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ON THE SYMMETRY OF THE LAPLACIAN SPECTRA OF SIGNED GRAPHS

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Abstract. We study the symmetry properties of the spectra of normalized Laplacians on signed graphs. We find a new machinery that generates symmetric spectra for signed graphs, which includes bipartiteness of unsigned graphs as a special case. Moreover, we prove a fundamental connection between the symmetry of the spectrum and the existence of damped two-periodic solutions for the discrete-time heat equation on the graph.

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1. Introduction

A signed graph refers to a graph together with a labeling of edges by a sign ± 1, so as to represent two different types of relationships between adjacent vertices. Such graphs originate from studies of social networks where one distinguishes between friend and foe type relations [Har55]. They are also appropriate models of electrical circuits with negative resistance or neuronal networks with inhibitory or excitatory connections.

Let $G(V, E, \mu)$ be a finite, undirected, weighted graph without isolated vertices, having the vertex set V, edge set E, and edge weight $\mu : E \to \mathbb{R}_+$. Two vertices x, y are called neighbours, denoted $x \sim y$, if $(x, y) \in E$. The weighted degree measure $m : V \to (0, \infty)$ on the vertices is defined by $m(x) = \sum_{y:y\sim x} \mu_{xy}$. It reduces to the combinatorial degree for the unweighted case, i.e. when $\mu = 1_E$, the constant function 1 on E. A function $\eta : E \to \{1, -1\}$ is called a sign function on E. We call $\Gamma = (G, \eta)$ a *signed graph* based on G with the sign function η. Let $\ell^2(V, m)$ denote the space of real functions on V equipped with the ℓ^2 norm with respect to the measure m. The *normalized signed Laplacian* $\Delta : \ell^2(V, m) \to \ell^2(V, m)$ for a signed graph Γ is defined by

$$\Delta f(x) = \frac{1}{m(x)} \sum_{y \in V : y \sim x} \mu_{xy}(f(x) - \eta_{xy} f(y)), \quad \forall f \in \ell^2(V, m). \tag{1}$$

One can show that Δ is a self-adjoint linear operator on $\ell^2(V, m)$. The eigenvalues of Δ will be referred to as the spectrum of the signed graph Γ and denoted by $\sigma(\Gamma)$. The spectrum is always contained in the interval $[0, 2]$. The spectral properties of signed graphs have been studied by several authors, e.g. [HLP03, Hou05, LL08, LL09, GKZ11, AT14, Bel14].

We say that the spectrum of a signed graph Γ is *symmetric* (w.r.t. the point $\{1\}$) if

$$\sigma(\Gamma) = 2 - \sigma(\Gamma),$$

that is, λ is an eigenvalue of the signed Laplacian Δ of Γ if and only if $2 - \lambda$ is. For unsigned graphs, i.e. for $\eta = 1_E$, it is well known that the spectrum is symmetric if and only if the graph is bipartite [Chu97]. For signed graphs, however, the symmetry of the spectrum and bipartiteness are not equivalent: While bipartite signed graphs do have symmetric spectra (see Lemma 5), we will see that the reverse implication does not hold, and there exist non-bipartite signed graphs which nevertheless have symmetric spectra. In fact, we will present a general machinery to produce symmetric spectra for signed graphs that has no counterpart in the unsigned case. This can be seen as one of the major differences between spectral theories of unsigned and signed graphs.

To state our result, we use the concept of a *switching function*, i.e., a function $\theta : V \to \{1, -1\}$: Given $\Gamma = (G, \eta)$, a switching function θ can be used to define a new graph $\Gamma^\theta = (G, \eta^\theta)$ having the sign function $\eta^\theta(xy) = \theta(x)\eta_{xy}\theta(y)$. For a signed graph $\Gamma = (G, \eta)$, we denote by $-\Gamma = (G, -\eta)$ the signed graph with the opposite sign function. We say that two signed weighted graphs $\Gamma = (V, E, \mu, \eta)$ and $\Gamma' = (V', E', \mu', \eta')$ are *isomorphic*, denoted $\Gamma \simeq \Gamma'$, if there is a bijective map $S : V \to V'$ such that $Sx \sim Sy$ iff $x \sim y$, and $\mu'_{SxSy} = \mu_{xy}$ and $\eta'_{SxSy} = \eta_{xy}$ for any $x \sim y$. That is, two signed graphs are isomorphic if they only differ up to a relabeling of vertices. Based on these notions we prove the following in Theorem 2:

Given $\Gamma = (G, \eta)$, if there is a switching function $\theta : V \to \{1, -1\}$ such that $\Gamma^\theta \simeq -\Gamma$, then the spectrum of Γ is symmetric.

Motivated by this new machinery, we provide a nontrivial example (Example 3) of a signed graph that possesses a symmetric spectrum but is not bipartite.

Although the symmetry of the spectrum of a signed graph is a purely algebraic property, we find that it plays an important role in the analysis and dynamics on the graph. We say that a function $f : (\mathbb{N} \cup \{0\}) \times V \to \mathbb{R}$ solves the discrete-time heat equation on (G, η) with the initial data $g : V \to \mathbb{R}$, if for any $n \in \mathbb{N} \cup \{0\}$ and $x \in V$,

$$\begin{cases} f(n+1, x) - f(n, x) = -\Delta f(n, x), \\ f(0, x) = g(x). \end{cases} \tag{2}$$

This definition mimics continuous-time heat equations on Euclidean domains or Riemannian manifolds. Among all solutions to the discrete-time heat equation, a special class, namely damped 2-periodic solutions (see Definition 2), will be of particular interest. These solutions are motivated from the well-known oscillatory solutions on unsigned graphs: For a bipartite unsigned graph $G(V, E, \mu)$ with bipartition $V = V_1 \cup V_2$, the solution to the discrete-time heat equation with initial data $f(0, \cdot) = 1_{V_1}$ (i.e. the characteristic function on V_1) is given by (see e.g. [Gri09, pp. 43])

$$f(n, \cdot) = \begin{cases} 1_{V_1}, & n \text{ even}, \\ 1_{V_2}, & n \text{ odd}. \end{cases} \tag{3}$$

That is, the solution oscillates between two phases, 1_{V_1} and 1_{V_2}, which justifies the name of 2-periodic solution. We generalize such solutions to signed graphs and also allow temporal damping with decay rate $\lambda \in [0, 1]$. (In the example of (3), $\lambda = 1$.) We show that this analytic property of the solutions, i.e. the periodicity of order two, is deeply connected to the symmetry of the spectrum of the signed graph. Precisely, we prove the following in Theorem 3:

Let (G, η) be a signed graph. Then u is a damped 2-periodic solution with decay rate λ $(\lambda \neq 0)$ if and only if

$$u = f + g,$$

where f and g are eigenfunctions corresponding to the eigenvalues $1-\lambda$ and $1+\lambda$, respectively, of the normalized Laplacian.

In the last section, we study the spectral properties of the motif replication of a signed graph. A subset of vertices, say Ω, of a signed graph $\Gamma = (G, \eta)$ is sometimes referred to as a motif. By motif replication we refer to the enlarged graph Γ^{Ω} that contains a replica of the subset Ω with all its connections and weights; see [AT14] or Section 5. Let Δ_{Ω} be the signed Laplacian on Ω with Dirichlet boundary condition whose spectrum is denoted by $\sigma(\Delta_{\Omega})$; see [BHJ12] for the unsigned case or Section 5. We prove in Theorem 4 that $\sigma(\Delta_{\Omega}) \subset \sigma(\Gamma^{\Omega})$. This follows from a discussion with Bauer-Keller [BK12] which obviously generalizes [AT14, Theorem 13]. As a consequence, if the subgraph Ω admits damped 2-periodic solutions, then so does the larger graph Γ^{Ω} after replication.

The paper is organized as follows. In the next section we introduce the concepts of signed graphs and normalized signed Laplacians, and study their spectral properties. In Section 3, we explore a general machinery to create symmetry in the spectrum. Section 4 is devoted to damped 2-periodic solutions of the discrete-time heat equation and their connection to the symmetry of the spectrum. The last section contains the spectral properties of motif replication.

2. Basic properties of signed graphs

In this section, we study the basic properties of signed graphs and the spectral properties of the normalized signed Laplacian. Let $G(V, E)$ be a finite (combinatorial) graph with the set of vertices V and the set of edges E where E is a symmetric subset of $V \times V$. A graph is called connected if for any $x, y \in V$ there is a finite sequence of vertices, $\{x_i\}_{i=0}^{n}$, such that

$$x = x_0 \sim x_1 \sim \cdots \sim x_n = y.$$

In this paper, we consider finite, connected, undirected graphs without isolated vertices.

We assign symmetric weights on edges,

$$\mu : E \to (0, \infty), \quad E \ni (x, y) \mapsto \mu_{xy}$$

which satisfies $\mu_{xy} = \mu_{yx}$ for any $x \sim y$, and call the triple $G(V, E, \mu)$ a *weighted graph*. The special case of $\mu = 1_E$ is also referred to as an unweighted graph. For any $x \in V$, the weighted degree of x is defined as

$$m(x) = \sum_{y \sim x} \mu_{xy}.$$

The weighted degree function $m : V \to (0, \infty)$, $x \mapsto m(x)$ can be understood as a measure on V. We denote by $\ell^2(V, m)$ the space of real functions on V equipped with an inner product with respect to the measure m, defined by $\langle u, v \rangle = \sum_{x \in V} u(x)v(x)m(x)$ for $u, v \in \ell^2(V, m)$.

Definition 1 (Weighted signed graphs)**.** Let $G(V,E,\mu)$ be a weighted graph. A symmetric function $\eta : E \to \{1,-1\}$, $E \ni (x,y) \mapsto \eta_{xy}$, is called a *sign function* on G. We refer to the quadruple $(V,E,\mu,\eta) = (G,\eta)$ as a *weighted signed graph.*

In the special case $\eta = 1_E$, (G,η) is called an unsigned graph. In the following, by signed graphs we always mean weighted signed graphs. For convenience, we define the *signed weight* of a signed graph (V,E,μ,η) by $\kappa = \eta\mu : E \to \mathbb{R}$, i.e. $\kappa_{xy} = \eta_{xy}\mu_{xy}$ for any $x \sim y$.

Let $\Gamma = (V,E,\mu,\eta)$ be a signed graph. The *normalized signed Laplacian* of Γ, denoted by Δ_Γ, is defined as

$$\Delta_\Gamma f(x) = \frac{1}{m(x)} \sum_{y\sim x} \mu_{xy}(f(x) - \eta_{xy} f(y)), \quad \forall\ f : V \to \mathbb{R}.$$

The adjacency matrix of a signed graph Γ is defined by

$$A_\Gamma(x,y) = \begin{cases} \kappa_{xy} = \eta_{xy}\mu_{xy}, & \text{if } x \sim y \\ 0, & \text{otherwise} \end{cases}$$

The degree matrix is defined as $D_\Gamma(x,y) := m(x)\delta_{xy}$, where $\delta_{xy} = 1$ if $y = x$, and 0 otherwise. Hence, the normalized Laplacian of the signed graph $\Gamma = (G,\eta)$ can be expressed as the matrix

$$\Delta_\Gamma = I - D_\Gamma^{-1} A_\Gamma,$$

or as an operator on $\ell^2(V,m)$,

$$\Delta_\Gamma f(x) = f(x) - \frac{1}{m(x)} \sum_{y\sim x} f(y)\kappa_{xy}.$$

for any $f : V \to \mathbb{R}$. We call $P_\Gamma = D_\Gamma^{-1} A_\Gamma$ the *generalized transition matrix* in analogy to the transition matrix of the simple random walk on an unsigned graph. One notices that the row sum of P_Γ is not necessarily equal to 1 for a generic signed graph, which is an obvious difference between signed and unsigned graphs. We will often omit the subscripts and simply write Δ and P for Δ_Γ and P_Γ, respectively unless we want to emphasize the underlying signed graph Γ.

One can show that $\Delta : \ell^2(V,m) \to \ell^2(V,m)$ is a bounded self-adjoint linear operator on a finite dimensional Hilbert space, and hence the spectrum is real and discrete. We denote by $\sigma(T)$ the spectrum of a linear operator T on $\ell^2(V,m)$. By the spectrum of a signed graph Γ, denoted by $\sigma(\Gamma)$, we mean the spectrum of the normalized signed Laplacian of Γ, $\sigma(\Delta_\Gamma)$. Since $\Delta = I - P$, we have $\sigma(\Delta) = 1 - \sigma(P)$, where the right hand side is understood as $\{1-\lambda : \lambda \in \sigma(P)\}$. By the Cauchy-Schwarz inequality, one can show that the operator norm of P on $\ell^2(V,m)$ is bounded by 1. Hence, $\sigma(P) \subset [-1,1]$, and consequently $\sigma(\Gamma) \subset [0,2]$ for any signed graph Γ.

Given a weighted graph $G(V,E,\mu)$, we let

$$\mathcal{G} = \{(G,\eta) \,|\, \eta \text{ is a sign function}\}$$

denote the set of all signed graphs with a common underlying weighted graph G. We distinguish two special cases where the edges have all positive or all negative signs, namely $(G,+) := (G,1_E)$ and $(G,-) := (G,-1_E)$, respectively.

Given a signed graph $\Gamma = (G, \eta)$, a function $\theta : V \to \{1, -1\}$ is called a *switching function* (on V). Using the switching function θ, one can define a new signed graph as

$$\Gamma^\theta = (G, \eta^\theta), \qquad \text{where} \quad \eta^\theta(xy) = \theta(x)\eta_{xy}\theta(y), \quad x \sim y. \tag{4}$$

For a switching function θ, let S^θ denote the diagonal matrix defined as $S^\theta(x, y) := \theta(x)\delta_{xy}$. Then the adjacency matrix of Γ^θ can be written as

$$A_{\Gamma^\theta} = S^\theta A_\Gamma S^\theta,$$

Note that the degree matrix is invariant under the switching operation, i.e. $D_{\Gamma^\theta} = D_\Gamma$, and $(S^\theta)^{-1} = S^\theta$. Clearly, $S^\theta D = DS^\theta$, and hence $\Delta_{\Gamma^\theta} = (S^\theta)^{-1}\Delta_\Gamma S^\theta$, which implies that the spectrum is invariant under the switching operation θ. This observation yields the following lemma.

Lemma 1. *Let (G, η) be a signed graph and $\theta : V \to \{1, -1\}$ a switching function. Then the switched signed graph Γ^θ defined in (4) has the same spectrum as Γ, i.e.*

$$\sigma(\Gamma^\theta) = \sigma(\Gamma).$$

Moreover, if $f : V \to \mathbb{R}$ is an eigenfunction of Δ_Γ corresponding to the eigenvalue λ, then the function $f^\theta : V \to \mathbb{R}$, defined by $f^\theta(x) = \theta(x)f(x)$ for $x \in V$, is an eigenfunction of Δ_{Γ^θ} corresponding to the eigenvalue λ.

Given a weighted graph $G(V, E, \mu)$ with a fixed labeling of vertices, we introduce an equivalence relation on the set of all signed graphs $\mathcal{G}$ based on G: Two signed graphs $\Gamma_1, \Gamma_2 \in \mathcal{G}$ are called equivalent, denoted $\Gamma_1 \sim \Gamma_2$, if there exists a switching function $\theta : V \to \{1, -1\}$ such that $\Gamma_1^\theta = \Gamma_2$. For $\Gamma \in \mathcal{G}$, we denote by $\overline{\Gamma}$ the equivalence class of Γ and by $\overline{\mathcal{G}} := \{\overline{\Gamma} : \Gamma \in \mathcal{G}\}$ the set of all equivalence classes.

For a finite signed graph $\Gamma = (G, \eta)$, we order the eigenvalues of the normalized Laplacian in a nondecreasing way:

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N \le 2,$$

where $N = |V|$. We always denote the smallest and largest eigenvalues by $\lambda_1(\Gamma)$ and $\lambda_N(\Gamma)$, respectively. The following result is well known for unsigned graphs (e.g. [Chu97]).

Lemma 2. *Let $G(V, E, \mu)$ be a finite connected unsigned graph. Then the following are equivalent:*

(i) *G is bipartite.*
(ii) *$\lambda_N(G) = 2$.*
(iii) *The spectrum of G is symmetric with respect to 1, i.e. $\sigma(G) = 2 - \sigma(G)$.*

Clearly, statement (iii) is equivalent to saying that the spectrum of the transition matrix $P = D^{-1}A$ is symmetric with respect to 0, i.e. $\sigma(P) = -\sigma(P)$.

In this section, we characterize property (ii) of Lemma 2 for signed graphs. Property (iii), i.e. the symmetry of the spectrum, will be postponed to the next section.

Let C be a cycle, i.e. $C = \{x_i\}_{i=0}^k$ such that $x_0 \sim x_1 \sim \cdots \sim x_k \sim x_0$, in a signed graph $\Gamma = (G, \eta)$. The sign of C is defined as

$$\operatorname{sign}(C) = \prod_{e \in C} \eta_e$$

where the product is taken over all edges e in the cycle. A signed graph $\Gamma = (G, \eta)$ is called *balanced* if every cycle in Γ has positive sign. The following characterization of balanced signed graphs is well known [Har55, Theorem 1], [Zas82], [Hou05, Corollary 2.4] and [LL09, Theorem 1].

Lemma 3. *Let $\Gamma = (G, \eta)$ be a signed graph. Then the following statements are equivalent:*

(a) Γ *is balanced.*

(b) $\Gamma \in \overline{(G, +)}$.

(c) There exists a partition of V, $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$, such that every edge connecting V_1 and V_2 has negative sign and every edge within V_1 or V_2 has positive sign.

(d) $\lambda_1(\Gamma) = 0$.

For a signed graph $\Gamma = (G, \eta)$, the *reverse signed graph* is defined as $-\Gamma := (G, -\eta)$. Clearly $(G, -) = -(G, +)$. A signed graph $\Gamma = (G, \eta)$ is called *antibalanced* if $-\Gamma$ is balanced. Furthermore, since $P_{-\Gamma} = -P_\Gamma$, we have

$$\sigma(-\Gamma) = 2 - \sigma(\Gamma). \tag{5}$$

Based on these observations, one obtains the following lemma; see also [LL09, Theorem 1].

Lemma 4. *Let $\Gamma = (G, \eta)$ be a signed graph. Then the following are equivalent:*

(a) Γ *is antibalanced.*

(b) $\Gamma \in \overline{(G, -)}$.

(c) There exists a partition of V, $V = V_1 \cup V_2$ ($V_1 \cap V_2 = \emptyset$), such that every edge connecting V_1 and V_2 has positive sign and each edge within V_1 or V_2 has negative sign.

(d) $\lambda_N(\Gamma) = 2$.

Remark 1. For any weighted graph,

$$\lambda_1(G, \eta) \geq 0 = \lambda_1(G, +),$$
$$\lambda_N(G, \eta) \leq 2 = \lambda_N(G, -),$$

where the equalities hold only for balanced or antibalanced graphs, respectively.

In the remainder of this section, we discuss the first eigenvalue and eigenvectors of signed graphs. As usual, the smallest eigenvalue of a finite signed graph Γ is characterized by the Rayleigh quotient

$$\lambda_1(\Gamma) = \inf_{f \neq 0} \frac{1}{2} \frac{\sum_{x,y \in V} \mu_{xy}(f(x) - \eta_{xy} f(y))^2}{\sum_{x \in V} f^2(x)\mu(x)}.$$

Is there any special property for the first eigenvalue and eigenvector? It is well known that the first eigenvalue of an unsigned weighted graph G is simple if the graph is connected. However, this is not the case for signed graphs:

Example 1. Let $(K_N,+)$ be an unsigned complete graph of N vertices and $\Gamma=(K_N,-)$. Then

$$\sigma(\Gamma)=\left\{\frac{N-2}{N-1},\dots,\frac{N-2}{N-1},2\right\},$$

where the multiplicity of the first eigenvalue is $N-1$.

Note that the multiplicity of the first eigenvalue of a signed graph can be quite large. The example above concerns antibalanced graphs. We next give an example of a generic signed graph, neither balanced nor antibalanced, whose first eigenvalue has multiplicity larger than 1.

Example 2. Let $\Gamma=(C_4,\eta)$ be a cycle graph of order 4 with edge signs $\{1,1,1,-1\}$. Clearly, it is neither balanced nor antibalanced. An explicit calculation shows that

$$\sigma(\Gamma)=\left\{1-\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{2},1+\frac{\sqrt{2}}{2},1+\frac{\sqrt{2}}{2}\right\},$$

where the multiplicity of the first eigenvalue is 2. Thus, the first eigenvalue of a signed graph may have high multiplicity.

A well-known fact is that the first eigenfunction of an unsigned graph can be chosen to be positive everywhere (in fact, it is a constant function). One then has the following natural question: If (G,η) is a signed graph with a positive first eigenfunction, should this graph be the unsigned weighted graph $(G,\eta)=(G,+)$? We provide a negative answer.

Theorem 1. *For any $\overline{\Gamma}\in\overline{\mathcal{G}}$, there exists a signed graph $\Gamma'\in\overline{\Gamma}$ such that the first eigenvector of $\Delta_{\Gamma'}$ is nonnegative everywhere. Moreover, if one of the first eigenvectors of Γ vanishes nowhere, then there exists a signed graph $\Gamma'\in\overline{\Gamma}$ such that the first eigenvector of $\Delta_{\Gamma'}$ is strictly positive everywhere.*

Proof. Let f be a first eigenvector of $\Delta(\Gamma)$. We define a switching function $\theta: V\to\mathbb{R}$ by

$$\theta(x)=\begin{cases}-1, & f(x)<0,\\ 1, & \text{otherwise.}\end{cases}$$

Clearly, $f^\theta:=\theta\cdot f$ is nonnegative everywhere. Then by Lemma 1, f^θ is the first eigenvector to Δ_{Γ^θ}. The signed graph $\Gamma'=\Gamma^\theta$ satisfies the assertions of the theorem. □

3. Symmetry of the spectrum

In this section, we study the symmetry of the spectra of signed graphs. Recall that for unsigned graphs bipartiteness is the only reason for the symmetry of the spectra. However, for signed graphs some new phenomena emerge.

A signed graph is called *bipartite* if its underlying graph is bipartite, i.e. there is a partition of V, $V=V_1\cup V_2$, such that any edge in E connects a vertex in V_1 to a vertex in V_2. One notices that the sign function plays no role in the definition of bipartiteness. By the same techniques as in the unsigned case, one can prove that the spectrum of a signed graph is symmetric if it is bipartite; see [AT14, Lemma 4].

Lemma 5. *If $\Gamma=(G,\eta)$ is a bipartite signed graph, then the spectrum of Γ is symmetric.*

The next proposition gives a characterization of bipartite signed graphs.

Proposition 1. *A signed graph $\Gamma=(G,\eta)$ is bipartite if and only if $\overline{\Gamma}=\overline{-\Gamma}$.*

Proof. $\Longrightarrow$: Let $\Gamma=(G,\eta)$ be a bipartite signed graph with bipartition V_1,V_2, i.e. $V=V_1\cup V_2$, $V_1\cap V_2=\emptyset$, $V_1,V_2\neq\emptyset$ and there is no edge in the induced subgraphs V_i, $i=1,2$. Set

$$\theta(x)=\begin{cases}1, & \text{if } x\in V_1,\\ -1, & \text{if } x\in V_2.\end{cases}$$

Then we have $\Gamma^\theta=-\Gamma$.

$\Longleftarrow$: By $\overline{\Gamma}=\overline{-\Gamma}$, there exists a switching function $\theta: V\to\{1,-1\}$ such that

$$\Gamma^\theta=-\Gamma. \tag{6}$$

Let $V_1=\{x\in V\,|\,\theta(x)=1\}$ and $V_2=\{x\in V\,|\,\theta(x)=-1\}$. Then there are no edges within the induced subgraphs V_1 and V_2. Indeed, if there were an edge in the subgraph V_1 or V_2, this would contradict (6). Hence we obtain a bipartition, $V_1\cup V_2$, of G. $\square$

Recall that for unsigned weighted graphs the symmetry of the spectrum is completely equivalent to the bipartiteness of the graph (see in Lemma 2(iii)). A main difference in signed graphs is that there are more structural conditions which may create symmetric spectra. In the following, we present a general machinery to produce symmetric spectrum for signed graphs that has no counterpart in the unsigned case.

As defined in the introduction, two signed graphs $(G,\eta),(G',\eta')$ are called isomorphic if they have same combinatorial, weighted, and signed graph structure. Hence

$$\sigma(G,\eta)=\sigma(G',\eta').$$

Now we are ready to prove one of the main results of this paper.

Theorem 2. *Let $\Gamma=(G,\eta)$ be a signed weighted graph. If there is a switching function $\theta: V\to\{1,-1\}$ such that $\Gamma^\theta\simeq-\Gamma$, then the spectrum of Γ is symmetric.*

Proof of Theorem 2. Combining Lemma 1, (5) and the invariance of the spectrum under the isomorphism, we have

$$\sigma(\Gamma)=\sigma(\Gamma^\theta)=\sigma(-\Gamma)=2-\sigma(\Gamma).$$

This proves the theorem. $\square$

Note that for bipartite signed graphs $\overline{\Gamma}=\overline{-\Gamma}$ by Proposition 1. This indicates that Lemma 5 is a special case of Theorem 2. In the following, inspired by Theorem 2, we provide an example of a signed graph with symmetric spectrum, although it is non-bipartite.

Example 3. Let Γ be the signed graph shown in Figure 1, with $+1$ and -1 edge weights as indicated. Consider the switching function θ given by

$$\theta(i)=\begin{cases}-1, & i=2,4,\\ 1, & \text{otherwise.}\end{cases}$$

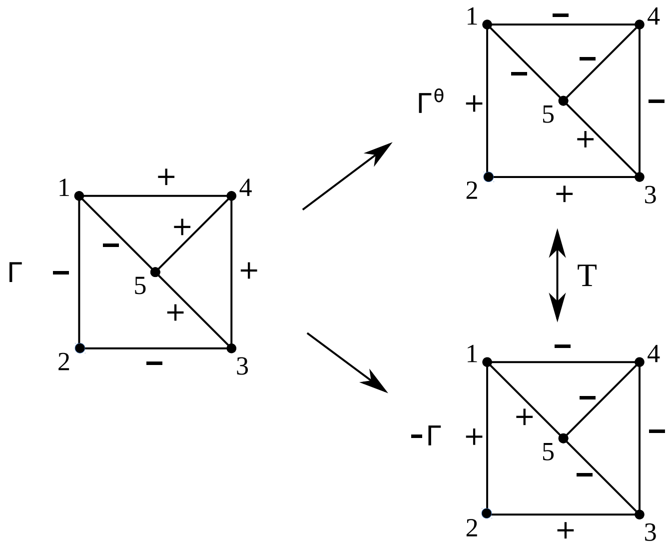

FIGURE 1. A non-bipartite signed graph with symmetric spectrum. The eigenvalues are $1 \pm \frac{\sqrt{4-\sqrt{3}}}{3}, 1 \pm \frac{\sqrt{4+\sqrt{3}}}{3}$, and 1.

Now the permutation $T = (1,3) \in S_5$, which is an isomorphism of signed graphs, transforms Γ^θ to $-\Gamma$. By Theorem 2, the spectrum of Γ is symmetric with respect to 1. Indeed, an explicit calculation gives

$$\sigma(\Gamma) = \left\{ 1 \pm \frac{\sqrt{4-\sqrt{3}}}{3}, 1 \pm \frac{\sqrt{4+\sqrt{3}}}{3}, 1 \right\}.$$

Thus, Γ is a nontrivial (i.e. non-bipartite) example for the symmetry of the spectrum.

4. Damped two-periodic solutions

In this section, we connect the symmetry of the spectrum of signed graphs to the existence of period-two oscillatory solutions for the discrete-time heat equation on the graph.

Let (G, η) be a finite signed graph. We say that a function $f : (\mathbb{N} \cup 0) \times V \to \mathbb{R}$ satisfies the discrete-time heat equation on (G, η) if

$$\begin{cases} f(n+1, x) - f(n, x) = -\Delta f(n, x), & n \in \mathbb{N} \cup \{0\}, \\ f(0, x) = g(x). \end{cases}$$

For simplicity, we denote the function $f_n : V \to \mathbb{R}$, $n \geq 0$, by $f_n(x) := f(n, x)$. Then the heat equation can be written as

$$\begin{cases} f_{n+1} = P f_n, \\ f_0 = g, \end{cases} \tag{7}$$

where $P = D^{-1}A$ is the generalized transition matrix. As usual, the notation $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ will denote the ℓ^2 norm and inner product, respectively, of the Hilbert space $\ell^2(V, m)$.

Definition 2. A function $u \neq 0$ is called a generic damped periodic solution of order two to the discrete-time heat equation (7), or a *damped* 2*-periodic solution* for short, if u is an eigenfunction of P^2 but not of P; that is, if there exists a constant $\lambda \geq 0$ such that $P^2u = \lambda^2 u$ and $Pu \neq \pm\frac{\|Pu\|}{\|u\|}u$. We call λ the *decay rate*.

Proposition 2. *Let u be a damped 2-periodic solution of (7) with decay rate λ. Then the following hold:*
(a) ku is a damped 2-periodic solution for any $k \neq 0$.
(b) $\lambda = \sqrt{\frac{\|P^2u\|}{\|u\|}} = \frac{\|Pu\|}{\|u\|}$.
(c) $0 < \lambda \leq 1$.

Proof. Statement (a) follows by definition. By $P^2u = \lambda^2 u$ and $u \neq 0$, we have $\lambda = \sqrt{\frac{\|P^2u\|}{\|u\|}}$. Moveover, noting that P is a self-adjoint operator with respect to the inner product on $\ell^2(V, m)$, we obtain

$$\|Pu\|^2 = \langle Pu, Pu\rangle = \langle P^2u, u\rangle = \lambda^2\|u\|^2.$$

This proves statement (b). For (c), to prove that $\lambda \neq 0$, we argue by contradiction. Suppose $\lambda = 0$. Then by statement (b) we have $\|Pu\| = 0$, and thus $Pu = 0$, which implies that u is an eigenvector of P pertaining to the eigenvalue 0. This, however, contradicts the definition of 2-periodic solutions. Finally, $\lambda \leq 1$ follows from the fact that all eigenvalues of P belong to the interval $[-1, 1]$ and $P^2u = \lambda^2 u$ by definition. □

For a 2-periodic oscillatory solution u with decay rate λ, we set $v = \frac{1}{\lambda}Pu$. (Notice that v is not parallel to u by definition). Hence we have the system of equations

$$\begin{cases} Pu = \lambda v, \\ Pv = \lambda u. \end{cases} \tag{8}$$

If we set u as the initial data of the discrete-time heat equation, then the solution to (7) reads

$$f_n = P^n u = \begin{cases} \lambda^n u, & n \text{ even}, \\ \lambda^n v, & n \text{ odd}. \end{cases} \tag{9}$$

Since the vectors u and v are linearly independent, the solution f_n oscillates between two phases, u and v, with an amplitude that decays exponentially at a rate λ, since $\lambda \leq 1$. This motivates the meaning of damped 2-periodic solution given in Definition 2.

We study the machinery to produce a 2-periodic solution. We will see that the existence of 2-periodic solutions is equivalent to certain symmetry of the spectrum of the normalized signed Laplacian operator. The next theorem states that all damped 2-periodic solutions are in fact encoded by the symmetry of the spectrum.

Theorem 3. *Let (G, η) be a signed graph. Then u is a damped 2-periodic solution with decay rate λ if and only if*

$$u = f + g,$$

where f and g are eigenfunctions corresponding to the eigenvalues $1 - \lambda$ and $1 + \lambda$, respectively, of the normalized Laplacian Δ, with $\lambda \neq 0$.

Proof. $\Leftarrow$: By the assumption and the fact that $\Delta = I - P$, f and g are eigenfunctions pertaining to the eigenvalues λ and $-\lambda$, respectively, of P. Hence $Pu = \lambda(f-g)$ and $P^2u = \lambda^2 u$. Since f, g are eigenvectors pertaining to different eigenvalues, $\langle f, g\rangle = 0$. It is easy to check that $u = f + g$ and $f - g$ are linearly independent. This yields that u is not an eigenvector to P, and proves the assertion.

$\Rightarrow$: Setting $v = \frac{1}{\lambda}Pu$, we have the system (8) for u and v. Define $f := \frac{1}{2}(u+v)$ and $g := \frac{1}{2}(u-v)$. It is easy to see that f and g are nonzero since u and v are linearly independent. By definition $u = f + g$. Direct calculation shows that $Pf = \lambda f$ and $Pg = -\lambda g$. This completes the proof. □

Remark 2. (a) We have the following corollary of Theorem 3: If there is a damped 2-periodic solution with decay rate λ on a signed graph Γ, then both $1-\lambda$ and $1+\lambda$ belong to the spectrum of Γ.

(b) For signed weighted graphs whose spectrum is symmetric with respect to 1, one can construct many 2-periodic solutions by virtue of Theorem 3.

Finally, as a consequence of Theorem 3, we have the following formulation. Let $\Gamma = (G, \eta)$ be a signed weighted graph. We set

$$\Lambda(\Gamma) := \{\lambda : \lambda \neq 1, \lambda \in \sigma(\Gamma), 2-\lambda \in \sigma(\Gamma)\}.$$

We denote by E_λ the eigenspace pertaining to the eigenvalue λ of Δ_Γ. Then the set of damped 2-periodic solutions, denoted by $\mathcal{P}$, has the representation

$$\mathcal{P} = \bigcup_{\lambda \in \Lambda(\Gamma)} E_\lambda \oplus E_{2-\lambda} \setminus (E_\lambda \oplus 0 \cup 0 \oplus E_{2-\lambda}).$$

5. Motif replication

In this section, we use the normalized Laplace operator with Dirichlet boundary conditions to study the spectral changes under motif replication.

The term *motif* refers to a subgraph Ω of a signed graph Γ. By motif replication we refer to the operation of appending an additional copy of Ω together with all its connections and corresponding weights, yielding the enlarged graph denoted by Γ^Ω. More precisely, let Ω be a subgraph on vertices $\{x_1, \ldots, x_n\}$ and let $\Omega' = \{x_1', \ldots, x_n'\}$ be an exact replica of Ω. Then the enlarged graph Γ^Ω, obtained by replicating Ω, is defined on the vertex set $V(\Gamma^\Omega) = V(\Gamma) \cup \Omega'$ with the signed edge weights given by

$$\kappa = \begin{cases} \kappa(x,y), & x, y \in V(\Gamma), \\ \kappa(x_i, x_j), & x_i', x_j' \in \Omega', \\ \kappa(x_i, y), & x_i' \in \Omega', y \in V(\Gamma) \setminus \Omega, \end{cases}$$

using the edge weights $\kappa(x,y)$ from the original graph Γ [AT14]. In particular, there are no edges between Ω and Ω' in the replicated graph Γ^Ω.

Let Ω be a finite subset of V and $\ell^2(\Omega, m)$ be the space of real-valued functions on Ω equipped with the ℓ^2 inner product. Note that every function $f \in \ell^2(\Omega, m)$ can be extended to a function $\tilde{f} \in \ell^2(V, m)$ by setting $\tilde{f}(x) = 0$ for all $x \in V \setminus \Omega$. The Laplace operator with Dirichlet boundary conditions $\Delta_\Omega : \ell^2(\Omega, m) \to \ell^2(\Omega, m)$ is defined as

$$\Delta_\Omega f = (\Delta \tilde{f})_{|\Omega}.$$

Thus, for $x \in \Omega$ the Dirichlet Laplace operator is given pointwise by

$$\begin{aligned}\Delta_\Omega f(x) &= f(x) - \frac{1}{m(x)} \sum_{y\in\Omega: y\sim x} \kappa_{xy} f(y) \\ &= \tilde{f}(x) - \frac{1}{m(x)} \sum_{y\sim x} \kappa_{xy} \tilde{f}(y).\end{aligned}$$

A simple calculation shows that Δ_Ω is a positive self-adjoint operator. We arrange the eigenvalues of the Dirichlet Laplace operator Δ_Ω in nondecreasing order, $\lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots \le \lambda_N(\Omega)$, where $N = |\Omega|$ denotes the cardinality of the set Ω.

We next prove that all eigenvalues of the Dirichlet Laplacian on Ω are preserved in the motif replication.

Theorem 4. *Let Ω be a motif in a signed weighted graph Γ and Γ^Ω be the new graph obtained after replicating Ω. Then $\sigma(\Delta_\Omega) \subset \sigma(\Gamma^\Omega)$.*

Proof. Let f be an eigenvector of the Dirichlet Laplacian Δ_Ω corresponding to eigenvalue λ. Then by direct calculation it can be seen that the following function is an eigenvector of the Laplacian on Γ^Ω:

$$f'(x) = \begin{cases} f(x), & x \in \Omega, \\ -f(x), & x \in \Omega', \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

where Ω' is the copy of Ω. This proves the theorem. □

Remark 3. By Theorem 4, if the motif Ω supports a damped 2-periodic solution u with respect to the Dirichlet boundary condition, then u', defined as in (10), is a damped 2-periodic solution for the replicated graph Γ^Ω. In this way, symmetric eigenvalues λ and 2-periodic solutions with decay rate λ are carried over to the larger graph after motif replication.

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