# Max-Planck-Institut
# für Mathematik
# in den Naturwissenschaften
# Leipzig

**Shadowing in linear skew product**

by

*Sergey Tikhomirov*

Preprint no.: 107 2014

# Shadowing in linear skew product

Sergey Tikhomirov

**Abstract**

We consider linear skew product with the full shift in the base and non-zero Lyapunov exponent in the fiber. We provide sharp estimate for the precision of shadowing for a typical pseudotrajectory of finite length. This result suggests that the high-dimensional analogue of Hammel-Yorke-Grebogi's conjecture [8, 9] concerning the interval of shadowability for a typical pseudotrajectory is not correct. The main technique is reduction of shadowing problem to the ruin problem for one-dimensional random walk.

## 1 Introduction

The theory of shadowing of approximate trajectories (pseudotrajectories) of dynamical systems is now a well-developed part of the global theory of dynamical systems (see, the monographs [13, 14] and [15] for overview of modern results). The shadowing problem is related to the following question: under which conditions, for any pseudotrajectory of $f$ there exists a close trajectory?

For a metric space $(G, \mathrm{dist})$, a continuous map $f : G \to G$, $d > 0$ and an interval $I = (a, b)$, where $a \in \mathbb{Z} \cup \{-\infty\}$, $b \in \mathbb{Z} \cup \{+\infty\}$ a sequence of points $\{y_k\}_{k \in I}$ is called a *d-pseudotrajectory* if the following inequalities hold

$$\mathrm{dist}(y_{k+1}, f(y_k)) < d, \quad k \in \mathbb{Z}, \quad k, k+1 \in I.$$

**Definition 1.** We say that $f$ has the *standard shadowing property* if for any $\varepsilon > 0$ there exists $d > 0$ such that for any $d$-pseudotrajectory $\{y_k\}_{k \in \mathbb{Z}}$ there exists a trajectory $\{x_k\}_{k \in \mathbb{Z}}$ such that

$$\mathrm{dist}(x_k, y_k) < \varepsilon, \quad k \in \mathbb{Z}. \tag{1}$$

In this case we say that pseudotrajectory $\{y_k\}$ is $\varepsilon$-*shadowed* by $\{x_k\}$.

The study of this problem was originated by Anosov [2] and Bowen [3]. This theory is closely related to the classical theory of structural stability.

Let $G$ be a smooth compact manifold of class $C^\infty$ without boundary with Riemannian metric dist and $f \in \mathrm{Diff}^1(G)$. It is well known that a diffeomorphism has shadowing property in a neighborhood of a hyperbolic set [2, 3] and a structurally stable diffeomorphism has shadowing property on the whole manifold [11, 19, 21]. At the same time, it is easy to give an example of a diffeomorphism that is not structurally stable but has standard shadowing property (see [16], for instance). Thus, structural stability is not equivalent to shadowing.

Relation between shadowing and structural stability was studied in several contexts. It is known that the $C^1$-interior of the set of diffeomorphisms having shadowing property coincides with the set of structurally stable diffeomorphisms [20] (see [17] for a similar result for orbital shadowing property). Abdenur and Diaz conjectured that a $C^1$-generic diffeomorphism with the shadowing property is structurally stable; they have proved this conjecture for so-called tame diffeomorphisms [1].

Analyzing the proofs of the first shadowing results by Anosov [2] and Bowen [3], it is easy to see that, in a neighborhood of a hyperbolic set, the shadowing property is Lipschitz (and the same holds in the case of a structurally stable diffeomorphism [14]).

**Definition 2.** We say that $f$ has the *Lipschitz shadowing property* if there exist $\varepsilon_0, L_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and $d$-pseudotrajectory $\{y_k\}_{k\in\mathbb{Z}}$ with $d = \varepsilon/L_0$ there exists a trajectory $\{x_k\}_{k\in\mathbb{Z}}$ such that inequalities (1) hold.

Recently [18] it was proved that a diffeomorphism $f \in C^1$ has Lipschitz shadowing property if and only if it is structurally stable (see [12, 16] for a similar results for periodic and variational shadowing properties).

In the present paper we are interested which type of shadowing one can have for non-hyperbolic diffeomorphisms. The following notion will be important for us [22]:

**Definition 3.** We say that $f$ has the *Finite Hölder shadowing property* with exponents $\theta \in (0, 1)$, $\omega \geq 0$ (FinHolSh$(\theta, \omega)$) if there exist constants $d_0, L, C > 0$ such that for any $d < d_0$ and $d$-pseudotrajectory $\{y_k\}_{k\in[0,Cd^{-\omega}]}$ there exists a trajectory $\{x_k\}_{k\in[0,Cd^{-\omega}]}$ such that

$$\mathrm{dist}(x_k, y_k) < Ld^\theta, \quad k \in [0, Cd^{-\omega}].$$

S. Hammel, J. Yorke and C. Grebogi based on results of numerical experiments conjectured the following [8, 9]:

**Conjecture 1.** *A typical dissipative map* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfies* $\mathrm{FinHolSh}(1/2, 1/2)$.

There are plenty of not structurally stable examples satisfying $\mathrm{FinHolSh}(1/2, 1/2)$, for instance [22, Example 1] and identical map.

In the present paper we study this conjecture for model example: linear skew product (see definition in section 2). We give lower and upper bounds for the precision of shadowing of finite length pseudotrajectories. Those bounds shows that depending on parameters of the skew product diffeomorphism might satisfy and not satisfy analog of Conjecture 1.

We expect that similarly to works [5, 6] such a skew product can be embedded into a diffeomorphism of a manifold of dimension 4. This will allow us to construct an open set of diffeomorphisms violating high-dimensional analog of Conjecture 1. However we do not claim such a construction and leave it out of the scope of the present paper.

Note that in [22] it was shown that Conjecture 1 cannot be improved (see also [10] for the discussion on Hölder shadowing for 1-dimensional maps):

**Theorem 1.** *If a diffeomorphism* $f \in C^2$ *satisfies* $\mathrm{FinHolSh}(\theta, \omega)$ *with*

$$\theta > 1/2, \quad \theta + \omega > 1$$

*then* $f$ *is structurally stable.*

The paper is organized as follows. In Section 2 we formulate exact statements of the results. In Section 3 we formulate a particular problem for random walks and prove its equivalence to the shadowing property. In Section 4 we give a proof of the main result.

# 2 Main Result

Let $\Sigma = \{0, 1\}^{\mathbb{Z}}$. Endow it with the standard metric dist and the standard probability measure $\nu$. For a sequence $\omega = \{\omega^i\} \in \Sigma$ denote by $t(\omega)$ the 0-th element of the sequence: $t(\omega) = \omega^0$. Define the "shift map" $\sigma : \Sigma \to \Sigma$ as the following

$$(\sigma(\omega))^i = \omega^{i+1}.$$

Consider the space $Q = \Sigma \times \mathbb{R}$. Endow $Q$ with the product measure $\mu = \nu \times \text{Leb}$ and the maximum metric:

$$\text{dist}((\omega, x), (\tilde{\omega}, \tilde{x})) = \max(\text{dist}(\omega, \tilde{\omega}), \text{dist}(x, \tilde{x})).$$

For $q \in Q$ and $a > 0$ denote by $B(a, q)$ the open ball of radius $a$ centered at $q$.

Fix $\lambda_0, \lambda_1 \in \mathbb{R}$, satisfying the following conditions

$$0 < \lambda_0 < 1 < \lambda_1, \quad \lambda_0 \lambda_1 \neq 1. \tag{2}$$

Consider map $f : Q \to Q$ defined as the following

$$f(\omega, x) = (\sigma(\omega), \lambda_{t(\omega)} x).$$

For $q \in Q$, $d > 0$, $N \in \mathbb{N}$ let $\Omega_{q,d,N}$ be the set of $d$-pseudotrajectories starting at $q$. If we consider $q_{k+1}$ being chosen at random in $B(d, q)$ uniformly with respect to measure $\mu$ then $\Omega_{q,d,N}$ forms a finite time Markov chain. This naturally endow $\Omega_{q,d,N}$ with a probability measure $P$. See also [7] for a similar concept for infinite pseudotrajectories.

For $\varepsilon > 0$ let $p(q, d, N, \varepsilon)$ be the probability of pseudotrajectory in $\Omega_{q,d,N}$ to be $\varepsilon$-shadowable. Note that corresponding event is measurable since it is an open subset of $\Omega_{q,d,N}$.

**Lemma 1.** *Let $q = (\omega, x)$, $\tilde{q} = (\omega, 0)$. For any $d, \varepsilon > 0$, $N \in \mathbb{N}$ the following equality holds:*

$$p(q, d, N, \varepsilon) = p(\tilde{q}, d, N, \varepsilon).$$

*Proof.* Consider $\{q_k = (\omega_k, x_k)\} \in \Omega_{q,d,N}$. Put $r_k := x_{k+1} - \lambda_{t(\omega_k)} x_k$. Consider a sequence $\{\tilde{q}_k = (\omega_k, \tilde{x}_k)\}$, where

$$\tilde{x}_0 = 0, \quad \tilde{x}_{k+1} = \lambda_{t(w_k)} x_k + r_k.$$

The following holds:

1. the correspondence $\{q_k\} \leftrightarrow \{\tilde{q}_k\}$ is one-to-one and preserves the probability measure;

2. for any $\varepsilon > 0$ pseudotrajectory $\{q_k\}$ is $\varepsilon$-shadowed by a trajectory of a point $(\omega, x)$ if and only if $\{\tilde{q}_k\}$ is $\varepsilon$-shadowed by a trajectory of a point $(\omega, x - x_0)$.

Those statements completes the proof of the lemma. $\square$

For $d, \varepsilon > 0$, $N \in \mathbb{N}$ define

$$p(d, N, \varepsilon) := \int_{\omega \in \Sigma} p((\omega, 0), d, N, \varepsilon) \mathrm{d}\nu.$$

Note that integral exists since for fixed $d$, $N$, $\varepsilon$ the value $p((\omega, 0), d, N, \varepsilon)$ depends only on finite number of entries of $\omega$. The quantity $p(d, N, \varepsilon)$ can be interpreted as the probability of a $d$-pseudotrajectory of length $N$ to be $\varepsilon$-shadowed.

The main result of the paper is the following:

**Theorem 2.** *For any $\lambda_0, \lambda_1 \in \mathbb{R}$ satisfying (2) there exists $\varepsilon_0 > 0$, $0 < c_0 < \infty$ such that for any $\varepsilon < \varepsilon_0$ the following holds*

1. *If $c < c_0$ then $\lim_{N \to \infty} p(\varepsilon / N^c, N, \varepsilon) = 0$;*

2. *If $c > c_0$ then $\lim_{N \to \infty} p(\varepsilon / N^c, N, \varepsilon) = 1$.*

**Remark 1.** Later (Lemma 2) we prove that for any $N \in \mathbb{N}$, $L > 0$, $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0)$ the following equality holds:

$$p(\varepsilon_1 / L, N, \varepsilon_1) = p(\varepsilon_2 / L, N, \varepsilon_2).$$

Hence statement of Theorem 2 actually does not depend on the value of $\varepsilon$.

**Remark 2.** Analog of Hammel-Grebogi-Yorke conjectured for map $f$ due to Remark 1 suggests that $p(\varepsilon / N, N, \varepsilon)$ is close to 1. Hence if $c_0 > 1$ then Hammel-Grebogi-Yorke conjecture is not satisfied. For the example of such parameters see Remark 3.

# 3 Equivalent Formulation

Let $a_0 = \ln \lambda_0$, $a_1 = \ln \lambda_1$. Consider the following random variable

$$\gamma = \begin{cases} a_0 & \text{with probability } 1/2, \\ a_1 & \text{with probability } 1/2. \end{cases}$$

Fix $N > 0$. Consider random walk $\{A_i\}_{i \in [0, \infty)}$ generated by $\gamma$ and independent uniformly distributed in $[-1, 1]$ variables $\{r_i\}_{i \in [0, \infty)}$.

Define sequence $\{z_i\}_{i\in[0,N]}$ by the following

$$z_0 = 0, \quad z_{i+1} = z_i + \frac{r_{i+1}}{e^{A_{i+1}}}. \tag{3}$$

For a given sequences $(\{A_i\}_{i\in[0,N]}, \{r_i\}_{i\in[0,N]})$ define

$$B(k,n) := \frac{e^{A_k+A_n}}{e^{A_k}+e^{A_n}}|z_n - z_k| = \frac{e^{A_n}}{e^{A_k}+e^{A_n}}\left|e^{A_k}z_n - e^{A_k}z_k\right|,$$

$$K(\{A_i\},\{r_i\}) := \max_{0\le k<n\le N} B(k,n),$$

$$s(N,L) := P(K(\{A_i\}_{i\in[0,N]}, \{r_i\}_{i\in[0,N]}) < L),$$

where $P(\cdot)$ is the probability of a certain event.

Below we prove the following lemma.

**Lemma 2.** *There exists $\varepsilon_0 > 0$, $L_0 > 0$ such that for any $d \ge 0$, $L > L_0$, $N \in \mathbb{N}$, satisfying $Ld < \varepsilon_0$ the following equality holds*

$$p(d, N, Ld) = s(N, L).$$

*Proof.* Let us choose $\varepsilon_0, L_0 > 0$ such that if $\operatorname{dist}(\omega, \tilde{\omega}) < \varepsilon_0$ then $t(\omega) = t(\tilde{\omega})$ and map $\sigma$ satisfies the Lipschitz shadowing property with constants $\varepsilon_0, L_0$.

Fix $d < d_0$, $N > 0$ and $L > L_0$ satisfying $Ld < \varepsilon_0$. Let us choose $\omega$ at random according to the probability measure $\nu$ and a pseudotajectory $\{q_k\} = \{(\omega_k, x_k)\} \in \Omega_{(\omega,0),d,N}$. Consider the sequences

$$\gamma_k = a_{t(\omega_k)}, \quad A_k = \sum_{i=0}^{k} \gamma_i, \quad r_k = (x_k - \lambda_{t(\omega_{k-1})}x_{k-1})/d.$$

Note that $r_k$ are independent uniformly distributed in $[-1, 1]$ and $\gamma_k$ are independent and distributed according to $\gamma$.

Below we prove that the sequence $\{q_k\}$ can be $Ld$-shadowed if and only if

$$L \ge K(\{A_i\}, \{r_i\}). \tag{4}$$

Assume that pseudotrajectory $(\omega_k, x_k)$ is $Ld$-shadowed by the exact trajectory $(\xi_k, y_k)$. By the choice of $\varepsilon_0$ the following equality holds

$$t(\omega_k) = t(\xi_k). \tag{5}$$

Now let us investigate behavior of the second coordinate. Note that

$$y_{k+1} = \lambda_{t(\xi_k)} y_k = e^{\gamma_k} y_k, \quad y_n = e^{A_n - A_k} y_k, \tag{6}$$

$$x_n = e^{A_n - A_k} x_k + e^{A_k}(z_n - z_k),$$

where $z_k$ are defined by (3). Hence

$$(y_n - x_n) = e^{A_n - A_k}(y_k - x_k) + e^{A_k}(z_n - z_k).$$

From this equality it is easy to deduce that

$$\max(|y_k - x_k|, |y_n - x_n|) \geq B(k, n).$$

and the equality holds if $(y_k - x_k) = -(y_n - x_n)$. Hence inequality (4) holds.

Now let us assume that (4) holds and prove that $(w_k, x_k)$ can be $Ld$-shadowed. Let us choose sequence $\{\xi_k\}$ which $Ld$-shadows $\{w_k\}$, then equalities (5) hold.

For any $y_0 \in \mathbb{R}$ we can find $y_k$ by the equation (6). Let us define function $F : \mathbb{R} \to \mathbb{R}$ by the following

$$F(y_0) = \max_{0 \leq k \leq N} |y_k - x_k|.$$

Since function $F$ is continuous it is easy to show that for some $y_0$ it has a minimum. Denote $L' := \min_{y_0 \in \mathbb{R}} F(y_0)$ and let $y_0$ is such that $L' = F(y_0)$. Let $D = \{k \in [0, N] : |y_k - x_k| = F(y_0)\}$. Let us consider two cases.

*Case 1.* For all $k \in D$ the value $y_k - x_k$ has the same sign. Without loss of generality we can assume that it is positive. Then for small enough $\delta > 0$ the inequality $F(y_0 - \delta) < F(y_0)$ holds, which contradicts to the choice of $y_0$.

*Case 2.* There exists indexes $k, n \in D$ such that the values $y_k - x_k$ and $y_n - x_n$ have different signs. Then $(y_k - x_k) = -(y_n - x_n)$ and hence $L' = B(k, n) \leq K(\{A_i\}, \{z_i\})$.

□

# 4 Proof of Theorem 2

Note that shadowing problems for maps $f$ and $f^{-1}$ are equivalent (up to a constant multiplier on $d$). In what follows we assume that $\lambda_0 \lambda_1 > 1$. Put

$$v := E(\gamma) = (a_0 + a_1)/2 > 0, \quad M := (\ln N)^2, \quad w := v/2.$$

In the proof of Theorem 2 we use the following statements.

**Lemma 3** (Large Deviation Principle, [23, Secion 3])**.** *There exists an increasing function $h : (0, \infty) \to (0, \infty)$ such that for any $\varepsilon > 0$ and $\delta > 0$ for large enough $n$ the following inequalities hold*

$$P\left(\frac{A_n}{n} - E(\gamma) < -\varepsilon\right) < e^{-(h(\varepsilon)-\delta)n}.$$

$$P\left(\frac{A_n}{n} - E(\gamma) < -\varepsilon\right) > e^{-(h(\varepsilon)+\delta)n}.$$

**Lemma 4** (Ruin Problem, [4, Chapter XII, §4, 5])**.** *Let $b$ be the unique positive root of the equation*

$$\frac{1}{2}\left(e^{-ba_0} + e^{-ba_1}\right) = 1.$$

*For any $\delta > 0$ for large enough $C > 0$ the following inequalities hold*

$$P(\exists i \geq 0 : A_i \leq -C) \leq e^{-C(b-\delta)}, \tag{7}$$

$$P(\exists i \geq 0 : A_i \leq -C) \geq e^{-C(b+\delta)}, \tag{8}$$

Put $c_0 = 1/b$. Due to Lemma 2 it is enough to prove the following:

(S1) If $c < c_0$ then $\lim_{N\to\infty} s(N, N^c) = 0$.

(S2) If $c > c_0$ then $\lim_{N\to\infty} s(N, N^c) = 1$.

**Remark 3.** For $\lambda_0 = 1/2$, $\lambda_1 = 3$ the inequalities $b < 1$, $c_0 > 1$ holds and hence due to Remark 2 the statement of Conjecture 1 does not hold. Similarly $c_0 > 1$ for $\lambda_0 = 1/3$, $\lambda_1 = 2$.

Below we prove items (S1) and (S2).

## 4.1 Proof of (S1)

Assume that $c < 1/b$. Let us choose $c_1 \in (c, 1/b)$ and $\delta > 0$, satisfying

$$c_1(b + \delta) < 1. \tag{9}$$

Consider the following events:

$$\begin{aligned} I &= \{\exists i \in [0, M] : A_i \le -c_1 \ln N;\ \text{and } A_{2M} \ge 0\}\,, \\ I_1 &= \{\exists i \in [0, M] : A_i \le -c_1 \ln N\}\,, \\ I_2 &= \{\exists i \in [0, M] : A_i \le -wM\}\,, \\ I_3 &= \{A_{2M} - A_M \le wM\}\,. \end{aligned}$$

The following holds.

$$P(I) \ge P(I_1) - P(I_2) - P(I_3), \tag{10}$$

$$\begin{aligned} P(I_1) &\ge P(\exists i \ge 0 : A_i \le -c_1 \ln N) - P(\exists i > M : A_i \le -c_1 \ln N) \\ &\ge e^{-c_1 \ln N (b+\delta)} - \sum_{i=M+1}^{N} P(A_i \le 0) \ge N^{-c_1(b+\delta)} - \sum_{i=M+1}^{N} e^{-ih(v)} \\ &\ge N^{-c_1(b+\delta)} - \frac{1}{1 - e^{-h(v)}} e^{-(M+1)h(v)} \ge N^{-c_1(b+\delta)} + o(N^{-2}). \end{aligned} \tag{11}$$

Similarly

$$P(I_2) \le \sum_{i=M+1}^{\infty} P(A_i \le 0) = o(N^{-2}), \tag{12}$$

$$P(I_3) \le e^{-Mh(v-w)} = o(N^{-2}). \tag{13}$$

Summarizing (10)-(13) we conclude that

$$P(I) \ge N^{-c_1(b+\delta)} + o(N^{-2}). \tag{14}$$

Assume that event $I$ has happened and let $i \in [0, M]$ be one of the indexes satisfying inequality $A_i < -c_1 \ln N$. Note that the following events are independent

$$J_1 = \{r_i \in [1/2; 1]\}, \quad J_2 = \left\{ z_{2M} - z_0 \ge \frac{r_i}{e^{A_i}} \right\}.$$

Hence

$$P\left( z_{2M} - z_0 \ge \frac{1}{2e^{A_i}} \right) \ge P(J_1)P(J_2) = 1/4 \cdot 1/2 = 1/8$$

and
$$P(B(0,2M) > N^{c_1}/4) \geq \frac{1}{8}P(I) = \frac{1}{8}N^{-c_1(b+\delta)} + o(N^{-2}).$$
Note that for large enough $N$ the inequality $N^c < N^{c_1}/4$ holds and hence

$$P(B(0,2M) > N^c) \geq \frac{1}{8}N^{-c_1(b+\delta)} + o(N^{-2}).$$

Similarly for any $k \in [0, N-2M]$:

$$P(B(k,k+2M) > N^c) \geq \frac{1}{8}N^{-c_1(b+\delta)} + o(N^{-2}).$$

Note that events in the last expression for $k = 0, 2M, 2\cdot 2M, \ldots ([N/(2M)]-1)2M$ are independent and hence

$$P\left(\exists k \in [0, N-2M] : B(k,k+2M) > N^c\right) \geq 1 - \left(1 - \left(\frac{1}{8}N^{-c_1(b+\delta)} + o(N^{-2})\right)\right)^{[N/(2M)]}. \quad (15)$$

Using (9) we conclude that

$$\left(\frac{1}{8}N^{-c_1(b+\delta)} + o(N^{-2})\right)[N/(2M)] \geq \left(\frac{1}{8}N^{-c_1(b+\delta)} + o(N^{-2})\right)\left(\frac{N}{2(\ln N)^2} - 1\right) = \frac{1}{16(\ln N)^2}N^{1-c_1(b+\delta)} + o(N^{-1}) \to_{N\to\infty} \infty$$

and hence

$$\left(1 - \left(\frac{1}{8}N^{-c_1(b+\delta)} + o(N^{-2})\right)\right)^{[N/(2M)]} \to_{N\to\infty} 0. \quad (16)$$

Relations (15), (16) implies that

$$P(K(\{A_i\}_{i\in[0,N]}, \{r_i\}_{i\in[0,N]}) > N^c) \to 1.$$

Hence
$$\lim_{N\to\infty} s(N, N^c) = 0.$$

## 4.2 Proof of (S2)

Let $c > 1/b$. Let us choose $c_1 \in (1/b, c)$ and $\delta > 0$, satisfying $c_1(b-\delta) > 1$.

Note that for any $n > k$ the following inequalities hold:

$$e^{A_k}|z_n - z_k| \le \sum_{i=k}^{n} e^{-(A_i - A_k)}$$

$$\frac{e^{A_n}}{e^{A_k} + e^{A_n}} \le 1.$$

Hence

$$K(\{A_i\}, \{r_i\}) \le \max_{0\le k<n\le N} \sum_{i=k}^{n} e^{-(A_i-A_k)} \le \max_{0\le k\le N} \sum_{i=k}^{N} e^{-(A_i-A_k)} =: D(\{A_i\}). \tag{17}$$

The following holds:

$$\begin{aligned} P(D(\{A_i\}) < N^c) &\ge 1 - P\left(\exists k \in [0,N] : \sum_{i=k}^{N} e^{-(A_i-A_k)} > N^c\right) \\ &\ge 1 - NP\left(\sum_{i=0}^{N} e^{-(A_i-A_k)} > N^c\right). \end{aligned}$$

Note that if $\sum_{i=0}^{N} e^{-(A_i-A_k)} > N^c$ then one of the following hold

$$\exists i \in [0, M] : e^{-A_i} > \frac{N^c}{2M},$$

$$\exists i \in [M, N] : e^{-A_i} > \frac{N^{c-1}}{2}.$$

Note that for large enough $N$ the following inequalities hold

$$\frac{N^c}{2M} > N^{c_1}, \quad N^{c-1}/2 > e^{-wM}$$

and hence (arguing similarly to the previous section) for large enough $N$

$$\begin{aligned} P\left(\sum_{i=0}^{N} e^{-(A_i-A_k)} > N^{c_1}\right) &\le P(\exists i \in [0,M] : A_i < -c_1 \ln N) + P(\exists i \in [M,N] : A_i < wM) \\ &\le e^{-(b-\delta)c_1 \ln N} + o(N^{-2}) = N^{-(b-\delta)c_1} + o(N^{-2}). \end{aligned}$$

Finally

$$P(D(\{A_i\}) \leq N^c) \geq 1 - N(N^{-(b-\delta)c_1} + o(N^{-2})) \to_{N\to\infty} 1,$$

and hence relations (17) imply

$$\lim_{N\to\infty} s(N, N^c) = 1.$$

# 5 Acknowledgement

The author is grateful to Jairo Bochi for the help in stating the problem and to Artem Sapozhnikov and Yinshan Chang for the help with basics in probability. The author is also grateful to the hospitality of Pontificia Universidade Catolica do Rio de Janeiro where the work was initiated. The work of the author was partially supported by by Chebyshev Laboratory (Department of Mathematics and Mechanics, St. Petersburg State University) under RF Government grant 11.G34.31.0026, JSC "Gazprom neft", by the Saint-Petersburg State University research grant 6.38.223.2014 and by the German-Russian Interdisciplinary Science Center (G-RISC) funded by the German Federal Foreign Office via the German Academic Exchange Service (DAAD).

# References

[1] F. Abdenur and L. J. Diaz, Pseudo-orbit shadowing in the $C^1$ topology, Discrete Contin. Dyn. Syst., **7**, 2003, 223-245.

[2] D. V. Anosov. On a class of invariant sets of smooth dynamical systems. Proc. 5th Int. Conf. on Nonlin. Oscill. **2**, Kiev, 1970, 39-45.

[3] R. Bowen. Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes Math., vol. 470, Springer, Berlin, 1975.

[4] Feller, William. An introduction to probability theory and its applications. Vol. II., John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[5] A. S. Gorodetskii, Yu. S. Ilyashenko. Some properties of skew products over a horseshoe and a solenoid. Proc. Steklov Inst. Math. 2000, no. 4 (231), 90-112.

[6] A. S. Gorodetskii. Regularity of central leaves of partially hyperbolic sets and applications. Izv. Math. 70 (2006), no. 6, 1093-1116

[7] Guo-Cheng Yuan; James A. Yorke. An open set of maps for which every point is absolutely nonshadowable. Proc. Amer. Math. Soc. 128 (2000), no. 3, 909-918.

[8] S. M. Hammel, J. A. Yorke, and C Grebogi. Do numerical orbits of chaotic dynamical processes represent true orbits. J. of Complexity **3** (1987), 136-145.

[9] S. M. Hammel, J. A. Yorke, C. Grebogi. Numerical orbits of chaotic processes represent true orbits. Bulletin of the American Mathematical Society **19** (1988), 465-469.

[10] A. Koropecki, E. Pujals. Consequences of the Shadowing Property in low dimensions. Ergodic Theory Dynam. Systems 34 (2014), no. 4, 1273-1309

[11] A. Morimoto. The method of pseudo-orbit tracing and stability of dynamical systems. Sem. Note **39**, Tokyo Univ., 1979.

[12] A. V. Osipov, S. Yu. Pilyugin, and S. B. Tikhomirov. Periodic shadowing and $\Omega$-stability. Regul. Chaotic Dyn. **15** (2010), 404-417.

[13] K. Palmer. Shadowing in Dynamical Systems. Theory and Applications. Kluwer, Dordrecht, 2000.

[14] S. Yu. Pilyugin. Shadowing in Dynamical Systems. Lecture Notes Math., vol. 1706, Springer, Berlin, 1999.

[15] S. Yu. Pilyugin Theory of pseudo-orbit shadowing in dynamical systems. Diff. Eqs. **47** (2011), 1929–1938.

[16] S. Yu. Pilyugin. Variational shadowing. Discr. Cont. Dyn. Syst., ser. B **14** (2010) 733-737.

[17] S. Yu. Pilyugin, A. A. Rodionova, and K. Sakai, Orbital and weak shadowing properties, Discrete Contin. Dyn. Syst., **9**, 2003, 287-308.

[18] S. Yu. Pilyugin, S. B. Tikhomirov. Lipschitz Shadowing implies structural stability. Nonlinearity **23** (2010), 2509-2515

[19] C. Robinson. Stability theorems and hyperbolicity in dynamical systems. Rocky Mount. J. Math. **7** (1977), 425-437.

[20] K. Sakai, Pseudo orbit tracing property and strong transversality of diffeomorphisms of closed manifolds, Osaka J. Math., **31**, 1994, 373-386.

[21] K. Sawada. Extended $f$-orbits are approximated by orbits. Nagoya Math. J. **79** (1980), 33-45.

[22] S. Tikhomirov. Holder Shadowing on finite intervals. Ergodic Theory and Dynamical Systems, available on CJO2014. doi:10.1017/etds.2014.7.

[23] S. R. S. Varadhan. Large deviations and applications. CBMS-NSF Regional Conference Series in Applied Mathematics, 46. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984.