Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Cheeger constants, structural balance, and spectral clustering analysis for signed graphs

by

Fatihcan M. Atay and Shiping Liu

Preprint no.: 111 2014

CHEEGER CONSTANTS, STRUCTURAL BALANCE, AND SPECTRAL CLUSTERING ANALYSIS FOR SIGNED GRAPHS

FATIHCAN M. ATAY AND SHIPING LIU

ABSTRACT. We introduce a family of multi-way Cheeger-type constants $\{h_k^\sigma, k = 1, 2, \ldots, N\}$ on a signed graph $\Gamma = (G, \sigma)$ such that $h_k^\sigma = 0$ if and only if Γ has k balanced connected components. These constants are switching invariant and bring together in a unified viewpoint a number of important graph-theoretical concepts, including the classical Cheeger constant, the non-bipartiteness parameter of Desai and Rao, the bipartiteness ratio of Trevisan, the dual Cheeger constant of Bauer and Jost on unsigned graphs, and the line index of imbalance of Harary (also called the frustration index) on signed graphs.

We then propose a corresponding spectral clustering algorithm for finding k almost-balanced subgraphs, each defining a sparse cut. We find that the proper metric for the clustering algorithm is the metric on a real projective space. Remarkably, this algorithm includes the traditional spectral clustering algorithm on unsigned graphs via spherical metrics as a special case. We verify the algorithm theoretically by proving higher-order signed Cheeger inequalities, and signed improved Cheeger inequalities concerning higher-order spectral gaps.

We also prove estimates of the extremal eigenvalues of signed Laplace matrix in terms of number of signed triangles (3-cycles).

1. Introduction

In this paper, we study the interaction between the spectra and the structural balance theory of signed graphs. Signed graphs and the idea of balance, introduced by Frank Harary [22] in 1953 and have since then been rediscovered in different contexts many times, are important models and tools for various research fields. The concepts were motivated and suggested by problems in social psychology [22, 23, 14] and have stimulated new methods for analyzing social networks [29, 48, 46], biological networks [45], logical programming [17], etc. Signed graphs also play important roles in various branches of mathematics, such as group theory, root systems (see [12] and the references therein), topology [11, 13], and even physics [9]. By relating signed graphs with 2-lifts of a graph, Bilu and Linial [10] reduce the problem of constructing expander graphs to finding a signature with small spectral radius. In a recent breakthrough work, Marcus, Spielman, and Srivastava [37, 38] show the existence of infinite families of regular bipartite Ramanujan graphs of every degree larger than 2, by proving a variant conjecture of Bilu and Linial about the existence of the signature of a given graph with very small spectral radius.

A signed graph $\Gamma = (G, \sigma)$ is an undirected graph $G = (V, E)$ with a signature $\sigma : E \to \{+1, -1\}$ on the edge set E. One can think of the vertex set V as a social group. A positive (resp., negative) edge between two vertices indicates that the two members are friends (resp., enemies). The sign of a cycle in G is defined as the product of the signs of all edges in it. Γ is called *balanced* if all cycles in G are positive. This is a crucial

2010 Mathematics Subject Classification 05C50, 05C22 (05C85, 39A12).

concept for a signed graph due to Harary [22]. Consider a group of three members a, b and c, in which a, b are enemies and a, c are enemies. Then the balance of the corresponding 3-cycle requires that b, c are friends. Hence, balance refers to a certain consistency in the relationship, often expressed as "the enemy of my enemy is my friend".

Observe that if we reverse the relations between a and b, a and c simultaneously, the 3-cycle is still balanced. In general, a switching operation of the signature σ can be defined. We call the operation of reversing the signs of all edges connecting a subset $S \subseteq V$ and its complement as *switching the subset* S. The sign of a cycle, and hence the property of being balanced, are switching invariant, i.e. preserved by switching any subset of V.

The properties of being balanced can be characterized by the spectrum of the *signed normalized Laplace matrix*

$$\Delta^\sigma := I - D^{-1}A^\sigma,$$

where I is the identity matrix, D is the diagonal degree matrix, i.e. $D_{uu} = d_u$, $\forall u \in V$, d_u is the degree of u in G, and A^σ is the signed adjacency matrix. The matrix Δ^σ appears naturally in the context of graph drawing and electrical networks [30]. It is known that the eigenvalues of Δ^σ can be listed (counting with multiplicity) as

$$0 \leq \lambda_1(\Delta^\sigma) \leq \lambda_2(\Delta^\sigma) \leq \cdots \leq \lambda_N(\Delta^\sigma) \leq 2,$$

where N is the cardinality of V. Moreover,

$$\Gamma \text{ has a balanced connected component } \Leftrightarrow \lambda_1(\Delta^\sigma) = 0, \tag{1.1}$$

see e.g. [49, 26, 34]. The eigenvalue $\lambda_k(\Delta^\sigma)$ is switching invariant for any $1 \leq k \leq N$ (see [50] or Proposition 1 below). We refer to [25, 33, 21, 5, 8, 41] for more results in the spectral theory of signed graphs.

In this paper, we define a Cheeger-type constant h_1^σ based on Harary's balance theorem (see Theorem 8) such that

$$\Gamma \text{ has a balanced connected component } \Leftrightarrow h_1^\sigma = 0. \tag{1.2}$$

In the following, we will refer to this Cheeger-type constant of a signed graph as a *signed Cheeger constant* for short. Similarly, we will also speak of signed inequalities and signed algorithms. The constant h_1^σ can be used to obtain a quantitative version of (1.1). (For previous results in this aspect, see [25, 8].) We prove that h_1^σ is switching invariant. This enables us to show that the signed Cheeger constant h_1^σ and its multi-way versions provide a common extension of the classical Cheeger constant [16, 20, 4, 3], the non-bipartiteness parameter of Desai and Rao [19] (after a modification), the bipartiteness ratio of Trevisan [47], and the dual Cheeger constant of Bauer and Jost [6]. Recall that on an unsigned graph, the Cheeger constant encodes the information of connectivity, while the latter three constants describe the deviation of the graph from being bipartite.

The introduction of the signed Cheeger constant further enables us to develop corresponding spectral clustering algorithms on signed networks. We propose an algorithm for finding k *almost-balanced subgraphs* of a signed graph $\Gamma = (G, \sigma)$. The novel point of this algorithm is that, after embedding the graph into the Euclidean space $\mathbb{R}^k$ via eigenfunctions, we find the proper metric for clustering points is a metric on the real projective space $P^{k-1}\mathbb{R}$ studied by the second named author [35]. Interestingly, when we take σ to be positive on all edges, this algorithm reduces to the traditional spectral clustering using a spherical metric (see e.g. [40, 36]) verified theoretically by Lee, Oveis Gharan, and Trevisan [32]. In other words, within the framework of signed graphs we find a unification of the traditional spectral clustering algorithm and the recent algorithm for finding k almost-bipartite subgraphs proposed in [35].

We further explore the related theoretical analysis of this algorithm. We extend the higher-order Cheeger [32], the higher-order dual Cheeger [35], and the improved Cheeger inequalities [31] from unsigned graphs to signed graphs in terms of our singed Cheeger constants.

Harary [23] defined a signed graph $\Gamma = (G, \sigma)$ to be *antibalanced* if its negation $-\Gamma := (G, -\sigma)$ is balanced. Thus, Γ is antibalanced if and only if every odd cycle in it is negative and every even cycle is positive. It is known that a connected signed graph is antibalanced if and only if $\lambda_N(\Delta^\sigma) = 2$ (see [34]). We obtain similar results concerning antibalance and the spectral gap $2 - \lambda_N(\Delta)$ via an antithetical dual signed Cheeger constant (see (1.7) below).

Finally, we prove estimates for extremal eigenvalues $\lambda_1(\Delta^\sigma)$, $\lambda_N(\Delta^\sigma)$ in terms of signed 3-cycles (we will speak of signed triangles in the following). By definition, the presence of positive (resp., negative) triangles implies that Γ can not be antibalanced (resp., balanced). Therefore, the number of signed triangles relate naturally to the spectral gaps $\lambda_1(\Delta^\sigma)$ and $2 - \lambda_N(\Delta^\sigma)$.

We now discuss those results in more detail.

1.1. **Signed Cheeger constants.** We introduce the notation and precise definitions. We say $u, v \in V$ are neighbors when $e = \{u, v\} \in E$, and write $u \sim v$. For ease of notation we write $\sigma(uv) := \sigma(\{u, v\})$ for the sign of an edge. In addition to the sign, we also assign a positive symmetric weight w_{uv} to every edge $e = \{u, v\} \in E$, and set $w_{uv} = 0$ if $e = \{u, v\} \notin E$. The degree d_u of a vertex u is defined as $d_u = \sum_{v \in V} w_{uv}$. We will restrict ourselves to signed simple graphs, i.e., the case when the underlying graph G has no self-loops and multi-edges. We also consider a general measure $\mu : V \to \mathbb{R}$ on the vertex set.

For any two subsets V_1, V_2 of V, we define $|E(V_1, V_2)| = \sum_{u \in V_1} \sum_{v \in V_2} w_{uv}$ and its signed versions E^+, E^-:

$$|E^{\pm}(V_1, V_2)| = \sum_{u \in V_1} \sum_{v \in V_2, \sigma(uv) = \pm 1} w_{uv}.$$

When $V_1 = V_2$, we write $|E(V_1)|$, $|E^{\pm}(V_1)|$ for short. Keep in mind that in this case every edge weight is counted twice. For a subset $S \subseteq V$, we define its volume as $\mathrm{vol}_\mu(S) = \sum_{u \in S} \mu(u)$.

Let (V_1, V_2) denote a sub-bipartition of V, i.e. $\emptyset \neq V_1 \cup V_2 \subseteq V$, $V_1 \cap V_2 = \emptyset$. We define the *signed bipartiteness ratio* of (V_1, V_2) to be

$$\beta^\sigma(V_1, V_2) = \frac{2|E^+(V_1, V_2)| + |E^-(V_1)| + |E^-(V_2)| + |E(V_1 \cup V_2, \overline{V_1 \cup V_2})|}{\mathrm{vol}_\mu(V_1 \cup V_2)}, \tag{1.3}$$

where $\overline{V_1 \cup V_2}$ is the complement of $V_1 \cup V_2$ in V.

Definition 1 (Signed Cheeger constant)**.** For a signed graph $\Gamma = (G, \sigma)$, *the Cheeger constant* $h_1^\sigma(\mu)$ is defined as

$$h_1^\sigma(\mu) = \min_{(V_1,V_2)} \beta^\sigma(V_1, V_2), \tag{1.4}$$

where the minimum is taken over all possible sub-bipartitions of V.

With this definition we have then the statement (1.2), because Harary's balance theorem (see Theorem 8) asserts that a signed graph is balanced if and only if there exists a partition V_1, V_2 of V such that $|E^+(V_1, V_2)| = |E^-(V_1)| = |E^-(V_2)| = 0$.

Moreover, we prove that $h_1^\sigma(\mu)$ is switching invariant (see Proposition 2). If the signature σ can be changed to σ' via switching operations, we say σ and σ' are switching equivalent, and write $\sigma \approx \sigma'$. We denote by σ_+ (resp., σ_-) the all positive (resp., all negative) signature. By definition, when $\sigma \approx \sigma_-$, the constant $h_1^\sigma(\mu)$ reduces to the bipartiteness ratio of Trevisan [47], or one minus the dual Cheeger constant of Bauer and Jost [6].

We further prove that (see Corollary 1)

$$h_1^\sigma(\mu) = \min_{\sigma',\,\sigma'\approx\sigma} \min_{\emptyset\neq S\subseteq V} \frac{|E^-(S)|(\sigma') + |E(S,\overline{S})|}{\mathrm{vol}(S)}, \tag{1.5}$$

where the negative edges counted in $|E^-(S)|(\sigma')$ is decided by the signature σ'. This implies that if $\sigma \approx \sigma_+$, the constant $h_1^\sigma(\mu)$ reduces to the one-way Cheeger constant (see [39, 32]) which trivially vanishes.

The constant $h_1^\sigma(\mu)$ can also be expressed from the following viewpoint (see Corollary 2),

$$h_1^\sigma(\mu) = \min_{\emptyset\neq S\subseteq V} \frac{2e^\sigma_{min}(S) + |E(S,\overline{S})|}{\mathrm{vol}(S)}, \tag{1.6}$$

where $e^\sigma_{min}(S)$ is the minimal number of edges that need to be removed from the induced subgraph of S to make it balanced. The quantity $e^\sigma_{min}(V)$ is the *line index of imbalance* of Γ introduced by Harary [24] (see also [1]), alternatively called the *frustration index* [9] and studied extensively, e.g. [2, 43, 8].

If $\sigma \approx \sigma_-$, after replacing $2e^\sigma_{min}(S)$ by $e^\sigma_{min}(S)$, the constant (1.6) reduces to the non-bipartiteness parameter of Desai and Rao [19]. The non-bipartiteness parameter was extended to signed graphs by Hou [25]. We see that our constant $h_1^\sigma(\mu)$ is larger than theirs in general.

Extending (1.4) in the spirit of [39, 32, 35], we can naturally define a family of multi-way signed Cheeger constant $\{h_k^\sigma(\mu), k = 1, 2, \ldots N\}$ (see Definition 2). The $h_1^\sigma(\mu)$ defined in (1.4) is the first one of this family. Furthermore, $h_2^\sigma(\mu)$ reduces to the classical Cheeger constant if $\sigma \approx \sigma_+$.

Hence, the signed Cheeger constants provide new insights into existing constants reflecting connectivity or bipartiteness of unsigned graphs in the language of switching within the framework of signed graphs, thus giving a unified viewpoint about connectivity and bipartiteness of the underlying graph via assigning signatures.

We also define a natural family of antithetical dual signed Cheeger constants $\{\widetilde{h}_k^\sigma(\mu), k = 1, 2, \ldots, N\}$ by $\widetilde{h}_k^\sigma(\mu) := h_k^{-\sigma}(\mu)$. Dually, we have

$$\Gamma \text{ has an antibalanced connected component } \Leftrightarrow \widetilde{h}_1^\sigma(\mu) = 0. \tag{1.7}$$

1.2. **Cheeger-type estimates.** We let μ_d denote the degree measure on V, i.e. $\mu_d(u) = d_u$, $\forall u \in V$.

1.2.1. *Results.* We prove the following signed Cheeger inequality.

Theorem 1. *Given a signed graph $\Gamma = (G, \sigma)$, we have*

$$\frac{\lambda_1(\Delta^\sigma)}{2} \leq h_1^\sigma(\mu_d) \leq \sqrt{2\lambda_1(\Delta^\sigma)}. \tag{1.8}$$

We further prove the higher-order versions of the signed Cheeger inequality (1.8).

Theorem 2. *There exists an absolute constant C such that for any signed graph $\Gamma = (G, \sigma)$ and any $k \in \{1, 2, \ldots, N\}$,*

$$\frac{\lambda_k(\Delta^\sigma)}{2} \leq h_k^\sigma(\mu_d) \leq Ck^3\sqrt{\lambda_k(\Delta^\sigma)}. \tag{1.9}$$

This is a generalization of the higher-order Cheeger and dual Cheeger inequalities for unsigned graphs by Lee, Oveis Gharan, and Trevisan [32] and the second named author [35].

A natural question is that when can we improve the order of $\lambda_1(\Delta^\sigma)$ on the right hand side of (1.8) to be 1. Extending the ideas of Kwok et al. [31], we answer this question by the following theorem.

Theorem 3. *Given a signed graph $\Gamma = (G, \sigma)$ and any $k \in \{1, 2, \ldots, N\}$,*

$$h_1^\sigma(\mu_d) < 16\sqrt{2}k\frac{\lambda_1(\Delta^\sigma)}{\sqrt{\lambda_k(\Delta^\sigma)}}. \tag{1.10}$$

In other words, when there exists a k such that the gap between λ_1 and λ_k is large, one can improve the order of λ_1 on the r.h.s. of (1.8) to be 1. Actually, a slightly stronger version of this result can be proved; see Theorem 13. We further have the following higher-order estimates.

Theorem 4. *There exists an absolute constant C such that for any signed graph $\Gamma = (G, \sigma)$ and any $1 \leq k \leq l \leq N$,*

$$h_k^\sigma(\mu_d) < Clk^6\frac{\lambda_k(\Delta^\sigma)}{\sqrt{\lambda_l(\Delta^\sigma)}}. \tag{1.11}$$

This generalizes the corresponding results for unsigned graphs given in [31] and [35].

The above estimates have two directions of extensions. On the one hand, they have their corresponding versions for *the non-normalized Laplace matrix* (or *Kirchhoff matrix*) $L^\sigma := D - A^\sigma$; see Theorems 11, 12, 14, and Corollary 3 and 4.

On the other hand, they can be easily translated into estimates for $\widetilde{h}_k^\sigma(\mu_d)$ and $2 - \lambda_{N-k+1}(\Delta^\sigma)$ by duality. This is due to the fact that $2 - \lambda_{N-k+1}(\Delta^\sigma) = \lambda_k(\Delta^{-\sigma})$ (see Lemma 1). For example, the dual version of Theorem 1 can be stated as below.

Theorem 5. *Given a signed graph $\Gamma = (G, \sigma)$, we have*

$$\frac{2 - \lambda_N(\Delta^\sigma)}{2} \leq \widetilde{h}_1^\sigma(\mu_d) \leq \sqrt{2(2 - \lambda_N(\Delta^\sigma))}. \tag{1.12}$$

We omit the dual versions of Theorems 2, 3 and 4 here. Actually, these results are nice demonstrations of a general antithetical duality principle discussed by Harary [23].

1.2.2. *Ideas for proofs.* The proofs of the above results are based on the crucial observation that the estimation of $\lambda_1(\Delta^\sigma)$ should be considered as a "mixture" of the estimates of the smallest and largest eigenvalues of unsigned graphs (for which the smallest eigenvalue trivially equals 0). This can be seen more clearly from the corresponding Rayleigh quotients. One can appeal either to the techniques for proving the Cheeger inequality for unsigned graphs [4, 3, 20] or to those for proving the dual Cheeger inequality [47, 6]. For the former strategy, one first needs to switch the signature to the one achieving the first minimum in (1.5). We adopt the latter strategy, which is stable under switching operations. We use local level dualities to bring the two extremal cases together in the proofs, as in Lemmas 5, 8, 11, Proposition 4(ii), and Claim 1.

Theorem 2 is a mixture of the higher-order Cheeger [32] and dual Cheeger [35] inequalities, the proofs of which utilize spectral clustering algorithms via metrics on spheres and real projective spaces, respectively. One might anticipate at first that the proper metrics for proving Theorem 2 are a mixture of those two kinds of metrics. It is somewhat surprising that the latter metrics [35] themselves are competent for the proof and provide unified spectral clustering algorithms. We will make this point more clear in the next subsection.

1.3. **Signed spectral clustering algorithm.** In order to prove Theorem 2, we develop a signed spectral clustering algorithm for finding k subsets whose induced subgraphs are nearly balanced. The connections among those k subsets, regardless of their signs, are very sparse. We explain the key points of this algorithm.

Let $\{\phi_1, \phi_2, \ldots, \phi_N\}$ be an orthonormal system of eigenfunctions corresponding to $\lambda_1(\Delta^\sigma), \lambda_2(\Delta^\sigma), \ldots, \lambda_N(\Delta$

(1) *Spectral embedding.* Using the first k eigenfunctions, we obtain a coordinate system for the vertices via the map
$$\Phi : V \to \mathbb{R}^k, \ \ v \mapsto (\phi_1(v), \phi_2(v), \ldots, \phi_k(v)).$$
(2) *Normalization.* We further map $\widetilde{V}_\Phi := \{v : \Phi(v) \neq 0\}$ to the unit sphere,
$$\Phi^{\mathrm{nor}} : \widetilde{V}_\Phi \to \mathbb{S}^{k-1}, \ \ v \mapsto \frac{\Phi(v)}{\|\Phi(v)\|}.$$
(3) *Clustering the points.* We use the following pseudometric d_Φ on $\widetilde{V}_\Phi$ studied in [35]
$$d_\Phi(u, v) := \min\left\{\|\Phi^{\mathrm{nor}}(u) + \Phi^{\mathrm{nor}}(v)\|, \|\Phi^{\mathrm{nor}}(u) - \Phi^{\mathrm{nor}}(v)\|\right\}, \tag{1.13}$$
where $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^k$.

Recall that the projective space $P^{k-1}\mathbb{R}$ is obtained from $\mathbb{S}^{k-1}$ by identifying the antipodal points,
$$Pr : \mathbb{S}^{k-1} \to P^{k-1}\mathbb{R} : x, -x \mapsto [x],$$
where x are the unit vectors in $\mathbb{R}^k$. The metric (1.13) is induced from the following metric on $P^{k-1}\mathbb{R}$,
$$d([x], [y]) := \min\{\|x + y\|, \|x - y\|\}, \ \forall [x], [y] \in P^{k-1}\mathbb{R}.$$

If $\sigma = \sigma_+$, we have $\lambda_1(\Delta^\sigma) = 0$ and ϕ_1 is the constant function $\phi_1 \equiv 1/\sqrt{\mathrm{vol}_{\mu_d}(V)}$. Therefore, Φ^{nor} maps all the vertices to the hemisphere $\{x \in \mathbb{S}^{k-1} : x_1 > 0\}$ and the metric (1.13) reduces to
$$d_\Phi(u, v) = \|\Phi^{\mathrm{nor}}(u) - \Phi^{\mathrm{nor}}(v)\|,$$

which is the spherical metric (or the radial projection distance) used in the traditional clustering algorithms verified by Lee, Oveis Gharan, and Trevisan [32]. Hence, the algorithm developed here is a natural extension of the traditional spectral clustering algorithm [36, 40] for unsigned graphs.

If, on the other hand, $\sigma = \sigma_-$, our algorithm reduces to finding k almost-bipartite subgraphs, since (G, σ_-) is balanced if and only if G is bipartite. This is exactly the one proposed in [35].

Theorem 2 provides the worst-case performance guarantee of the algorithm described above. Further, Theorems 3 and 4 suggest that the well-known eigengap heuristic [36, 31] for the traditional algorithm still holds for signed networks. That is, in case that $\lambda_k(\Delta^\sigma)$ is small and $\lambda_{k+1}(\Delta^\sigma)$ is large, it is better to cluster the data into k almost-balanced subgraphs.

We remark that if we use the last k eigenfunctions $\phi_{N-k+1}, \phi_{N-k+2}, \ldots, \phi_N$ instead of the first k eigenfunctions in the step of spectral embedding, we will obtain an algorithm for finding k subsets whose induced subgraphs are nearly antibalanced, each defining a sparse cut.

1.3.1. *Further related work.* For any signed graph (or subgraph), one can continue to do the next-level clustering. Roughly speaking, the objective is to find two subsets whose signed bipartiteness ratio is small. The heuristics of the spectral method for such clustering was discussed in [30, 28, 15]. Actually, the proof of Theorem 1 (especially Lemma 6 below) provides a theoretical guarantee for their heuristic arguments. We can achieve this clustering by the threshold sets $V_f(t) := \{u \in V : f(u) \geq t\}$ and $V_f(-t) := \{u \in V : f(u) \leq -t\}$ of a certain function f.

There are studies about another kind of multi-way clustering of signed networks, called the correlation clustering. It aims at finding k non-trivial disjoint subsets $V_1, V_2, \ldots, V_k$ such that edges connecting two vertices from the same subset are almost all positive and edges connecting two vertices from different subsets are almost all negative. Heuristic spectral algorithms for such clustering were studied in, e.g., [29, 30, 28, 46, 45, 15]; for non-spectral algorithms, see e.g. [44].

1.4. **Signed Triangles.** We denote the number of signed triangles $\sharp^+(u,v), \sharp^-(u,v)$ including an edge $\{u,v\}$ by

$$\sharp^{\pm}(u,v) := \sharp\{u' | u' \sim u, u' \sim v, \sigma(uv)\sigma(vu')\sigma(u'u) = \pm 1\}.$$

Note that the quantities $\sharp^+(u,v), \sharp^-(u,v)$ are switching invariant and their unsigned counterpart has interesting close relation with the coarse Ricci curvature of the underlying graph G [27, 7]. We prove the following theorem.

Theorem 6. *Given a signed graph $\Gamma = (G, \sigma)$, we have*

$$\frac{w^2}{W} \frac{\min_{u\sim v} \sharp^-(u,v)}{\max_u d_u} \leq \lambda_1(\Delta^\sigma) \leq \cdots \leq \lambda_N(\Delta^\sigma) \leq 2 - \frac{w^2}{W} \frac{\min_{u\sim v} \sharp^+(u,v)}{\max_u d_u}, \tag{1.14}$$

where $w = \min_{u\sim v} w_{uv}$ and $W = \max_{u\sim v} w_{uv}$.

This result is obtained by considering the iterated matrix $\Delta^\sigma[2]$ (see (7.1) below), extending an idea of Bauer, Jost and the second named author [7] for unsigned case. For the signed non-normalized Laplace matrix L^σ, similar estimates hold.

Theorem 7. *Given a signed unweighted graph $\Gamma = (G, \sigma)$, we have*

$$\lambda_N(L^\sigma) \leq \max_{u \sim v}\{d_u + d_v - \sharp^+(u, v)\}. \tag{1.15}$$

However, the proof for this case follows from different ideas, which are adapted from Das [18]. In Theorem 15 we present the corresponding results for weighted graphs. This result improves the estimate $\lambda_N(L^\sigma) \leq \max_{u\sim v}\{d_u + d_v\}$ by Hou, Li and Pan [26]. In fact, Theorem 7 answers the question asked in their paper [26, remark after Theorem 3.5].

2. Preliminaries

2.1. Harary's balance theorem and bipartition.

The following structure theorem for balance was proved in [22].

Theorem 8 (Harary's Balance Theorem)**.** *A signed graph Γ is balanced if and only if there exists a bipartition of V into two disjoint subsets V_1 and V_2 (one of which may be empty) such that each positive edge connects two vertices of the same subset and each negative edge connects two vertices of different subsets.*

By reversing the signature, Harary gave the antithetical dual result for antibalance.

Theorem 9. [23] *A signed graph Γ is antibalanced if and only if there exists a bipartition of V into two disjoint subsets V_1 and V_2 (one of which may be empty), such that each negative edge connects two vertices of the same subset and each positive edge connects two vertices of different subsets.*

2.2. Switching equivalence.

A function $\theta : V \to \{+1, -1\}$ is called a switching function. Switching the signature of $\Gamma = (G, \sigma)$ by θ refers to the operation of changing σ to σ^θ via

$$\sigma^\theta(uv) := \theta(u)\sigma(uv)\theta(v), \ \forall \ \{u, v\} \in E.$$

Two signatures σ and σ' are called to be switching equivalent if there exists a switching function θ such that $\sigma' = \sigma^\theta$. We write $\sigma \approx \sigma'$ in this case. Switching equivalence is an equivalence relation on signatures of a fixed underlying graph. We call the corresponding equivalent classes the *switching classes*, and denote the switching class of σ by $[\sigma]$.

Stated differently, switching σ by θ means reversing the signs of all edges between the set $V_\theta^- := \{u \in V : \theta(u) = -1\}$ and its complement. Therefore, we also refer to this operation as *switching the subset* V_θ^- of V. Given $v \in V$, define $\theta_v(u) = -1$ if $u = v$ and $+1$ otherwise. A vertex switching at v, i.e. switching the vertex v, means switching σ by θ_v. Note that $\theta = \prod_{v \in V_\theta^-} \theta_v$; thus, switching a subset of V is equivalent to switching every vertex in it one after another.

Zaslavsky [49] proved the following useful characterization.

Theorem 10 (Zaslavsky's switching lemma)**.** *A signed graph $\Gamma = (G, \sigma)$ is balanced if and only if σ is switching equivalent to the all-positive signature, and it is antibalanced if and only if σ is switching equivalent to the all-negative signature.*

A significant invariant of a switching class is the spectrum (see e.g. [50]). Let $D(\theta)$ be the diagonal matrix with $D(\theta)_{uu} = \theta(u)$. It is then easily checked that

$$A^{\sigma^\theta} = D(\theta)^{-1} A^\sigma D(\theta). \tag{2.1}$$

Therefore, we have the following fundamental property for the spectrum of Δ^σ or L^σ.

Proposition 1. *The spectrum of Δ^σ or L^σ for a signed graph $\Gamma = (G, \sigma)$ is switching invariant.*

For more details and history about switching, see [50] and the references therein.

2.3. **Basic spectral theory.** The operator form of Δ^σ can be expressed by its action on any function $f : V \to \mathbb{R}$ and any $u \in V$ as

$$\Delta^\sigma f(u) = \frac{1}{\mu_d(u)} \sum_{v, v\sim u} w_{uv}(f(u) - \sigma(uv)f(v)). \tag{2.2}$$

Replacing μ_d above by the constant measure $\mu_1 \equiv 1$ yields the operator form for L^σ. For a general measure μ, we denote the corresponding inner product of two functions $f, g : V \to \mathbb{R}$ by

$$(f, g)_\mu = \sum_{u\in V} \mu(u)f(u)g(u).$$

The signed Rayleigh quotient of a map $\Phi : V \to \mathbb{R}^k$ is given by

$$\mathcal{R}^\sigma(\Phi) = \frac{\sum_{u\sim v} w_{uv}\|\Phi(u) - \sigma(uv)\Phi(v)\|^2}{\sum_{u\in V} \mu(u)\|\Phi(u)\|^2}. \tag{2.3}$$

We also define a dual version of the Rayleigh quotient of Φ by

$$\widetilde{\mathcal{R}}^\sigma(\Phi) = \frac{\sum_{u\sim v} w_{uv}\|\Phi(u) + \sigma(uv)\Phi(v)\|^2}{\sum_{u\in V} \mu(u)\|\Phi(u)\|^2}. \tag{2.4}$$

The Courant-Fisher-Weyl min-max principle says that the k-th eigenvalue λ_k of Δ^σ (or L^σ) satisfies

$$\lambda_k = \min_{\substack{f_1, f_2, \ldots, f_k \not\equiv 0 \\ (f_i, f_j)_\mu = 0, \forall i \neq j}} \max_{\substack{f \not\equiv 0 \\ f \in \mathrm{span}\{f_1, f_2, \ldots, f_k\}}} \mathcal{R}^\sigma(f). \tag{2.5}$$

In particular, we have

$$\lambda_1(\Delta^\sigma) = \min_{f \not\equiv 0} \mathcal{R}^\sigma(f), \text{ and } 2 - \lambda_N(\Delta^\sigma) = \min_{f \not\equiv 0} \widetilde{\mathcal{R}}^\sigma(f). \tag{2.6}$$

Lemma 1. *For any $1 \leq k \leq N$, it holds that $2 - \lambda_{N-k+1}(\Delta^\sigma) = \lambda_k(\Delta^{-\sigma})$.*

This follows immediately from the fact that $\widetilde{\mathcal{R}}^\sigma(f) = \mathcal{R}^{-\sigma}(f)$. The support of a map Φ is defined as

$$\mathrm{supp}(\Phi) := \{u \in V : \Phi(u) \neq 0\}.$$

By (2.5), one can derive the following lemma (see e.g. [31]).

Lemma 2. *For any k disjointly supported functions $f_1, f_2, \ldots, f_k : V \to \mathbb{R}$,*

$$\lambda_k \leq 2 \max_{1\leq i\leq k} \mathcal{R}^\sigma(f_i). \tag{2.7}$$

3. (Multi-way) Signed Cheeger constants

In this section, we discuss the properties of the signed Cheeger constant $h_1^\sigma(\mu)$ and define the corresponding multi-way signed Cheeger constants.

First, we prove the switching invariance of $h_1^\sigma(\mu)$.

Proposition 2. *Let $\Gamma = (G, \sigma)$ be a signed graph. For any switching function $\theta : V \to \{+1, -1\}$,*

$$h_1^\sigma(\mu) = h_1^{\sigma^\theta}(\mu). \tag{3.1}$$

This property is a direct corollary of the following lemma.

Lemma 3. *For any switching function $\theta : V \to \{+1, -1\}$ and any sub-bipartition (V_1, V_2), there exists a sub-bipartition (V_1', V_2'), such that $V_1' \cup V_2' = V_1 \cup V_2$, and*

$$\beta^{\sigma^\theta}(V_1', V_2') = \beta^\sigma(V_1, V_2). \tag{3.2}$$

Proof. We only need to prove the lemma for a vertex switching at $u \in V$, that is, a switching of σ by θ_u. If $u \in \overline{V_1 \cup V_2}$, the vertex switching at u does not change the signed bipartiteness ratio; hence choosing $V_1' = V_1$ and $V_2' = V_2$ gives (3.2). Suppose, on the other hand, that $u \in V_1 \cup V_2$. W.l.o.g., we suppose $u \in V_1$. After the vertex switching at u, we have

$$\begin{aligned}&(\beta^{\sigma^{\theta_u}}(V_1, V_2) - \beta^\sigma(V_1, V_2))\mathrm{vol}_\mu(V_1 \cup V_2)\\ =&2\sum_{\substack{v\in V_2\\ \sigma(uv)=-1}} w_{uv} - 2\sum_{\substack{v\in V_2\\ \sigma(uv)=+1}} w_{uv} + 2\sum_{\substack{v\in V_1\\ \sigma(uv)=+1}} w_{uv} - 2\sum_{\substack{v\in V_1\\ \sigma(uv)=-1}} w_{uv}.\end{aligned}$$

Then we move u from V_1 to V_2, i.e. we choose $V_1' = V_1 \setminus \{u\}$ and $V_2' = V_2 \cup \{u\}$. Now we calculate

$$\begin{aligned}&(\beta^{\sigma^{\theta_u}}(V_1', V_2') - \beta^{\sigma^{\theta_u}}(V_1, V_2))\mathrm{vol}_\mu(V_1' \cup V_2')\\ =&-2\sum_{\substack{v\in V_1\\ \sigma^{\theta_u}(uv)=-1}} w_{uv} + 2\sum_{\substack{v\in V_1\\ \sigma^{\theta_u}(uv)=+1}} w_{uv} - 2\sum_{\substack{v\in V_2\\ \sigma^{\theta_u}(uv)=+1}} w_{uv} + 2\sum_{\substack{v\in V_2\\ \sigma^{\theta_u}(uv)=-1}} w_{uv}.\end{aligned}$$

Combining the above two equalities, we arrive at (3.2). □

We recall the bipartiteness ratio of Trevisan [47] given by

$$\begin{aligned}\beta :=& \min_{(V_1,V_2)} \frac{2|E^+(V_1,V_2)| + |E^-(V_1)| + |E^-(V_2)| + |E(V_1\cup V_2, \overline{V_1\cup V_2})|}{\mathrm{vol}_\mu(V_1\cup V_2)}\\ =& 1 - \max_{(V_1,V_2)} \frac{2|E(V_1,V_2)|}{\mathrm{vol(V_1\cup V_2)}} := 1 - \overline{h},\end{aligned}$$

where $\overline{h}$ is the dual Cheeger constant of Bauer and Jost [6]. It is easy to see that if $\sigma \approx \sigma_-$, i.e. if $\Gamma = (G, \sigma)$ is antibalanced, then $h_1^\sigma(\mu)$ reduces to $\beta = 1 - \overline{h}$.

The expansion (or conductance) of a subset $S \subseteq V$ is defined as

$$\rho(S) := \frac{|E(S, \overline{S})|}{\mathrm{vol}_\mu(S)}. \tag{3.3}$$

We define a signed expansion of $S \subseteq V$ in Γ to be

$$\rho^\sigma(S) := \frac{|E^-(S)| + |E(S, \overline{S})|}{\mathrm{vol}_\mu(S)}. \tag{3.4}$$

We have the following relations between $h_1^\sigma(\mu)$ and signed expansions.

Corollary 1. *Let $\Gamma = (G, \sigma)$ be a signed graph. Then,*

$$h_1^\sigma(\mu) = \min_{\sigma' \in [\sigma]} \min_{\emptyset \neq S \subseteq V} \rho^{\sigma'}(S). \tag{3.5}$$

Proof. We denote by $(V_1, V_2)_S$ a bipartition of S, i.e., $V_1 \cup V_2 = S$, $V_1 \cap V_2 = \emptyset$. We claim that

$$\min_{(V_1,V_2)_S} \beta^\sigma(V_1, V_2) = \min_{\sigma' \in [\sigma]} \rho^{\sigma'}(S). \tag{3.6}$$

Let V_1^0, V_2^0 be the bipartition of S which achieves the minimum in the l.h.s. of (3.6). Suppose that σ is changed to be σ_0 by switching the subset V_1^0. Then the proof of Lemma 3 gives

$$\beta^\sigma(V_1^0, V_2^0) = \beta^{\sigma_0}(S, \emptyset) = \rho^{\sigma_0}(S) \geq \min_{\sigma' \in [\sigma]} \rho^{\sigma'}(S). \tag{3.7}$$

Moreover, the inequality above can only be an equality. For otherwise, there would exist a $\sigma' \in [\sigma]$, such that

$$\beta^{\sigma'}(S, \emptyset) = \rho^{\sigma'}(S) < \beta^\sigma(V_1^0, V_2^0).$$

By Lemma 3, we could then find a bipartition V_1', V_2' of S, such that

$$\beta^{\sigma'}(S, \emptyset) = \beta^\sigma(V_1', V_2') < \beta^\sigma(V_1^0, V_2^0),$$

which is a contradiction. Hence (3.6) holds. Then (3.5) follows directly. □

Therefore, when $\sigma \approx \sigma_+$, i.e. when $\Gamma = (G, \sigma)$ is balanced, $h_1^\sigma(\mu)$ reduces to the one-way Cheeger constant, which trivially vanishes.

Desai and Rao [19] introduced the non-bipartiteness parameter

$$\alpha := \min_{\emptyset \neq S \subseteq V} \frac{e_{\min}(S) + |E(S, \overline{S})|}{\mathrm{vol}_\mu(S)}, \tag{3.8}$$

where $e_{\min}(S)$ is the minimum number of edges that need to be removed from the induced subgraph of S to make it bipartite. Hou [25] extend this notion to a signed graph $\Gamma = (G, \sigma)$ as

$$\alpha^\sigma := \min_{\emptyset \neq S \subseteq V} \frac{e^\sigma_{\min}(S) + |E(S, \overline{S})|}{\mathrm{vol}_\mu(S)}, \tag{3.9}$$

where $e^\sigma_{\min}(S)$ is the same as in (1.6). By definition, (G, σ_-) is balanced if and only if G has no odd cycles, i.e. G is bipartite. Therefore, $\alpha^{\sigma_-} = \alpha$.

For a subset S, we modify the above notion as

$$\overline{\alpha}^\sigma(S) := \frac{2e^\sigma_{\min}(S) + |E(S, \overline{S})|}{\mathrm{vol}_\mu(S)}. \tag{3.10}$$

Corollary 2. *Let $\Gamma = (G, \sigma)$ be a signed graph. Then*

$$h_1^\sigma(\mu) = \min_{\emptyset \neq S \subseteq V} \overline{\alpha}^\sigma(S). \tag{3.11}$$

Thus the constant $h_1^\sigma(\mu)$ is larger than α^σ in general. Corollary 2 follows directly from the following lemma.

Lemma 4. *For any $\emptyset \neq S \subseteq V$, we have*

$$\min_{(V_1,V_2)_S} \beta^\sigma(V_1,V_2) = \min_{\sigma'\in[\sigma]} \rho^{\sigma'}(S) = \overline{\alpha}^\sigma(S). \tag{3.12}$$

Proof. The first equality follows from (3.6). To prove the second equality, let Γ_S denote the induced signed graph of S. Also let σ_0 be the signature that achieves $\min_{\sigma'\in[\sigma]}\rho^{\sigma'}(S)$. It is easy to see that

$$2e^\sigma_{min}(S) \leq |E^-(S)|(\sigma').$$

Therefore, we obtain $\overline{\alpha}^\sigma(S) \leq \min_{\sigma'\in[\sigma]}\rho^{\sigma'}(S)$.

Let Γ'_S be the balanced graph obtained from Γ_S by deleting $e^\sigma_{\min}(S)$ edges. By Theorem 8, there exists a bipartition V_1, V_2 of S such that

$$|E^+_{|\Gamma'_S}(V_1,V_2)| = |E^-_{|\Gamma'_S}(V_1)| = |E^-_{|\Gamma'_S}(V_2)| = 0.$$

This implies

$$2e^\sigma_{\min}(S) = 2|E^+_{|\Gamma_S}(V_1,V_2)| + |E^-_{|\Gamma_S}(V_1)| + |E^-_{|\Gamma_S}(V_2)|.$$

Hence $\overline{\alpha}^\sigma(S) \geq \min_{(V_1,V_2)_S}\beta^\sigma(V_1,V_2)$. This proves the second equality. □

Remark 1. *The equality $2e^\sigma_{\min}(S) = \min_{\sigma'\in[\sigma]}|E^-(S)|(\sigma')$ seems to be folklore for experts; see Theorem 3.3 in [51]. We include a proof here for completeness.*

We can compare our constants with the *degree of balance* $b(\Gamma)$ of a signed graph Γ introduced by Cartwright and Harary [14]. In [14], they also aimed at quantifying the deviation of a signed graph from being balanced. Their constant $b(\Gamma)$ is defined as

$$b(\Gamma) := \frac{\text{the number of positive cycles of } \Gamma}{\text{the number of cycles of } \Gamma}. \tag{3.13}$$

Observe $b(\Gamma) \in [0,1]$. Smaller values of $1-b(\Gamma)$ imply that Γ is closer to being balanced. Consider the signed graph $\Gamma = (\mathcal{C}_N, \sigma)$ where $\mathcal{C}_N$ is the unweighted cycle graph on N vertices and σ is the signature such that Γ has exactly one negative edge. Intuitively, Γ is close to being balanced. Actually, we have

$$1-b(\Gamma) = 1 \text{ and } h_1^\sigma(\mu_d) = \frac{1}{N}. \tag{3.14}$$

This shows that the constant $h_1^\sigma(\mu_d)$ is finer than $b(\Gamma)$.

We now define the multi-way signed Cheeger constants.

Definition 2. Given $1 \leq k \leq N$, the k-way signed Cheeger constant $h_k^\sigma(\mu)$ of a signed graph $\Gamma = (G,\sigma)$ is defined as

$$h_k^\sigma(\mu) := \min_{\{(V_{2i-1},V_{2i})\}_{i=1}^k} \max_{1\leq i\leq k} \beta^\sigma(V_{2i-1},V_{2i}). \tag{3.15}$$

where the minimum is taken over the space of all possible k pairs of disjoint sub-bipartitions $(V_1,V_2),(V_3,V_4),\ldots,(V_{2k-1},V_{2k})$. To ease the notation, we denote this space by $\mathrm{Pair}(k)$ and call every element of $\mathrm{Pair}(k)$ a k-sub-bipartition of V.

Note that we have the monotonicity $h_k^\sigma(\mu) \leq h_{k+1}^\sigma(\mu)$. Moreover, Theorem 8 implies the following property.

Proposition 3. *For a signed graph $\Gamma = (G, \sigma)$ and $1 \leq k \leq N$, we have $h_k^\sigma(\mu) = 0$ if and only if Γ has k balanced connected components.*

Roughly speaking, the k-way signed Cheeger constant is a "mixture" of the k-way Cheeger constant $h_k(\mu)$ introduced by Miclo [39] (see also [32]) and the k-way dual Cheeger constant $\overline{h}_k(\mu)$ in [35] for unsigned graphs.

By Lemma 3, $h_k^\sigma(\mu)$ is switching invariant for any $1 \leq k \leq N$. Furthermore, Lemma 4 implies the following equivalent expressions for $h_k^\sigma(\mu)$:

$$\begin{aligned} h_k^\sigma(\mu) &= \min_{\sigma' \in [\sigma]} \min_{\{S_i\}_{i=1}^k} \max_{1 \leq i \leq k} \rho^{\sigma'}(S_i) \\ &= \min_{\{S_i\}_{i=1}^k} \max_{1 \leq i \leq k} \overline{\alpha}^\sigma(S_i), \end{aligned}$$

where the minimum $\min_{\{S_i\}_{i=1}^k}$ is taken over the space of all possible k-subpartitions, $S_1, S_2, \ldots, S_k$, where $S_i \neq \emptyset$ for any $1 \leq i \leq k$.

4. Signed Cheeger inequality

In this section, we prove Theorem 1. The lower bound estimate in (1.8) is easier. For any (V_1, V_2), we choose a particular function given by,

$$f(u) = \begin{cases} 1, & \text{if } u \in V_1; \\ -1, & \text{if } u \in V_2; \\ 0, & \text{otherwise.} \end{cases}$$

We calculate

$$\begin{aligned} \mathcal{R}^\sigma(f) &= \frac{4|E^+(V_1, V_2)| + 2|E^-(V_1)| + 2|E^-(V_2)| + |E(V_1 \cup V_2, \overline{V_1 \cup V_2})|}{\mathrm{vol}_\mu(V_1 \cup V_2)} \\ &\leq 2\beta^\sigma(V_1, V_2). \end{aligned}$$

Then (2.6) implies $\lambda_1 \leq 2h_1^\sigma(\mu)$.

The upper bound estimate in (1.8) is essential. To prove it, we adapt an idea of Trevisan [47] for proving the dual Cheeger inequality for unsigned graphs.

Given a non-zero function $f : V \to \mathbb{R}$ and a real number $t \geq 0$, we define the following subsets of V,

$$V_f(t) := \{u \in V : f(u) \geq t\}, \qquad V_f(-t) := \{u \in V : f(u) \leq -t\}.$$

Suppose $\max_{u \in V} f(u)^2 = 1$. For any $t \in [0, 1]$, we define the vector $Y(t) \in \{-1, 0, 1\}^V$ as

$$Y_u(t) = \begin{cases} 1, & \text{if } u \in V_f(\sqrt{t}); \\ -1, & \text{if } u \in V_f(-\sqrt{t}); \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma is crucial for our purpose.

Lemma 5. *For any $\{u,v\}\in E$, we have*

$$\int_0^1 |Y_u(t)-\sigma(uv)Y_v(t)|\,dt \le |f(u)-\sigma(uv)f(v)|(|f(u)|+|f(v)|). \tag{4.1}$$

Proof. First observe that we only need to prove that the inequality

$$\int_0^1 |Y_u(t)-Y_v(t)|\,dt \le |f(u)-f(v)|(|f(u)|+|f(v)|), \tag{4.2}$$

holds for any two real numbers $f(u), f(v)\in[-1,1]$, since we can then apply (4.2) to the two numbers $g(u):=f(u)$ and $g(v):=\sigma(uv)f(v)\in[-1,1]$ to obtain (4.1).

Now we prove (4.2). W.l.o.g., we suppose $|f(u)|\ge|f(v)|$.

Case 1: $f(u)$ and $f(v)$ have different signs. We have

$$|Y_u(t)-Y_v(t)| = \begin{cases} 2, & \text{if } t\le f(v)^2; \\ 1, & \text{if } f(v)^2<t\le f(u)^2; \\ 0, & \text{if } t>f(u)^2. \end{cases}$$

Therefore,

$$\begin{aligned}\int_0^1 |Y_u(t)-Y_v(t)|\,dt &= f(u)^2+f(v)^2 \le (|f(u)|+|f(v)|)^2 \\ &\le |f(u)-f(v)|(|f(u)|+|f(v)|).\end{aligned}$$

Case 2: $f(u)$ and $f(v)$ have the same sign. We have

$$|Y_u(t)-Y_v(t)| = \begin{cases} 0, & \text{if } t\le f(v)^2; \\ 1, & \text{if } f(v)^2<t\le f(u)^2; \\ 0, & \text{if } t>f(u)^2. \end{cases}$$

Therefore,

$$\begin{aligned}\int_0^1 |Y_u(t)-Y_v(t)|\,dt &= f(u)^2-f(v)^2 \\ &= |f(u)-f(v)|\,(|f(u)|+|f(v)|).\end{aligned}$$

□

Furthermore, one can check $\int_0^1 |Y_u(t)|\,dt = f(u)^2$. Then we obtain

$$\begin{aligned} I := &\frac{\int_0^1 \sum_{u\sim v} w_{uv}|Y_u(t)-\sigma(uv)Y_v(t)|dt}{\int_0^1\sum_{u\in V}\mu(u)|Y_u(t)|dt} \\ \le &\frac{\sum_{u\sim v} w_{uv}|f(u)-\sigma(uv)f(v)|(|f(u)|+|f(v)|)}{\sum_u \mu(u)f(u)^2} \\ \le &\frac{\sqrt{\sum_{u\sim v} w_{uv}|f(u)-\sigma(uv)f(v)|^2}\sqrt{\sum_{u\sim v} w_{uv}(|f(u)|+|f(v)|)^2}}{\sum_u \mu(u)f(u)^2}. \end{aligned}$$

Observing that

$$\sum_{u\sim v} w_{uv}(|f(u)|+|f(v)|)^2 \le 2\max_u\left\{\frac{\sum_{v,v\sim u} w_{uv}}{\mu(u)}\right\}\sum_u \mu(u)f(u)^2, \tag{4.3}$$

we arrive at

$$I \leq \sqrt{\max_u \left\{ \frac{\sum_{v,v\sim u} w_{uv}}{\mu(u)} \right\}} \sqrt{2\mathcal{R}^\sigma(f)}.$$

In the following, we write $d_\mu^w := \max_u \left\{ \frac{\sum_{v,v\sim u} w_{uv}}{\mu(u)} \right\}$ for short. It is easy to check that

$$\sum_{u\sim v} w_{uv}|Y_u(t) - \sigma(uv)Y_v(t)| = 2|E^+(V_f(\sqrt{t}), V_f(-\sqrt{t}))| + |E^-(V_f(\sqrt{t}))|$$
$$+ |E^-(V_f(-\sqrt{t}))| + |E(V_f(\sqrt{t}) \cup V_f(-\sqrt{t}), \overline{V_f(\sqrt{t}) \cup V_f(-\sqrt{t})})|,$$

and $\sum_u \mu(u)|Y_u(t)| = \mathrm{vol}_\mu(V_f(\sqrt{t}) \cup V_f(-\sqrt{t}))$. Hence, we have proved the following lemma.

Lemma 6. *For any non-zero function $f : V \to \mathbb{R}$, there exists a $t' \in [0, \max_{u\in V} f^2(u)]$ such that*

$$\beta^\sigma(V_f(\sqrt{t'}), V_f(-\sqrt{t'})) \leq \sqrt{2d_\mu^w \mathcal{R}^\sigma(f)}. \tag{4.4}$$

The upper bound estimate of (1.8) is then a direct corollary of Lemma 6 by taking the function to be the first eigenfunction ϕ_1 and $\mu = \mu_d$.

By assigning $\mu = \mu_1$, we obtain an estimate for signed non-normalized Laplace matrix.

Theorem 11. *Let $\Gamma = (G, \sigma)$ be a signed graph. Then*

$$\frac{\lambda_1(L^\sigma)}{2} \leq h_1^\sigma(\mu_1) \leq \sqrt{2d_{max}\lambda_1(L^\sigma)}. \tag{4.5}$$

where $d_{max} = \max_{u\in V} d_u$.

Remark 2. *If we adapt a constructive method of Hou [25] (see Theorem 3.4 there), which is extended from Desai and Rao [19], we can obtain slightly stronger estimates. In fact, the following lemma can be proved.*

Lemma 7. *For any non-zero function $f : V \to \mathbb{R}$, there exist two subsets V_1, $V_2 \subseteq \mathrm{supp}(f)$ such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 \neq \emptyset$, and*

$$\mathcal{R}^\sigma(f) \geq d_\mu^w - \sqrt{(d_\mu^w)^2 - \beta^\sigma(V_1, V_2)}.$$

This implies $\lambda_1(\Delta^\sigma) \geq 1 - \sqrt{1 - h_1^\sigma(\mu)^2} \geq h_1^\sigma(\mu)^2/2$.

5. Higher-order signed Cheeger inequalities

In this section, we prove Theorem 2.

Again, the lower bound estimate in (1.9) is easier. Given $\{(V_{2i-1}, V_{2i})\}_{i=1}^k \in \mathrm{Pair}(k)$, we define for each i,

$$f_i(u) = \begin{cases} 1, & \text{if } u \in V_{2i-1}; \\ -1, & \text{if } u \in V_{2i}; \\ 0, & \text{otherwise.} \end{cases}$$

Let $f = \sum_{i=1}^k a_i f_i, a_1, \ldots, a_k \in \mathbb{R}$ be a function in $\mathrm{span}\{f_1, \ldots, f_k\}$. Recall

$$\mathcal{R}^\sigma(f) = \frac{\sum_{u\sim v} w_{uv}(f(u) - \sigma(uv)f(v))}{\sum_u \mu(u)f(u)^2}.$$

We have

$$\sum_u \mu(u)f(u)^2 = \sum_{i=1}^k a_i^2 \sum_u \mu(u)f_i(u)^2 = \sum_{i=1}^k a_i^2 \mathrm{vol}_\mu(V_{2i-1}\cup V_{2i}).$$

It is also not hard to check that

$$\begin{aligned}\sum_{u\sim v} w_{uv}(f(u)-\sigma(uv)f(v)) \leq & 2\sum_{i=1}^k a_i^2(2|E^+(V_{2i-1},V_{2i})| + |E^-(V_{2i-1})| \\ & + |E^-(V_{2i-1})| + |E(V_{2i-1}\cup V_{2i}, \overline{V_{2i-1}\cup V_{2i}})|).\end{aligned}$$

Hence,

$$\max_{a_1,\ldots,a_k} \mathcal{R}^\sigma(f) \leq 2\max_{1\leq i\leq k}\beta^\sigma(V_{2i-1},V_{2i}). \tag{5.1}$$

By (2.5), we arrive at $\lambda_k \leq 2h_k^\sigma(\mu)$.

We now prove the remaining upper bound estimate of (1.9). Let $\phi_1,\phi_2,\ldots,\phi_N$ be an orthonormal system of eigenfunctions corresponding to $\lambda_1,\lambda_2,\ldots,\lambda_N$, respectively. We construct the map

$$\Phi : V\to\mathbb{R}^k : v\mapsto(\phi_1(v),\phi_2(v),\ldots,\phi_k(v)).$$

Since $\lambda_i = \mathcal{R}^\sigma(\phi_i)$, $i=1,2,\ldots,k$, we have

$$\lambda_k \geq \frac{\sum_{u\sim v} w_{uv}\|\Phi(u)-\sigma(uv)\Phi(v)\|^2}{\sum_{u\in V}\mu(u)\|\Phi(u)\|^2} = \mathcal{R}^\sigma(\Phi). \tag{5.2}$$

In the following, we try to find k disjointly supported maps $\{\Psi_i\}_{i=1}^k$ by localizing Φ, such that $\mathcal{R}^\sigma(\Psi_i)$ can be bounded above by $\mathcal{R}^\sigma(\Phi_i)$ (up to a polynomial of k). Recall the pseudometric space $(\widetilde{V}_\Phi, d_\Phi)$ from Section 1.3. In order to localize Φ, we need the following cut-off function: Given $S_i\subseteq V$ and $\epsilon>0$, define

$$\theta_i(v) = \begin{cases} 0, & \text{if } \Phi(v)=0; \\ \max\left\{0, 1-\frac{d_\Phi(v,S_i\cap\widetilde{V}_\Phi)}{\epsilon}\right\}, & \text{otherwise,}\end{cases}$$

through which we can localize Φ as

$$\Psi_i := \theta_i\Phi : V\to\mathbb{R}^k.$$

Observe that $\Psi_i|_{S_i} = \Phi|_{S_i}$ and

$$\mathrm{supp}(\Psi_i)\subseteq N_\epsilon(S_i\cap\widetilde{V}_\Phi, d_\Phi) := \{v\in\widetilde{V}_\Phi : d_\Phi(v,S_i\cap\widetilde{V}_\Phi)<\epsilon\}.$$

We have the following important lemma which is an extension of Lemma 5.3 in [35] and Lemma 3.3 in [32].

Lemma 8. *Given $0<\epsilon<2$, Let Ψ_i be defined as above. Then for any $\{u,v\}\in E$,*

$$\|\Psi_i(u)-\sigma(uv)\Psi_i(v)\| \leq \left(1+\frac{2}{\epsilon}\right)\|\Phi(u)-\sigma(u,v)\Phi(v)\|. \tag{5.3}$$

Proof. If either $\Phi(u)$ or $\Phi(v)$ vanishes, (5.3) follows from the fact that $|\theta_i|\leq 1$. So we only need to prove (5.3) for $u,v\in\widetilde{V}_\Phi$.

We calculate

$$\|\Psi_i(u) - \sigma(uv)\Psi_i(v)\| = \|\theta_i(u)F(u) - \sigma(uv)\theta_i(v)F(v)\|$$
$$\leq |\theta_i(u)|\|\Phi(u) - \sigma(uv)\Phi(v)\| + |\theta_i(u) - \theta_i(v)|\|\Phi(v)\|. \tag{5.4}$$

We claim that

$$|\theta_i(u) - \theta_i(v)|\|\Phi(v)\| \leq \frac{2}{\epsilon}\|\Phi(u) - \sigma(uv)\Phi(v)\|. \tag{5.5}$$

Note that (5.3) follows immediately from (5.4) and (5.5). Hence, the remaining task is proving (5.5). By a similar argument as in the beginning of the proof for Lemma 5, we only need to show

$$|\theta_i(u) - \theta_i(v)|\,\|\Phi(v)\| \leq \frac{2}{\epsilon}\|\Phi(u) - \Phi(v)\| \tag{5.6}$$

for any two vectors $\Phi(u), \Phi(v) \in \mathbb{R}^k \setminus \{0\}$. This is proved in the following two cases.

Case 1: If $\{u, v\}$ satisfies $d_\Phi(u, v) = \left\|\frac{F(u)}{\|F(u)\|} - \frac{F(v)}{\|F(v)\|}\right\|$, then $\langle\Phi(u), \Phi(v)\rangle \geq 0$, where $\langle\cdot, \cdot\rangle$ stands for the inner product of vectors in $\mathbb{R}^k$. Hence,

$$|\theta_i(u) - \theta_i(v)|\|\Phi(v)\| \leq \frac{1}{\epsilon}d_\Phi(u, v)\|\Phi(v)\| = \frac{1}{\epsilon}\left\|\frac{\|\Phi(v)\|}{\|\Phi(u)\|}\Phi(u) - \Phi(v)\right\|$$
$$\leq \frac{1}{\epsilon}\|\Phi(u) - \Phi(v)\| + \frac{1}{\epsilon}\left\|\frac{\|\Phi(v)\|}{\|\Phi(u)\|}\Phi(u) - \Phi(u)\right\|$$
$$\leq \frac{1}{\epsilon}\|\Phi(u) - \Phi(v)\| + \frac{1}{\epsilon}\left|\|\Phi(v)\| - \|\Phi(u)\|\right| \leq \frac{2}{\epsilon}\|\Phi(u) - \Phi(v)\|.$$

Case 2: If $\{u, v\} \in E$ satisfies $d_\Phi(u, v) = \left\|\frac{F(u)}{\|F(u)\|} + \frac{F(v)}{\|F(v)\|}\right\|$, then $\langle\Phi(u), \Phi(v)\rangle \leq 0$. Thus,

$$|\theta_i(u) - \theta_i(v)|\|\Phi(v)\| \leq \|\Phi(v)\| \leq \|\Phi(u) - \Phi(v)\|.$$

□

Applying the padded random partition theory to $(\widetilde{V}_\Phi, d_\Phi)$ (see e.g. [35]), we can find k disjoint subsets of $\widetilde{V}_\Phi$ with good properties.

Lemma 9. *There exist k non-empty, mutually disjoint subsets $S_1, S_2, \ldots, S_k \subseteq \widetilde{V}_\Phi$ and an absolute constant $C_0 > 1$, such that*

- *for any $1 \leq i \neq j \leq k$,*
$$d_\Phi(S_i, S_j) \geq \frac{1}{C_0 k^{\frac{5}{2}}}; \tag{5.7}$$
- *for any $1 \leq i \leq k$,*
$$\sum_{u \in S_i} \mu(u)\|\Phi(u)\|^2 \geq \frac{1}{2k}\sum_{u \in V}\mu(u)\|\Phi(u)\|^2. \tag{5.8}$$

Since the signature plays no role in this lemma, we refer to [35, Section 6] for the proof. Combining Lemma 8 and Lemma 9, we arrive at the following result.

Lemma 10. *For any $k \in \{1, 2, \dots, N\}$, there exist k disjointly supported functions $\psi_1, \psi_2, \dots, \psi_k : V \to \mathbb{R}$ such that for each $1 \le i \le k$,*

$$\mathcal{R}^\sigma(\psi_i) \le Ck^6 \mathcal{R}^\sigma(F), \tag{5.9}$$

where C is an absolute constant.

Proof. Let $\{\theta_i\}_{i=1}^k$ be the k cut-off functions corresponding to $\{S_i\}_{i=1}^k$ obtained in Lemma 9, and set $\epsilon = 1/(2C_0 k^{\frac{5}{2}})$. For each i, we define $\Psi_i = \theta_i F$. By Lemma 8,

$$\sum_{u\sim v} w_{uv} \|\Psi_i(u) - \sigma(uv)\Psi_i(v)\|^2 \le \left(1 + \frac{2}{\epsilon}\right)^2 \sum_{u\sim v} w_{uv} \|\Phi(u) - \sigma(uv)\Phi(v)\|^2.$$

By Lemma 9,

$$\sum_{u\in V} \mu(u) \|\Psi_i(u)\|^2 \ge \frac{1}{2k} \sum_{u\in V} \mu(u) \|\Phi(u)\|^2.$$

Therefore,

$$\mathcal{R}^\sigma(\Psi_i) \le 2k(1 + 2C_0 k^{\frac{5}{2}})^2 \mathcal{R}^\sigma(\Phi) \le Ck^6 \mathcal{R}^\sigma(\Phi). \tag{5.10}$$

If we write $\Psi_i = (\Psi_i^1, \Psi_i^2, \dots, \Psi_i^k) : V \to \mathbb{R}^k$, we can always find $j_0 \in \{1, 2, \dots, k\}$ such that

$$\mathcal{R}^\sigma(\Psi_i^{j_0}) \le \mathcal{R}^\sigma(\Psi_i).$$

Choosing $\psi_i := \Psi_i^{j_0}$, $1 \le i \le k$, completes the proof. $\square$

Assigning $\mu = \mu_d$, Lemma 6, (5.2) and Lemma 10 imply the lower bound estimate in (1.9). If we assign $\mu = \mu_1$ instead, we obtain the following estimate for L^σ.

Theorem 12. *There exists an absolute constant C such that for any signed graph $\Gamma = (G, \sigma)$ and any $k \in \{1, 2, \dots, N\}$,*

$$\frac{\lambda_k(L^\sigma)}{2} \le h_k^\sigma(\mu_1) \le Ck^3 \sqrt{d_{max} \lambda_k(L^\sigma)}. \tag{5.11}$$

6. Signed Cheeger constants and higher order spectral gaps

In this section, we prove Theorem 3 and Theorem 4. Actually, we shall prove the following slightly stronger result.

Theorem 13. *Given a signed graph $\Gamma = (G, \sigma)$ and $k \in \{1, 2, \dots, N\}$, at least one of the following holds,*

$$\text{(i). } h_1^\sigma(\mu_d) \le 8k\lambda_1(\Delta^\sigma); \quad \text{(ii). } h_1^\sigma(\mu_d) < 16\sqrt{2}k \frac{\lambda_1(\Delta^\sigma)}{\sqrt{\lambda_k(\Delta^\sigma)}}. \tag{6.1}$$

Theorem 3 is a direct corollary of this theorem and the fact that $0 \le \lambda_k(\Delta^\sigma) \le 2$.

We first prove the following crucial lemma, which should be compared with Lemma 6.

Lemma 11. *For any non-zero function $f : V \to \mathbb{R}$, there exists a $t' \in [0, \max_{u\in V} |f(u)|]$, such that*

$$\beta^\sigma(V_f(t'), V_f(-t')) \le \frac{\sum_{u\sim v} w_{uv} |f(u) - \sigma(uv) f(v)|}{\sum_u \mu(u) |f(u)|}. \tag{6.2}$$

Proof. We can assume $\max_{u\in V}|f(u)|=1$ since the r.h.s. of (6.2) in invariant under scaling of f. For $t\in[0,1]$, we define a vector $X(t)\in\{-1,0,1\}^V$ by

$$X_u(t)=\begin{cases}1, & \text{if } f(u)\geq t;\\ -1, & \text{if } f(u)\leq -t;\\ 0, & \text{otherwise.}\end{cases} \tag{6.3}$$

We claim that, for any $\{u,v\}\in E$,

$$\int_0^1 |X_u(t)-\sigma(uv)X_v(t)|dt=|f(u)-\sigma(uv)f(v)|. \tag{6.4}$$

Similar to Lemma 5, we only need to prove

$$\int_0^1 |X_u(t)-X_v(t)|dt=|f(u)-f(v)| \tag{6.5}$$

for any two numbers $f(u),f(v)\in[-1,1]$. W.l.o.g. suppose $|f(u)|\geq|f(v)|$. If $f(u)$ and $f(v)$ have different signs,

$$|X_u(t)-X_v(t)|=\begin{cases}2, & \text{if } t\leq|f(v)|;\\ 1, & \text{if } |f(v)|<t\leq|f(u)|;\\ 0, & \text{if } t>|f(u)|.\end{cases} \tag{6.6}$$

Then, $\int_0^1|X_u(t)-X_v(t)|\,dt=|f(u)|+|f(v)|=|f(u)-f(v)|$. If, on the other hand, $f(u)$ and $f(v)$ have the same sign,

$$|X_u(t)-X_v(t)|=\begin{cases}0, & \text{if } t\leq|f(v)|;\\ 1, & \text{if } |f(v)|<t\leq|f(u)|;\\ 0, & \text{if } t>|f(u)|.\end{cases} \tag{6.7}$$

Then $\int_0^1|X_u(t)-X_v(t)|dt=|f(u)|-|f(v)|=|f(u)-f(v)|$. Therefore (6.4) holds.

Furthermore, $\int_0^1|X_u(t)|dt=|f(u)|$. By a similar argument as in the proof of Lemma 6, there exists a $t'\in[0,1]$ such that

$$\begin{aligned}\beta^\sigma(V_f(t'),V_f(-t'))&\leq\frac{\int_0^1\sum_{u\sim v}w_{uv}|X_u(t)-\sigma(uv)X_v(t)|dt}{\int_0^1\sum_u\mu(u)|X_u|dt}\\&=\frac{\sum_{u\sim v}w_{uv}|f(u)-\sigma(uv)f(v)|}{\sum_u\mu(u)|f(u)|}.\end{aligned}$$

□

For any non-zero function $f:V\to\mathbb{R}$, we define a step function approximation: For $k\in\mathbb{N}$, let

$$0=t_0\leq t_1\leq\cdots\leq t_{2k} \tag{6.8}$$

be a sequence of real numbers with $t_{2k}=\max_{u\in V}|f(u)|$. Define the step function approximation g by

$$g(u)=\psi_{-t_{2k},\dots,-t_1,0,t_1,\dots,t_{2k}}(f(u)):=\arg\min_{t\in\{-t_{2k},\dots,0,\dots,t_{2k}\}}|f(v)-t|. \tag{6.9}$$

That is, $g(u)$ equals one of the constants $\{-t_{2k},\dots,0,\dots,t_{2k}\}$ that is closest to $f(u)$.

We further construct an auxiliary function $F : V \to \mathbb{R}$. First, define $\eta : [-t_{2k}, t_{2k}] \to \mathbb{R}$ by

$$\eta(x) = |x - \psi_{-t_{2k},\ldots,-t_1,0,t_1,\ldots,t_{2k}}(x)|. \tag{6.10}$$

Note that $\eta(-x) = -\eta(x)$. Then for each $u \in V$, we assign

$$F(u) := \int_0^{f(u)} \eta(x)\, dx. \tag{6.11}$$

The function F has the following properties.

Proposition 4. *(i). For any $u \in V$,*

$$|F(u)| \geq \frac{1}{8k}|f(v)|^2. \tag{6.12}$$

(ii). For any $\{u,v\} \in E$,

$$\begin{aligned}|F(u) - \sigma(uv)F(v)| \leq &\frac{1}{2}|f(u) - \sigma(uv)f(v)|(|f(u) - \sigma(uv)f(v)| \\ &+ |f(u) - g(u)| + |f(v) - g(v)|).\end{aligned} \tag{6.13}$$

Proof. (i). First observe that $F(u)$ and $f(u)$ share the same sign. Since $\eta(-x) = -\eta(x)$, we can assume $F(u) > 0$ and $f(u) > 0$ for our purposes. Then the proof can be done as in [31, Claim 3.3]. Since the argument is not long, we recall it here. Suppose $f(u) \in [t_i, t_{i+1}]$ for some i. Then by the Cauchy-Schwarz inequality,

$$\begin{aligned} f(u)^2 &= \left(\sum_{j=0}^{i=1}(t_{j+1} - t_j) + (f(u) - t_i)\right)^2 \\ &\leq 2k\left(\sum_{j=0}^{i=1}(t_{j+1} - t_j)^2 + (f(u) - t_i)^2\right). \end{aligned}$$

Using the definition,

$$\begin{aligned} F(u) &= \sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} \eta(x)\, dx + \int_{t_i}^{f(u)} \eta(x)\, dx \\ &\geq \sum_{j=0}^{i-1}\frac{1}{4}(t_{j+1} - t_j)^2 + \frac{1}{4}(f(u) - t_i)^2 = \frac{1}{8k}f(u)^2. \end{aligned}$$

(ii). Observing the fact that $\int_0^{\sigma(uv)f(v)} \eta(x)\, dx = \sigma(uv)\int_0^{f(v)} \eta(x)\, dx$, we only need to prove (6.13) for the case $\sigma(uv) = +1$. W.l.o.g., suppose that $|f(u)| \geq |f(v)|$. If $f(u)$ and $f(v)$

have different signs, say $f(u) \geq 0$, $f(v) \leq 0$, then, recalling the fact $\eta(-x) = -\eta(x)$,

$$\begin{aligned} |F(u) - F(v)| &= \int_0^{f(u)} \eta(x)\, dx + \int_0^{-f(v)} \eta(x)\, dx \\ &\leq \int_0^{f(u)} x\, dx + \int_0^{-f(v)} x\, dx \\ &= \frac{1}{2}(f(u)^2 + f(v)^2) \leq \frac{1}{2}|f(u) - f(v)|^2. \end{aligned}$$

If, on the other hand, $f(u)$ and $f(v)$ have the same sign, we can assume $f(u) > 0$ and $f(v) > 0$ since $\eta(-x) = -\eta(x)$ and the step function approximation of $-f$ is $-g$. Then,

$$|F(u) - F(v)| = \int_{f(v)}^{f(u)} \eta(x)\, dx \leq |f(u) - f(v)| \cdot \max_{f(v) \leq x \leq f(u)} \eta(x).$$

By definition,

$$\begin{aligned} \max_{f(v) \leq x \leq f(u)} \eta(x) &\leq \min\{|x - g(u)|, |x - g(v)|\} \leq \frac{1}{2}(|x - g(u)| + |x - g(v)|) \\ &\leq \frac{1}{2}\left(|x - f(u)| + |f(u) - g(u)| + |x - f(v)| + |f(v) - g(v)|\right) \\ &= \frac{1}{2}\left(|f(u) - f(v)| + |f(u) - g(u)| + |f(v) - g(v)|\right). \end{aligned}$$

This proves (6.13). □

Using the above properties and Lemma 11, we can derive the following lemma.

Lemma 12. *For any non-zero functions $f : V \to \mathbb{R}$ and an step function approximation g of it constructed from $0 = t_0 \leq t_1, \ldots, t_{2k}$ as above, there exists a $t' \in [0, \max_{u \in V} |f(u)|]$ such that*

$$\beta^\sigma(V_f(t'), V_{-f}(t')) \leq 4k\mathcal{R}^\sigma(f) + 4\sqrt{2}k\frac{\|f - g\|_\mu}{\|f\|_\mu}\sqrt{d_\mu^w \mathcal{R}^\sigma(f)}, \tag{6.14}$$

where $\|f\|_\mu^2 := \sum_{u \in V} \mu(u) f(u)^2$.

Note that the notation $\|\cdot\|_{\mu_1} = \|\cdot\|$ reduces to the Euclidean norm.

Proof. Applying Lemma 11 to the function F, we find a $\bar{t} \in [0, \max_u |F(u)|]$ such that

$$\begin{aligned} &\beta^\sigma(V_F(\bar{t}), V_F(-\bar{t})) \leq \frac{\sum_{u \sim v} w_{uv} |F(u) - \sigma(uv) F(v)|}{\sum_u \mu(u) |F(u)|} \\ &\leq 4k\mathcal{R}^\sigma(f) \\ &+ 4k\frac{\sum_{u \sim v} w_{uv} |f(u) - \sigma(uv) f(v)| (|f(u) - g(u)| + |f(v) - g(v)|)}{\sum_u \mu(u) f(u)^2} \\ &\leq 4k\mathcal{R}^\sigma(f) + 4k\sqrt{\mathcal{R}^\sigma(f)}\frac{\sqrt{\sum_{u \sim v} w_{uv} (|f(u) - g(u)| + |f(v) - g(v)|)^2}}{\sqrt{\sum_u \mu(u) f(u)^2}}, \end{aligned}$$

where we have used Proposition 4 in the second inequality and the Cauchy-Schwarz inequality in the last inequality. The same technique used in (4.3) yields

$$\sum_{u\sim v} w_{uv}|(|f(u)-g(u)|+|f(v)-g(v)|)^2 \leq 2d_\mu^w\|f-g\|_\mu^2. \tag{6.15}$$

Inserting (6.15) into the above calculations and using the fact that $f(u)\geq f(v)$ if and only if $F(u)\geq F(v)$, the proof is completed. □

Now we are prepared for the proof of Theorem 13.

Lemma 13. *For any non-zero function $f:V\to\mathbb{R}$ and any $1\leq k\leq N$, there exists a $t'\in[0,\max_{u\in V}|f(u)|]$, such that at least one of the following estimates holds:*

(i) $\beta^\sigma(V_f(t'),V_f(-t'))\leq 8k\mathcal{R}^\sigma(f)$;

(ii) *there exists k disjointly supported functions $f_1,f_2,\ldots,f_k:V\to\mathbb{R}$ such that for each $1\leq i\leq k$,*

$$\mathcal{R}^\sigma(f_i)<256d_\mu^w k^2\frac{\mathcal{R}^\sigma(f)^2}{\beta^\sigma(V_f(t'),V_f(-t'))^2}. \tag{6.16}$$

Proof. Denote $M:=\max_u|f(u)|$. We construct $2k+1$ real numbers $t_0\leq t_1\leq\cdots\leq t_{2k}\leq M$ as follows: Take $t_0=0$. Suppose that we have already fixed $t_0,t_1,\ldots,t_{i-1}$. Now we try to find $t_i\in[t_{i-1},M]$ such that

$$\sum_{u:-t_i\leq f(u)<-t_{i-1}}\mu(u)|f(u)-\psi_{-t_i,-t_{i-1}}(f(u))|^2 + \sum_{u:t_{i-1}<f(u)\leq t_i}\mu(u)|f(u)-\psi_{t_{i-1},t_i}(f(u))|^2=C, \tag{6.17}$$

where

$$C=\frac{\beta^\sigma(V_f(t'),V_f(-t'))^2\|f\|_\mu^2}{256k^3d_\mu^w\mathcal{R}^\sigma(f)}.$$

Recall that $\psi_{t_{i-1},t_i}(f(u))$ is the closest one of $\{t_{i-1},t_i\}$ to $f(u)$, and note that the l.h.s. of (6.17) is continuous and non-decreasing w.r.t. t_i. If we can find such constants satisfying (6.17), we take the smallest one of them as t_i; otherwise, we set $t_i=M$. This procedure is considered to be successful if $t_{2k}=M$.

If the procedure succeeds, we define a step function g as in (6.9). Then by definition,

$$\|f-g\|_\mu^2\leq 2kC. \tag{6.18}$$

Applying Lemma 12, we arrive at the inequality

$$\beta^\sigma(V_f(t'),V_f(-t'))\leq 4k\mathcal{R}^\sigma(f)+\frac{\beta^\sigma(V_f(t'),V_f(-t'))}{2}.$$

Therefore $h(\Delta)\leq 8k\mathcal{R}^\sigma(f)=8k\lambda_1(\Delta)$. Hence, Theorem 13 (i) holds.

If, on the other hand, the procedure fails, we have $t_{2k}<M$. Then we define the $2k$ disjointly supported functions,

$$f_i(u)=\begin{cases} -|f(u)-\psi_{-t_{i-1},t_i}(f(u))|, & \text{if } -t_i\leq f(u)\leq -t_{i-1};\\ |f(u)-\psi_{t_{i-1},t_i}(f(v))| & \text{if } t_{i-1}<f(u)\leq t_i;\\ 0 & \text{otherwise.}\end{cases} \tag{6.19}$$

for $1 \leq i \leq k$. Recall that (6.17) ensures $\|f_i\|^2 = C$. Next we estimate the Rayleigh quotient for these functions.

Claim 1. *For any* $\{u, v\} \in E$,

$$\sum_{i=1}^{2k} |f_i(u) - \sigma(uv) f_i(v)|^2 \leq |f(u) - \sigma(uv) f(v)|^2. \tag{6.20}$$

As in Lemma 5, we only need to prove the claim when $\sigma(uv) = +1$.

Case 1: u, v lie in the support of the same function f_i. In this case,

$$\sum_{i=1}^{2k} |f_i(u) - f_i(v)|^2 = |f_i(u) - f_i(v)|.$$

If $f(u)$ and $f(v)$ have the same sign, say $f(u) \geq 0, f(v) \geq 0$, then

$$\begin{aligned} |f_i(u) - f_i(v)|^2 &= \big||f(u) - \psi_{t_{i-1},t_i}(f(u))| - |f(v) - \psi_{t_{i-1},t_i}(f(v))|\big|^2 \\ &\leq |f(u) - f(v)|^2. \end{aligned}$$

If $f(u)$ and $f(v)$ have different signs, say $f(u) > 0, f(v) < 0$, then

$$\begin{aligned} |f_i(u) - f_i(v)|^2 &= \big||f(u) - \psi_{t_{i-1},t_i}(f(u))| + |f(v) - \psi_{-t_i,-t_{i-1}}(f(v))|\big|^2 \\ &\leq \big||f(u) - \psi_{t_{i-1},t_i}(f(u))| + |-f(v) - \psi_{t_{i-1},t_i}(-f(v))|\big|^2 \\ &\leq (|f(u) - t_{i-1}| + |-f(v) - t_{i-1}|)^2 \leq |f(u) - f(v)|^2. \end{aligned}$$

Case 2: $u \in \mathrm{supp}(f_i), v \in \mathrm{supp}(f_j)$, where $i \neq j$. We can assume $j > i$. Then,

$$\sum_{i=1}^{2k} |f_i(u) - f_i(v)|^2 = |f_i(u)|^2 + |f_j(v)|^2.$$

If $f(u)$ and $f(v)$ have the same sign, say $f(u) \geq 0, f(v) \geq 0$, then

$$\begin{aligned} |f_i(u)|^2 + |f_j(v)|^2 &= |f(u) - \psi_{t_{i-1},t_i}(f(u))|^2 + |f(v) - \psi_{t_{j-1},t_j}(f(v))|^2 \\ &\leq |f(u) - t_i|^2 + |f(v) - t_i|^2 \leq |f(u) - f(v)|^2. \end{aligned}$$

If $f(u)$ and $f(v)$ have different signs, say $f(u) \geq 0, f(v) \leq 0$, then

$$\begin{aligned} |f_i(u)|^2 + |f_j(v)|^2 &= |f(u) - \psi_{t_{i-1},t_i}(f(u))|^2 + |f(v) - \psi_{-t_j,-t_{j-1}}(f(v))|^2 \\ &= |f(u) - \psi_{t_{i-1},t_i}(f(u))|^2 + |-f(v) - \psi_{t_{j-1},t_j}(-f(v))|^2 \\ &\leq |f(u) - t_{i-1}|^2 + |f(v) - t_{j-1}|^2 \leq |f(u) - f(v)|^2. \end{aligned}$$

Hence, the claim is proved. Now using Claim 1, we calculate

$$\begin{aligned}\sum_{i=1}^{2k}\mathcal{R}^\sigma(f_i) &= \frac{1}{C}\sum_{i=1}^{2k}\sum_{u\sim v}w_{uv}(f_i(u)-\sigma(uv)f_i(v))^2\\ &\leq \frac{1}{C}\sum_{u\sim v}w_{uv}|f(u)-\sigma(uv)f(v)|^2\\ &= 256k^3d_\mu^w\frac{\mathcal{R}^\sigma(f)^2}{\beta^\sigma(V_f(t'),V_f(-t'))^2}.\end{aligned}$$

Then, we can find k functions of $\{f_1, f_2, \ldots, f_{2k}\}$, relabeling them as $f_1, f_2, \ldots, f_k$, such that (6.16) holds for $1 \leq i \leq k$. □

Proof of Theorem 13. Combining Lemma 2 and Lemma 13 yields Theorem 13. □

Proof of Theorem 4. Combining Lemma 10 and Lemma 13, we find an absolute constant C such that for $1 \leq k \leq l \leq N$, at least one the following inequalities holds:

$$(i)\ h_k^\sigma(\mu_d) \leq Clk^6\lambda_k(\Delta^\sigma); \quad (ii)\ h_k^\sigma(\mu_d) \leq Clk^6\lambda_k(\Delta^\sigma)/\sqrt{\lambda_l(\Delta^\sigma)}. \tag{6.21}$$

Theorem 4 then follows. □

When $\mu = \mu_1$, we have the following results for L^σ.

Theorem 14. *Given a signed graph $\Gamma = (G, \sigma)$ and $k \in \{1, 2, \ldots, N\}$, at least one of the following holds:*

$$(i).\ h_1^\sigma(\mu_1) \leq 8k\lambda_1(L^\sigma); \quad (ii).\ h_1^\sigma < 16\sqrt{2d_{\max}}k\frac{\lambda_1(L^\sigma)}{\sqrt{\lambda_k(L^\sigma)}}. \tag{6.22}$$

Recalling that $\lambda_k(L^\sigma) \leq 2d_{\max}$, we further obtain the following corollaries.

Corollary 3. *For a signed graph Γ and $1 \leq k \leq N$,*

$$h_1^\sigma(\mu_1) < 16\sqrt{2d_{max}}k\frac{\lambda_1(L^\sigma)}{\sqrt{\lambda_k(L^\sigma)}}. \tag{6.23}$$

Corollary 4. *There exists an absolute constant C such that for any signed graph Γ and $1 \leq k \leq l \leq N$, we have*

$$h_k^\sigma(\mu_1) < C\sqrt{d_{max}}lk^6\frac{\lambda_k(L^\sigma)}{\sqrt{\lambda_l(L^\sigma)}}. \tag{6.24}$$

7. Signed triangles and the spectral gaps λ_1 and $2-\lambda_N$

In this section, we prove Theorem 6 and Theorem 7. We will present a proof of the weighted version of Theorem 7.

Proof of Theorem 6. We consider an iterated matrix

$$\Delta^\sigma[2] = I - (D^{-1}A^\sigma)^2. \tag{7.1}$$

Then, for any function $f : V \to \mathbb{R}$ and any $u \in V$,

$$\Delta^\sigma[2]f(u) = f(u) - \frac{1}{d_u}\sum_v\sum_{u', u'\sim u, u'\sim v}\frac{w_{u'u}w_{u'v}}{d_{u'}}\sigma(u'u)\sigma(u'v)f(v).$$

Let f_N be the corresponding eigenfunction of $\lambda_N(\Delta^\sigma)$. Then,

$$\frac{(f_N, \Delta^\sigma[2]f_N)_\mu}{(f_N, \Delta^\sigma f_N)_\mu} = \frac{(f_N, [1-(1-\lambda_N(\Delta^\sigma))^2]f_N)_\mu}{(f_N, \lambda_N(\Delta^\sigma)f_N)_\mu} = 2-\lambda_N(\Delta^\sigma). \tag{7.2}$$

Note $\lambda_N(\Delta^\sigma)(f_N, f_N)_\mu \neq 0$, hence the above expression is proper. Furthermore,

$$(f_N, \Delta^\sigma f_N)_\mu = \sum_{u\sim v}(f_N(u) - \sigma(uv)f_N(v))^2, \tag{7.3}$$

and

$$\begin{aligned}
&(f_N, \Delta^\sigma[2]f_N)_\mu \\
&= \sum_u f_N(u) \sum_v \sum_{u', u'\sim u, u'\sim v} \frac{w_{u'u}w_{u'v}}{d_{u'}}(f_N(u) - \sigma(u'u)\sigma(u'v)f_N(v)) \\
&= \sum_{(u,v)} \sum_{u', u'\sim u, u'\sim v} \frac{w_{u'u}w_{u'v}}{d_{u'}}(f_N(u) - \sigma(u'u)\sigma(u'v)f_N(v))^2 \\
&\geq \sum_{u\sim v} \sum_{\substack{u', u'\sim u, u'\sim v\\ \sigma u'u\sigma(u'v)=\sigma(uv)}} \frac{w_{u'u}w_{u'v}}{d_{u'}}(f_N(u) - \sigma(u'u)\sigma(u'v)f_N(v))^2.
\end{aligned}$$

In the above, $\sum_{(u,v)}$ stands for the summation over unordered pair of vertices u, v. Inserting the above estimate and (7.3) into (7.2), we obtain

$$\begin{aligned}
2-\lambda_N(\Delta^\sigma) &\geq \frac{\sum_{u\sim v}\sum_{\substack{u', u'\sim u, u'\sim v\\ \sigma u'u\sigma(u'v)=\sigma(uv)}} \frac{w_{u'u}w_{u'v}}{d_{u'}}(f_N(u) - \sigma(u'u)\sigma(u'v)f_N(v))^2}{\sum_{u\sim v} w_{uv}(f_N(u)-\sigma(uv)f_N(v))^2} \\
&\geq \frac{w^2}{W}\min_{u\sim v} \sum_{\substack{u', u'\sim u, u'\sim v\\ \sigma(u'u)\sigma(u'v)=\sigma(uv)}} \frac{1}{d_{u'}} \geq \frac{w^2}{W}\frac{\min_{u\sim v}\sharp^+(u,v)}{\max_u d_u}.
\end{aligned}$$

Using Lemma 1, the lower bound estimate for $\lambda_1(\Delta^\sigma)$ follows from duality. □

Next, we prove Theorem 7, that is, the analogous result for the matrix L^σ. The techniques we used above do not work for L^σ. We will employ different ideas, which are adapted from Das [18] and Rojo [42]. In fact, we shall prove the following result.

Theorem 15. *Given a signed graph $\Gamma = (G, \sigma)$, we have*

$$\begin{aligned}
\lambda_N(L^\sigma) \leq \frac{1}{2}\max_{u\sim v}\{d_u + d_v &+ \sum_{u', u'\sim u, u'\not\sim v} w_{u'u} + \sum_{\substack{u', u'\sim u, u'\sim v\\ \sigma(u'u)\sigma(u'v)=-\sigma(uv)}} w_{u'u} \\
&+ \sum_{u', u'\sim v, u'\not\sim u} w_{u'v} + \sum_{\substack{u', u'\sim u, u'\sim v\\ \sigma(u'u)\sigma(u'v)=-\sigma(uv)}} w_{u'v} \\
&+ \sum_{\substack{u', u'\sim u, u'\sim v\\ \sigma(u'u)\sigma(u'v)=\sigma(uv)}} |w_{u'u} - w_{u'v}|\}.
\end{aligned} \tag{7.4}$$

Remark 3. *Theorem 7 is a direct corollary of this theorem.*

Proof of Theorem 15. Let f_N be the eigenfunction corresponding to $\lambda_N(L^\sigma)$. W.l.o.g., we suppose $f_N(u) = \max_{u'\in V} |f_N(u')|$, and

$$\sigma(uv)f_N(v) = \min_{u',u'\sim u} \sigma(uu')f_N(u').$$

First, observe $f_N(u) - \sigma(uv)f_N(v) \neq 0$. (Because otherwise $\sigma(uu')f_N(u') = f_N(u)$ for any u' such that $u' \sim u$, which implies $\lambda_N(L^\sigma)f_N(u) = L^\sigma f_N(u) = 0$, a contradiction.) Then calculate

$$\begin{aligned}
&\lambda_N(L^\sigma)(f_N(u) - \sigma(uv)f_N(v)) = L^\sigma f_N(u) - \sigma(uv)L^\sigma f_N(v) \\
&= d_u f_N(u) - \sum_{u',u'\sim u} w_{u'u}\sigma(uu')f_N(u') - d_v\sigma(uv)f_N(v) \\
&+ \sum_{u',u'\sim v} w_{u'v}\sigma(uv)\sigma(u'v)f_N(u').
\end{aligned}$$

For ease of notation, we will adopt the simplified notations

$$\sum_{u'}^{u} := \sum_{u',u'\sim u,u'\not\sim v}, \qquad \sum_{u',\pm}^{u,v} := \sum_{u',u'\sim u,u'\sim v,\ \sigma(u'u)\sigma(u'v)=\pm\sigma(uv)}.$$

We continue to estimate:

$$\begin{aligned}
&\lambda_N(L^\sigma)(f_N(u) - \sigma(uv)f_N(v)) \\
&\leq d_u f_N(u) - \sigma(uv)f_N(v)(\sum_{u'}^{u} + \sum_{u',-}^{u,v})w_{u'u} - d_v\sigma(uv)f_N(v) \\
&+ f_N(u)(\sum_{u'}^{v} w_{u'v} + \sum_{u',-}^{u,v})w_{u'v} + \sum_{u',+}^{u,v}(w_{u'v} - w_{u'u})\sigma(u'u)f_N(u') \\
&= \frac{1}{2}(f_N(u) - \sigma(uv)f_N(v))(d_u + d_v + (\sum_{u'}^{u} + \sum_{u',-}^{u,v})w_{u'u} + (\sum_{u'}^{v} w_{u'v} + \sum_{u',-}^{u,v})w_{u'v}) \\
&+ \frac{1}{2}(f_N(u) + \sigma(uv)f_N(v))(d_u - (\sum_{u'}^{u} + \sum_{u',-}^{u,v})w_{u'u}) \\
&- \frac{1}{2}(f_N(u) + \sigma(uv)f_N(v))(d_v - (\sum_{u'}^{v} + \sum_{u',-}^{u,v})w_{u'v}) \\
&+ \sum_{u',+}^{u,v}(w_{u'v} - w_{u'u})\sigma(u'u)f_N(u')
\end{aligned}$$

Observing that $d_u - (\sum_{u'}^{u} + \sum_{u',-}^{u,v})w_{u'u} = \sum_{u',+}^{u,v} w_{u'u}$, we have

$$\begin{aligned}
&\lambda_N(L^\sigma)(f_N(u) - \sigma(uv)f_N(v)) \\
&\leq \frac{1}{2}(f_N(u) - \sigma(uv)f_N(v))(d_u + d_v + (\sum_{u'}^{u} + \sum_{u',-}^{u,v})w_{u'u} + (\sum_{u'}^{v} w_{u'v} + \sum_{u',-}^{u,v})w_{u'v}) \\
&+ \frac{1}{2}\sum_{u',+}^{u,v}(w_{u'v} - w_{u'u})\left(f_N(u) + \sigma(uv)f_N(v) - 2\sigma(u'u)f_N(u')\right). \qquad (7.5)
\end{aligned}$$

For the latter term above, we further estimate

$$\begin{aligned}
&\sum_{u',+}^{u,v}(w_{u'v} - w_{u'u})\left(f_N(u) + \sigma(uv)f_N(v) - 2\sigma(u'u)f_N(u')\right) \\
&\leq \sum_{u',+}^{u,v} |w_{u'v} - w_{u'u}|(f_N(u) - \sigma(u'u)f_N(u')) \\
&+ \sum_{u',+}^{u,v} |w_{u'v} - w_{u'u}|(\sigma(u'u)f_N(u') - \sigma(uv)f_N(v)) \\
&= (f_N(u) - \sigma(uv)f(v))\sum_{u',+}^{u,v} |w_{u'v} - w_{u'u}|.
\end{aligned}$$

Inserting the above estimation into (7.5), we arrive at

$$\begin{aligned}
&\lambda_N(L^\sigma)(f_N(u) - \sigma(uv)f_N(v)) \\
&\leq \frac{1}{2}(f_N(u) - \sigma(uv)f(v))(d_u + d_v + (\sum_{u'}^{u} + \sum_{u',-}^{u,v})w_{u'u} + (\sum_{u'}^{v} w_{u'v} + \sum_{u',-}^{u,v})w_{u'v}) \\
&+ \frac{1}{2}(f_N(u) - \sigma(uv)f(v))\sum_{u',+}^{u,v} |w_{u'v} - w_{u'u}|.
\end{aligned}$$

This completes the proof. □

Acknowledgements

The authors are very grateful to Norbert Peyerimhoff for suggesting the crucial idea leading to Proposition 2. The main part of this work was conceived when both authors were visiting the *Zentrum für interdisziplinäre Forschung (ZiF)* of Bielefeld University. The authors thank the hospitality of ZiF and the financial support from ZiF Cooperation group "Discrete and Continuous Models in the Theory of Networks". SL was supported by the EPSRC Grant EP/K016687/1. FMA acknowledges the support of the European Commission under grant #318723 (MatheMACS).

References

[1] R. P. Abelson and M. J. Rosenberg, Symbolic psycho-logic: A model of attitudinal cognition, Behavioral Sci. 3 (1958), 1-13.
[2] J. Akiyama, D. Avis, V. Chvátal and H. Era, Balancing signed graphs. Discrete Appl. Math. 3 (1981), no. 4, 227-233.
[3] N. Alon, Eigenvalues and expanders, Theory of computing (Singer Island, Fla., 1984). Combinatorica 6 (1986), no. 2, 83-96.
[4] N. Alon and V. Milman, λ_1, isoperimetric inequalities for graphs, and superconcentrators, J. Combin. Theory Ser. B 38 (1985), no. 1, 73-88.
[5] F. M. Atay and H. Tunçel, On the spectrum of the normalized Laplacian for signed graphs: Interlacing, contraction, and replication, Linear Algebra Appl. 442 (2014), 165-177.
[6] F. Bauer and J. Jost, Bipartite and neighborhood graphs and the spectrum of the normalized graph Laplacian, Comm. Anal. Geom. 21 (2013), no. 4, 787-845.
[7] F. Bauer, J. Jost and S. Liu, Ollivier-Ricci curvature and the spectrum of the normalized graph Laplace operator, Math. Res. Lett. 19 (2012), no. 6, 1185-1205.
[8] F. Belardo, Balancedness and the least eigenvalue of Laplacian of signed graphs, Linear Algebra Appl. 446 (2014), 133-147.
[9] I. Bieche, R. Maynard, R. Rammal and J. P. Uhry, On the ground states of the frustration model of a spin glass by a matching method of graph theory, J. Phys. A: Math. Gen. 13 (1980), 2553-2576.
[10] Y. Bilu and N. Linial, Lifts, discrepancy and nearly optimal spectral gap, Combinatorica 26 (2006), no. 5, 495-519.
[11] P. J. Cameron, Cohomological aspects of two-graphs, Math. Z. 157 (1977), 101-119.
[12] P. J. Cameron, J. J. Seidel and S.V. Tsaranov, Signed graphs, root lattices and Coxeter groups, J. Algebra 164 (1994), no. 1, 173-209.
[13] P. J. Cameron and A. L. Wells, Jr., Signatures and signed switching classes, J. Combin. Theory Ser. B 40 (1986), no. 3, 344-361.
[14] D. Cartwright and F. Harary, Structural balance: a generalization of Heider's theory, Psychol. Rev. 63 (1956), no. 5, 277-293.
[15] N. Cesa-Bianchi, C. Gentile, F. Vitale and G. Zappella, A correlation clustering approach to Link classification in signed networks, Proceedings of the 25th conference on learning theory (COLT 2012), 2012.
[16] J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian, Problems in analysis (Papers dedicated to Salomon Bochner, 1969), pp. 195-199. Princeton Univ. Press, Princeton, N. J., 1970.
[17] S. Costantini and A. Provetti, Graph representations of logic programs: properties and comparison, Proceedings of the Sixth Latin American Workshop on Non-monotonic Reasoning, 1-14, 2010.
[18] K. Ch. Das, An improved upper bound for Laplacian graph eigenvalues, Linear Algebra Appl. 368 (2003), 269-278.
[19] M. Desai and V. Rao. A characterization of the smallest eigenvalue of a graph, J. Graph Theory 18 (1994), no. 2, 181-194.
[20] J. Dodziuk, Difference equations, isoperimetric inequality and transience of certain random walks, Trans. Amer. Math. Soc. 284 (1984), no. 2, 787-794.
[21] K. A. Germina, S. Hameed K. and T. Zaslavsky, On products and line graphs of signed graphs, their eigenvalues and energy, Linear Algebra Appl. 435 (2011), no. 10, 2432-2450.
[22] F. Harary, On the notion of balance of a signed graph, Michigan Math. J. 2 (1953), no. 2, 143-146.
[23] F. Harary, Structural duality, Behavioral Sci. 2 (1957), no. 4, 255-265.
[24] F. Harary, On the measurement of structural balance, Behavioral Sci. 4 (1959), 316-323.
[25] Y.-P. Hou, Bounds for the least Laplacian eigenvalue of a signed graph, Acta Math. Sin. (Engl. Ser.) 21 (2005), no. 4, 955-960.
[26] Y.-P. Hou, J.-S. Li and Y.-L. Pan, On the Laplacian eigenvalues of signed graphs, Linear Multilinear Algebra 51 (2003), no. 1, 21-30.
[27] J. Jost and S. Liu, Ollivier's Ricci curvature, local clustering and curvature-dimension inequalities on graphs, Discrete Comput. Geom. 51 (2014), no. 2, 300-322.

[28] J. Kunegis, Applications of structural balance in signed social networks, arXiv:1402.6865, Febuary 2014.
[29] J. Kunegis, A. Lommatzsch and C. Bauckhage, The Slashdot Zoo: Mining a social network with negative edges, Proceedings of the 18th International Conference on World Wide Web, 741-750, ACM, 2009.
[30] J. Kunegis, S. Schmidt, A. Lommatzsch, J. Lerner, E. W. De Luca and S. Albayrak, Spectral analysis of signed graphs for clustering, prediction and visualization, SIAM International Conference on Data Mining-SDM, 559-570, 2010.
[31] T.-C. Kwok, L.-C. Lau, Y.-T. Lee, S. Oveis Gharan and L. Trevisan, Improved Cheeger's inequality: Analysis of spectral partitioning algorithms through higher order spectral gap, STOC'13-Proceedings of the 2013 ACM Symposium on Theory of Computing, 11-20, ACM, New York, 2013.
[32] J. R. Lee, S. Oveis Gharan and L. Trevisan, Multi-way spectral partitioning and higher-order Cheeger inequalities, STOC'12-Proceedings of the 2012 ACM Symposium on Theory of Computing, 1117-1130, ACM, New York, 2012.
[33] H.-H. Li and J.-S. Li, An upper bound on the Laplacian spectral radius of the signed graphs, Discuss. Math. Graph Theory 28 (2008), no. 2, 345-359.
[34] H.-H. Li and J.-S. Li, Note on the normalized Laplacian eigenvalues of signed graphs, Australas. J. Combin. 44 (2009), 153-162.
[35] S. Liu, Multi-way dual Cheeger constants and spectral bounds of graphs, Adv. Math. 268 (2015), 306-338.
[36] U. von Luxburg, A tutorial on spectral clustering, Statistics and computing, 17 (2007), no. 4, 395-416.
[37] A. W. Marcus, D. A. Spielman and N. Srivastava, Interlacing families I: bipartite Ramanujan graphs of all degrees, Proceedings of FOCS, 529-537, 2013; to appear in Ann. of Math..
[38] A. W. Marcus, D. A. Spielman and N. Srivastava, Ramanujan graphs and the solution of the Kadison-Singer problem, arXiv: 1408.4421, to appear in Proceedings of ICM 2014.
[39] L. Miclo, On eigenfunctions of Markov processes on trees, Probab. Theory Related Fields 142 (2008), no. 3-4, 561-594.
[40] A. Ng, M. Jordan and Y. Weiss, On spectral clustering: analysis and an algorithm, In T. Dietterich, S. Becker and Z. Ghahramani (Eds.), Advances in Neural Information Processing System, vol. 14, MIT Press, 849-856, 2001.
[41] N. Reff, New bounds for the Laplacian spectral radius of a signed graph, arXiv:1103.4629, March 2011.
[42] O. Rojo, A nontrivial upper bound on the largest Laplacian eigenvalue of weighted graphs, Linear Algebra Appl. 420 (2007), 625-633.
[43] P. Solé and T. Zaslavsky, A coding approach to signed graphs, SIAM J. Discrete Math. 7 (1994), no. 4, 544-553.
[44] C. Swamy, Correlation clustering: maximizing agreements via semidefinite programming, Proceedings of the Fifteenth Annual ACM-SIAM Symposium on Discrete Algorithms, 526-527 (electronic), ACM, New York, 2004.
[45] P. Symeonidis, N. Iakovidou, N. Mantas and Y. Manolopoulos, From biological to social networks: Link prediction based on multi-way spectral clustering, Data & Knowledge Engineering 87 (2013), 226-242.
[46] P. Symeonidis and N. Mantas, Spectral clustering for link prediction in social networks with positive and negative links, Soc. Netw. Anal. Min. 3 (2013), 1433-1447.
[47] L. Trevisan, Max cut and the smallest eigenvalue, STOC'09-Proceedings of the 2009 ACM International Symposium on Theory of Computing, 263-271, ACM, New York, 2009; SIAM J. Comput. 41 (2012), no. 6, 1769-1786.
[48] L. Wu, X. Ying, X. Wu, A. Lu and Z.-H. Zhou, Examining spectral space of complex networks with positive and negative links, Int. J. Social Network Mining 1 (2012), no. 1, 91-111.
[49] T. Zaslavsky, Signed graphs, Discrete Appl. Math. 4 (1982), no. 1, 47-74.
[50] T. Zaslavsky, Matrices in the theory of signed simple graphs, Advances in discrete mathematics and applications: Mysore, 2008, 207-229, Ramanujan Math. Soc. Lect. Notes Ser., 13, Ramanujan Math. Soc., Mysore, 2010.
[51] T. Zaslavsky, Balance and clustering in signed graphs, Slides from lectures at the C.R. Rao Advanced Institute of Mathematics, Statistics and Computer Science, Univ. of Hyderabad, India, 26 July 2010.

Max Planck Institute for Mathematics in the Sciences, 04103 Leipzig, Germany.
E-mail address: fatay@mis.mpg.de

Department of Mathematical Sciences, Durham University, DH1 3LE Durham, United Kingdom
E-mail address: shiping.liu@durham.ac.uk