^k u = O(1) = o(x^{\varrho})$ as $x \searrow 0$ since $\varrho < 0$.

□

Proof of Lemma 7.5. Throughout the proof, estimates may depend on α, β, and k.

By first approximating in x as in Remark 4.2, then cutting of in time t, and then convolving, we can without loss of generality assume that $v = v(t,x) \in C_0^{\infty}([0,\infty)_t \times (0,\infty)_x)$. We further note that it suffices to consider the case of $k = 0$ (since we can apply it to $D^{\ell} v$, $\ell = 0, \cdots, k$, and then sum). We then note that we only need to treat the case of $\alpha = -\frac{1}{2}$ (apply to $x^{-\frac{1}{2}-\alpha} v$), that is,

$$\int_0^{\infty} t^{2\beta-2} \, |v|^2_{-\frac{1}{2}+\beta} \, \mathrm{d}t \lesssim \int_0^{\infty} \left(|\partial_t v|^2_{-\frac{1}{2}} + |Dv|^2_{\frac{1}{2}} \right) \mathrm{d}t,$$

which by definition of the norms $|\cdot|_{\alpha}$ (cf. (4.1b)) and of $D = x\partial_x$ follows from

$$\iint_{(0,\infty)^2} t^{-2(1-\beta)} x^{-2\beta} v^2 \, \mathrm{d}x \, \mathrm{d}t \lesssim \iint_{(0,\infty)^2} \left((\partial_t v)^2 + (\partial_x v)^2 \right) \mathrm{d}x \, \mathrm{d}t. \tag{7.36}$$

We note that (7.36) amounts to an *anisotropic* type of Hardy's inequality with *critical* scaling (meaning that both sides are invariant under isotropic rescaling of (t,x)) ; in particular, it is wrong for $\beta = \frac{1}{2}$, since the left hand side of (7.36) then dominates $\iint_{(0,\infty)^2} (t^2 + x^2)^{-1} v^2 \, \mathrm{d}x \, \mathrm{d}t$ and the *isotropic* critical Hardy inequality is known to fail.

We proceed as for the proof of the isotropic Hardy inequality: For the weight function

$$\omega := t^{2\beta-2} x^{-2\beta}, \tag{7.37}$$

we construct a vector field $q = (q_t, q_x)$ in time-space with the following properties

$$\omega \lesssim \partial_t q_t + \partial_x q_x, \tag{7.38a}$$

$$q_t^2 + q_x^2 \lesssim \omega, \tag{7.38b}$$

$$q_t = 0 \quad \text{for } t = 0. \tag{7.38c}$$

We first argue why (7.38) yields (7.36). Indeed, using (7.38a) we have

$$\iint_{(0,\infty)^2} \omega v^2 \, \mathrm{d}x \, \mathrm{d}t \lesssim \iint_{(0,\infty)^2} (\partial_t q_t + \partial_x q_x) v^2 \, \mathrm{d}x \, \mathrm{d}t.$$

Because v satisfies $v = 0$ in a neighborhood of $x = 0$, $x = \infty$, $t = \infty$, and because of (7.38c), we may integrate the last term by parts:

$$\iint_{(0,\infty)^2} \omega v^2 \,\mathrm{d}x \,\mathrm{d}t \lesssim \iint_{(0,\infty)^2} (q_t \partial_t v + q_x \partial_x v) v \,\mathrm{d}x \,\mathrm{d}t.$$

Using Cauchy-Schwarz' inequality and inserting (7.38b) yields as desired

$$\begin{aligned}\iint_{(0,\infty)^2} \omega v^2 \,\mathrm{d}x \,\mathrm{d}t &\lesssim \left(\iint_{(0,\infty)^2} (q_t^2 + q_x^2) v^2 \,\mathrm{d}x \,\mathrm{d}t\right)^{\frac{1}{2}} \left(\iint_{(0,\infty)^2} \left((\partial_t v)^2 + (\partial_x v)^2\right) \,\mathrm{d}x \,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\lesssim \left(\iint_{(0,\infty)^2} \omega v^2 \,\mathrm{d}x \,\mathrm{d}t\right)^{\frac{1}{2}} \left(\iint_{(0,\infty)^2} \left((\partial_t v)^2 + (\partial_x v)^2\right) \,\mathrm{d}x \,\mathrm{d}t\right)^{\frac{1}{2}}.\end{aligned}$$

We now construct q explicitly:

$$q := \begin{cases} \left(\varepsilon t^{2\beta-1} x^{-2\beta}, -t^{\beta-1} x^{-\beta}\right) & \text{for } x \geq t \\ \left(\varepsilon t^{\beta-1} x^{-\beta}, -t^{2\beta-2} x^{-2\beta+1}\right) & \text{for } x \leq t \end{cases}, \tag{7.39}$$

where $\varepsilon \geq 0$ will be adjusted. We note that (q_t, q_x) is continuous and that

$$\begin{aligned}\partial_t q_t + \partial_x q_x &= \begin{Bmatrix} \varepsilon(2\beta-1) t^{2\beta-2} x^{-2\beta} + \beta t^{\beta-1} x^{-\beta-1} & \text{for } x > t \\ \varepsilon(\beta-1) t^{\beta-2} x^{-\beta} + (2\beta-1) t^{2\beta-2} x^{-2\beta} & \text{for } x < t \end{Bmatrix} \\ &\overset{(7.37)}{=} \omega \begin{Bmatrix} \varepsilon(2\beta-1) + \beta t^{-\beta+1} x^{\beta-1} & \text{for } x > t \\ \varepsilon(\beta-1) t^{-\beta} x^{\beta} + (2\beta-1) & \text{for } x < t \end{Bmatrix} \\ &\overset{\beta \in [0,1]}{\geq} \min\left\{\varepsilon(2\beta-1), \varepsilon(\beta-1) + (2\beta-1)\right\} \omega.\end{aligned}$$

Because by assumption $2\beta - 1 > 0$, we see that this implies (7.38a) for $0 < \varepsilon \ll 1$. Rewriting (7.39) as

$$q = t^{\beta-1} x^{-\beta} \begin{Bmatrix} \left(\varepsilon t^{\beta} x^{-\beta}, -1\right) & \text{for } x \geq t \\ \left(\varepsilon, -t^{\beta-1} x^{-\beta+1}\right) & \text{for } x \leq t \end{Bmatrix}$$

and using $\beta \in [0,1]$, we learn (7.38b). Finally, (7.38c) can be directly read off from (7.39), using once more $2\beta - 1 > 0$. □

Proof of Proposition 7.6. Throughout the proof estimates may depend on k, ℓ, and δ.

Existence. Since for finite time intervals I we can extend f by 0, it suffices to consider the case $I = (0, \infty)$. By the identification (4.7) we can, by the standard theory of Sobolev spaces, approximate every f by a sequence of functions $f_n \in C_0^\infty\left((0,\infty)^2\right)$ with respect to $|||\cdot|||_1$. Furthermore in Lemma B.3 we prove that u_0 can be approximated in the norm $|||\cdot|||_0$ by a sequence of functions $u_{0,n} \in C_0^0([0,\infty)) \cap C^\infty((0,\infty))$ which satisfy (G_0). Proving the assertions of the proposition for $u_{0,n}$ and f_n instead of u_0 and f, the general case can be treated by linearly extending the solution operator, the latter being bounded through the maximal regularity estimate (7.2). Hence in the following we assume without loss of generality $f \in C_0^\infty((0,\infty)^2)$ and $u_0 \in C_0^0([0,\infty)) \cap C^\infty((0,\infty))$ satisfying (G_0).

For $j \in \mathbb{N}_0$ and $h > 0$, we define $t_j^h := jh$, $f_j^h := \frac{1}{h}\int_{t_j^h}^{t_{j+1}^h} f(t)\,\mathrm{d}t$, and $u_0^h := u_0$. By Lemma 7.1, there exists a solution u_1^h of

$$x\frac{u_1^h - u_0^h}{h} + p(D)u_1^h = f_0^h$$

which satisfies (G_0) and (G_∞). Since both $\frac{f_1^h}{x}$ and $x\frac{u_1^h}{x} = u_1^h$ satisfy (G_∞), and both f_1^h and xu_1^h satisfy (G_0), Lemma 7.1 can be iteratively applied: For $j \geq 1$, we let u_{j+1}^h be the solution of

$$x\frac{u_{j+1}^h - u_j^h}{h} + p(D)u_{j+1}^h = f_j^h. \tag{7.40a}$$

By applying $p(D-1)$ to (7.40a), using the commutator relation $[D,x] = x$ and setting $v_j^h := p(D)u_j^h$, we arrive at

$$x\frac{v_{j+1}^h - v_j^h}{h} + p(D-1)v_{j+1}^h = p(D-1)f_j^h. \tag{7.40b}$$

Let $k, \ell \geq 0$. Since $-\frac{1}{2} \pm \delta$ and $\beta - 1 \pm \delta$ belong to the coercivity range of $p(D)$ (cf. Corollary 5.4), applying (7.19), we obtain

$$\begin{aligned}&\left(\left|D^{\ell+2}v_{j+1}^h\right|_{\pm\delta}^2 + C_1\left|v_{j+1}^h\right|_{\pm\delta}^2\right) - \left(\left|D^{\ell+2}v_j^h\right|_{\pm\delta}^2 + C_1\left|v_j^h\right|_{\pm\delta}^2\right) + \frac{h}{C_2}\left|v_{j+1}^h\right|_{\ell+4,\frac{1}{2}\pm\delta}^2 \\ &\leq C_2 h\left|f_j^h\right|_{\ell+4,\frac{1}{2}\pm\delta}^2,\end{aligned} \tag{7.41a}$$

respectively

$$\begin{aligned}&(t_j^h)^{2\beta-1}\left(\left|D^{k+2}v_{j+1}^h\right|_{\beta-\frac{1}{2}\pm\delta}^2 + C_1\left|v_{j+1}^h\right|_{\beta-\frac{1}{2}\pm\delta}^2\right) - (t_j^h)^{2\beta-1}\left(\left|D^{k+2}v_j^h\right|_{\beta-\frac{1}{2}\pm\delta}^2 + C_1\left|v_j^h\right|_{\beta-\frac{1}{2}\pm\delta}^2\right) \\ &+ (t_j^h)^{2\beta-1}\frac{h}{C_2}\left|v_{j+1}^h\right|_{k+4,\beta\pm\delta}^2 \leq C_2(t_j^h)^{2\beta-1}h\left|f_j^h\right|_{k+4,\beta\pm\delta}^2.\end{aligned} \tag{7.41b}$$

We sum (7.41) over j from 0 to J and reorder the resulting expression, demonstrating

$$\begin{aligned}&\left(\left|D^{\ell+2}v_{J+1}^h\right|_{\pm\delta}^2 + C_1\left|v_{J+1}^h\right|_{\pm\delta}^2\right) + \frac{h}{C_2}\sum_{j=0}^J\left|v_{j+1}^h\right|_{\ell+4,\frac{1}{2}\pm\delta}^2 \\ &\leq \left(\left|D^{\ell+2}v_0\right|_{\pm\delta}^2 + C_1\left|v_0\right|_{\pm\delta}^2\right) + C_2 h\sum_{j=0}^J\left|f_j^h\right|_{\ell+4,\frac{1}{2}\pm\delta}^2,\end{aligned} \tag{7.42a}$$

and

$$
\begin{aligned}
&(t_J^h)^{2\beta-1}\left(\left|D^{k+2}v_{J+1}^h\right|^2_{\beta-\frac12\pm\delta}+C_1\left|v_{J+1}^h\right|^2_{\beta-\frac12\pm\delta}\right)+\frac{h}{C_2}\sum_{j=0}^{J}(t_j^h)^{2\beta-1}\left|v_{j+1}^h\right|^2_{k+4,\beta\pm\delta}\\
&-h\sum_{j=1}^{J}\frac{(t_{j+1}^h)^{2\beta-1}-(t_j^h)^{2\beta-1}}{h}\left(\left|D^{k+2}v_j^h\right|^2_{\beta-\frac12\pm\delta}+C_1\left|v_j^h\right|^2_{\beta-\frac12\pm\delta}\right)\\
&\le C_2h\sum_{j=0}^{J}(t_j^h)^{2\beta-1}\left|f_j^h\right|^2_{k+4,\beta\pm\delta}.
\end{aligned}
\tag{7.42b}
$$

Taking $J\to\infty$, further using the interpolation inequality (B.1a) of Lemma B.1, equation (7.40b), and the fact that the right hand side of (7.42) monotonically increases, we arrive at

$$
\sup_{j\ge0}\left|v_{j+1}^h\right|^2_{\ell+2,\pm\delta}+h\sum_{j=0}^{\infty}\left|v_{j+1}^h\right|^2_{\ell+4,\frac12\pm\delta}\lesssim|v_0|^2_{\ell+2,\pm\delta}+h\sum_{j=0}^{\infty}\left|f_j^h\right|^2_{\ell+4,\frac12\pm\delta}
\tag{7.43a}
$$

and

$$
\begin{aligned}
&\sup_{j\ge0}(t_j^h)^{2\beta-1}\left|v_{j+1}^h\right|^2_{k+2,\beta-\frac12\pm\delta}+h\sum_{j=0}^{\infty}(t_j^h)^{2\beta-1}\left|v_{j+1}^h\right|^2_{k+4,\beta\pm\delta}\\
&-h\sum_{j=1}^{\infty}\frac{(t_{j+1}^h)^{2\beta-1}-(t_j^h)^{2\beta-1}}{h}\left|v_j^h\right|^2_{k+2,\beta-\frac12\pm\delta}\lesssim h\sum_{j=0}^{\infty}(t_j^h)^{2\beta-1}\left|f_j^h\right|^2_{k+4,\beta\pm\delta}.
\end{aligned}
\tag{7.43b}
$$

We interpolate u_j^h and v_j^h linearly in time t:

$$
u_h:=\sum_{j=0}^{\infty}\frac1h\left(\left(t_{j+1}^h-t\right)u_j^h+\left(t-t_j^h\right)u_{j+1}^h\right)\mathbb{1}_{[t_j^h,t_{j+1}^h)},
\tag{7.44a}
$$

$$
v_h:=p(D)u_h=\sum_{j=0}^{\infty}\frac1h\left(\left(t_{j+1}^h-t\right)v_j^h+\left(t-t_j^h\right)v_{j+1}^h\right)\mathbb{1}_{[t_j^h,t_{j+1}^h)}.
\tag{7.44b}
$$

Additionally, we define $f_h(t)$ as piecewise constant by averaging $f(t)$ on time intervals of length h:

$$
f_h:=\sum_{j=0}^{\infty}f_j^h\,\mathbb{1}_{[t_j^h,t_{j+1}^h)}=\sum_{j=0}^{\infty}\frac1h\int_{t_j^h}^{t_{j+1}^h}f(t)\,\mathrm{d}t.
$$

We then have

$$
x\partial_tv_h=x\frac{v_{j+1}^h-v_j^h}{h}\overset{(7.40b)}{=}p(D-1)f_j^h-p(D-1)v_{j+1}^h\quad\text{on }[t_j^h,t_{j+1}^h)
\tag{7.45}
$$

and furthermore

$$\left|\frac{1}{h}\left(\left(t_{j+1}^h - t\right) v_j^h + \left(t - t_j^h\right) v_{j+1}^h\right)\right|_{k+4,\beta\pm\delta}^2 \lesssim \left|v_j^h\right|_{k+4,\beta\pm\delta}^2 + \left|v_{j+1}^h\right|_{k+4,\beta\pm\delta}^2$$

for $t \in \left[t_j^h, t_{j+1}^h\right)$, so that estimates (7.43) turn into

$$\begin{aligned}&\sup_{t\geq 0} |v_h(t)|_{\ell+2,\pm\delta}^2 + \int_0^\infty \left(|\partial_t v_h(t)|_{\ell,-\frac{1}{2}\pm\delta}^2 + |v_h(t)|_{\ell+4,\frac{1}{2}\pm\delta}^2\right) \mathrm{d}t \\ &\lesssim |v_0|_{\ell+2,\pm\delta}^2 + \int_0^\infty |f_h(t)|_{\ell+4,\frac{1}{2}\pm\delta}^2 \,\mathrm{d}t,\end{aligned} \tag{7.46a}$$

$$\begin{aligned}&\sup_{t\geq 0} t^{2\beta-1} |v_h(t)|_{k+2,\beta-\frac{1}{2}\pm\delta}^2 + \int_0^\infty t^{2\beta-1}\left(|\partial_t v_h(t)|_{k,\beta-1\pm\delta}^2 + |v_h(t)|_{k+4,\beta\pm\delta}^2\right) \mathrm{d}t \\ &\lesssim \int_0^\infty t^{2\beta-1} |f_h(t)|_{k+4,\beta\pm\delta}^2 \,\mathrm{d}t + (2\beta-1)\int_0^\infty t^{2\beta-2}|v_h(t)|_{k+2,\beta-\frac{1}{2}\pm\delta}^2 \,\mathrm{d}t,\end{aligned} \tag{7.46b}$$

where we used that $t \mapsto t^{2\beta-1}$ is monotonically increasing and the fundamental theorem of calculus in the form

$$h\frac{(t_{j+1}^h)^{2\beta-1} - (t_j^h)^{2\beta-1}}{h} = (2\beta-1)\int_{t_j^h}^{t_{j+1}^h} t^{2\beta-2}\,\mathrm{d}t.$$

In order to absorb the last term on the right hand side of (7.46b), we apply the interpolation result of Lemma 7.5 (with $\alpha = -\frac{1}{2} \pm \delta$):

$$\int_0^\infty t^{2\beta-2}|v_h(t)|_{k+2,\beta-\frac{1}{2}\pm\delta}^2 \,\mathrm{d}t \lesssim \int_0^\infty |\partial_t v_h(t)|_{k+2,-\frac{1}{2}\pm\delta}^2 \,\mathrm{d}t + \int_0^\infty |v_h(t)|_{k+3,\frac{1}{2}\pm\delta}^2 \,\mathrm{d}t.$$

Hence the last term on the right hand side of (7.46b) can be absorbed into the left hand side of (7.46a) provided $\ell \geq k+2$ and $\ell + 4 \geq k+3$, i.e. $\ell \geq k+2$. For later purposes, we choose $\ell = k+3$. Summing up, we obtain

$$\begin{aligned}&\sup_{t\geq 0} |v_h(t)|_{k+5,\pm\delta}^2 + \int_0^\infty \left(|\partial_t v_h(t)|_{k+3,-\frac{1}{2}\pm\delta}^2 + |v_h(t)|_{k+7,\frac{1}{2}\pm\delta}^2\right) \mathrm{d}t \\ &+ \sup_{t\geq 0} t^{2\beta-1}|v_h(t)|_{k+2,\pm\delta}^2 + \int_0^\infty t^{2\beta-1}\left(|\partial_t v_h(t)|_{k,\beta-1\pm\delta}^2 + |v_h(t)|_{k+4,\beta\pm\delta}^2\right) \mathrm{d}t \\ &\lesssim |v_0|_{k+5,\pm\delta}^2 + \int_0^\infty |f_h(t)|_{k+7,\frac{1}{2}\pm\delta}^2 \,\mathrm{d}t + \int_0^\infty t^{2\beta-1}|f_h(t)|_{k+4,\beta\pm\delta}^2 \,\mathrm{d}t \\ &\lesssim \|\!|u_0\|\!|_0 + \|\!|f_h\|\!|_1\end{aligned} \tag{7.47}$$

where in the last inequality we have used that $v_0 = p(D)u_0$ and Lemma 7.2.

For every $j \geq 0$ we know that u_j^h satisfies (G_0), hence in particular $u_j^h(x) = a_j^h + b_j^h x^\beta + O(x)$ as $x \searrow 0$. We now use the same linear interpolation in time as in (7.44) to define

$$a_h(t) := \sum_{j=0}^{\infty} \frac{1}{h} \left(\left(t_{j+1}^h - t\right) a_j^h + \left(t - t_j^h\right) a_{j+1}^h \right) \mathbb{1}_{[t_j^h, t_{j+1}^h)}, \tag{7.48a}$$

$$b_h(t) := \sum_{j=0}^{\infty} \frac{1}{h} \left(\left(t_{j+1}^h - t\right) b_j^h + \left(t - t_j^h\right) b_{j+1}^h \right) \mathbb{1}_{[t_j^h, t_{j+1}^h)}. \tag{7.48b}$$

Since $a_h(t) = \lim_{x \searrow 0} u_h(t,x)$ and $b_h(t) = \lim_{x \searrow 0} (u(t,x) - a(t)) x^{-\beta}$, by using Lemma 7.2, we see that $|||u_h||| < \infty$ and that the left hand side of (7.47) controls $|||u_h|||$ (cf. (4.10)). Hence we obtain the estimate

$$|||u_h||| \lesssim |||u_0|||_0 + |||f_h|||_1. \tag{7.49}$$

By the definition of f_h one sees that $|||f_h - f|||_1 \to 0$ as $h \searrow 0$. In particular $|||f_h|||_1$ is bounded so that by (7.49) also $|||u_h|||$ is. Hence in particular, u_h is locally uniformly integrable on $(0,\infty)^2$, so that, up to a subsequence, it converges weakly to some locally integrable u. By the weak lower-semicontinuity of $||| \cdot |||$, the maximal regularity estimate (7.2) also holds in the limit. Likewise, the distributional form of equation (7.1) is preserved under weak convergence. Finally, H_t^1-convergence locally in $(0,\infty)_x$ implies C_t^0 convergence of u, whence the initial condition is fulfilled classically.

Uniqueness. Due to linearity we may assume $f \equiv 0$ and $u_0 \equiv 0$. Since $v := p(D)u = p(D)(u - a)$, from the definition of $||| \cdot |||$ by (4.10) and by Lemma 7.2, we see in particular that

$$\sup_{t \in I} |v|_{5,\delta}^2 + \int_I |v|_{7,\frac{1}{2}+\delta}^2 \, \mathrm{d}t < \infty. \tag{7.50}$$

We pass to the logarithmic variable $s := \ln x$ and test $x \partial_t v + p(D-1)v = 0$ with $\eta_n^2 v$, where $\eta_n(s) = \eta\left(\frac{s}{n}\right)$ and η is a symmetric cut off:

$$(x \eta_n^2 v, \partial_t v)_{\frac{1}{2}+\delta} + (\eta_n^2 v, p(D-1)v)_{\frac{1}{2}+\delta} = 0.$$

We rewrite this as

$$(x \eta_n^2 v, \partial_t v)_{\frac{1}{2}+\delta} + (\eta_n v, p(D-1)\eta_n v)_{\frac{1}{2}+\delta} = R_n,$$

where the remainder term R_n comes from permuting η_n with $p(D-1)$ and therefore contains at least one spatial derivative on η_n so that $R_n \lesssim \frac{1}{n} |v|_{3,\frac{1}{2}+\delta}^2$. From equation (7.1) (with $f \equiv 0$) and (7.50) we get $\int_I |\partial_t v|_{3,-\frac{1}{2}+\delta}^2 \, \mathrm{d}t < \infty$, whence in particular

$$(x \eta_n^2 v, \partial_t v)_{\frac{1}{2}+\delta} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (x \eta_n^2 v, v)_{\frac{1}{2}+\delta}.$$

Since $\frac{1}{2}+\delta$ is in the coercivity range of $p(D-1)$ (cf. Lemma 5.2 *(d)* and Corollary 5.4), we arrive at

$$\sup_{t\in I}\frac{1}{2}\left|\eta_n v(t)\right|_\delta^2+\lambda'\int_I\left|\eta_n v\right|_{2,\frac{1}{2}+\delta}^2\,\mathrm{d}t\le\frac{1}{2}\left|\eta_n v_{|t=0}\right|_\delta^2+\int_I R_n\,\mathrm{d}t$$

for some $\lambda'>0$. Since $v_{|t=0}\equiv 0$, trivially $\left|\eta_n v_{t=0}\right|_\delta^2=0$ and passing to the limit $n\to\infty$ also $\int_I R_n\,\mathrm{d}t\to 0$ as $n\to\infty$. Hence by dominated convergence $\sup_{t\in I}\frac{1}{2}\left|v(t)\right|_\delta^2\le 0$ implying $v=p(D)u=0$. As the kernel of $p(D)$ is spanned by 1, x^β, $x^{-\frac{3}{2}}$, and $x^{-\beta-\frac{1}{2}}$, we know that

$$u(t,x)=a_1(t)+a_2(t)x^\beta+a_3(t)x^{-\frac{3}{2}}+a_4(t)x^{-\beta-\frac{1}{2}}.$$

By the definition of $\|\!|\cdot|\!\|$ (cf. (4.10)) we have $\sup_{t\in I}|u|_{k+8,-\delta}^2\le\|\!|u|\!\|<\infty$. Therefore, $a_1(t)\equiv a_2(t)\equiv 0$ (since $-\delta<0,\beta$) and $a_3(t)\equiv a_4(t)\equiv 0$ (since $-\delta>-\beta-\frac{1}{2},-\frac{3}{2}$). □

8. The Nonlinear Problem

In this section we prove Theorem 3.1. The main ingredient for the proof is the following estimate on the nonlinearity $\mathcal{N}(u)$ defined in (1.9f):

Lemma 8.1. *Let $I=(0,\tau)\subseteq(0,\infty)$ be an interval, $0<\delta<\min\left\{1-\beta,\beta-\frac{1}{2}\right\}$, and $k\in\mathbb{N}_0$. For all smooth functions $u,u_1,u_2:I\times(0,\infty)\to\mathbb{R}$, the following estimates for the nonlinearity $\mathcal{N}(u)$ hold:*

$$\|\!|\mathcal{N}(u)|\!\|_{1,I}\lesssim_{k,\delta}\max_{j=2,\cdots,7}\|\!|u|\!\|_I^j, \tag{8.1a}$$

$$\|\!|\mathcal{N}(u_1)-\mathcal{N}(u_2)|\!\|_{1,I}\lesssim_{k,\delta}\max_{j=1,\cdots,6}\left(\|\!|u_1|\!\|_I+\|\!|u_2|\!\|_I\right)^j\|\!|u_1-u_2|\!\|_I, \tag{8.1b}$$

Proof. Throughout the proof, all estimates depend on k and δ.

Preliminaries. We recall the norm for the solution (cf. (4.10)):

$$\begin{aligned}\|\!|u|\!\|_I^2\ge&\sup_{t\in I}\left(|u-a|_{k+9,\delta}^2+|u|_{k+9,-\delta}^2\right)\\&+\int_I\left(t^{2\beta-1}\left|u-a-bx^\beta\right|_{k+8,\beta+\delta}^2+|u-a|_{k+11,\frac{1}{2}+\delta}^2\right)\mathrm{d}t\\&+\int_I\left(t^{2\beta-1}|u-a|_{k+8,\beta-\delta}^2+|u-a|_{k+11,\frac{1}{2}-\delta}^2\right)\mathrm{d}t.\end{aligned}\tag{8.2}$$

We dropped the time-weighted contribution to the trace norm, since we do not need it in the nonlinear estimate. The norm for the right hand side (cf. (2.19)) reads

$$\|\!|f|\!\|_{1,I}^2=\int_I\left(t^{2\beta-1}|f|_{k+4,\beta+\delta}^2+|f|_{k+7,\frac{1}{2}+\delta}^2+t^{2\beta-1}|f|_{k+4,\beta-\delta}^2+|f|_{k+7,\frac{1}{2}-\delta}^2\right)\mathrm{d}t. \tag{8.3}$$

We start by deriving uniform estimates in time and space (cf. (4.8) for the definition of norms), that is,

$$\|u-a\|_{k+8,(0,1]} \lesssim |u-a|_{k+9,\delta} \quad \text{and} \quad \|u\|_{k+8,[1,\infty)} \lesssim |u|_{k+9,-\delta}\,. \tag{8.4}$$

As the arguments are the same, we focus on proving the first inequality. It suffices to establish $\|u-a\|_{(0,1]} \lesssim |u-a|_{1,\delta}$. Choosing $\tilde{u}(s) := u(e^s)$, $\tilde{\eta}(s) := \eta(e^s)$, and a smooth cut off $\eta : (0,\infty) \to \mathbb{R}$ with $\eta(x) \equiv 1$ for $x \leq 1$ and $\eta(x) \equiv 0$ for $x \geq 2$, we infer

$$\begin{aligned}\|u-a\|_{(0,1]}^2 \leq \|\eta(u-a)\|^2 = \sup_{s\in\mathbb{R}} |(\tilde{\eta}(s)\,(\tilde{u}(s)-a))|^2 &\lesssim \sum_{\ell=0}^{1} \int_{-\infty}^{\infty} \left(\partial_s^\ell\,(\tilde{\eta}(s)\,(\tilde{u}(s)-a))\right)^2 \mathrm{d}s \\ &\lesssim |u-a|_{1,0,(0,2)}^2 \lesssim |u-a|_{1,\delta}^2\,.\end{aligned}$$

Estimate (8.4) with (4.13) (cf. Lemma 4.3) and (8.2) implies $\sup_{t\in I} \|u-a\|_{k+8}^2 \lesssim |||u|||_I^2$. For later reference, we combine this with (4.13) to

$$\sup_{t\in I} \left(\|u-a\|_{k+8}^2 + |a|^2\right) + \int_I t^{2\beta-1}\,|b|^2\,\mathrm{d}t \lesssim |||u|||_I^2. \tag{8.5}$$

Proof of estimate (8.1a). We denote by $\mathcal{M}_{\mathrm{sym}}$ the symmetrization of $\mathcal{M}$. Because of multi-linearity of $\mathcal{M}_{\mathrm{sym}}$, $\mathcal{N}(u)$ is a linear combination of terms of the form

$$\mathcal{M}_{\mathrm{sym}}(u,u,w_3,w_4,w_5), \quad \text{where } w_3,w_4,w_5 \in \{u,1\}.$$

Once again by multi-linearity, these terms can be written as linear combinations of terms of the form

$$\mathcal{M}_{\mathrm{sym}}(w_1,w_2,w_3,w_4,w_5), \quad \text{where } w_1,w_2 \in \{u-a,a\} \text{ and } w_3,w_4,w_5 \in \{u-a,a,1\}.$$

However, since

$$\mathcal{M}_{\mathrm{sym}}(1,\cdots,1) \overset{(1.9c)}{=} \frac{1}{5}p(D)1 \overset{(1.9d)}{=} 0, \tag{8.6}$$

the only spatially non-constant term $u-a$ has to appear at least once as one of the five arguments, so that the latter reduces to

$$\mathcal{M}_{\mathrm{sym}}(u-a,w_2,w_3,w_4,w_5), \quad \text{where } w_2 \in \{u-a,a\} \text{ and } w_3,w_4,w_5 \in \{u-a,a,1\}. \tag{8.7}$$

We recall that $\mathcal{M}_{\mathrm{sym}}$ distributes 4 derivations D onto its five arguments. Hence by the product rule, $D^\ell\mathcal{N}(u)$, $\ell \leq k+4$ or $\ell \leq k+7$ respectively, is a linear combination of products $w_1 \times \cdots \times w_5$ with

$$\begin{aligned} w_1 &= D^{\ell_1}(u-a) \quad \text{with } \ell_1 \leq \ell+4, \\ w_2 &\in \{D^{\ell_2}(u-a),a\} \quad \text{with } \ell_2 \leq \frac{1}{2}(\ell+4) \leq k+8, \\ w_3,w_4,w_5 &\in \{D^{\ell_3}(u-a),a,1\} \quad \text{with } \ell_3 \leq \frac{1}{3}(\ell+4) \leq k+8. \end{aligned}$$

In view of (8.2), we see that $w_2, \cdots, w_5$ can be controlled by sup-bounds in time, and only w_1 requires an L^2-bound in time. This yields an estimate on the "subcritical" part of norm, i.e., the part which is unaffected by the x^β-behavior near $x = 0$:

$$
\begin{aligned}
&\int_I \left(t^{2\beta-1} \left|\mathcal{N}(u)\right|^2_{k+4,\beta-\delta} + \left|\mathcal{N}(u)\right|^2_{k+7,\frac{1}{2}\pm\delta} \right) \mathrm{d}t \\
&\lesssim \quad \int_I \left(t^{2\beta-1} \left|u-a\right|^2_{k+8,\beta-\delta} + \left|u-a\right|^2_{k+11,\frac{1}{2}\pm\delta} \right) \mathrm{d}t \\
&\qquad \times \sup_{t\in I} \left(\|u-a\|^2_{k+8} + |a|^2 \right) \left(1 + \left(\sup_{t\in I} \left(\|u-a\|^2_{k+8} + |a|^2 \right) \right)^3 \right) \\
&\overset{(8.2),(8.5)}{\lesssim} \quad |||u|||^4_I \left(1 + |||u|||^6_I \right).
\end{aligned} \tag{8.8}
$$

For the "supercritical" part of the nonlinear estimate, we have to go one step further in the decomposition. Because of linearity in the first argument, the terms in (8.7) can be written as linear combinations of terms of the following two forms:

$$
\mathcal{M}_{\mathrm{sym}} \left(u - a - bx^\beta, w_2, w_3, w_4, w_5 \right) \text{ with } w_2 \in \{u-a, a\},\ w_3, w_4, w_5 \in \{u-a, a, 1\}, \tag{8.9}
$$

$$
\mathcal{M}_{\mathrm{sym}} (bx^\beta, w_2, w_3, w_4, w_5) \text{ with } w_2 \in \{u-a, a\},\ w_3, w_4, w_5 \in \{u-a, a, 1\}. \tag{8.10}
$$

However, since

$$
\mathcal{M}_{sym}(x^\beta, 1, \cdots, 1) \overset{(1.9c)}{=} \frac{1}{5} p(D) x^\beta \overset{(1.9d)}{=} 0, \tag{8.11}
$$

the spatially non-constant term $u-a$ has to appear at least once in the last four arguments in (8.10), so that the latter reduces to

$$
\mathcal{M}_{\mathrm{sym}} \left(bx^\beta, u-a, w_3, w_4, w_5 \right) \quad \text{where } w_3, w_4, w_5 \in \{u-a, a, 1\}. \tag{8.12}
$$

We recall that $\mathcal{M}_{\mathrm{sym}}$ distributes 4 derivations D onto its five arguments. Hence by the product rule, $D^\ell \mathcal{N}(u)$ for $\ell \le k+4$, is a linear combination of products $w_1 \times \cdots \times w_5$ of two types, depending on whether they come from (8.9) or from (8.12). For terms of the type (8.9), we obtain

$$
\begin{aligned}
w_1 &= D^{\ell_1}(u - a - bx^\beta) \quad \text{with } \ell_1 \le \ell + 4 \le k + 8 \\
w_2 &\in \{D^{\ell_2}(u-a), a\} \quad \text{with } \ell_2 \le \ell + 4 \le k + 8, \\
w_3, w_4, w_5 &\in \{D^{\ell_3}(u-a), a, 1\} \quad \text{with } \ell_3 \le \frac{1}{2}(\ell + 4) \le k + 8,
\end{aligned}
$$

so that

$$
\begin{aligned}
|w_1 \times \cdots \times w_5|^2_{\beta+\delta} \lesssim &\left| u - a - bx^\beta \right|^2_{k+8,\beta+\delta} \\
&\times \left(\|u-a\|^2_{k+8} + |a|^2 \right) \left(1 + \left(\|u-a\|^2_{k+8} + |a|^2 \right)^3 \right).
\end{aligned} \tag{8.13}
$$

Since $D^\ell x^\beta = \beta^\ell x^\beta$, the terms of the type (8.12) are given by linear combination of products $w_1 \times \cdots \times w_5$ of the form

$$\begin{aligned} w_1 &= b, \\ w_2 &= x^\beta D^{\ell_1}(u-a) \quad \text{with } \ell_1 \le \ell + 4 \le k+8, \\ w_3, w_4, w_5 &\in \{D^{\ell_2}(u-a), a, 1\} \quad \text{with } \ell_2 \le \frac{1}{2}(\ell+4) \le k+8, \end{aligned}$$

so that here we obtain using $\left|x^\beta \cdot\right|_{\beta+\delta} = |\cdot|_\delta$

$$|w_1 \times \cdots \times w_5|_{\beta+\delta}^2 \lesssim |b|^2 \, |u-a|_{k+8,\delta}^2 \left(1 + \left(\|u-a\|_{k+8}^2 + |a|^2\right)^3\right). \tag{8.14}$$

From (8.13) and (8.14) we obtain, using the weighted L^2-bound in time on the first term and the sup-bounds in time for the remaining ones,

$$\begin{aligned} &\int_I t^{2\beta-1} \, |\mathcal{N}(u)|_{k+4,\beta+\delta}^2 \, \mathrm{d}t \\ &\lesssim \quad \int_I t^{2\beta-1} \left(\left|u-a-bx^\beta\right|_{k+8,\beta+\delta}^2 + |b|^2\right) \mathrm{d}t \\ &\qquad \times \sup_{t\ge 0} \left(\|u-a\|_{k+8}^2 + |a|^2 + |u-a|_{k+8,\delta}^2\right) \left(1 + \left(\|u-a\|_{k+8}^2 + |a|^2\right)^3\right) \\ &\overset{(8.2),(8.5)}{\lesssim} \quad \|u\|_I^4 \left(1 + \|u\|_I^6\right). \end{aligned} \tag{8.15}$$

In view of the definition (8.3), (8.1a) follows from adding (8.8) and (8.15), and taking the square root.

Proof of estimate (8.1b). Because of multi-linearity of $\mathcal{M}_{\text{sym}}$, $\mathcal{N}(u_1) - \mathcal{N}(u_2)$ is a linear combination of terms of the form

$$\mathcal{M}_{\text{sym}}(u_1 - u_2, w_2, w_3, w_4, w_5), \quad \text{where } w_2 \in \{u_1, u_2\}, \ w_3, w_4, w_5 \in \{u_1, u_2, 1\}.$$

Once again by multi-linearity, these terms can be written as linear combinations of terms of the form

$$\mathcal{M}_{\text{sym}}(w_1, w_2, w_3, w_4, w_5), \quad \text{where} \begin{cases} w_1 \in \{(u_1 - a_1) - (u_2 - a_2), a_1 - a_2\}, \\ w_2 \in \{u_1 - a_1, u_2 - a_2, a_1, a_2\}, \\ w_3, w_4, w_5 \in \{u_1 - a_1, u_2 - a_2, a_1, a_2, 1\}. \end{cases}$$

Note that now we cannot argue that $(u_1 - a_1) - (u_2 - a_2)$ always appears once, whence the second possible form of w_1. The arguments for the case $w_1 = u_1 - a_1 - (u_2 - a_2)$ are identical to the ones above. Hence we are left with the case

$$\mathcal{M}_{\text{sym}}(a_1 - a_2, w_2, w_3, w_4, w_5), \quad \text{where} \begin{cases} w_2 \in \{u_1 - a_1, u_2 - a_2, a_1, a_2\}, \\ w_3, w_4, w_5 \in \{u_1 - a_1, u_2 - a_2, a_1, a_2, 1\}. \end{cases}$$

In fact, since spatially non-constant terms have two appear at least once (cf. (8.6)),

$$\mathcal{M}_{\mathrm{sym}}(a_1 - a_2, w_2, w_3, w_4, w_5), \quad \text{where } \begin{cases} w_2 \in \{u_1 - a_1, u_2 - a_2\} \\ w_3, w_4, w_5 \in \{u_1 - a_1, u_2 - a_2, a_1, a_2, 1\}. \end{cases} \tag{8.16}$$

For the subcritical part, as above we see that $D^\ell \mathcal{N}(u)$, for $\ell \le k+4$ respectively $\ell \le k+7$, is a linear combination of products $w_1 \times \cdots \times w_7$ of the form

$$\begin{aligned} w_1 &= a_1 - a_2, \\ w_2 &\in \{D^{\ell_1}(u_i - a_i)\} \quad \text{with } \ell_1 \le \ell + 4, \\ w_3, w_4, w_5 &\in \{D^{\ell_2}(u_i - a_i), a_i, 1\} \quad \text{with } \ell_2 \le \frac{1}{2}(\ell+4) \le k+8. \end{aligned}$$

Hence the subcritical part of the estimate follows similarly to (8.13), using the L^2-bound in time for w_2 and the sup-bounds on the remaining terms. For the supercritical part we further decompose (8.16) into

$$\mathcal{M}_{\mathrm{sym}}(a_1 - a_2, w_2, w_3, w_4, w_5), \quad \text{where } \begin{cases} w_2 \in \{u_1 - a_1 - b_1 x^\beta, u_2 - a_2 - b_2 x^\beta\} \\ w_3, w_4, w_5 \in \{u_1 - a_1, u_2 - a_2, a_1, a_2, 1\} \end{cases} \tag{8.17}$$

and

$$\mathcal{M}_{\mathrm{sym}}(a_1 - a_2, w_2, w_3, w_4, w_5), \quad \text{where } \begin{cases} w_2 \in \{b_1 x^\beta, b_2 x^\beta\},\ w_3 \in \{u_1 - a_1, u_2 - a_2\}, \\ w_4, w_5 \in \{u_1 - a_1, u_2 - a_2, a_1, a_2, 1\}, \end{cases} \tag{8.18}$$

where due to (8.11) without loss of generality w_3 is non-constant. In case (8.17), as above we see that $D^\ell \mathcal{N}(u)$, for $\ell \le k+4$, is a linear combination of products $w_1 \times \cdots \times w_5$ of the form

$$\begin{aligned} w_1 &= a_1 - a_2, \\ w_2 &\in \{D^{\ell_1}(u_i - a_i - b_i x^\beta)\} \text{ with } \ell_1 \le \ell + 4 \le k+8, \\ w_3, w_4, w_5 &\in \{D^{\ell_2}(u_i - a_i), a_i, 1\} \text{ with } \ell_2 \le \ell + 4 \le k+8, \end{aligned}$$

which is estimated similarly to (8.15), using the weighted L^2-bound in time on w_2. In case (8.18), as above we see that $D^\ell \mathcal{N}(u)$, for $\ell \le k+4$, is a linear combination of products $w_1 \times \cdots \times w_5$ of the form

$$\begin{aligned} w_1 &= a_1 - a_2, \\ w_2 &\in \{b_i\}, \\ w_3 &\in \{x^\beta D^{\ell_1}(u_i - a_i)\} \text{ with } \ell_1 \le \ell + 4 \le k+8, \\ w_4, w_5 &\in \{D^{\ell_2}(u_i - a_i), a_i, 1\} \text{ with } \ell_2 \le \frac{1}{2}(\ell+4) \le k+8, \end{aligned}$$

which is also estimated similarly to (8.14), using the weighted L^2-bound in time on w_2. □

We are now ready to prove our main result:

Proof of Theorem 3.1. Throughout the proof, all estimates depend on k and δ.

Existence. Let $\varepsilon > 0$ yet to be determined and $u_0 : (0,\infty) \to \mathbb{R}$ be locally integrable with $|||u_0|||_0 < \varepsilon$. We define the space

$$S := \left\{ u : (0,\infty)^2 \to \mathbb{R} \text{ locally integrable} : |||u||| < \eta,\, u_{|t=0} = u_0 \right\}$$

for some $\eta > 0$ to be specified later. Note that by Remark 7.7 the boundary value $u_{|t=0}$ is well defined. Let T denote the solution operator of Proposition 7.6. Then the nonlinear parabolic equation (1.8) is equivalent to the fixed point equation

$$u = \mathcal{T}(u) := T\mathcal{N}(u).$$

We will now show that the mapping $\mathcal{T} : S \to S$ is a contraction for $\varepsilon > 0$ sufficiently small. For this, note that by the maximal regularity estimate (7.2) of Proposition 7.6 and the nonlinear estimate (8.1a) of Lemma 8.1 we can show for $u \in S$

$$|||\mathcal{T}(u)||| = |||T\mathcal{N}(u)||| \overset{(7.2)}{\lesssim} |||u_0|||_0 + |||\mathcal{N}(u)|||_1 \overset{(8.1a)}{\lesssim} |||u_0|||_0 + \max_{j=2,\cdots,5} |||u|||^j \lesssim \varepsilon + \eta^2.$$

for $\eta \leq 1$. Choosing $\eta \ll 1$ and $\varepsilon \ll \eta$, we can infer from (8) that $\mathcal{T}$ maps S into itself. For $u_1, u_2 \in S$ we can estimate by the maximal regularity estimate (7.2) of Proposition 7.6 and estimate (8.1b) for the nonlinearity (cf. Lemma 8.1)

$$\begin{aligned} |||\mathcal{T}(u_1) - \mathcal{T}(u_2)||| &= |||T(\mathcal{N}(u_1) - \mathcal{N}(u_2))||| \overset{(7.2)}{\lesssim} |||\mathcal{N}(u_1) - \mathcal{N}(u_2)|||_1 \\ &\overset{(8.1b)}{\lesssim} \sup_{j=1,\cdots,6} (|||u_1||| + |||u_2|||)^j \cdot |||u_1 - u_2||| \lesssim \eta |||u_1 - u_2|||. \end{aligned}$$

for $\eta \leq 1$. This shows that $\mathcal{T}$ is a contraction for $\eta \ll 1$. We apply the contraction mapping theorem, showing the existence part of the theorem.

Uniqueness. Let u denote the above constructed solution and w denote another solution of (1.8). Suppose that there exists a $t > 0$ such that $u(t)$ and $w(t)$ differ. Then, by continuity in time (cf. Lemma B.4), there exists a maximal $t^* \geq 0$ such that

(a) $u(t,x) = w(t,x)$ for all $t \in (0, t^*]$ and $x > 0$,

(b) $u(t)$ and $w(t)$ differ for $t > t^*$ sufficiently small.

We can use $u(t^*) = w(t^*)$ as initial data and obtain for $I := (t^*, t^* + \tau)$

$$|||u - w|||_I = |||\mathcal{T}(u) - \mathcal{T}(w)|||_I \overset{(7.2),(8.1b)}{\lesssim} \max_{j=1,\cdots,6} (|||u|||_I + |||w|||_I)^j \cdot |||u - w|||_I. \qquad (8.19)$$

By Lemma B.4 we know $|||u|||_I \to |||u(t^*)|||_0$ and $|||w|||_I \to |||u(t^*)|||_0$ as $\tau \searrow 0$. Since $|||u(t^*)|||_0 \leq |||u||| \leq \eta$, we obtain from (8.19) that $|||u - w|||_I \leq 0$ for $\tau \ll 1$ and $\eta \ll 1$. This shows $u = w$ for $t \in I = (t^*, t^* + \tau)$, which is a contradiction to the maximality of t^*. □

A. Coordinate Transformations

In this appendix we discuss details of the coordinate transformations that have not been given in the introduction.

A.1. Derivation of (1.8)

Differentiating the hodograph transform, equation (1.6), with respect to time t, we obtain:

$$\partial_t h + \partial_z h \partial_t Z = 0. \tag{A.1}$$

Furthermore we can make use of the transformation of derivatives

$$\partial_z = \frac{1}{\partial_x Z}\partial_x = F\partial_x, \tag{A.2}$$

where we have introduced $F := \frac{1}{\partial_x Z}$ (cf. (1.7a)). Using (1.6), (A.1), and (A.2) in the thin-film equation (1.1a), we get

$$-\partial_t Z F \partial_x x^{\frac{3}{2}} + F\partial_x x^3 (F\partial_x)^3 x^{\frac{3}{2}} = 0,$$

i.e.

$$-\partial_t Z + \frac{2}{3} x^{-\frac{1}{2}} \partial_x x^3 (F\partial_x)^3 x^{\frac{3}{2}} = 0. \tag{A.3}$$

Using furthermore that $\partial_t F = -\frac{1}{(\partial_x Z)^2}\partial_t \partial_x Z = -F^2 \partial_x \partial_t Z$, applying $F^2\partial_x$ to (A.3) we obtain

$$F_t + \frac{2}{3} F^2 \partial_x x^{-\frac{1}{2}} \partial_x x^3 (F\partial_x)^3 x^{\frac{3}{2}} = 0.$$

Multiplying by x and carrying out the derivative $\partial_x x^{\frac{3}{2}}$, this equation can be rewritten as

$$xF_t + xF^2 \partial_x x^{-\frac{1}{2}} \partial_x x^3 F \partial_x F \partial_x x^{\frac{1}{2}} F = 0.$$

We introduce the logarithmic derivative $D := x\partial_x$ (cf. (1.9a)). Together with the commutation relation $Dx^\mu = x^\mu (D + \mu)$, we obtain the problem

$$xF_t + \mathcal{M}(F, \cdots, F) = 0 \quad \text{for } x, t > 0, \tag{A.4a}$$

$$F_{|t=0} = F_0 \quad \text{at } t = 0 \text{ and for } x > 0, \tag{A.4b}$$

where we introduced the 5-linear form

$$\mathcal{M}(F_1, \cdots, F_5) := F_1 F_2 D \left(D + \frac{3}{2}\right) F_3 \left(D - \frac{1}{2}\right) F_4 \left(D + \frac{1}{2}\right) F_5. \tag{A.5}$$

For the new unknown $F(t,x)$, the traveling-wave solution

$$H_{\mathrm{TW}} = x^{\frac{3}{2}} = \left(z + \frac{3}{8}t\right)^{\frac{3}{2}} \quad \text{or} \quad Z_{\mathrm{TW}} = x - \frac{3}{8}t$$

translates into $F_{\mathrm{TW}} = 1$. Since the traveling-wave solution is a solution of (A.4a), we necessarily have $\mathcal{M}(1, \cdots, 1) = 0$, which can also be validated by the definition of $\mathcal{M}$, as the logarithmic derivative D appears as a single factor. Therefore it seems reasonable to consider perturbations of the traveling wave, i.e. to set $u := F - 1$ (cf. (1.7b)). We can now define the linearization of (A.4a) through

$$\mathcal{L}u := \mathcal{M}(u, 1, \cdots, 1) + \cdots + \mathcal{M}(1, \cdots, 1, u). \tag{A.6}$$

An elementary calculation shows that $\mathcal{L} = p(D)$, with the polynomial $p(\zeta)$ defined in (1.9d). Now, using the definition of the nonlinearity (1.9f), we obtain the parabolic initial value problem (1.8).

A.2. The Speed of the Contact Line

Let us now give an argument for the boundary behavior (3.2) of the vertically averaged speed $V(t, Z(t,x))$ of the fluid: For that we notice that in view of the hodograph transform (1.6) and the transformation of derivatives (A.2), the speed $V = h\partial_z^3 h$ transforms into

$$V(t, Z(t,x)) = x^{\frac{3}{2}}(F\partial_x)^3 x^{\frac{3}{2}} = \tilde{\mathcal{M}}(F,F,F) \tag{A.7a}$$

with

$$\tilde{\mathcal{M}}(F_1, F_2, F_3) := \frac{3}{2}F_1\left(D - \frac{1}{2}\right)F_2\left(D + \frac{1}{2}\right)F_3. \tag{A.7b}$$

Let $\tilde{\mathcal{M}}_{\mathrm{sym}}$ denote the symmetrization of $\tilde{\mathcal{M}}$. In view of (A.7), writing $F = 1 + u = (1 + a) + (u - a)$ where $a(t) = \lim_{x\searrow 0} u(t,x)$, and making use of the multi-linearity of $\tilde{\mathcal{M}}_{\mathrm{sym}}$, we obtain

$$V(t, Z(t,x)) = \partial_t Z(t,x) = A(t) + B(t,x) + C(t,x),$$

where

$$\begin{aligned} A &= -\frac{3}{8}(1+a)^3, \\ B &= 3(1+a)^2\tilde{\mathcal{M}}_{\mathrm{sym}}(u-a,1,1), \\ C &= 3(1+a)\tilde{\mathcal{M}}_{\mathrm{sym}}(u-a,u-a,1) + \tilde{\mathcal{M}}_{\mathrm{sym}}(u-a,u-a,u-a). \end{aligned}$$

We further note that $B(t)$ can be simplified as

$$B = \frac{3}{2}(1+a)^2\tilde{p}(D)(u-a),$$

with the second order polynomial

$$\tilde{p}(\zeta) = \zeta^2 + \frac{1}{2}\zeta - \frac{3}{4} = \left(\zeta + \beta + \frac{1}{2}\right)(\zeta - \beta).$$

The fact that $\tilde{p}(\zeta)$ has the root β together with the insight that the leading power of C is $x^{2\beta}$ shows that (3.2) holds true.

A.3. The Singular Expansion of h

Using (3.1) in (1.7) and expanding, we obtain

$$\partial_x Z(t,x) = \frac{1}{1+a(t)} - \frac{b(t)}{(1+a(t))^2} x^\beta + o(x^\beta) \quad \text{as } x \searrow 0$$

almost everywhere in $t > 0$. Integrating this expression, we conclude

$$\tilde{x} \overset{(3.4b)}{=} Z(t,x) - Z_0(t) = x\left(\frac{1}{1+a(t)} - \frac{b(t)}{(1+\beta)(1+a(t))^2} x^\beta + o(x^\beta)\right) \quad \text{as } x \searrow 0$$

almost everywhere in $t > 0$. We may invert this expression and get

$$x = (1+a(t))\tilde{x}\left(1 + \frac{b(t)(1+a(t))^{\beta-1}}{1+\beta}\tilde{x}^\beta + o\left(\tilde{x}^\beta\right)\right) \quad \text{as } \tilde{x} \searrow 0 \tag{A.8}$$

almost everywhere in $t > 0$. Inserting (A.8) into (1.6) and expanding the resulting expression, we are left with (3.4). Since the contact line obeys $Z_0(t) = z_0 + \int_0^t V_0(t')\,\mathrm{d}t'$, (3.3) implies that $Z_0(t) = z_0 - \frac{3}{8}\int_0^t (1+a(t'))^3\,\mathrm{d}t'$.

B. Interpolation and Approximation

B.1. Interpolation Inequalities

Lemma B.1 (interpolation inequalities)**.** *Suppose $\alpha \in \mathbb{R}$. Then, for all $k, \ell, m \in \mathbb{N}_0$ with $k \le \ell \le m$, we have*

$$\left|(D+\alpha)^\ell v\right|_\alpha \lesssim \left|(D+\alpha)^k v\right|_\alpha^{\frac{m-\ell}{m-k}} \left|(D+\alpha)^m v\right|_\alpha^{\frac{\ell-k}{m-k}}, \tag{B.1a}$$

$$|v|_{\ell,\alpha} \lesssim |v|_{k,\alpha}^{\frac{m-\ell}{m-k}} |v|_{m,\alpha}^{\frac{\ell-k}{m-k}}, \tag{B.1b}$$

for all locally integrable $v : (0,\infty) \to \mathbb{R}$ for which both factors on the right hand sides are finite. The constants in (B.1) *only depend on k, ℓ, m, and α.*

Proof. Keeping the identification of the norms $|\cdot|_{k,\alpha}$ with the usual Sobolev norms $\|\cdot\|_{W^{k,2}(\mathbb{R})}$ in mind (cf. (4.7)), inequalities (B.1b) are the standard interpolation inequalities. For inequality (B.1a) we use

$$\left|(D+\alpha)^\ell v\right|_\alpha^2 = \int_0^\infty x^{-2\alpha}((D+\alpha)^\ell v(x))^2 \frac{\mathrm{d}x}{x} = \int_{-\infty}^\infty \left(\partial_s^\ell \tilde{v}(s)\right)^2 \mathrm{d}s = \left\|\partial_s^\ell \tilde{v}\right\|_{L^2(\mathbb{R})}^2,$$

where we put $\tilde{v}(s) := e^{-\alpha s} v(e^s)$. Again, inequality (B.1a) follows from the standard homogeneous interpolation inequalities. ☐

Lemma B.2 (trace estimate)**.** *Let $k, \ell \in \mathbb{N}_0$, $\alpha \in \mathbb{R}$, $I = (0,\tau) \subseteq (0,\infty)$, and $\beta \in \mathbb{R}$. Then for all locally integrable $v = v(t,x) : I \times (0,\infty) \to \mathbb{R}$ we have*

$$\sup_{t\in I} t^{2\beta-1} |v(t)|^2_{k+\ell,\alpha+\frac{1}{2}} \lesssim \int_I t^{2\beta-1} \left(|\partial_t v(t)|^2_{k,\alpha} + |v(t)|^2_{k+2\ell,\alpha+1} \right) \mathrm{d}t + \int_I t^{2\beta-2} |v(t)|^2_{k+\ell,\alpha+\frac{1}{2}} \, \mathrm{d}t, \tag{B.2a}$$

and for $\beta = \frac{1}{2}$ and $I = (0,\infty)$

$$\sup_{t\geq 0} |v(t)|^2_{k+\ell,\alpha+\frac{1}{2}} \lesssim \int_0^\infty \left(|\partial_t v(t)|^2_{k,\alpha} + |v(t)|^2_{k+2\ell,\alpha+1} \right) \mathrm{d}t. \tag{B.2b}$$

where the constants in (B.2) only depend on β, k, ℓ, and α.

We note that within this lemma we do not fix the value of β (unlike in the rest of the paper, where $\beta = \frac{\sqrt{13}-1}{4}$).

Proof. Throughout the proof, estimates may depend on β, k, ℓ, and α.

Proof of (B.2a)*.* By writing $w(t) = x^{-\alpha-\frac{1}{2}} v(t)$ and using $Dw(t) = x^{-\alpha-\frac{1}{2}}(D-\alpha-\frac{1}{2})v(t)$, we see that $|v(t)|_{m,\alpha+\frac{1}{2}+\gamma} \sim |w(t)|_{m,\gamma}$ for $\gamma \in \left\{0, \frac{1}{2}\right\}$ and $|\partial_t v(t)|_{k,\alpha} \sim |\partial_t w(t)|_{k,-\frac{1}{2}}$. Replacing w by $D^m w$, it is enough to prove the homogeneous version, that is,

$$\sup_{t\in I} t^{2\beta-1} \left|D^\ell w(t)\right|_0^2 \lesssim \int_I t^{2\beta-1} \left(|\partial_t w(t)|^2_{-\frac{1}{2}} + \left|D^{2\ell} w(t)\right|^2_{\frac{1}{2}} \right) \mathrm{d}t + \int_I t^{2\beta-2} \left|D^\ell w(t)\right|_0^2 \mathrm{d}t.$$

By the symmetry of $(\cdot,\cdot)_0$ (cf. (4.4) and Remark 4.2), it suffices to establish

$$\sup_{t\in I} t^{2\beta-1} \left|D^\ell w(t)\right|_0^2 \lesssim \int_I t^{2\beta-1} \left| \left(\partial_t D^\ell w(t), D^\ell w(t)\right)_0 \right| \mathrm{d}t \; + \int_I t^{2\beta-2} \left|D^\ell w(t)\right|_0^2 \mathrm{d}t.$$

Setting $f(t) := t^{\beta-\frac{1}{2}} D^\ell w(t)$ this follows from

$$\sup_{t\in I} |f(t)|_0^2 \lesssim \int_I \left| (\partial_t f(t), f(t))_0^2 \right| \mathrm{d}t + \int_I \frac{1}{t} |f(t)|_0^2 \, \mathrm{d}t.$$

By cut off and convolution it suffices to assume that $f \in C_0^\infty(I_t \times (0,\infty)_x)$ and therefore we only need to show

$$\sup_{t\in I} |f(t)|_0^2 \lesssim \int_I \left| \frac{\mathrm{d}}{\mathrm{d}t} |f(t)|_0 \right| \mathrm{d}t + \int_I \frac{1}{t} |f(t)|_0^2 \, \mathrm{d}t. \tag{B.3}$$

Since $\int_I \frac{1}{t} |f(t)|_0^2 \, \mathrm{d}t \geq \frac{1}{\tau} \int_I |f(t)|_0^2 \, \mathrm{d}t$, by scaling t with τ and setting $g(t) := |f(t)|_0^2$, (B.3) follows from the basic estimate

$$\sup_{0\leq t\leq 1} g(t) \lesssim \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}t} g(t) \right| \mathrm{d}t + \int_0^1 g(t) \, \mathrm{d}t$$

which is a consequence of the fundamental theorem of calculus.

Proof of (B.2a). First we note that we can assume that $v \in C_0^\infty([0,\infty)_t \times (0,\infty)_x)$ (by first approximating in x as in Remark 4.2, then cutting off at $t = \infty$, and then convolving). Instead of (B.3), we may establish the simpler

$$\sup_{t \geq 0} |f(t)|_0^2 \lesssim \int_0^\infty \left| \frac{\mathrm{d}}{\mathrm{d}t} |f(t)|_0^2 \right| \mathrm{d}t$$

which, by again setting $g(t) := |f(t)|_0^2$, reduces to

$$\sup_{t \geq 0} g(t) \lesssim \int_0^\infty \left| \frac{\mathrm{d}}{\mathrm{d}t} g(t) \right| \mathrm{d}t.$$

Since $g(t) \equiv 0$ for $t \gg 1$, the estimate follows from the fundamental theorem of calculus by integrating from $t = \infty$. □

B.2. Approximation Results

Lemma B.3. *Suppose $k \in \mathbb{N}_0$ and $\delta > 0$. Then for each locally integrable $u_0 : (0,\infty) \to \mathbb{R}$ with $|||u_0|||_0 < \infty$ there exists a sequence $(u_n)_n$ in $C^\infty((0,\infty)) \cap C_0^0([0,\infty))$ with $u_n(x) = a_n + b_n x^\beta$ for $x \ll_n 1$ (where $a_n, b_n \in \mathbb{R}$) and $|||u_n - u_0|||_0 \to 0$ as $n \to \infty$. In particular u_n satisfies (G_0) (cf. Definition 6.1).*

Proof. We set $v_0 := p(D)u_0$. Since $|v_0|_{k+5,\pm\delta} \lesssim |||u_0|||_0 < \infty$ and using the identification (4.7), we know that there exists a sequence $(v_n)_n \in C_0^\infty((0,\infty))$ with $|v_n - v_0|_{k+5,\pm\delta} \to 0$ as $n \to \infty$. Next we solve the ODE

$$p(D)\tilde{u}_n = v_n \tag{B.4}$$

in the following way:

(a) *Global solutions satisfying* (G_0). Since $v_n(x) \equiv 0$ for $x \ll_n 1$, the family $\tilde{u}_n^{(1)}(x) = a_{1,n} + a_{2,n}x^\beta$ with $a_{1,n}, a_{2,n} \in \mathbb{R}$ solves (B.4) for $x \ll_n 1$ and satisfies (G_0). Furthermore, by standard ODE theory we can extend $u_n^{(1)}$ to a family of solutions of (B.4) in $(0,\infty)$ of the form $\tilde{u}_n^{(1)}(x) = a_{1,n} + a_{2,n}x^\beta + w_n(x)$, where w_n is a global particular solution of (B.4) with $w_n(x) \equiv 0$ for $x \ll_n 1$.

(b) *Local solutions for $x \gg_n 1$.* For $x \gg_n 1$ we have $v_n(x) \equiv 0$. Hence the two-parameter family $\tilde{u}_n^{(2)}(x) = a_{3,n}x^{-\frac{3}{2}} + a_{4,n}x^{-\beta-\frac{1}{2}}$ (with $a_{3,n}, a_{4,n} \in \mathbb{R}$) of algebraically decaying functions solves (B.4) for $x \gg_n 1$.

The family

$$u_n(x) = a_{1,n} + a_{2,n}x^\beta + a_{3,n}x^{-\beta-\frac{1}{2}} + a_{4,n}x^{-\frac{3}{2}} + w_n(x) \quad \text{with } a_{j,n} \in \mathbb{R} \tag{B.5}$$

spans the affine space of all solutions to (B.4) for $x \gg_n 1$. In particular, there exists a set of parameters $a_{j,n} \in \mathbb{R}$ such that $u_n(x)$ defined through (B.5) coincides with $\tilde{u}_n^{(2)}(x)$ for

$x \gg_n 1$ and hence decays algebraically as $x \to \infty$. This property still remains true if we subtract $a_{3,n}x^{-\frac{3}{2}}$ and $a_{4,n}x^{-\beta-\frac{1}{2}}$. But then

$$\tilde{u}_n(x) := u_n(x) - a_{3,n}x^{-\frac{3}{2}} - a_{4,n}x^{-\beta-\frac{1}{2}} = a_{1,n} + a_{2,n}x^{\beta} + w_n(x)$$

is both: algebraically decaying and obeying (G_0) (since it is of the form in *(a)*).

These decay properties also ensure that the solution obeys $|||\tilde{u}_n|||_0 < \infty$. By Lemma 7.2 we know that

$$|||\tilde{u}_n - \tilde{u}_m|||_0 \lesssim |v_n - v_m|_{k+5,-\delta} + |v_n - v_m|_{k+5,\delta}\,.$$

Hence $(\tilde{u}_n)_n$ is a Cauchy sequence with respect to $|||\cdot|||_0$ and we denote by $\tilde{u}_0$ its limit. Since (B.4) is preserved under this convergence, we obtain $p(D)\tilde{u}_0 = v_0 = p(D)u_0$. Hence we know that $u_0 - \tilde{u}_0 \in \operatorname{span}\left\{1, x^{\beta}, x^{-\frac{3}{2}}, x^{-\beta-\frac{1}{2}}\right\}$. As on the one hand $|||u_0 - \tilde{u}_0|||_0 \le |||u_0|||_0 + |||\tilde{u}_0|||_0 < \infty$ and on the other hand

$$|||a_1 + a_2x^{\beta} + a_3x^{-\beta-\frac{1}{2}} + a_4x^{-\frac{3}{2}}||| = \infty$$

for each non-zero tuple $(a_j)_{j=1}^4$, we have $u_0 = \tilde{u}_0$ and hence $|||\tilde{u}_n - u_0||| \to 0$ as $n \to \infty$.

Now take a smooth cut off $\eta \in C^{\infty}((0,\infty))$ with $\eta(x) \equiv 1$ for $x \le 1$ and $\eta(x) \equiv 0$ for $x \ge 2$. We set $\eta_k(x) := \eta\left(\frac{x}{k}\right)$. Then $|||\eta_k\tilde{u}_n - \tilde{u}_n|||_0 \to 0$ as $k \to \infty$ by dominated convergence. Then there exists a sub-sequence k_n such that $u_n(x) := \eta_{k_n}(x)\tilde{u}_n(x)$ fulfills the assertions of the lemma. □

Lemma B.4. *Let $k \in \mathbb{N}_0$ and $0 < \delta < \min\left\{1-\beta, \beta - \frac{1}{2}\right\}$. Suppose $u : (0,\infty)^2 \to \mathbb{R}$ is locally integrable with $|||u||| < \infty$. Then there exists a sequence $(u_n)_n$ in $C^{\infty}((0,\infty)^2) \cap C_0^0([0,\infty)^2)$ with*

(a) for every $t \in [0,\infty)$ we have $u_n(t,x) = a_n(t) + b_n(t)x^{\beta}$ for $x \ll_n 1$, where $a_n(t), b_n(t) \in \mathbb{R}$ are smooth functions of t (in particular $u_n(t)$ satisfies (G_0), cf. Definition 6.1);

(b) $|||u - u_n|||_I \to 0$ as $n \to \infty$.

Furthermore, $I \ni \tau \mapsto |||u|||_{(0,\tau)}$ is continuous and bounded with the limit $|||u|||_{(0,\tau)} \to |||u(0)|||_0$ as $\tau \searrow 0$. The trace $a(t) := \lim_{x\searrow 0} u(t,x)$ is continuous and bounded with $\lim_{t\to\infty} a(t) = 0$.

Proof. Throughout the proof estimates may depend on k and δ.

Similar to the proof of Lemma B.3, we can approximate each term of the sum in the definition of $|||\cdot|||$ (cf. (2.17)) point wise in time by a sequence of functions satisfying (G_0) and with support away from $t = \infty$. Therefore, passing to $v := p(D)u$, we may apply the estimates of Lemma 7.2 (and analogous estimates for $\partial_t u$) which yield

$$\begin{aligned}\|\!|u|\!\|^2 \sim \sup_{t\geq 0}\Big(& t^{2\beta-1}|v|^2_{k+2,\beta-\frac{1}{2}+\delta} + |v|^2_{k+5,\delta} + t^{2\beta-1}|v|^2_{k+2,\beta-\frac{1}{2}-\delta} + |v|^2_{k+5,-\delta}\Big)\\ &+ \int_0^\infty \Big(t^{2\beta-1}|\partial_t v|^2_{k,\beta-1+\delta} + |\partial_t v|^2_{k+3,-\frac{1}{2}+\delta} + t^{2\beta-1}|\partial_t v|^2_{k,\beta-1-\delta} + |\partial_t v|^2_{k+3,-\frac{1}{2}-\delta}\Big)\,\mathrm{d}t\\ &+ \int_0^\infty \Big(t^{2\beta-1}|v|^2_{k+4,\beta+\delta} + |v|^2_{k+7,\frac{1}{2}+\delta} + t^{2\beta-1}|v|^2_{k+4,\beta-\delta} + |v|^2_{k+7,\frac{1}{2}-\delta}\Big)\,\mathrm{d}t.\end{aligned} \tag{B.6}$$

We claim that

$$\|\!|u|\!\| \sim \|\!|v|\!\|_* \tag{B.7a}$$

where

$$\begin{aligned}\|\!|v|\!\|_*^2 := &\int_0^\infty \Big(t^{2\beta-1}|\partial_t v|^2_{k,\beta-1+\delta} + |\partial_t v|^2_{k+3,-\frac{1}{2}+\delta} + t^{2\beta-1}|\partial_t v|^2_{k,\beta-1-\delta} + |\partial_t v|^2_{k+3,-\frac{1}{2}-\delta}\Big)\,\mathrm{d}t\\ &+ \int_0^\infty \Big(t^{2\beta-1}|v|^2_{k+4,\beta+\delta} + |v|^2_{k+7,\frac{1}{2}+\delta} + t^{2\beta-1}|v|^2_{k+4,\beta-\delta} + |v|^2_{k+7,\frac{1}{2}-\delta}\Big)\,\mathrm{d}t\\ &+ \int_0^\infty t^{2\beta-2}\Big(|v|^2_{k+2,\beta-\frac{1}{2}-\delta} + |v|^2_{k+2,\beta-\frac{1}{2}+\delta}\Big)\,\mathrm{d}t.\end{aligned} \tag{B.7b}$$

Applying the trace estimate (cf. Lemma B.2), we obtain $\|\!|u|\!\| \lesssim \|\!|v|\!\|_*$. On the other hand, by Lemma 7.5 the last term on the right-hand side of (B.7b) is estimated by

$$\int_0^\infty t^{2\beta-2}|v|^2_{k+2,\beta-\frac{1}{2}\pm\delta}\,\mathrm{d}t \lesssim \left(\int_0^\infty |\partial_t v|^2_{k+2,-\frac{1}{2}\pm\delta}\,\mathrm{d}t + \int_0^\infty |v|^2_{k+3,\frac{1}{2}\pm\delta}\,\mathrm{d}t\right),$$

hence in view of (B.6), $\|\!|v|\!\|_* \lesssim \|\!|u|\!\|$ and (B.7a) holds.

By approximation (first approximating in x as in Remark 4.2, then cutting off at $t=\infty$, and then convolving), we can approximate v by a sequence of functions $v_n = v_n(t,x) \in C_0^\infty([0,\infty)_t \times (0,\infty)_x)$ with respect to $\|\!|\cdot|\!\|_*$. Then, as in the proof of Lemma B.3, we can solve the equation $p(D)\tilde{u}_n = v_n$ which uniquely determines a sequence of functions $\tilde{u}_n \in C^\infty([0,\infty)_t \times (0,\infty)_x)$ such that $\tilde{u}_n(t,x) \equiv 0$ for $t \gg_n 1$ and for all $t \geq 0$ $\tilde{u}_n(t)$ satisfies $\tilde{u}_n(t,x) = a_n(t) + b_n(t)x^\beta$ for $x \ll_n 1$ and decays as $O\left(x^{-\frac{1}{2}-\beta}\right)$ as $x \to \infty$. Since the right hand side of the equation $p(D)\tilde{u}_n = v_n$ depends smoothly on time t, so does the solution $\tilde{u}_n$. By (B.7a) this sequence satisfies the Cauchy property:

$$\|\!|\tilde{u}_n - \tilde{u}_m|\!\| \lesssim \|\!|v_n - v_m|\!\|_* \to 0 \quad \text{as} \quad m,n \to \infty.$$

The limit $\tilde{u}$ of this sequence meets $p(D)\tilde{u} = v = p(D)u$, whence

$$\tilde{u} - u \in \operatorname{span}\left\{1, x^\beta, x^{-\frac{1}{2}-\beta}, x^{-\frac{3}{2}}\right\}.$$

Together with $|||\tilde{u} - u||| < \infty$, this implies $\tilde{u} = u$. Now take a smooth cut off $\eta \in C^\infty([0, \infty))$ with $\eta(x) \equiv 1$ for $x \le 1$ and $\eta(x) \equiv 0$ for $x \ge 2$ and set $\eta_k(x) := \eta\left(\frac{x}{k}\right)$. Then there exists a sequence k_n so that $u_n(t,x) := \tilde{u}_n(t,x)\eta_{k_n}(x)$ fulfills the assertions of the lemma.

We now turn our attention to the continuity and limit properties: In view of $|||\tilde{u}_n - u||| \to 0$ as $n \to \infty$ and since the constructed approximation sequence has compact support in time, by estimate (4.13) of Lemma 4.3 the limit $a(t) \to 0$ as $t \to \infty$ follows. Likewise, we have

$$\sup_{t\ge 0} |||u - u_n|||_0^2 = \sup_{t \ge 0} \left(|(u-a) - (u_n - a_n)|^2_{k+9,\delta} + |u - u_n|^2_{k+9,-\delta} \right) \to 0 \tag{B.8}$$

as $n \to \infty$ and

$$\sup_{t\ge 0} t^{2\beta-1} \left|(u-a) - (u_n - a_n)\right|^2_{k+6,\beta-\frac{1}{2}\pm\delta} \to 0 \quad \text{as } n \to \infty. \tag{B.9}$$

As continuity is preserved under uniform convergence, (B.8) implies that the value $u(0)$ is well-defined with $\lim_{\tau\searrow 0} |||u(\tau) - u(0)|||_0 = 0$. By the reverse triangle inequality, also $\tau \mapsto \sup_{t\in[0,\tau)} |||u(\tau)|||_0^2$ is continuous with the limit

$$\lim_{\tau \searrow 0} \sup_{t\in[0,\tau)} |||u(\tau)|||_0^2 = |||u(0)|||_0^2. \tag{B.10}$$

From (B.9), by the same arguments we also have that

$$t \mapsto t^{2\beta-1} \left|u - a\right|^2_{k+6,\beta-\frac{1}{2}\pm\delta}$$

is continuous. By the trace estimate (B.2a) of Lemma B.2 in combination with Lemma 7.2[4], we additionally have

$$\begin{aligned} \sup_{t\in[0,\tau)} t^{2\beta-1} \left|u(t) - a(t)\right|^2_{k+6,\beta-\frac{1}{2}\pm\delta} \lesssim \int_0^\tau t^{2\beta-1} \left(|\partial_t v(t)|^2_{k+4,\beta-1\pm\delta} + |v(t)|^2_{k+4,\beta\pm\delta} \right) \mathrm{d}t \\ + \int_0^\tau t^{2\beta-2} \left|v(t)\right|^2_{k+2,\beta-\frac{1}{2}\pm\delta} \mathrm{d}t. \end{aligned} \tag{B.11}$$

Since by (B.6) and (B.7) the right hand side of (B.11) is bounded for $\tau = \infty$, Lebesgue's theorem on dominated convergence leads to

$$\sup_{t\in[0,\tau)} t^{2\beta-1} \left|u(t) - a(t)\right|^2_{k+6,\beta-\frac{1}{2}\pm\delta} \to 0 \quad \text{as } \tau \searrow 0. \tag{B.12}$$

Finally, the L^2-parts in the norm $|||u|||_{[0,\tau)}$ are continuous in time and converge to 0 as $\tau \searrow 0$ by Lebesgue's theorem on dominated convergence. Together with (B.10) and (B.12), this proves that $|||u|||_{[0,\tau)} \to |||u(0)|||_0$ as $\tau \searrow 0$. □

[4]This lemma is applicable e.g. by approximating u with the sequence u_n.

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References

[1] P. G. de Gennes, Wetting: statics and dynamics, Rev. Mod. Phys. 57 (1985) 827–863. doi:10.1103/RevModPhys.57.827.
URL http://link.aps.org/doi/10.1103/RevModPhys.57.827

[2] A. Oron, S. H. Davis, S. G. Bankoff, Long-scale evolution of thin liquid films, Rev. Mod. Phys. 69 (1997) 931–980. doi:10.1103/RevModPhys.69.931.
URL http://link.aps.org/doi/10.1103/RevModPhys.69.931

[3] L. Giacomelli, M. Gnann, F. Otto, Regularity of source-type solutions to the thin-film equation with zero contact angle and mobility exponent between 3/2 and 3, European Journal of Applied Mathematics 24 (2013) 735–760. doi:10.1017/S0956792513000156.
URL http://journals.cambridge.org/article_S0956792513000156

[4] F. Bernis, L. A. Peletier, S. M. Williams, Source type solutions of a fourth order nonlinear degenerate parabolic equation, Nonlinear Anal. 18 (3) (1992) 217–234. doi:10.1016/0362-546X(92)90060-R.
URL http://dx.doi.org/10.1016/0362-546X(92)90060-R

[5] L. Giacomelli, H. Knüpfer, F. Otto, Smooth zero-contact-angle solutions to a thin-film equation around the steady state, J. Differential Equations 245 (6) (2008) 1454–1506. doi:10.1016/j.jde.2008.06.005.
URL http://dx.doi.org/10.1016/j.jde.2008.06.005

[6] L. Giacomelli, H. Knüpfer, A free boundary problem of fourth order: classical solutions in weighted Hölder spaces, Comm. Partial Differential Equations 35 (11) (2010) 2059–2091. doi:10.1080/03605302.2010.494262.
URL http://dx.doi.org/10.1080/03605302.2010.494262

[7] D. John, Uniqueness and stability near stationary solutions for the thin-film equation in multiple space dimensions with small initial lipschitz perturbations, Phd thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn (2013).

[8] L. Giacomelli, F. Otto, Rigorous lubrication approximation, Interfaces Free Bound. 5 (4) (2003) 483–529.

[9] H. Knüpfer, N. Masmoudi, Well-posedness and uniform bounds for a nonlocal third order evolution operator on an infinite wedge, Comm. Math. Phys. 320 (2) (2013) 395–424. doi:10.1007/s00220-013-1708-z.
URL http://dx.doi.org/10.1007/s00220-013-1708-z

[10] H. Knüpfer, N. Masmoudi, Darcy flow on a plate with prescribed contact angle – well-posedness and lubrication approximation, submitted.

[11] H. Knüpfer, Well-posedness for the Navier slip thin-film equation in the case of partial wetting, Comm. Pure Appl. Math. 64 (9) (2011) 1263–1296. doi:10.1002/cpa.20376.
URL http://dx.doi.org/10.1002/cpa.20376

[12] H. Knüpfer, Well-posedness for a class of thin-film equations with general mobility in the regime of partial wetting, preprint.

[13] F. Bernis, A. Friedman, Higher order nonlinear degenerate parabolic equations, J. Differential Equations 83 (1) (1990) 179–206. doi:10.1016/0022-0396(90)90074-Y.
URL http://dx.doi.org/10.1016/0022-0396(90)90074-Y

[14] E. Beretta, M. Bertsch, R. Dal Passo, Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation, Arch. Rational Mech. Anal. 129 (2) (1995) 175–200. doi:10.1007/BF00379920.
URL http://dx.doi.org/10.1007/BF00379920

[15] A. L. Bertozzi, M. Pugh, The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions, Comm. Pure Appl. Math. 49 (2) (1996) 85–123. doi:10.1002/(SICI)1097-0312(199602)49:2<85::AID-CPA1>3.3.CO;2-V.
URL http://dx.doi.org/10.1002/(SICI)1097-0312(199602)49:2<85::AID-CPA1>3.3.CO;2-V

[16] R. Dal Passo, H. Garcke, G. Grün, On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions, SIAM J. Math. Anal. 29 (2) (1998) 321–342 (electronic). doi:10.1137/S0036141096306170.
URL http://dx.doi.org/10.1137/S0036141096306170

[17] G. Grün, Droplet spreading under weak slippage—existence for the Cauchy problem, Comm. Partial Differential Equations 29 (11-12) (2004) 1697–1744. doi:10.1081/PDE-200040193.
URL http://dx.doi.org/10.1081/PDE-200040193

[18] F. Bernis, Finite speed of propagation and continuity of the interface for thin viscous flows, Adv. Differential Equations 1 (3) (1996) 337–368.

[19] F. Bernis, Finite speed of propagation for thin viscous flows when $2 \leq n < 3$, C. R. Acad. Sci. Paris Sér. I Math. 322 (12) (1996) 1169–1174.

[20] J. Hulshof, A. E. Shishkov, The thin film equation with $2 \leq n < 3$: finite speed of propagation in terms of the L^1-norm, Adv. Differential Equations 3 (5) (1998) 625–642.

[21] M. Bertsch, R. Dal Passo, H. Garcke, G. Grün, The thin viscous flow equation in higher space dimensions, Adv. Differential Equations 3 (3) (1998) 417–440.

[22] R. Dal Passo, L. Giacomelli, G. Grün, A waiting time phenomenon for thin film equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 30 (2) (2001) 437–463.
URL http://www.numdam.org/item?id=ASNSP_2001_4_30_2_437_0

[23] J. A. Carrillo, G. Toscani, Long-time asymptotics for strong solutions of the thin film equation, Comm. Math. Phys. 225 (3) (2002) 551–571. doi:10.1007/s002200100591.
URL http://dx.doi.org/10.1007/s002200100591

[24] G. Grün, Droplet spreading under weak slippage: the optimal asymptotic propagation rate in the multi-dimensional case, Interfaces Free Bound. 4 (3) (2002) 309–323. doi:10.4171/IFB/63.
URL http://dx.doi.org/10.4171/IFB/63

[25] G. Grün, Droplet spreading under weak slippage: the waiting time phenomenon, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2) (2004) 255–269. doi:10.1016/S0294-1449(03)00033-7.
URL http://dx.doi.org/10.1016/S0294-1449(03)00033-7

[26] L. Giacomelli, G. Grün, Lower bounds on waiting times for degenerate parabolic equations and systems, Interfaces Free Bound. 8 (1) (2006) 111–129. doi:10.4171/IFB/137.
URL http://dx.doi.org/10.4171/IFB/137

[27] J. Fischer, Optimal lower bounds on asymptotic support propagation rates for the thin-film equation, J. Differential Equations 255 (10) (2013) 3127–3149. doi:10.1016/j.jde.2013.07.028.
URL http://dx.doi.org/10.1016/j.jde.2013.07.028

[28] J. Fischer, Upper bounds on waiting times for the thin-film equation: The case of weak slippage, Arch. Rational Mech. Anal. (2013) in pressdoi:10.1007/s00205-013-0690-0.
URL http://dx.doi.org/10.1007/s00205-013-0690-0

[29] F. Otto, Lubrication approximation with prescribed nonzero contact angle, Comm. Partial Differential Equations 23 (11-12) (1998) 2077–2164. doi:10.1080/03605309808821411.
URL http://dx.doi.org/10.1080/03605309808821411

[30] M. Bertsch, L. Giacomelli, G. Karali, Thin-film equations with "partial wetting" energy: existence of weak solutions, Phys. D 209 (1-4) (2005) 17–27. doi:10.1016/j.physd.2005.06.012.
URL http://dx.doi.org/10.1016/j.physd.2005.06.012

[31] L. Ansini, L. Giacomelli, Doubly nonlinear thin-film equations in one space dimension, Arch. Ration. Mech. Anal. 173 (1) (2004) 89–131. doi:10.1007/s00205-004-0313-x.
URL http://dx.doi.org/10.1007/s00205-004-0313-x

[32] L. Giacomelli, A. Shishkov, Propagation of support in one-dimensional convected thin-film flow, Indiana Univ. Math. J. 54 (4) (2005) 1181–1215. doi:10.1512/iumj.2005.54.2532.
URL http://dx.doi.org/10.1512/iumj.2005.54.2532

[33] H. Koch, Non-eudlidean singular intergrals and the porous medium equation, Habilitation thesis, Ruprecht-Karls-Universität Heidelberg, Heidelberg (1999).

[34] D. G. Aronson, J. L. Vázquez, Eventual C^∞-regularity and concavity for flows in one-dimensional porous media, Arch. Rational Mech. Anal. 99 (4) (1987) 329–348. doi:10.1007/BF00282050.
URL http://dx.doi.org/10.1007/BF00282050

[35] S. Angenent, Local existence and regularity for a class of degenerate parabolic equations, Math. Ann. 280 (3) (1988) 465–482. doi:10.1007/BF01456337.
URL http://dx.doi.org/10.1007/BF01456337

[36] P. Daskalopoulos, R. Hamilton, Regularity of the free boundary for the porous medium equation, J. Amer. Math. Soc. 11 (4) (1998) 899–965. doi:10.1090/S0894-0347-98-00277-X.
URL http://dx.doi.org/10.1090/S0894-0347-98-00277-X

[37] J. Prüss, G. Simonett, Maximal regularity for degenerate evolution equations with an exponential weight function, in: Functional analysis and evolution equations, Birkhäuser, Basel, 2008, pp. 531–545. doi:10.1007/978-3-7643-7794-6_32.
URL http://dx.doi.org/10.1007/978-3-7643-7794-6_32