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by

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Unextendible maximally entangled bases in $\mathbb{C}^d \bigotimes \mathbb{C}^d$

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We investigate the unextendible maximally entangled bases in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ and present a 30-number UMEB construction in $\mathbb{C}^6 \bigotimes \mathbb{C}^6$. For higher dimensional case, we show that for a given N-number UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$, there is a $\widetilde{N}$-number, $\widetilde{N} = (qd)^2 - (d^2 - N)$, UMEB in $\mathbb{C}^{qd} \bigotimes \mathbb{C}^{qd}$ for any $q \in \mathbb{N}$. As an example, for $\mathbb{C}^{12n} \bigotimes \mathbb{C}^{12n}$ systems, we show that there are at least two sets of UMEBs which are not equivalent.

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I. INTRODUCTION

Einstein, Podolsky, and Rosen (EPR) proposed a thought experiment which demonstrated that quantum mechanics is not a complete theory of nature [1, 2], quantum entanglement has been shown to be tightly related to some fundamental problems in quantum mechanics such as reality and nonlocality. It was quite surprising when it was found that there are sets of product states which nevertheless display a form of nonlocality [3, 4]. It was shown that there are sets of orthogonal product vectors in $\mathbb{C}^m \bigotimes \mathbb{C}^n$ such that there are no further product states which are orthogonal to all the state in the set, even though the space spanned by the set is smaller than nm. A set of states satisfying such property is called unextendible product bases (UPBs). Many useful applications have been obtained ever since the concept of UPBs in multipartite quantum systems was introduced [5–7]. It was shown that the UPBs are not distinguishable by local measurements and classical communication, and the space complementary to a UPB contains bound entanglement [5].

In 2009, S. Bravyi and J. A. Smolin generalized the notion of the UPB to unextendible maximally entangled basis [8]: a set of orthonormal maximally entangled states in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ consisting of fewer than d^2 vectors which have no additional maximally entangled vectors that are orthogonal to all of them. The authors proved that there do not exist UMEBs for $d = 2$, and constructed a 6-member UMEB for $d = 3$ and a 12-member UMEB for $d = 4$.

In Ref. [9], B. Chen and S.M. Fei studied the UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$ ($\frac{d'}{2} < d < d'$). They constructed a d^2-member UMEBs, and left an opem problem for the existence of UMEBs in the case of $\frac{d'}{2} \geq d$. Recently, we give an explicit construction of UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^{d'} (d < d')$ [10]. We show that the states in the complementary space of the UMEBs have Schmidt numbers less than d.

In his paper, we study the unsolved problem of UMEBs in $\mathbb{C}^d \bigotimes \mathbb{C}^d$. We start with the construction of a 30-member UMEB in $\mathbb{C}^6 \bigotimes \mathbb{C}^6$. Then we generalized the example to higher dimension case. We show that for an given N-number UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$, there is a $\widetilde{N}$-number, $\widetilde{N} = (qd)^2 - (d^2 - N)$, UMEB in $\mathbb{C}^{qd} \bigotimes \mathbb{C}^{qd}$ for any $q \in \mathbb{N}$. For $\mathbb{C}^{12n} \bigotimes \mathbb{C}^{12n}$ systems, we show that there are at least two sets of UMEBs which are not equivalent.

II. UMEBS IN $\mathbb{C}^d \bigotimes \mathbb{C}^d$

A set of states $\{|\phi_a\rangle \in \mathbb{C}^d \bigotimes \mathbb{C}^d : a = 1, 2, \cdots, n, n < d^2\}$ is called an n-number UMEB if and only if (i) $|\phi_a\rangle$, $a = 1, 2, \cdots, n$, are maximally entangled; (ii) $\langle\phi_a|\phi_b\rangle = \delta_{ab}$; (iii) if $\langle\phi_a|\psi\rangle = 0$ for all $a = 1, 2, \cdots, n$, then $|\psi\rangle$ cannot be maximally entangled.

Here under computational basis a maximally entangled state $|\phi_a\rangle$ can be expressed as

$$|\phi_a\rangle = (I \otimes U_a)\,\frac{1}{\sqrt{d}}\sum_{i=1}^{d} |i\rangle \otimes |i\rangle, \qquad (1)$$

where I is the $d \times d$ identity matrix, U_a is any unitary matrix. According to (1), a set of unitary matrices $\{U_a \in M_d(\mathbb{C}) | a = 1, ..., n\}$ gives an n-number UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ if and only if
(i) $n < d^2$;
(ii) $Tr(U_a^\dagger U_b) = d\,\delta_{ab}, \quad \forall a, b = 1, \cdots, n$;
(iii) For any $U \in M_d(\mathbb{C})$, if $Tr(U_a^\dagger U) = 0$, $\forall\, a = 1, \cdots, n$, then U cannot be unitary.

Two n-number UMEBs $\{U_a\}_{a=1}^n$ and $\{V_a\}_{a=1}^n$ in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ are said to be equivalent if there exist $\sigma \in S_n$ and $U \in U(d)$ such that $UU_aU^\dagger = V_{\sigma(a)}$ for $a = 1, .., n$, where S_n is the permutation group of n elements.

In the following we present a 30-member UMEB in $\mathbb{C}^6 \bigotimes \mathbb{C}^6$. Set

$$U_{nm} \triangleq \sum_{k=0}^{2} e^{\frac{2\pi\sqrt{-1}}{3}kn} |k \oplus m\rangle\langle k|,$$

$$U_{nm}^{\pm} = \delta_{\pm} \otimes U_{nm} \quad n, m = 1, 2, 3,$$

and

$$U_i^{\pm} = \eta_{\pm} \otimes U_i \quad i = 1, 2, 3, 4, 5, 6,$$

where $k \oplus m$ denotes the number $k+m$ mod d,

$$\delta_\pm = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}, \quad \eta_\pm = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix},$$

$\{U_i\}_{i=1}^6$ are the unitary matrices constructed in $\mathbb{C}^3 \bigotimes \mathbb{C}^3$ Ref. [8]:

$$U_i = I - (1 - e^{i\theta})|\psi_i\rangle\langle\psi_i|, \quad i = 1, 2, ..., 6,$$

where

$$\begin{aligned} |\psi_{1,2}\rangle &= \frac{1}{\sqrt{1+\alpha^2}}(|0\rangle \pm |1\rangle), \\ |\psi_{3,4}\rangle &= \frac{1}{\sqrt{1+\alpha^2}}(|1\rangle \pm |2\rangle), \\ |\psi_{5,6}\rangle &= \frac{1}{\sqrt{1+\alpha^2}}(|2\rangle \pm |0\rangle), \end{aligned}$$

with $\alpha = (1+\sqrt{5})/2$.

We now prove that $\{U_{nm}^\pm,\ U_i^\pm,\ n, m = 1, 2, 3;\ i = 1, ..., 6\}$ give rise to a 30-member UMEB in $\mathbb{C}^6 \bigotimes \mathbb{C}^6$.

(1) Since $\{U_{nm}\}$ and $\{U_i\}$ are unitary, it is easily seen that $\{U_{nm}^\pm, U_i^\pm\}$ are also unitary.

(2) To prove the orthogonality of these unitary states, we consider three different cases:
(i) inner product between two elements in $\{U_{nm}^\pm\}$,

$$\begin{aligned} Tr((\delta_+ \otimes U_{nm})^\dagger(\delta_\pm \otimes U_{\widetilde{n}\widetilde{m}})) &= \pm Tr(\eta_\pm \otimes U_{nm}^\dagger U_{\widetilde{n}\widetilde{m}}) \\ &= 6\delta_{+\pm}\delta_{n\widetilde{n}}\delta_{m\widetilde{m}}; \end{aligned}$$

(ii) inner product between two elements in $\{U_i^\pm\}$, $Tr((\eta_+ \otimes U_i)^\dagger \eta_\pm \otimes U_{\widetilde{i}}) = Tr(\eta_+\eta_\pm)Tr(U_i^\dagger U_{\widetilde{i}}) = 6\delta_{+\pm}\delta_{i\widetilde{i}}$;
(iii) the inner product between one elements in $\{U_{nm}^\pm\}$ and the one in $\{U_i^\pm\}$, $Tr((\delta_{pm} \otimes U_{nm})^\dagger \eta_\pm \otimes U_i) = Tr(\delta_\pm \eta_\pm)Tr(U_{nm}^\dagger U_i) = 0$.

(3) Assume that $U \in M_6(\mathbb{C})$ satisfy:

$$Tr(U^\dagger U_{nm}^\pm) = 0 \ \text{ and } \ Tr(U^\dagger U_i^\pm) = 0.$$

Let $V_1 = \text{span}\{U_{nm}^\pm\}$, $\text{dim}V_1 = 18$. Denote

$$V_2 = \left\{ \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} | A, B \in M_3(\mathbb{C}) \right\},$$

$\text{dim}V_2 = 18$. Since the canonical inner product

$$Tr\left(\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}^\dagger \begin{pmatrix} \mathbf{0} & U_{nm} \\ \pm U_{nm} & \mathbf{0} \end{pmatrix} \right) = 0,$$

one has $V_1^\perp = V_2$. Now let $V_3 = \text{span}\{U_{nm}^\pm, U_i^\pm\}$. We have $\text{dim}V_3 = 30$ and $V_3^\perp \subset V_1^\perp = V_2$. Therefore $U \in V_3^\perp$, and the matrix U has the form $U = diag(W_1, W_2)$, where $W_1, W_2 \in M_3(\mathbb{C})$. As U satifies

$$Tr\left(\begin{pmatrix} W_1 & \mathbf{0} \\ \mathbf{0} & W_2 \end{pmatrix}^\dagger \begin{pmatrix} U_i & \mathbf{0} \\ \mathbf{0} & \pm U_i \end{pmatrix} \right) = 0,$$

i.e. $Tr({W_1}^\dagger U_i) \pm Tr({W_2}^\dagger U_i) = 0$, we have $Tr({W_1}^\dagger U_i) = Tr({W_2}^\dagger U_i) = 0$ for $i = 1, 2, \cdots, 6$, which implies that $W_1, W_2 \notin U(3)$. Hence $U \notin U(6)$. Therefore we conclude that $\{U_{nm}^\pm, U_i^\pm\}$ is a 30-member UMEB in $\mathbb{C}^6 \bigotimes \mathbb{C}^6$. $\square$

Now we show that for any UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$, there will be an UMEB in $\mathbb{C}^{qd} \bigotimes \mathbb{C}^{qd}$ for any $q \in \mathbb{N}$.

Theorem 1. If there is an N-number UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$, then for any $q \in \mathbb{N}$, there is a $\widetilde{N}$-number, $\widetilde{N} = (qd)^2 - (d^2 - N)$, UMEB in $\mathbb{C}^{qd} \bigotimes \mathbb{C}^{qd}$.

Proof: Denote

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$W = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta_q & \zeta_q^2 & \cdots & \zeta_q^{q-1} \\ 1 & \zeta_q^2 & \zeta_q^4 & \cdots & \zeta_q^{2(q-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_q^{q-1} & \zeta_q^{2(q-1)} & \cdots & \zeta_q^{(q-1)^2} \end{pmatrix},$$

where $\zeta_q = e^{\frac{2\pi\sqrt{-1}}{q}}$ and

$$U_{nm} = \sum_{k=0}^{d-1} e^{\frac{2\pi\sqrt{-1}}{d}kn}|k \oplus m\rangle\langle k|, \ m, n = 0, 1, \cdots, d-1.$$

In the following for any $q \times q$ matrix M with entries m_{ij}, we define $M^i = \text{diag}(m_{i+1,1}, m_{i+1,2}, ..., m_{i+1,q})$, $i \in \{0, 1, \cdots, q-1\}$.

Let $\{U_n\}$, $n = 1, 2, \cdots, N < d^2$, be the set of unitary matrices that give rise to the UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$. Set

$$U_{nm}^{ij} = (W^i S^j) \otimes U_{nm},$$

where $i, j = 0, \cdots, q-1, m, n = 0, \cdots, d-1$, and

$$U_n^i = W^i \otimes U_n, i = 0, 1, \cdots, q-1, n = 1, 2, \cdots, N < d^2.$$

Let $\widetilde{N}$ denote the number of matrices in $\{U_{nm}^{ij}, U_n^i\}$. We have

$$\widetilde{N} = q(q-1)d^2 + qN = (qd)^2 - (d^2 - N) < q^2 d^2.$$

Next we prove that $\{U_{nm}^{ij}, U_n^i\}$ give a $\widetilde{N}$-member UMEB in $\mathbb{C}^{qd} \bigotimes \mathbb{C}^{qd}$.

(1) Since W^i, S^j, U_{nm} are all unitary, so are $\{U_{nm}^{ij}, U_n^i\}$. So the given set of matrices satisfy the first condition of UMEB.

(2) In order to prove the orthogonality of the related basic states, we need to check the inner products between two elements in $\{U_{nm}^{ij}\}$, between two elements in $\{U_n^i\}$, and between one in $\{U_{nm}^{ij}\}$ and the other one in $\{U_n^i\}$. It is direct to verify that

(i) $Tr((U_{nm}^{ij})^\dagger U_{\widetilde{n}\widetilde{m}}^{\widetilde{i}\widetilde{j}}) = qd\delta_{\widetilde{i}i}\delta_{j\widetilde{j}}\delta_{n\widetilde{n}}\delta_{m\widetilde{m}}$;

(ii) $Tr((W^i \otimes U_n)^\dagger (W^{\widetilde{i}} \otimes U_{\widetilde{n}})) = qd\delta_{\widetilde{i}i}\delta_{n\widetilde{n}}$;

(iii) $Tr((U_{nm}^{ij})^\dagger U_{\widetilde{n}}^{\widetilde{i}}) = Tr((S^j)^\dagger (W^i)^\dagger W^{\widetilde{i}} \otimes U_{nm}^\dagger U_n) = 0$.

(3) Let $V_1 = \text{span}\{U_{nm}^{ij}\}$ be a subspace of $M_{qd}(\mathbb{C})$, $\dim V_1 = q(q-1)d^2$. Denote

$$V_2 = \{diag(A_1, A_2, ..., A_q) | A_i \in M_d(\mathbb{C}),\ i = 1, 2, ..., q\}.$$

It is seens that $\dim V_2 = qd^2$. For any matrix $A \in V_2$ and $i, j \in \{0, 1, \cdots, q-1\}$, $m,\ n \in \{0, 1, \cdots, d-1\}$, we have $Tr(A^\dagger U_{nm}^{ij}) = 0$. Thus for any matrix $A \in V_2$ and $B \in V_1$, $Tr(A^\dagger B) = 0$. Namely, $V_2 \subseteq V_1^\perp$. Accounting to the dimensions of V_1, V_2 and $M_{qd}(\mathbb{C})$, we obtain $V_1^\perp = V_2$. Set $V_3 = \text{span}\{U_{nm}^{ij}, U_n^i\}$. Clearly, $V_3^\perp \subset V_1^\perp$. Hence any $U \in V_3^\perp$ has the following form

$$U = diag\,(W_1, W_2, ..., W_q) \quad \text{where } W_i \in M_d(\mathbb{C}).$$

In addition, from $Tr(U^\dagger U_n^i) = 0$, for $i = 1, ..., q$, we have

$$Tr(diag\,(W_1, W_1, ..., W_q)\, diag\,(w_{i1}U_n, w_{i2}U_n, ..., w_{iq}U_n)) = 0,$$

i.e.,

$$w_{i1}Tr(W_1^\dagger U_n) + \cdots + w_{iq}Tr(W_q^\dagger U_n) = 0,\ i = 1, ..., q. \quad (2)$$

Noting that

$$\det(W) = \det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta_q & \zeta_q^2 & \cdots & \zeta_q^{q-1} \\ 1 & \zeta_q^2 & \zeta_q^4 & \cdots & \zeta_q^{2(q-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_q^{q-1} & \zeta_q^{2(q-1)} & \cdots & \zeta_q^{(q-1)^2} \end{pmatrix} \neq 0,$$

from equation (2) we obtain $Tr(W_1^\dagger U_n) = \cdots = Tr(W_q^\dagger U_n) = 0$, for $n = 1, \cdots, N$. Therefore $W_i \notin U(d)$, and hence $U \notin U(qd)$. From (1), (2) and (3), we conclude that $\{U_{nm}^{ij}, U_n^i\}$ is an $\widetilde{N}$-member UMEB in $\mathbb{C}^{qd} \bigotimes \mathbb{C}^{qd}$. $\square$

In [8] a 6-member UMEB for $d = 3$ and a 12-member UMEB for $d = 4$ have been constructed. We have constructed in this paper a 30-member UMEB for $d = 6$. From our theorem, for $d = 12$, one con construct $\widetilde{N} = (qd)^2 - (d^2 - N)$-number UMEBs in $\mathbb{C}^{qd} \bigotimes \mathbb{C}^{qd}$, by respectively taking $N = 3, 4, 6$ and $q = 4, 3, 2$. Therefore in $\mathbb{C}^{12} \bigotimes \mathbb{C}^{12}$ there are three ways to construct UMEBs from the UMEBs of dimension 3,4 and 6. In the following we show that at least two of the three UMEBs obtained in this way are not equivalent.

Let $\{U_1, ..., U_6\}$ be the 6-member UMEB in $\mathbb{C}^3 \bigotimes \mathbb{C}^3$ presented in [8]. We note that the eigenvalues of $U_1, ..., U_6$ are all $\{1, 1, e^{\sqrt{-1}\theta}\}$, where $\cos\theta = -\frac{7}{8}$. But in Ref. [11] one can see that $\cos^2(\frac{n}{m}\pi) \in \mathbb{Q}$ if and only if $\cos^2(\frac{n}{m}\pi) \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. So $\cos^2\theta = \frac{49}{64}$ implies that θ is not of the form $\frac{n}{m}\pi$. Hence for any $n \in \mathbb{N}$, $(e^{\sqrt{-1}\theta})^n \neq 1$. Since $U_n^i = W^i \otimes U_n$, the eigenvalues of U_n^i are $\{1, ..., \zeta_4^{3i}, 1, ..., \zeta_4^{3i}, e^{\sqrt{-1}\theta}, ..., e^{\sqrt{-1}\theta}\zeta_4^{3i}\}$. If we consider the order of the eigenvalue, then the order of $e^{\sqrt{-1}\theta}\zeta_4^{3i}$ is infinite. The orders of the eigenvalues of $U_{m,n}^{i,j}$ are all less or equal than 12. Similarly, we can calculate the orders of eigenvalues of $U_n^i, U_{m,n}^{i,j}$ derived from the UMEB in $\mathbb{C}^4 \bigotimes \mathbb{C}^4$ as above. The minimal and maximal order of the eigenvalues of $U_n^i, U_{m,n}^{i,j}$ are presented in Table I. By the definition of equivalence between two UMEBs, they should share the the same eigenvalues. There are 12 elements of the UMEB derived from $\mathbb{C}^3 \bigotimes \mathbb{C}^3$ with infinite order eigenvalues, but all the elements of the UMEB derived from $\mathbb{C}^4 \bigotimes \mathbb{C}^4$ only have finite order eigenvalues. Hence they are not equivalent.

Moreover, the above conclusion can be generalized to $\mathbb{C}^{12n} \bigotimes \mathbb{C}^{12n}$. One can show that in $\mathbb{C}^{12n} \bigotimes \mathbb{C}^{12n}$, there exist two sets of UMEBs which are not equivalent.

TABLE I: Order of eigenvalues of UMEBs in $\mathbb{C}^{12} \bigotimes \mathbb{C}^{12}$

	$O_{min}(U_n^i)$	$O_{max}(U_n^i)$	$O_{min}(U_{m,n}^{i,j})$	$O_{max}(U_{m,n}^{i,j})$
$\mathbb{C}^3 \bigotimes \mathbb{C}^3$	1	∞	1	12
$\mathbb{C}^4 \bigotimes \mathbb{C}^4$	1	12	1	12

TABLE II: Results about UMEBs in $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$

condition	number in UMEB	*reference*
$d = d' = 2$	*none*	[8]
$d = d' = 3$	6	[8]
$d = d' = 4$	12	[8]
$d < d' < 2d$	d^2	[9]
$d' = qd + r, 0 < r < d$	qd^2	[10]
$d' > d$	$d(d'-1)$	[10]
$d = d' = 3n$	$d(d-1)$	This paper
$d = d' = 4n$	$d(d-1)$	This paper

III. CONCLUSION

We have studied the UMEBs in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ and presented a 30-number UMEB construction in $\mathbb{C}^6 \bigotimes \mathbb{C}^6$. By using approach in [10], we have presented the construction of an UMEB in $\mathbb{C}^{qd} \bigotimes \mathbb{C}^{qd}$ from an UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$. In particular, we can obtain UMEBs in $\mathbb{C}^{3n} \bigotimes \mathbb{C}^{3n}$ and $\mathbb{C}^{4n} \bigotimes \mathbb{C}^{4n}$ from the results in [8]. By analysing the order of the eigenvalues of UMEB in $\mathbb{C}^{12} \bigotimes \mathbb{C}^{12}$ derived from the UMEBs in $\mathbb{C}^3 \bigotimes \mathbb{C}^3$ and in $\mathbb{C}^4 \bigotimes \mathbb{C}^4$, it has been shown that the two sets of UMEBs in $\mathbb{C}^{12} \bigotimes \mathbb{C}^{12}$, obtained from our theorem, are not equivalent. Similarly there are two sets of UMEBs in $\mathbb{C}^{12n} \bigotimes \mathbb{C}^{12n}$ which are not equivalent. As a summary, Table II shows the known results about the UMEBs $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$.

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