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by

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Abstract

We study Rayleigh Bénard convection based on the Boussinesq approximation. We are interested in upper bounds on the Nusselt number Nu, the upwards heat transport, in terms of the Rayleigh number Ra, that characterizes the relative strength of the driving mechanism and the Prandtl number Pr, that characterizes the strength of the inertial effects. We show that, up to logarithmic corrections, the upper bound $\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{3}}$ from [1] persists as long as $\mathrm{Pr} \gtrsim \mathrm{Ra}^{\frac{1}{3}}$ and then crosses over to $\mathrm{Nu} \lesssim \mathrm{Pr}^{-\frac{1}{2}}\mathrm{Ra}^{\frac{1}{2}}$. This result improves the one of Wang [2] by going beyond the perturbative regime $\mathrm{Pr} \gg \mathrm{Ra}$. The proof uses a new way to estimate the transport nonlinearity in the Navier Stokes equations capitalizing on the no-slip boundary condition. It relies on a new Calderón-Zygmund estimate for the non-stationary Stokes equations in L^1 with a borderline Muckenhoupt weight.

Keywords. Rayleigh-Bénard convection, Navier-Stokes equations, no-slip boundary condition, finite Prandtl number, Nusselt number, maximal regularity for non-stationary Stokes equations,

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1 Introduction

1.1 Background

Rayleigh-Bénard convection is the buoyancy-driven flow of a fluid heated from below and cooled from above. This model of thermal convection, besides having some important application in geophysics, astrophysics, meteorology, oceanography and engineering, is a paradigm for nonlinear and chaotic dynamics, pattern formation and turbulence.
A fluid is enclosed between two rigid parallel plates separated by a vertical distance h and held at different temperatures $T = T_{\text{bottom}}$ and $T = T_{\text{top}}$ at height 0 and h respectively, with $T_{\text{bottom}} > T_{\text{top}}$.
In a $d-$dimensional container we follow the evolution equations of the velocity vector field $u(x,t)$, the temperature scalar field $T(x,t)$ and the pressure scalar field $p(x,t)$ where we indicate with x the $d-$dimensional spatial variable and with t the time variable. In what follows we specify with x' the first $d-1$ horizontal components and with z the vertical component of the vector x. In the Oberbeck-Boussinesq approximation, where variations of the density of the fluid ρ are ignored except in the buoyancy term, u, T and p are governed by

$$\left\{\begin{array}{rcll} \partial_t T + u\cdot\nabla T - \chi\Delta T &=& 0 & \text{for} \quad 0<z<h\,, \\ \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p &=& g\alpha T e_z & \text{for} \quad 0<z<h\,, \\ \nabla\cdot u &=& 0 & \text{for} \quad 0<z<h\,, \\ u &=& 0 & \text{for} \quad z\in\{0,h\}\,, \\ u &=& u_0 & \text{for} \quad t=0\,, \\ T &=& T_{bottom} & \text{for} \quad z=0\,, \\ T &=& T_{top} & \text{for} \quad z=h\,, \end{array}\right. \tag{1}$$

where e_z is the upward unit normal vector and the parameters appearing are the kinematic viscosity ν, the acceleration of gravity g, the thermal expansion coefficient α and the thermometric conductivity χ. We notice that $\chi = \frac{\kappa}{\rho_0 c_p}$ where κ is the thermal conductivity, ρ_0 is the constant density and c_p is the specific heat at constant pressure (see [3], chapter 5).
The temperature, which is set to be higher at the bottom plate than at the top plate, diffuses $(-\Delta T)$ and is advected by the velocity $(u\cdot\nabla T)$. The fluid, which to leading order is incompressible $(\nabla\cdot u = 0)$ expands in response to a change in temperature (the origin of the term αT), which in presence of gravity leads to the buoyancy term $g\alpha T e_z$. This accelerates the

fluid ($\partial_t u + (u\cdot\nabla)u$ in Eulerian coordinates). The acceleration is eventually balanced by viscosity ($-\nu\Delta u$) in conjunction with the no-slip boundary condition ($u = 0$ at $z = 0$ and $z = h$). The pressure p appears as a Lagrangian multiplier to enforce the divergence-free condition. Periodicity in the horizontal variables $x' \in [0, L)^{d-1}$ is imposed for all the functions.
If the driving forces, as measured by the Rayleigh number defined below, are small, the pure conduction solution ($T = T_{\text{bottom}} - (T_{\text{bottom}} - T_{\text{top}})\frac{z}{h}$, $u = 0$) is the global attractor. Above an explicitly known critical Rayleigh number, the conduction solution is unstable and the global attractor consists of stationary *convection rolls* (see [4] for historical context). As the Rayleigh number increases further, the stationary convection rolls become unstable. For sufficiently high Rayleigh number, the temperature field features boundary layer, from which plumes detach. This is what, in this context, is called the turbulent regime.

The quantity of interest is the averaged upward heat flux. An inspection of the advection-diffusion equation for the temperature shows that the heat flux is given by $\rho_0 c_p(Tu - \chi\nabla T)$ [1]. The appropriately non-dimensionalized measure of the time-space average of the upward heat flux is given by the Nusselt number Nu defined through

$$\mathrm{Nu} = \frac{h}{\rho_0 c_p \chi(T_{bottom} - T_{top})} \lim_{t_0\uparrow\infty} \frac{1}{t_0}\int_0^{t_0} \frac{1}{h}\int_0^h \frac{1}{L^{d-1}}\int_{[0,L)^{d-1}} \rho_0 c_p(Tu - \chi\nabla T)\cdot e_z\, dx'\, dz\, dt. \tag{2}$$

Note that the Nusselt number is normalized by the term $\rho_0 c_p \frac{\chi(T_{bottom}-T_{top})}{h}$, which is the vertical heat flux associated to the pure conductive state $u = 0$ and $T = T_{bottom} - (T_{bottom} - T_{top})\frac{z}{h}$. In other words, the pure conductive solution gives rise to $\mathrm{Nu} = 1$.
For generic initial data Nu is thought to satisfy a "similarity law", that is, to be a function of the aspect ratio $\frac{L}{h}$ of the container (provided the artificial period L is assimilated to the lateral width of the container) and the two non-dimensional parameters Ra and Pr : The Rayleigh number Ra is defined as

$$\mathrm{Ra} = \frac{g\alpha(T_{\text{bottom}} - T_{\text{top}})h^3}{\nu\chi}$$

and the Prandtl number Pr is the ratio of two diffusivities

$$\mathrm{Pr} = \frac{\nu}{\chi}.$$

[1] Writing the equation of the temperature in divergence form we isolate the term $Tu - \nabla T$, which turns out to have the dimensions of heat flux after multiplication with $\rho_0 c_p$.

While the Rayleigh number is a system parameter which expresses the relative strength of the driving mechanisms (temperature differences, thermal expansion and gravity), the Prandtl number depends only on the fluid (and its absolute temperature), and is thus more arduous to vary in the experiments. It can vary from very small number (e.g. 0.015 for mercury) to large and very large numbers (e.g. 13.4 for seawater and 10^{24} for Earth's mantle). In the non-dimensionalization (3), the aspect ratio can be assimilated to the artificial lateral periodicity L. This paper does not address the dependency on the aspect ratio: We will focus on (upper) bounds that are independent of the period L, which amounts to neglecting the (limiting) effects of the lateral boundary conditions. Likewise, this paper does not address the specifics of two-dimensional flows: Our analysis is in fact oblivious to the dimension d, which in particular amounts to allowing for turbulent boundary layers.

There are classical heuristic arguments in favor of two (different) scaling laws for Nu in terms of Ra and Pr:

Spiegel in [5] realized that in the regime in which Nu is independent of the quantities characterizing both dissipative mechanisms, namely the kinematic viscosity ν and the thermometric conductivity χ, the only possible scaling is $\mathrm{Nu} \sim (\mathrm{Pr}\,\mathrm{Ra})^{\frac{1}{2}}$. Indeed, by the above definitions of Ra, Pr, and Nu, this scaling translates into

$$\lim_{t_0\uparrow\infty}\frac{1}{t_0}\int_0^{t_0}\frac{1}{h}\int_0^h\frac{1}{L^{d-1}}\int_{[0,L)^{d-1}}\tfrac{T-T_{top}}{T_{bottom}-T_{top}}u\cdot e_z dx' dz dt \sim (g\alpha(T_{bottom}-T_{top})h)^{\frac{1}{2}}\,,$$

where we neglected the conductive contribution to the heat flux and used incompressibility (together with the no-flux boundary conditions) in form of $\int_{[0,L)^{d-1}} u\cdot e_z dx' = 0$. Now the left-hand side is an average upward velocity of the warm fluid parcels whereas the right-hand side assumes the form of $(\text{acceleration}\times\text{height})^{\frac{1}{2}}$, where the acceleration $g\alpha(T_{bottom}-T_{top})$ is the (relative) upward acceleration of warm fluid parcels due to their density reduction of $\alpha(T_{bottom}-T_{top})$ in the presence of gravity g. In this sense, this scaling is the scaling of free fall, or rather, free rise (in (13) we give a more mathematical heuristic argument for this scaling regime starting from the non-dimensionalized equations).

When the inertia of the fluid is neglected, i. e. $\mathrm{Pr}=\infty$, Malkus [6] proposed the following heuristic argument in favor of the scaling $\mathrm{Nu}\sim\mathrm{Ra}^{\frac{1}{3}}$: It starts with the observation that for Ra $\gg 1$, in a boundary layer of (to be determined) thickness $\delta \ll h$, the temperature drops from T_{bottom} to its average $\frac{1}{2}(T_{bottom} + T_{top})$ and the flow is suppressed. By definition of Nu, this yields

$$\mathrm{Nu} \approx \frac{h}{2\delta},$$

so that Nu is linked to the relative size of the (thermal) boundary layer. Incidentally, an upper bound motivated by this relation is the starting point in the rigorous treatment, cf (17). Here comes the crucial argument of a *marginally stable* boundary layer: The actual size δ is expected to be proportional to the largest height h^* of container in which the pure conductive solution $T = T_{bottom} + (T_{top} - T_{bottom})\frac{z}{h^*}$ and $u = 0$ is stable. This critical height h^* (critical in both the sense of linear and nonlinear stability) is explicitly known. Here it suffices to appeal to a dimensional argument which yields that the critical Rayleigh number Ra^* must be universal, so that $\mathrm{Ra}^* \sim 1$, which in view of its definition translates into $h^* \sim (\frac{g\alpha(T_{bottom}-T_{top})}{\nu\chi})^{-\frac{1}{3}}$ so that we obtain

$$\delta \sim \mathrm{Ra}^{-\frac{1}{3}} h.$$

The combination yields the desired $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}$. We note that this implies

$$\lim_{t_0 \uparrow \infty} \frac{1}{t_0} \int_0^{t_0} \frac{1}{h} \int_0^h \frac{1}{L^{d-1}} \int_{[0,L)^{d-1}} (Tu - \chi \nabla T) \cdot e_z \, dx' dz dt \sim \left(\frac{g\alpha\chi^2 (T_{bottom} - T_{top})^4}{\nu} \right)^{\frac{1}{3}},$$

so that in this regime the heat transport is independent of the container height h (see (12) for a more mathematical heuristic argument).

Many more scaling regimes for Nu in the Pr-Ra-plane have been proposed on experimental and theoretical grounds in the physics literature. By means of mixing length theory, Kraichnan in [7] not only reproduced the scalings of Malkus and Spiegel for big and very small Pr respectively, but also suggested a third scaling $\mathrm{Nu} \sim \mathrm{Pr}^{-\frac{1}{4}} \mathrm{Ra}^{\frac{1}{2}}$ for big Ra and moderately low Pr. A fairly complete theory has been worked out in [8]. It is based on global balance laws (which we also use in our rigorous treatment (see Section 1.3) on distinguishing the cases of the dissipation dominantly taking place in the bulk or in the boundary layer) and on assumptions on the structure of both the thermal and the viscous boundary layer (which becomes relevant for $\mathrm{Pr} < \infty$). However, these statements are more speculative when the viscous boundary layer is turbulent rather than laminar.

Despite the complexity of the phenomenon of Rayleigh-Bénard convection in the turbulent regime, there are rigorous upper bounds of Nu in terms of Ra and Pr. In their 1996 paper [9], Constantin & Doering among other results gave an easy argument for $\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{2}}$ for all values of Pr. On the one hand, this bound is suboptimal for $\mathrm{Pr} \gg 1$ (as our result implies), which is not surprising since the argument is oblivious to replacing the no-slip boundary condition by a no-stress boundary condition. On the other hand, it does not capture the Spiegel scaling of $\mathrm{Nu} \sim (\mathrm{PrRa})^{\frac{1}{2}}$ in the inviscid limit $\mathrm{Pr} \ll 1$. In their seminal 1999 paper [1], Constantin & Doering considered the case of $\mathrm{Pr} = \infty$ and proved $\mathrm{Nu} \lesssim (\mathrm{Ra} \ln^2 \mathrm{Ra})^{\frac{1}{3}}$. They obtained this bound by combining global balance laws with the maximum principle for the temperature and a (logarithmically failing) maximal regularity estimate for the (quasi)-stationary Stokes equations in L^∞_x. The present paper follows and extends the strategy laid out in this work. In 2006, Charles Doering, Maria Westdickenberg (née Reznikoff) and the last author [10] obtained the same bound (with a slightly improved power of the logarithm) with help of a strategy that is inspired by the above heuristic argument of marginal stability of the boundary layer, namely the background field method. However, this method is not capable of giving the optimal bound: On the one hand, this saddle point method cannot give a better bound than $(\mathrm{Ra} \ln^{\frac{1}{15}} \mathrm{Ra})^{\frac{1}{3}}$ [11]; on the other hand, the combination of arguments in [1] and [10] yields the doubly logarithmic bound $\mathrm{Nu} \lesssim (\mathrm{Ra} \ln\ln \mathrm{Ra})^{\frac{1}{3}}$, cf. [12].

In the case of $\mathrm{Pr} < \infty$, the lack of instantaneous *slaving* of the velocity field to the temperature field increases the difficulty in bounding the convection term $\int \langle u^z T \rangle dz$ in the definition of the Nusselt number and the background field method turns out to be no longer fruitful. Besides [9], there is only one other rigorous result for $\mathrm{Pr} < \infty$: Wang [2] proved by a perturbative argument that the Constantin & Doering 1999 bound $\mathrm{Nu} \lesssim (\mathrm{Ra} \ln^2 \mathrm{Ra})^{\frac{1}{3}}$ persists for $\mathrm{Pr} \gg \mathrm{Ra}$ (see (20) for an argument why this is the classical scaling regime). Like Wang's argument, ours treats the convective nonlinearity $(u \cdot \nabla)u$ perturbatively. However, there is a difference: We perturb around the *non-stationary* Stokes equations and gain access to Ra-Pr-regimes where the effective *Reynolds* number Re is allowed to be large. In fact we work with Leray's solution and thus only appeal to the global energy estimate on the level of the Navier-Stokes equations, whereas Wang's regime is limited by the use of the small-data regularity theory for the Navier-Stokes equations and thus $\mathrm{Re} \ll 1$, which in his analysis translates into $\mathrm{Pr} \gg \mathrm{Ra}$. Our more robust strategy allows us to show that the Constantin & Doering 1999 bound $\mathrm{Nu} \lesssim (\mathrm{Ra}\ln^2\mathrm{Ra})^{\frac{1}{3}}$ (in its slightly improved form of $\mathrm{Nu} \lesssim (\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}}$) persists in the much larger regime $\mathrm{Pr} \gtrsim (\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}}$ and then crosses over to $\mathrm{Nu} \lesssim (\frac{\mathrm{Ra}\ln\mathrm{Ra}}{\mathrm{Pr}})^{\frac{1}{2}}$, which can be seen as an interpolation between the marginal stability bound and the Constantin & Doering 1996 bound as Pr decreases from large $\mathrm{Pr} = (\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}}$ to moderate $\mathrm{Pr} = 1$. Loosely speaking our analysis just requires small Re in the thermal boundary layer, not in the entire container, for the $(\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}}$ scaling to persist.

1.2 Main Results

Measuring lengths in units of the layer depth h, time in units of the thermal diffusion h^2/χ, and temperature on a scale where $T_{\mathrm{top}} = 0$ and $T_{\mathrm{bottom}} = 1$, we non-dimensionalize the problem (1) and consider

$$\begin{cases} \partial_t T + u\cdot\nabla T - \Delta T = 0 & \text{for} \quad 0<z<1\,, \\ \frac{1}{\mathrm{Pr}}(\partial_t u + (u\cdot\nabla)u) - \Delta u + \nabla p = \mathrm{Ra}Te_z & \text{for} \quad 0<z<1\,, \\ \nabla\cdot u = 0 & \text{for} \quad 0<z<1\,, \\ u = 0 & \text{for} \quad z\in\{0,1\}\,, \\ u = u_0 & \text{for} \quad t=0\,, \\ T = 1 & \text{for} \quad z=0\,, \\ T = 0 & \text{for} \quad z=1\,. \end{cases} \tag{3}$$

In terms of the dimensionless variables in equation (3) the Nusselt number turns to be

$$\mathrm{Nu} = \limsup_{t_0\uparrow\infty} \frac{1}{t_0}\int_0^{t_0}\int_0^1 \frac{1}{L^{d-1}}\int_{[0,L)^{d-1}} (Tu - \nabla T)\cdot e_z dx' dz dt,$$

where we consider the limit superior in order to avoid the cases in which the limit does not exist.
For the problem (3) we establish the following result

Theorem 1 (Bounds on the Nusselt number).
Provided the initial data satisfy $T_0 \in [0,1]$, $\int |u_0| dx < \infty$ and for $\mathrm{Ra} \gg 1$

$$\mathrm{Nu} \le C \begin{cases} (\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}} & \text{for} \quad \mathrm{Pr} \ge (\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}}, \\ \left(\mathrm{Pr}^{-1}\mathrm{Ra}\ln\mathrm{Ra}\right)^{\frac{1}{2}} & \text{for} \quad \mathrm{Pr} \le (\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}}, \end{cases} \tag{4}$$

where C depends only on the dimension d.

In our analysis, the crucial no-slip boundary condition is both a blessing and a curse, as we shall presently explain. The no-slip boundary condition is a *blessing* because, via Hardy's inequality, it gives us good control of the convective nonlinearity $(u\cdot\nabla)u$ near the boundary in terms of the average viscous dissipation $\int_0^1\langle|\nabla u|^2\rangle dz$ [2], which is the physically relevant quantity (and the only bound at hand for the Leray solution) see (15). This yields control of $(u\cdot\nabla)u$ in an L^1-type space with the weight $\frac{1}{z(1-z)}$. Hence a maximal regularity theory for the non-stationary Stokes equations with respect to this norm is required. Since this norm is borderline for Calderón-Zygmund estimates (both because the exponent and the weight are borderline), maximal regularity "fails logarithmically" and can only be recovered under bandedness assumptions (i. e. a restriction to a packet of wave numbers in Fourier space) — this is the source of the logarithm. It is in this maximal regularity theory where the *curse* of the no-slip boundary condition appears: As opposed to the no-stress boundary condition in the half space, the no-slip boundary condition does not allow for an extension by reflection to the whole space, and thereby the use of simple kernels or Fourier methods also in the normal variable. The difficulty coming from the the no-slip boundary condition in the non-stationary Stokes equations when deriving maximal regularity estimates is of course well-known; many techniques have been developed to derive Calderón-Zygmund estimates despite this difficulty. In the *half space* Solonnikov in [13] has constructed a solution formula for (5) with zero initial data via the Oseen an Green tensors. An easier and more compact representation of the solution to the problem (5) with zero forcing term and non-zero initial value was later given by Ukai in [14] by using a different method. Indeed he could write an explicit formula of the solution operator as a composition of Riesz' operators and solutions operator for the heat and Laplace's equation. This formula is an effective tool to get $L^p - L^q$ $(1 < q, p < \infty)$ estimates for the solution and its derivatives. In the case of *exterior domains*, Maremonti and Solonnikov [15] derive $L^p - L^q$ $(1 < q, p < \infty)$ estimates for (5), going through estimates for the extended solution in the half space and in the whole space. In particular in the half space they propose a decomposition of (5) with non-zero divergence equation. The book of Galdi [16] provides with a complete treatment of the classical theory and results on the non-stationary Stokes equations and Navier-Stokes equations.

The new element here is that we need maximal regularity in the *borderline*

[2]See Section 1.4 for notations.

space $L^1(dtdx'\frac{1}{z(1-z)}dz)$, and in $L^\infty_z(L^1_{t,x'})$ (the latter space coming from the original argument in 1999 paper of Constantin & Doering and pertaining to the buoyancy term). As mentioned, these borderline Calderón-Zygmund estimates can only hold under bandedness assumption.

Theorem 2 (Maximal regularity in the strip)**.**
There exists $R_0 \in (0,\infty)$ depending only on d and L such that the following holds. Let u, p, f satisfy

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= f \qquad \text{for} \quad 0<z<1\,, \\ \nabla\cdot u &= 0 \qquad \text{for} \quad 0<z<1\,, \\ u &= 0 \qquad \text{for} \quad z\in\{0,1\}\,, \\ u &= 0 \qquad \text{for} \quad t=0\,. \end{cases} \tag{5}$$

Assume f is horizontally band-limited , i.e

$$\mathcal{F}' f(k',z,t) = 0 \text{ unless } 1\le R|k'|\le 4 \text{ where } R<R_0. \tag{6}$$

Then,

$$||(\partial_t-\partial_z^2)u'||_{(0,1)}+||\nabla'\nabla u'||_{(0,1)}+||\partial_t u^z||_{(0,1)}+||\nabla^2 u^z||_{(0,1)}+||\nabla p||_{(0,1)} \lesssim ||f||_{(0,1)}, \tag{7}$$

where $||\cdot||_{(0,1)}$ denotes the norm

$$||f||_{(0,1)} := ||f||_{(R,(0,1))} = \inf_{f=f_0+f_1}\left(\sup_{0<z<1}\langle|f_0|\rangle + \int_0^1 \langle|f_1|\rangle\frac{dz}{(1-z)z}\right), \tag{8}$$

where f_0 and f_1 satisfy the bandedness assumption (6).

See Section 1.4 for notation.
Our analysis offers two insights:

It turns out that for the maximal regularity estimate for the quantity of interest, namely the second vertical derivative $\partial_z^2 u^z$ of the vertical velocity component $u^z = u\cdot e_z$, bandedness only in the *horizontal* variable x' is required. This is extremely convenient, since the horizontal Fourier transform (or rather, series), with help of which bandedness is expressed, is compatible with the lateral periodic boundary conditions.

It turns out that maximal regularity is naturally expressed in terms of the *interpolation* between the two norms of interest $L^1(dtdx'\frac{1}{z(1-z)}dz)$ and $L^\infty_z(L^1_{t,x'})$.

[3] This way, one avoids the logarithm one would expect to be the price of the borderline weight $\frac{1}{z(1-z)}$. It is a pleasant coincidence that the norm $L^\infty_z(L^1_{t,x'})$ arises for two unrelated reasons: It is needed to estimate the buoyancy term Te_z driving the Navier-Stokes equations and it is the natural partner of $L^1(dtdx'\frac{1}{z(1-z)}dz)$ in the maximal regularity estimate.

Aside from their application to this problem (see Section 3.5) all the estimates in Theorem 1 might have an independent interest since they show the full extent of what one can obtain under the horizontal bandedness assumption only.

Acknowledgments: We thank Charles Doering for many stimulating discussions on Rayleigh-Bénard convection in general, and, more specifically, for bringing the Pr-scaling to our attention.

1.3 Idea of the proof

We first notice that for the maximum principle applied to the temperature equation we have

$$\text{if } T_0 \in [0,1] \ \text{ then } \ ||T||_{L^\infty} \leq 1\,, \tag{9}$$

which furnishes us an a-priori bound on the temperature T.
One can verify that Nu as defined in (32) satisfies

$$\mathrm{Nu} = \langle Tu^z - \partial_z T\rangle \ \ \forall z \in (0,1) \tag{10}$$

(see [12]) and thus in particular

$$\mathrm{Nu} = \int_0^1 \langle (Tu - \nabla T)\cdot e_z\rangle dz. \tag{11}$$

As a side remark, we now argue that the scaling laws predicted by Malkus and Spiegel, respectively, can be simply deduced by rescaling the equations in the limiting case when the viscosity term wins over the inertial term and vice versa. On one hand, if we assume that the inertial term is negligible (setting $\mathrm{Pr} = \infty$) the equation (3) reduces to

$$\left\{\begin{array}{rcl} \partial_t T + u\cdot\nabla T - \Delta T &=& 0\,,\\ -\Delta u + \nabla p &=& \mathrm{Ra}Te_z\,,\\ \nabla\cdot u &=& 0\,. \end{array}\right.$$

[3]See Section 1.4 for notations.

Rescaling this equation according to

$$x = \mathrm{Ra}^{-\frac{1}{3}}\hat{x},\ t = \mathrm{Ra}^{-\frac{2}{3}}\hat{t},\ u = \mathrm{Ra}^{\frac{1}{3}}\hat{u},\ p = \mathrm{Ra}^{\frac{2}{3}}\hat{p}\ \text{ and thus }\ \mathrm{Nu} = \mathrm{Ra}^{\frac{1}{3}}\widehat{\mathrm{Nu}} \tag{12}$$

we end up with the parameters-free system

$$\begin{cases} \partial_{\hat{t}}T + \hat{u}\cdot\hat{\nabla}T - \hat{\Delta}T &= 0\,, \\ -\Delta\hat{u} + \nabla\hat{p} &= Te_z\,, \\ \hat{\nabla}\cdot\hat{u} &= 0\,, \end{cases}$$

which naturally lives in the half space. Since for the latter system, it is natural to expect that the heat flux is universal, i.e. $\widehat{\mathrm{Nu}} \approx 1$, we obtain $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}$.
On the other hand, if we rewrite the system (3) neglecting the diffusivity and the viscosity term

$$\begin{cases} \partial_t T + u\cdot\nabla T &= 0\,, \\ \frac{1}{\mathrm{Pr}}(\partial_t u + (u\cdot\nabla)u) + \nabla p &= \mathrm{Ra}Te_z\,, \\ \nabla\cdot u &= 0\,, \end{cases}$$

and we rescale according to

$$t = \frac{1}{(\mathrm{PrRa})^{\frac{1}{2}}}\hat{t},\ u = (\mathrm{PrRa})^{\frac{1}{2}}\hat{u},\ p = \mathrm{Ra}\hat{p}\ \text{ and thus }\ \mathrm{Nu} = (\mathrm{PrRa})^{\frac{1}{2}}\widehat{\mathrm{Nu}} \tag{13}$$

we end up with the system

$$\begin{cases} \partial_{\hat{t}}T + \hat{u}\cdot\nabla T &= 0\,, \\ \frac{1}{\mathrm{Pr}}(\partial_{\hat{t}}\hat{u} + (\hat{u}\cdot\nabla)\hat{u}) + \nabla\hat{p} &= \mathrm{Ra}Te_z\,, \\ \nabla\cdot\hat{u} &= 0\,. \end{cases}$$

Imitating the previous argument we can conclude that $\mathrm{Nu} \sim \mathrm{Pr}^{\frac{1}{2}}\mathrm{Ra}^{\frac{1}{2}}$.

From the equations of motion we can easily derive useful properties and representations of the Nusselt number: Testing the equation of the temperature with T and using (10) for $z = 0$ we get

$$\mathrm{Nu} = \int_0^1 \langle|\nabla T|^2\rangle dz, \tag{14}$$

while testing the Navier Stokes equations with u, using the incompressibility and the boundary conditions for u, we have the energy inequality

$$\int_0^1 \langle|\nabla u|^2\rangle dz \leq \mathrm{Ra}(\mathrm{Nu}-1) \tag{15}$$

for Leray solutions.
Property (10) together with the boundary conditions for T and the maximum principle (9) yield

$$\mathrm{Nu} \leq \frac{1}{\delta}\int_0^\delta \langle Tu^z\rangle dz + \frac{1}{\delta}\,, \tag{16}$$

where the vertical average is taken over an arbitrary but small boundary layer of thickness δ. Moreover, using once more the maximum principle for the temperature (9) we get

$$\mathrm{Nu} \leq \frac{1}{\delta}\int_0^\delta \langle|u^z|\rangle dz + \frac{1}{\delta}\,. \tag{17}$$

The first rigorous upper bound in the case $\mathrm{Pr} < \infty$ was derived in 1996 by Constantin and Doering [9]. Exploiting the vanishing of the normal velocity at the boundaries they could prove

$$\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{2}},$$

where with the symbol $\lesssim$ we denote the symbol $\leq$ up to universal constants. One hopes to get a better estimate exploiting the incompressibility condition which, in conjunction with the full no-slip boundary condition, implies

$$\partial_z u^z = 0 \text{ at } z = 0, 1.$$

To illustrate this idea in a simpler situation, we first consider the case of $\mathrm{Pr} = \infty$ as treated by Constantin and Doering (following the argument in [12]). Starting with inequality (17) and using Jensen's inequality in the form of $\left|\frac{d^2}{dz^2}\langle|u^z|\rangle\right| \leq \langle|\partial_z^2 u|\rangle$ we get

$$Nu \leq \delta^2 \sup_z \langle|\partial_z^2 u^z|\rangle + \frac{1}{\delta}\,. \tag{18}$$

Note that for $\mathrm{Pr} = \infty$, the velocity is instantaneously slaved to the temperature via the stationary Stokes equations

$$\begin{cases} -\Delta u + \nabla p &= \mathrm{Ra} T e_z\,, \\ \nabla \cdot u &= 0\,, \end{cases}$$

with the no-slip boundary condition. Loosely speaking, the theory of maximal regularity states that for any "reasonable " norm $||\cdot||$, one has

$$||\nabla^2 u|| \lesssim ||\mathrm{Ra} T e_z||.$$

However the Calderón-Zygmund theory fails for $||\cdot|| = \sup_z \langle|\cdot|\rangle$. Nevertheless, Constantin and Doering proved that, up to logarithms, one indeed has

$$\sup_z \langle|\partial_z^2 u^z|\rangle \lesssim \sup_z \langle|\mathrm{Ra} T e_z|\rangle \, .$$

By the maximum principle for the temperature (9), the last bound implies

$$\sup_z \langle|\partial_z^2 u^z|\rangle \lesssim \mathrm{Ra}$$

and inserting this result into (18), we get

$$\mathrm{Nu} \lesssim \delta^2 \mathrm{Ra} + \frac{1}{\delta} \, .$$

The minimum occurs for $\delta \sim \mathrm{Ra}^{-\frac{1}{3}}$ and the resulting bound is, up to logarithmic corrections, $\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{3}}$.

In this paper we will consider the case $\mathrm{Pr} < \infty$.
We perturb the non-stationary Stokes equations, bringing *only* the nonlinear term to the right hand side

$$\frac{1}{\mathrm{Pr}} \partial_t u - \Delta u + \nabla p = \mathrm{Ra} T e_z - \frac{1}{\mathrm{Pr}} (u \cdot \nabla) u. \tag{19}$$

As a side remark we now argue why

$$\mathrm{Pr} \gg \mathrm{Ra} \tag{20}$$

amounts to the classical perturbative regime for Navier-Stokes equations. The classical perturbative argument goes as follows: One seeks a norm $||\cdot||$ in which the maximal regularity estimate for the non-stationary Stokes equations (19) holds, yielding

$$||\nabla^2 u|| \lesssim ||\mathrm{Ra} T e_z - \frac{1}{\mathrm{Pr}} (u \cdot \nabla) u||.$$

This norm has to be strong enough to control the convective nonlinearity in the sense of

$$||(u\cdot\nabla)u|| \lesssim ||\nabla^2 u||^2.$$

In the application the norm should be also sufficiently weak so that (9) translates into

$$||\mathrm{Ra}Te_z|| \lesssim \mathrm{Ra}.$$

The combination yields the estimate

$$||\nabla^2 u|| \lesssim \mathrm{Ra} + \frac{1}{\mathrm{Pr}}||\nabla^2 u||^2,$$

which is nontrivial only when $\mathrm{Pr} \gg \mathrm{Ra}$.
Our analysis does not attempt to buckle via the classical perturbation argument but instead uses the dissipation bound (15) to estimate the convective nonlinearity: By the Cauchy-Schwarz inequality, Hardy's inequality and capitalizing once more on the no-slip boundary condition we have

$$\int_0^1 \langle|(u\cdot\nabla)u|\rangle\frac{dz}{z} \lesssim \left(\int_0^1\langle|u|^2\rangle\frac{dz}{z^2}\right)^{\frac{1}{2}}\left(\int_0^1\langle|\nabla u|^2\rangle dz\right)^{\frac{1}{2}} \lesssim \int_0^1\langle|\nabla u|^2\rangle dz, \tag{21}$$

which, using (15), implies

$$\int_0^1 \langle|(u\cdot\nabla)u|\rangle\frac{dz}{z} \lesssim \mathrm{NuRa}. \tag{22}$$

It is this estimate that motivates the maximal regularity theory in the norm $||\cdot|| = \int_0^\infty\langle\cdot\rangle\frac{dz}{z}$.

In order to bound the right hand side of (17) we split the solution to the Navier-Stokes equations u as

$$u = u_{CD} + u_{NL} + u_{IV},$$

where u_{CD} satisfies the non-stationary Stokes equations with the buoyancy force as right hand side [4]

$$\left\{\begin{array}{rclll} \frac{1}{\mathrm{Pr}}\partial_t u_{CD} - \Delta u_{CD} + \nabla p_{CD} & = & \mathrm{Ra}Te_z & \text{for} & 0<z<1\,,\\ \nabla\cdot u_{CD} & = & 0 & \text{for} & 0<z<1\,,\\ u_{CD} & = & 0 & \text{for} & z\in\{0,1\}\,,\\ u_{CD} & = & 0 & \text{for} & t=0\,, \end{array}\right. \tag{23}$$

[4]The stationary version of this problem was already analyzed by Constantin and Doering in the seminal paper of 1999. This motivates the subscript CD.

u_{NL} satisfies the non-stationary Stokes equations with the nonlinear term as right hand side [5]

$$\left\{\begin{array}{rcll} \frac{1}{\mathrm{Pr}}\partial_t u_{NL} - \Delta u_{NL} + \nabla p_{NL} &=& -\frac{1}{\mathrm{Pr}}(u\cdot\nabla)u & \text{for} \quad 0<z<1\,, \\ \nabla\cdot u_{NL} &=& 0 & \text{for} \quad 0<z<1\,. \\ u_{NL} &=& 0 & \text{for} \quad z\in\{0,1\}\,, \\ u_{NL} &=& 0 & \text{for} \quad t=0 \end{array}\right. \tag{24}$$

and u_{IV} satisfies the non-stationary Stokes equations with zero forcing term and non-zero initial values [6]

$$\left\{\begin{array}{rcll} \frac{1}{\mathrm{Pr}}\partial_t u_{IV} - \Delta u_{IV} + \nabla p_{IV} &=& 0 & \text{for} \quad 0<z<1\,, \\ \nabla\cdot u_{IV} &=& 0 & \text{for} \quad 0<z<1\,, \\ u_{IV} &=& 0 & \text{for} \quad z=0\,, \\ u_{IV} &=& u_0 & \text{for} \quad t=0\,. \end{array}\right. \tag{25}$$

Inserting the decomposition $u = u_{CD} + u_{NL} + u_{IV}$ into the bound (17) for the Nusselt number, we have

$$\begin{array}{rcl} \mathrm{Nu} &\leq& \frac{1}{\delta}\int_0^\delta \langle |u^z|\rangle dz + \frac{1}{\delta} \\ &\leq& \sup_{z\in(0,\delta)}\langle |u^z_{CD}|\rangle + \delta\int_0^\delta \langle |\partial_z^2 u^z_{NL}|\rangle dz + \delta^{-\frac{1}{2}}\left(\int_0^\delta \langle |u^z_{IV}|^2\rangle dz\right)^{\frac{1}{2}} + \frac{1}{\delta} \\ &\leq& \delta^2\left(\sup_{z\in(0,\delta)}\langle |\partial_z^2 u^z_{CD}|\rangle + \int_0^\delta \langle |\partial_z^2 u^z_{NL}|\rangle \frac{dz}{z}\right) + \delta^{-\frac{1}{2}}\left(\int_0^\delta \langle |u^z_{IV}|^2\rangle dz\right)^{\frac{1}{2}} + \frac{1}{\delta}. \end{array} \tag{26}$$

We notice that

$$\int_0^\delta \langle |u^z_{IV}|^2\rangle dz = 0\,. \tag{27}$$

Indeed testing the equation (25) with u_{IV} we find that

$$\int_0^{t_0}\int_{x'}\int_z |\nabla u_{IV}(x',z,t)|^2 dz dx' dt \leq \int_{x'}\int_z |u_0(x',z)|^2 dz dx'$$

and in turn by the Poincaré inequality and passing to limits we get

$$\int_0^1 \langle |u_{IV}|^2\rangle dz = 0.$$

[5] The subscript NL stands for non-linear. Indeed only in this equation the non-linear term of the Navier Stokes equations is appearing as right hand side.

[6] The subscript IV stands for initial value.

On the one hand, for equation (23) we expect the following logarithmically failing maximal regularity bound

$$\sup_{z\in(0,1)} \langle|\partial_z^2 u_{CD}^z|\rangle \lesssim \mathrm{Ra}, \tag{28}$$

just as for the case of $\mathrm{Pr}=\infty$.
On the other hand, the problem of bounding the term $\int_0^1 \langle|\partial_z^2 u_{NL}^z|\rangle \frac{dz}{z}$ in (26) requires new techniques. Nevertheless we expect

$$\int_0^\delta \langle|\partial_z^2 u_{NL}^z|\rangle \frac{dz}{z} \lesssim \frac{1}{\mathrm{Pr}} \int_0^1 \langle|(u\cdot\nabla)u|\rangle \frac{dz}{z}\,, \tag{29}$$

up to logarithmic corrections. Using (22) we obtain

$$\int_0^\delta \langle|\partial_z^2 u_{NL}^z|\rangle \frac{dz}{z} \lesssim \frac{1}{\mathrm{Pr}} \mathrm{NuRa}\,. \tag{30}$$

Inserting (27), (28) and (30) into the bound (26) for the Nusselt number and ignoring logarithmic correction factors, we get

$$\mathrm{Nu} \lesssim \delta^2 \mathrm{Ra}(1+\frac{1}{\mathrm{Pr}}\mathrm{Nu}) + \frac{1}{\delta}.$$

After choosing $\delta \sim \left(\mathrm{Ra}(1+\frac{\mathrm{Nu}}{\mathrm{Pr}})\right)^{-\frac{1}{3}}$ and applying Young's inequality, we have

$$\mathrm{Nu} \approx \mathrm{Ra}^{\frac{1}{3}} + \left(\frac{\mathrm{Ra}}{\mathrm{Pr}}\right)^{\frac{1}{2}}\,,$$

which implies, up to logarithms,

$$\mathrm{Nu} \lesssim \begin{cases} \mathrm{Ra}^{\frac{1}{3}} & \text{for } \mathrm{Pr} \geq \mathrm{Ra}^{\frac{1}{3}}\,, \\ \mathrm{Pr}^{-\frac{1}{2}}\mathrm{Ra}^{\frac{1}{2}} & \text{for } \mathrm{Pr} \leq \mathrm{Ra}^{\frac{1}{3}}\,. \end{cases} \tag{31}$$

1.4 Notations

The $(d-1)-$dimensional torus:
We denote with $[0,L)^{d-1}$ the $(d-1)-$dimensional torus of lateral size L.
The spatial vector:

$$x = (x', z) \in [0, L)^{d-1} \times \mathbb{R}\,.$$

The velocity vector field:

$$u = (u', u^z) \in \mathbb{R}^d \text{ where } u' \in \mathbb{R}^{d-1} \text{ and } u^z \in \mathbb{R}.$$

The horizontal average:

$$\langle \cdot \rangle' = \frac{1}{L^{d-1}} \int_{[0,L)^{d-1}} \cdot \quad dx'\,.$$

Long-time and horizontal average:

$$\langle \cdot \rangle = \limsup_{t_0 \to \infty} \frac{1}{t_0} \int_0^{t_0} \langle \cdot \rangle' dt\,. \tag{32}$$

Convolution in the horizontal direction:

$$f *_{x'} g(x') = \int_{[0,L)^{d-1}} f(x' - \widetilde{x'}) g(\widetilde{x'}) d\widetilde{x'}\,.$$

Convolution in the whole space:

$$f * g(x) = \int_{\mathbb{R}} \int_{[0,L)^{d-1}} f(x' - \widetilde{x'}, z - \tilde{z}) g(\widetilde{x'}, \tilde{z}) d\widetilde{x'} d\tilde{z}\,.$$

Horizontally band-limited function:
A function $g = g(x', z, t)$ is called *horizontally band-limited* with bandwidth R if it satisfies the *bandedness assumption*

$$\mathcal{F}' g(k', z, t) = 0 \text{ unless } 1 \le R|k'| \le 4 \text{ where } R < R_0. \tag{33}$$

Interpolation norm:

$$||f||_{(0,1)} = ||f||_{R;(0,1)} = \inf_{f = f_1 + f_2} \left\{ \sup_{z \in (0,1)} \langle |f_1| \rangle + \int_{(0,1)} \langle |f_2| \rangle \frac{dz}{z(1-z)} \right\}\,,$$

$$||f||_{(0,\infty)} = ||f||_{R;(0,\infty)} = \inf_{f = f_1 + f_2} \left\{ \sup_{z \in (0,\infty)} \langle |f_1| \rangle + \int_{(0,\infty)} \langle |f_2| \rangle \frac{dz}{z} \right\}\,,$$

$$||f||_{(-\infty,1)} = ||f||_{R;(-\infty,1)} = \inf_{f = f_1 + f_2} \left\{ \sup_{z \in (-\infty,1)} \langle |f_1| \rangle + \int_{(-\infty,1)} \langle |f_2| \rangle \frac{dz}{1-z} \right\}\,.$$

where f_0, f_1 satisfy the bandedness assumption (33).
Horizontal Fourier transform:

$$\mathcal{F}'f(k',z,t) = \int e^{-ik'\cdot x'} f(x',z,t)dx' \,.$$

where k' is the dual variable of x'.

Throughout the paper we will denote with $\lesssim$ the inequality up to universal constants.

2 Proof of Theorem 1

Without loss of generality we will assume $u_0 = 0$ since we have already seen that the contribution of u_{IV} to the Nusselt number is zero (see (27)).

Let us fix a smooth cut-off function ψ in Fourier space satisfying

$$\psi(k') = \begin{cases} 1 & 0 \le |k'| \le \frac{7}{2}\,, \\ 0 & \qquad |k'| \ge 4\,. \end{cases}$$

Consider the function $\zeta(k') = \psi(k') - \psi(\frac{7}{2}k')$ and define $\zeta_j(k') = \zeta(2^{-j}k')$. Notice that ζ_j is supported in $(2^j, 2^{j+2})$.
Following a Littlewood-Paley-type decomposition we construct three operators $\mathbb{P}_<$, $\mathbb{P}_j$ and $\mathbb{P}_>$, which act at the level of Fourier space by multiplication by cut off functions that localize to small, intermediate and large wavelengths respectively:

$$\mathcal{F}'\mathbb{P}_< f = \zeta_< \mathcal{F}'f,$$
$$\mathcal{F}'\mathbb{P}_j f = \zeta_j \mathcal{F}'f,$$
$$\mathcal{F}'\mathbb{P}_> f = \zeta_> \mathcal{F}'f,$$

where $\zeta_< = \sum_{j<j_1} \zeta_j$ and $\zeta_> = \sum_{j>j_2} \zeta_j$ with $j_1 < j_2$ to be determined. We notice that the operator $\mathbb{P}_j : L^p \to L^p$ for $1 \le p < \infty$ is bounded.
Inserting the decomposition into the bound for the Nusselt number (16) we get

$$\begin{aligned} \mathrm{Nu} &\le \tfrac{1}{\delta}\int_0^\delta \langle Tu^z\rangle dz + \tfrac{1}{\delta} \\ &= \tfrac{1}{\delta}\int_0^\delta \langle T\mathbb{P}_< u^z\rangle dz + \sum_{j=j_1}^{j_2} \tfrac{1}{\delta}\int_0^\delta \langle T\mathbb{P}_j u^z\rangle dz + \tfrac{1}{\delta}\int_0^\delta \langle T\mathbb{P}_> u^z\rangle + \tfrac{1}{\delta}\,. \end{aligned} \tag{34}$$

At first, let us focus on the second term in (34) arising from the intermediate wavelengths. In order to bound this term we will need the maximal regularity estimate stated in Proposition 2. For this purpose, rewrite the Navier-Stokes equations in (3) as non-stationary Stokes equations with the nonlinear term and the buoyancy term in the right hand side

$$\left\{\begin{array}{rcll} \frac{1}{\mathrm{Pr}}\partial_t\mathbb{P}_j u-\Delta\mathbb{P}_j u+\nabla\mathbb{P}_j p &=& \mathrm{Ra}\mathbb{P}_j Te_z-\frac{1}{\mathrm{Pr}}\mathbb{P}_j(u\cdot\nabla)u & \text{for}\quad 0<z<1\,,\\ \nabla\cdot\mathbb{P}_j u &=& 0 & \text{for}\quad 0<z<1\,,\\ \mathbb{P}_j u &=& 0 & \text{for}\quad z\in\{0,1\}\,,\\ \mathbb{P}_j u &=& 0 & \text{for}\quad t=0\,. \end{array}\right. \tag{35}$$

Observe that the application of the operator "horizontal filtering", namely $\mathbb{P}_j$, preserves the no-slip boundary condition at $z=0,1$ and it commutes with ∂_z and all the differential operators that act in the vertical direction. Using the maximum principle for the temperature (9), the Poincaré inequality in the $z-$variable and considering a generic decomposition of $\partial_z^2\mathbb{P}_j u^z=h_0+h_1$ we have

$$\begin{aligned} \frac{1}{\delta}\int_0^\delta\langle T\mathbb{P}_j u^z\rangle dz &\leq \frac{1}{\delta}\int_0^\delta\langle|\mathbb{P}_j u^z|\rangle dz\\ &\leq \delta\int_0^\delta\langle|\partial_z^2\mathbb{P}_j u^z|\rangle dz\\ &\leq \delta\left(\int_0^\delta\langle|h_0|\rangle dz+\int_0^\delta\langle|h_1|\rangle dz\right)\\ &\leq \delta^2\left(\sup_{0<z<1}\langle|h_0|\rangle+\int_0^1\langle|h_1|\rangle\frac{dz}{z(1-z)}\right)\,. \end{aligned}$$

Passing to the infimum over all the possible decompositions of $\partial_z^2\mathbb{P}_j u^z$ we get

$$\frac{1}{\delta}\int_0^\delta\langle T\mathbb{P}_j u^z\rangle dz\leq\delta^2||\partial_z^2\mathbb{P}_j u^z||_{(0,1)}\,.$$

We notice that $\mathbb{P}_j u$ satisfies the linear Stokes equations (35) and for $j>j_1$ it satisfies the bandedness assumption provided

$$j_1\gtrsim\log_2 R_0^{-1}\,. \tag{36}$$

Therefore by the maximal regularity estimate (7) applied to $\mathbb{P}_j u$ we have

$$||\partial_z^2\mathbb{P}_j u^z||_{(0,1)}\leq\sup_{0<z<1}\langle|\mathrm{Ra}\mathbb{P}_j Te_z|\rangle+\int_0^1\langle|\frac{1}{\mathrm{Pr}}\mathbb{P}_j(u\cdot\nabla)u|\rangle\frac{dz}{z(1-z)}\,.$$

Applying again the maximum principle (9) to the first term of the right hand side we find

$$\sup_{0<z<1} \langle |\mathrm{Ra}\mathbb{P}_j T e_z| \rangle \leq \mathrm{Ra}\,.$$

To estimate the nonlinear part we apply the Cauchy-Schwarz inequality and Hardy's inequality

$$\begin{aligned}
&\int_0^1 \langle | \frac{1}{\mathrm{Pr}} \mathbb{P}_j (u\cdot\nabla)u|\rangle \frac{dz}{z(1-z)} \\
\leq\ & \frac{1}{\mathrm{Pr}} \int_0^1 \langle |\mathbb{P}_j (u\cdot\nabla)u|\rangle \frac{dz}{z} + \frac{1}{\mathrm{Pr}} \int_0^1 \langle |\mathbb{P}_j (u\cdot\nabla)u|\rangle \frac{dz}{1-z} \\
\lesssim\ & \frac{1}{\mathrm{Pr}} \left(\int_0^1 \frac{1}{z^2} \langle |\mathbb{P}_j u|^2\rangle dz + \int_0^1 \frac{1}{(1-z)^2} \langle |\mathbb{P}_j u|^2\rangle dz \right)^{\frac{1}{2}} \left(\int_0^1 \langle |\nabla \mathbb{P}_j u|^2\rangle dz \right)^{\frac{1}{2}} \\
\lesssim\ & \frac{1}{\mathrm{Pr}} \left(\int_0^1 \langle |\partial_z \mathbb{P}_j u|^2\rangle dz \right)^{\frac{1}{2}} \left(\int_0^1 \langle |\nabla \mathbb{P}_j u|^2\rangle dz \right)^{\frac{1}{2}} \\
\lesssim\ & \frac{1}{\mathrm{Pr}} \int_0^1 \langle |\nabla u|^2\rangle dz \\
\overset{(15)}{\leq}\ & \frac{1}{\mathrm{Pr}} \mathrm{Ra}(\mathrm{Nu}-1)\,.
\end{aligned}$$

Summing up over all the intermediate wavelengths we obtain

$$\sum_{j=j_1}^{j_2} \frac{1}{\delta} \int_0^\delta \langle T \mathbb{P}_j u^z \rangle dz \lesssim (j_2 - j_1)\delta^2 \left(\mathrm{Ra} + \frac{1}{\mathrm{Pr}} \mathrm{Ra}(\mathrm{Nu}-1) \right)\,. \tag{37}$$

We now turn to the first term appearing on the right hand side of (34), contribution of the small wavelengths. By using the Cauchy-Schwarz inequality, the divergence-free condition, the horizontal bandedness assumption in form of (119) and the Poincaré inequality in the $z-$variable, we obtain

$$
\begin{aligned}
\frac{1}{\delta}\int_0^\delta \langle T\mathbb{P}_< u^z\rangle dz &= \frac{1}{\delta}\int_0^\delta \langle (T-1)\mathbb{P}_< u^z\rangle dz \qquad (38)\\
&\lesssim \frac{1}{\delta}\left(\int_0^\delta \langle |T-1|^2\rangle dz\right)^{\frac{1}{2}}\left(\int_0^\delta \langle |\mathbb{P}_< u^z|^2\rangle dz\right)^{\frac{1}{2}}\\
&\lesssim \delta\frac{1}{\delta}\left(\int_0^\delta \langle |\partial_z T|^2\rangle dz\right)^{\frac{1}{2}}\delta\left(\int_0^\delta \langle |\partial_z\mathbb{P}_< u^z|^2\rangle dz\right)^{\frac{1}{2}}\\
&\lesssim \delta\left(\int_0^\delta \langle |\nabla T|^2\rangle dz\right)^{\frac{1}{2}}\left(\int_0^\delta \langle |\mathbb{P}_<\nabla'\cdot u'|^2\rangle dz\right)^{\frac{1}{2}}\\
&\lesssim \delta\left(\int_0^\delta \langle |\nabla T|^2\rangle dz\right)^{\frac{1}{2}}2^{j_1}\left(\int_0^\delta \langle |u'|^2\rangle dz\right)^{\frac{1}{2}}\\
&\lesssim \delta\left(\int_0^\delta \langle |\nabla T|^2\rangle dz\right)^{\frac{1}{2}}2^{j_1}\delta\left(\int_0^\delta \langle |\partial_z u'|^2\rangle dz\right)^{\frac{1}{2}}\\
&\lesssim 2^{j_1}\delta^2\left(\int_0^\delta \langle |\nabla T|^2\rangle dz\right)^{\frac{1}{2}}\left(\int_0^\delta \langle |\nabla' u|^2\rangle dz\right)^{\frac{1}{2}}\\
&\lesssim 2^{j_1}\delta^2\left(\int_0^1 \langle |\nabla T|^2\rangle dz\right)^{\frac{1}{2}}\left(\int_0^1 \langle |\nabla u|^2\rangle dz\right)^{\frac{1}{2}}\\
&\overset{(14)\&(15)}{\lesssim} 2^{j_1}\delta^2\mathrm{Ra}^{\frac{1}{2}}\mathrm{Nu}\,, \qquad (39)
\end{aligned}
$$

where in (38) we used the fact that $\langle u^z\rangle = 0$. Finally, we turn to the third term in (34), which represents the contribution from the large wavelengths. In order to estimate this term we use the Cauchy-Schwarz inequality, the Poincaré inequality in the $z-$variable and the horizontal bandedness assumption in form of (117) applied to T

$$\begin{aligned}\frac{1}{\delta}\int_0^\delta\langle\mathbb{P}_>Tu^z\rangle dz &\leq \frac{1}{\delta}\left(\int_0^\delta\langle|\mathbb{P}_>T|^2\rangle dz\right)^{\frac12}\left(\int_0^\delta\langle|u^z|^2\rangle dz\right)^{\frac12}\\ &\leq \frac{1}{\delta}\frac{1}{2^{j_2}}\left(\int_0^\delta\langle|\nabla' T|^2\rangle dz\right)^{\frac12}\delta\left(\int_0^\delta\langle|\partial_z u^z|^2\rangle dz\right)^{\frac12}\\ &\leq \frac{1}{2^{j_2}}\left(\int_0^1\langle|\nabla T|^2\rangle dz\right)^{\frac12}\left(\int_0^\delta\langle|\nabla u|^2\rangle dz\right)^{\frac12}\\ &\overset{(14)\&(15)}{\lesssim} \frac{1}{2^{j_2}}\mathrm{Ra}^{\frac12}\mathrm{Nu}\,. \end{aligned} \tag{40}$$

Putting the three estimates (37),(39) and (40) together, we have the following bound on the Nusselt number

$$\mathrm{Nu}\lesssim (j_2-j_1)\delta^2\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)\mathrm{Ra}+\left(\delta^2 2^{j_1}+\frac{1}{2^{j_2}}\right)\mathrm{Ra}^{\frac12}\mathrm{Nu}+\frac{1}{\delta}\,.$$

In the last inequality we impose $2^{-j_2}=2^{j_1}\delta^2$. In turn, observe that $2^{-j_2}=2^{-\frac{(j_2-j_1)}{2}}\delta$ and therefore

$$\mathrm{Nu}\lesssim (j_2-j_1)\delta^2\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)\mathrm{Ra}+2^{-\frac{(j_2-j_1)}{2}}\delta\mathrm{Ra}^{\frac12}\mathrm{Nu}+\frac{1}{\delta}\,. \tag{41}$$

Observe that, on the one hand, we want the second term of the right hand side to be absorbed in the left hand side, therefore we impose

$$1\approx 2^{-\frac{(j_2-j_1)}{2}}\delta\mathrm{Ra}^{\frac12}$$

and, on the other hand, we require all the terms in the right hand side to be of the same size

$$(j_2-j_1)\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)\mathrm{Ra}\approx\frac{1}{\delta^3}. \tag{42}$$

From these two conditions we deduce

$$(j_2-j_1)2^{\frac32(j_2-j_1)}\approx\mathrm{Ra}^{\frac12}\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)^{-1},$$

which is of the form $x \log_a x \approx y$ with $x = a^{(j_2 - j_1)}$ and $y = \mathrm{Ra}^{\frac{1}{2}} \left(\frac{\mathrm{Nu}}{\mathrm{Pr}} + 1\right)^{-1}$ for $a > 1$. This implies that, asymptotically, $x \approx \frac{y}{\log_a y}$ and therefore

$$j_2 - j_1 \approx \log_a \left(\frac{\mathrm{Ra}^{\frac{1}{2}} \left(\frac{\mathrm{Nu}}{\mathrm{Pr}} + 1\right)^{-1}}{\log_a(\mathrm{Ra}^{\frac{1}{2}} \left(\frac{\mathrm{Nu}}{\mathrm{Pr}} + 1\right)^{-1})} \right) \approx \ln \mathrm{Ra}.$$

Inserting this back into (42), we are led to the natural choice of δ

$$\delta = \left(\left(\frac{\mathrm{Nu}}{\mathrm{Pr}} + 1 \right) \mathrm{Ra} \ln \mathrm{Ra} \right)^{-\frac{1}{3}},$$

which give us the bound

$$\mathrm{Nu} \lesssim \left(\left(\frac{\mathrm{Nu}}{\mathrm{Pr}} + 1 \right) \mathrm{Ra} \ln \mathrm{Ra} \right)^{\frac{1}{3}}.$$

Applying the triangle inequality [7]

$$\mathrm{Nu} \lesssim (\mathrm{Ra} \ln \mathrm{Ra})^{\frac{1}{3}} + \left(\left(\frac{\mathrm{Nu}}{\mathrm{Pr}} \right) \mathrm{Ra} \ln \mathrm{Ra} \right)^{\frac{1}{3}}$$

and Young's inequality, we finally obtain

$$\mathrm{Nu} \lesssim (\mathrm{Ra} \ln \mathrm{Ra})^{\frac{1}{3}} + \left(\frac{\mathrm{Ra} \ln \mathrm{Ra}}{\mathrm{Pr}} \right)^{\frac{1}{2}}.$$

In conclusion we get the following bound on the Nusselt number

$$\mathrm{Nu} \lesssim \begin{cases} (\mathrm{Ra} \ln \mathrm{Ra})^{\frac{1}{3}} & \text{for } \mathrm{Pr} \geq (\mathrm{Ra} \ln \mathrm{Ra})^{\frac{1}{3}}, \\ \left(\frac{\mathrm{Ra}}{\mathrm{Pr}} \ln \mathrm{Ra}\right)^{\frac{1}{2}} & \text{for } \mathrm{Pr} \leq (\mathrm{Ra} \ln \mathrm{Ra})^{\frac{1}{3}}. \end{cases}$$

[7] Note that for $0 < p < 1$ we have

$$||f + g||_p \leq 2^{\frac{1}{p} - 1}(||f||_p + ||g||_p)$$

3 Maximal regularity in the strip

3.1 From the strip to the half space

Let us consider the non-stationary Stokes equations

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= f \qquad \text{for} \quad 0 < z < 1\,, \\ \nabla \cdot u &= 0 \qquad \text{for} \quad 0 < z < 1\,, \\ u &= 0 \qquad \text{for} \quad z \in \{0,1\}\,, \\ u &= 0 \qquad \text{for} \quad t = 0\,. \end{cases}$$

In order to prove the maximal regularity estimate in the strip we extend the problem (5) in the half space. By symmetry, it is enough to consider for the moment the extension to the upper half space.
Consider the localization $(\tilde{u}, \tilde{p}) := (\eta u, \eta p)$ where

$$\eta(z) \text{ is a cut-off function for } [0, \frac{1}{2}) \text{ in } [0,1)\,. \tag{43}$$

Extending $(\tilde{u}, \tilde{p})$ by zero they can be viewed as functions in the upper half space. The couple $(\tilde{u}, \tilde{p})$ satisfies

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} + \nabla \tilde{p} &= \tilde{f} \qquad \text{for} \quad z > 0\,, \\ \nabla \cdot \tilde{u} &= \tilde{\rho} \qquad \text{for} \quad z > 0\,, \\ \tilde{u} &= 0 \qquad \text{for} \quad z = 0\,, \\ \tilde{u} &= 0 \qquad \text{for} \quad t = 0\,, \end{cases} \tag{44}$$

where

$$\tilde{f} := \eta f - 2(\partial_z \eta)\partial_z u - (\partial_z^2 \eta) u + (\partial_z \eta) p e_z, \qquad \tilde{\rho} := (\partial_z \eta) u^z\,. \tag{45}$$

3.2 Maximal regularity in the upper half space

In the half space, taking advantages from the explicit representation of the solution via Green functions, we prove the regularity estimates which will be crucial in the proof of Theorem 2.

Proposition 1 (Maximal regularity in the upper half space)**.**
Consider the non-stationary Stokes equations in the upper half-space

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= f \qquad \text{for} \quad z > 0\,, \\ \nabla \cdot u &= \rho \qquad \text{for} \quad z > 0\,, \\ u &= 0 \qquad \text{for} \quad z = 0\,, \\ u &= 0 \qquad \text{for} \quad t = 0\,. \end{cases} \tag{46}$$

Suppose that f and ρ are horizontally band-limited , i.e

$$\mathcal{F}'f(k',z,t)=0 \textit{ unless } 1\le R|k'|\le 4 \textit{ where } R\in(0,\infty)\,, \tag{47}$$

and

$$\mathcal{F}'\rho(k',z,t)=0 \textit{ unless } 1\le R|k'|\le 4 \textit{ where } R\in(0,\infty)\,. \tag{48}$$

Then

$$\begin{aligned} & ||\partial_t u^z||_{(0,\infty)}+||\nabla^2 u^z||_{(0,\infty)}+||\nabla p||_{(0,\infty)}+||(\partial_t-\partial_z^2)u'||_{(0,\infty)}+||\nabla'\nabla u'||_{(0,\infty)} \\ \lesssim\; & ||f||_{(0,\infty)}+||(-\Delta')^{-\frac12}\partial_t\rho||_{(0,\infty)}+||(-\Delta')^{-\frac12}\partial_z^2\rho||_{(0,\infty)}+||\nabla\rho||_{(0,\infty)}, \end{aligned}$$

where $||\cdot||_{(0,\infty)}$ denotes the norm

$$||f||_{(0,\infty)}:=||f||_{R;(0,\infty)}\inf_{f=f_0+f_1}\left(\sup_{0<z<\infty}\langle|f_0|\rangle+\int_0^\infty\langle|f_1|\rangle\frac{dz}{z}\right)\,, \tag{49}$$

where f_0 and f_1 satisfy the bandedness assumption (47).

The first ingredient to establish Proposition 1 is a suitable representation of the solution operator $(f=(f',f^z),\rho)\to u=(u',u^z)$ of the Stokes equations with the no-slip boundary condition. In the case of no-slip boundary condition the Laplace operator has to be factorized as $\Delta=\partial_z^2+\Delta'=(\partial_z+(-\Delta')^{\frac12})(\partial_z-(-\Delta')^{\frac12})$. In this way the solution operator to the Stokes equations with the no-slip boundary condition (46) can be written as the fourfold composition of solution operators to three more elementary boundary value problems:

- Backward fractional diffusion equation (50):

$$\left\{\begin{array}{rcll} (\partial_z-(-\Delta')^{\frac12})\phi &=& \nabla\cdot f-(\partial_t-\Delta)\rho & \text{for}\quad z>0\,,\\ \phi &\to& 0 & \text{for}\quad z\to\infty. \end{array}\right. \tag{50}$$

- Heat equation (51):

$$\left\{\begin{array}{rcll} (\partial_t-\Delta)v^z &=& (-\Delta')^{\frac12}(f^z-\phi)-\nabla'\cdot f'+(\partial_t-\Delta)\rho & \text{for}\quad z>0,\\ v^z &=& 0 & \text{for}\quad z=0\,,\\ v^z &=& 0 & \text{for}\quad t=0\,. \end{array}\right. \tag{51}$$

- Forward fractional diffusion equation (52):

$$\begin{cases} (\partial_z + (-\Delta')^{\frac{1}{2}})u^z &= v^z & \text{for} \quad z > 0\,, \\ u^z &= 0 & \text{for} \quad z = 0\,. \end{cases} \tag{52}$$

- Heat equation (53):

$$\begin{cases} (\partial_t - \Delta)v' &= (1 + \nabla'(-\Delta')^{-1}\nabla' \cdot)f' & \text{for} \quad z > 0\,, \\ v' &= 0 & \text{for} \quad z = 0\,, \\ v' &= 0 & \text{for} \quad t = 0\,. \end{cases} \tag{53}$$

Finally set

$$u' = v' - \nabla'(-\Delta')^{-1}(\rho - \partial_z u^z)\,. \tag{54}$$

In order to prove the validity of the decomposition we need to argue that

$$(\partial_t - \Delta)u - f \text{ is irrotational },$$

which reduces to prove that

$$(\partial_t - \Delta)u' - f' \text{ is irrotational in } x'$$

and

$$\partial_z((\partial_t - \Delta)u' - f') = \nabla'((\partial_t - \Delta)u^z - f^z)\,. \tag{55}$$

Let us consider for simplicity $\rho = 0$. The first statement follows easily from the definition. Indeed by definition (54) and equation (53),

$$(\partial_t - \Delta)u' - f' = \nabla'((-\Delta')^{-1}\nabla' \cdot f' + (-\Delta')^{-1}\partial_z u^z).$$

Let us now focus on (55), which by using (54) and (53) can be rewritten as

$$\partial_z \nabla'((-\Delta')^{-1}\nabla' \cdot f' + (-\Delta')^{-1}(\partial_t - \Delta)\partial_z u^z) = \nabla'((\partial_t - \Delta)u^z - f^z)\,.$$

Because of the periodic boundary conditions in the horizontal direction, the latter is equivalent to

$$\partial_z(-\Delta')((-\Delta')^{-1}\nabla' \cdot f' + (-\Delta')^{-1}(\partial_t - \Delta)\partial_z u^z) = (-\Delta')((\partial_t - \Delta)u^z - f^z),$$

that, after factorizing $\Delta = (\partial_z - (-\Delta')^{\frac{1}{2}})(\partial_z + (-\Delta')^{\frac{1}{2}})$, turns into

$$(\partial_z - (-\Delta')^{\frac{1}{2}})(\partial_t - \Delta)(\partial_z + (-\Delta')^{\frac{1}{2}})u^z = (-\Delta')f^z - \partial_z \nabla' \cdot f'\,.$$

One can easily check that the identity holds true by applying (52), (51) and (50). The no-slip boundary condition is trivially satisfied, indeed by (52) we have $u^z = 0$ and $\partial_z u^z = 0$. The combination of (54) with $\partial_z u^z = 0$ gives $u' = 0$.

For each step of the decomposition of the Navier Stokes equations we will derive maximal regularity-type estimates. These are summed up in the following

Proposition 2.

1. *Let ϕ, f, ρ satisfy the problem (50) and assume f, ρ are horizontally band-limited, i.e*

$$\mathcal{F}' f(k', z, t) = 0 \text{ unless } 1 \le R|k'| \le 4$$

and

$$\mathcal{F}' \rho(k', z, t) = 0 \text{ unless } 1 \le R|k'| \le 4.$$

Then,

$$||\phi||_{(0,\infty)} \lesssim ||f||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}} \partial_t \rho||_{(0,\infty)} + ||\nabla \rho||_{(0,\infty)} \, .$$

2. *Let v^z, f, ϕ, ρ satisfy the problem (51) and assume f, ϕ, ρ are horizontally band-limited, i.e*

$$\mathcal{F}' f(k', z, t) = 0 \text{ unless } 1 \le R|k'| \le 4 \, ,$$

$$\mathcal{F}' \phi(k', z, t) = 0 \text{ unless } 1 \le R|k'| \le 4$$

and

$$\mathcal{F}' \rho(k', z, t) = 0 \text{ unless } 1 \le R|k'| \le 4 \, .$$

Then,

$$\begin{aligned} & ||\nabla v^z||_{(0,\infty)} + ||(-\Delta)^{-\frac{1}{2}} (\partial_t - \partial_z^2) v^z||_{(0,\infty)} \\ \lesssim \; & ||f||_{(0,\infty)} + ||\phi||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}} \partial_t \rho||_{(0,\infty)} \\ + \; & ||(-\Delta)^{-\frac{1}{2}} \partial_z^2 \rho||_{(0,\infty)} + ||\nabla \rho||_{(0,\infty)} \, . \end{aligned} \tag{56}$$

3. *Let u^z, v^z satisfy the problem (52) and assume v^z is horizontally band-limited, i.e*

$$\mathcal{F}'v^z(k',z,t) = 0 \text{ unless } 1 \le R|k'| \le 4\,.$$

Then,

$$\begin{aligned} & ||\partial_t u^z||_{(0,\infty)} + ||\nabla^2 u^z||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z(\partial_t - \partial_z^2)u^z||_{(0,\infty)} \\ \lesssim\ & ||\nabla v^z||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)}\,. \end{aligned} \tag{57}$$

4. *Let v', f', satisfy the problem (53) and assume f' is horizontally band-limited, i.e*

$$\mathcal{F}'f(k',z,t) = 0 \text{ unless } 1 \le R|k'| \le 4\,.$$

Then,

$$||\nabla'\nabla v'||_{(0,\infty)} + ||(\partial_t - \partial_z^2)v'||_{(0,\infty)} \lesssim ||f'||_{(0,\infty)}\,. \tag{58}$$

3.3 Proof of Proposition 1

By an easy application of Proposition 2, we will now prove the maximal regularity estimate on the upper half space.

Proof of Proposition 1.
From Proposition 2 we have the following bound for the vertical component of the velocity u

$$\begin{aligned} & ||\partial_t u^z||_{(0,\infty)} + ||\nabla^2 u^z||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z(\partial_t - \partial_z^2)u^z||_{(0,\infty)} \\ \overset{(57)}{\lesssim}\ & ||\nabla v^z||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)} \\ \overset{(56)}{\lesssim}\ & ||f||_{(0,\infty)} + ||\phi||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\rho||_{(0,\infty)} + ||(-\Delta)^{-\frac{1}{2}}\partial_z^2\rho||_{(0,\infty)} + ||\nabla\rho||_{(0,\infty)} \\ \overset{(1)}{\lesssim}\ & ||f||_{(0,\infty)} + ||(-\Delta')^{\frac{1}{2}}\partial_t\rho||_{(0,\infty)} + ||(-\Delta)^{-\frac{1}{2}}\partial_z^2\rho||_{(0,\infty)} + ||\nabla\rho||_{(0,\infty)}\,. \end{aligned}$$

Instead for the horizontal components of the velocity u' we have

$$
\begin{aligned}
& ||(\partial_t - \partial_z^2)u'||_{(0,\infty)} + ||\nabla'\nabla u'||_{(0,\infty)} \\
\overset{(54)}{\lesssim} \quad & ||(\partial_t - \partial_z^2)v'||_{(0,\infty)} + ||\nabla'\nabla v'||_{(0,\infty)} \\
+ \quad & ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)\rho||_{(0,\infty)} + ||\nabla\rho||_{(0,\infty)} \\
+ \quad & ||(-\Delta')^{-\frac{1}{2}}\partial_z(\partial_t - \partial_z^2)u^z||_{(0,\infty)} + ||\partial_z\nabla u^z||_{(0,\infty)} \\
\overset{(56),(57),(58)}{\lesssim} \quad & ||f||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\rho||_{(0,\infty)} + ||(-\Delta)^{-\frac{1}{2}}\partial_z^2\rho||_{(0,\infty)} + ||\nabla\rho||_{(0,\infty)}\,.
\end{aligned}
$$

Summing up we obtain

$$
\begin{aligned}
& ||\partial_t u^z||_{(0,\infty)} + ||\nabla^2 u^z||_{(0,\infty)} + ||(\partial_t - \partial_z^2)u'||_{(0,\infty)} + ||\nabla'\nabla u'||_{(0,\infty)} \\
\lesssim \quad & ||f||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\rho||_{(0,\infty)} + ||(-\Delta)^{-\frac{1}{2}}\partial_z^2\rho||_{(0,\infty)} + ||\nabla\rho||_{(0,\infty)}\,.
\end{aligned}
\tag{59}
$$

The bound for the ∇p follows by equations (46) and applying (59). □

3.4 Proof of Proposition 2

This section is devoted to the proof of Proposition 2, which rely on a series of Lemmas (Lemma 1, Lemma 2 and Lemma 3) that we state here and prove in Section 4.

The following Lemmas contain the basic maximal regularity estimates for the three auxiliary problems. These estimates, together with the bandedness assumption in the form of (120), (121) and (122) will be the main ingredients for the proof of Proposition 2.

Lemma 1.
Let u, f satisfy the problem

$$
\begin{cases}
(\partial_z - (-\Delta')^{\frac{1}{2}})u = f & \text{for} \quad z > 0\,, \\
u \to 0 & \text{for} \quad z \to \infty
\end{cases}
\tag{60}
$$

and assume f to be horizontally band-limited, i.e

$$
\mathcal{F}'f(k', z, t) = 0 \quad \textit{unless} \quad 1 \leq R|k'| \leq 4\,.
$$

Then,

$$
||\nabla u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}\,. \tag{61}
$$

Lemma 2.
Let $u, f, g = g(x', t)$ satisfy the problem

$$\left\{ \begin{array}{rcllr} (\partial_z + (-\Delta')^{\frac{1}{2}})u & = & f & \text{for} & z > 0\,, \\ u & = & g & \text{for} & z = 0 \end{array} \right. \tag{62}$$

and define the constant extension $\tilde{g}(x', z, t) := g(x', t)$. Assume f and g to be horizontally band-limited, i.e

$$\mathcal{F}' f(k', z, t) = 0 \quad \textit{unless} \quad 1 \leq R|k'| \leq 4$$

and

$$\mathcal{F}' g(k', z, t) = 0 \quad \textit{unless} \quad 1 \leq R|k'| \leq 4\,.$$

Then

$$||\nabla u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)} + ||\nabla' \tilde{g}||_{(0,\infty)}\,. \tag{63}$$

Remark 1. *Clearly if $g = 0$ in Lemma 2, then we have*

$$||\nabla u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}\,. \tag{64}$$

Lemma 3.
Let u, f satisfy the problem

$$\left\{ \begin{array}{rcllr} (\partial_t - \Delta)u & = & f & \text{for} & z > 0\,, \\ u & = & 0 & \text{for} & z = 0\,, \\ u & = & 0 & \text{for} & t = 0 \end{array} \right. \tag{65}$$

and assume f to be horizontally band-limited, i.e

$$\mathcal{F}' f(k', z, t) = 0 \quad \textit{unless} \quad 1 \leq R|k'| \leq 4\,.$$

Then,

$$||(\partial_t - \partial_z^2)u||_{(0,\infty)} + ||\nabla' \nabla u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}\,. \tag{66}$$

Proof of Proposition 2.

1. Subtracting the quantity $(\partial_z - (-\Delta')^{\frac{1}{2}})(f^z + \partial_z \rho)$ from both sides of equation (50) and then multiplying the new equation by $(-\Delta)^{-\frac{1}{2}}$ we get

$$\begin{aligned} & (\partial_z - (-\Delta')^{\frac{1}{2}})(-\Delta')^{-\frac{1}{2}}(\phi - f^z - \partial_z \rho) \\ = \; & \nabla' \cdot (-\Delta')^{-\frac{1}{2}} f' + f^z - (-\Delta')^{-\frac{1}{2}} \partial_t \rho + \partial_z \rho - (-\Delta')^{\frac{1}{2}} \rho \,. \end{aligned}$$

From the basic estimate (61) we obtain

$$\begin{aligned} & \|\nabla'(-\Delta')^{-\frac{1}{2}}(\phi - f^z - \partial_z \rho)\|_{(0,\infty)} \lesssim \|\nabla' \cdot (-\Delta')^{-\frac{1}{2}} f'\|_{(0,\infty)} \\ + \; & \|f^z\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}} \partial_t \rho\|_{(0,\infty)} + \|\partial_z \rho\|_{(0,\infty)} + \|(-\Delta')^{\frac{1}{2}} \rho\|_{(0,\infty)} \,. \end{aligned}$$

Thanks to the bandedness assumption in the form of (120) and (121) we have

$$\begin{aligned} & \|\phi - f^z - \partial_z \rho\|_{(0,\infty)} \\ \lesssim \; & \|f'\|_{(0,\infty)} + \|f^z\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}} \partial_t \rho\|_{(0,\infty)} + \|\partial_z \rho\|_{(0,\infty)} + \|\nabla' \rho\|_{(0,\infty)} \end{aligned}$$

and from this we obtain easily the desired estimate (1).

2. After multiplying the equation (51) by $(-\Delta')^{-\frac{1}{2}}$, the application of (66) to $(-\Delta')^{-\frac{1}{2}} v^z$ yields

$$\begin{aligned} & \|(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2) v^z\|_{(0,\infty)} + \|(-\Delta')^{-\frac{1}{2}} \nabla' \nabla v^z\|_{(0,\infty)} \\ \lesssim \; & \|f^z\|_{(0,\infty)} + \|\phi\|_{(0,\infty)} + \|\nabla' \cdot (-\Delta')^{-\frac{1}{2}} f'\|_{(0,\infty)} \\ + \; & \|(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2) \rho\|_{(0,\infty)} + \|(-\Delta')^{\frac{1}{2}} \rho\|_{(0,\infty)} \,. \end{aligned}$$

The estimate (56) follows after observing (121) and applying the triangle inequality to the second to last term on the right hand side.

3. We need to estimate the the three terms on the right hand side of (57) separately. We start with the term $\nabla^2 u^z$: since $\|\nabla^2 u^z\|_{(0,\infty)} \leq$

$||\nabla'\nabla u^z||_{(0,\infty)} + ||\partial_z^2 u^z||_{(0,\infty)}$, we tackle the term $\nabla'\nabla u^z$ and $\partial_z^2 u^z$ separately. First multiply by ∇' the equation (52). An application of the estimate (64) to $\nabla' u^z$ yields

$$||\nabla\nabla' u^z||_{(0,\infty)} \lesssim ||\nabla' v^z||_{(0,\infty)}. \tag{67}$$

Now multiplying the equation (52) by ∂_z^2

$$\partial_z^2 u^z = -(-\Delta')^{\frac{1}{2}}\partial_z u^z + \partial_z v^z = -\Delta' u^z - (-\Delta')^{\frac{1}{2}} v^z + \partial_z v^z \tag{68}$$

and using the bandedness assumption in the form (121) we have

$$\begin{aligned} ||\partial_z^2 u^z||_{(0,\infty)} &\leq ||\nabla'^2 u^z||_{(0,\infty)} + ||\nabla v^z||_{(0,\infty)} \\ &\overset{(67)}{\leq} ||\nabla v^z||_{(0,\infty)}\,. \end{aligned} \tag{69}$$

The second term of (57), i.e $(-\Delta')^{-\frac{1}{2}}\partial_z(\partial_t - \partial_z^2)u^z$, can be bounded in the following way: We multiply the equation (52) by $(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)$

$$\begin{cases} (\partial_z + (-\Delta')^{\frac{1}{2}})(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)u^z = (-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z & \text{for} \quad z > 0, \\ (-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)u^z = (-\Delta')^{-\frac{1}{2}}\partial_z v^z & \text{for} \quad z = 0, \end{cases}$$

where we have used that at $z = 0$

$$(\partial_t - \partial_z^2)u^z = -\partial_z^2 u^z \overset{(68)}{=} \partial_z v^z.$$

Applying (63) to $(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)u^z$ and using the bandedness assumption in the form of (120),

$$||\nabla(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)u^z||_{(0,\infty)} \lesssim ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)} + ||\partial_z v^z||_{(0,\infty)}\,. \tag{70}$$

Finally we can bound the last term of (57), i.e $\partial_t u^z$: We observe that $\partial_t u^z = (\partial_t - \partial_z^2)u^z + \partial_z^2 u^z$ thus

$$||\partial_t u^z||_{(0,\infty)} \leq ||(\partial_t - \partial_z^2)u^z||_{(0,\infty)} + ||\partial_z^2 u^z||_{(0,\infty)}\,. \tag{71}$$

For the first term in the right hand side of (71) we notice that

$$\begin{aligned}
||(\partial_t - \partial_z^2)u^z||_{(0,\infty)} &\overset{(120)}{\leq} ||(-\Delta')^{-\frac{1}{2}}\nabla'(\partial_t - \partial_z^2)u^z||_{(0,\infty)} \\
&\overset{(70)}{\lesssim} ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)} + ||\partial_z v^z||_{(0,\infty)} \\
&\lesssim ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)} + ||\nabla v^z||_{(0,\infty)} \, .
\end{aligned}$$

The second term on the right hand side of (71) is bounded in (69). Thus we have the following bound for $\partial_t u$

$$||\partial_t u^z||_{(0,\infty)} \leq ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)} + ||\nabla v^z||_{(0,\infty)} \, . \tag{72}$$

Putting together all the above we obtain the desired estimate.

4. From the defining equation (53), the basic estimate (66) and the bandedness assumption in form of (122), we get

$$||(\partial_t - \partial_z^2)v'||_{(0,\infty)} + ||\nabla'\nabla v'||_{(0,\infty)} \lesssim ||f'||_{(0,\infty)} \, .$$

□

3.5 Proof of Theorem 2

Let u, p, f be the solutions of the non-stationary Stokes equations in the strip $0 < z < 1$ (5). Then $\tilde{u} = \eta u, \tilde{p} = \eta p$ (with η defined in (43) satisfy (44), namely

$$\begin{cases}
\partial_t \tilde{u} - \Delta \tilde{u} + \nabla \tilde{p} = \tilde{f} & \text{for} \quad z > 0 \, , \\
\nabla \cdot \tilde{u} = \tilde{\rho} & \text{for} \quad z > 0 \, , \\
\tilde{u} = 0 & \text{for} \quad z = 0 \, , \\
\tilde{u} = 0 & \text{for} \quad t = 0 \, ,
\end{cases}$$

where

$$\tilde{f} := \eta f - 2(\partial_z \eta)\partial_z u - (\partial_z^2 \eta)u + (\partial_z \eta)p e_z, \qquad \tilde{\rho} := (\partial_z \eta)u^z \, . \tag{73}$$

Since, by assumption f, ρ are horizontally band-limited , then also $\tilde{f}$ and $\tilde{\rho}$ satisfy the horizontal bandedness assumption (47) and (48) respectively. We can therefore apply Proposition 1 to the upper half space problem (44) and get

$$\begin{aligned} & ||(\partial_t-\partial_z^2)\tilde u'||_{(0,\infty)}+||\nabla'\nabla\tilde u'||_{(0,\infty)}+||\partial_t\tilde u^z||_{(0,\infty)}+||\nabla^2\tilde u^z||_{(0,\infty)}+||\nabla\tilde p||_{(0,\infty)}\\ \lesssim\ & ||\tilde f||_{(0,\infty)}+||(-\Delta')^{-\frac12}\partial_t\tilde\rho||_{(0,\infty)}+||(-\Delta')^{-\frac12}\partial_z^2\tilde\rho||_{(0,\infty)}+||\nabla\tilde\rho||_{(0,\infty)}\,. \end{aligned}$$

By symmetry, we also have the same maximal regularity estimates in the lower half space. Indeed, let $\tilde{\tilde u},\tilde{\tilde p}$ satisfy the equation

$$\left\{\begin{array}{rcllr} \partial_t\tilde{\tilde u}-\Delta\tilde{\tilde u}+\nabla\tilde{\tilde p} &=& \tilde{\tilde f} & \text{for} \quad z<1\,, \\ \nabla\cdot\tilde{\tilde u} &=& \tilde{\tilde \rho} & \text{for} \quad z<1\,, \\ \tilde{\tilde u} &=& 0 & \text{for} \quad z=1\,, \\ \tilde{\tilde u} &=& 0 & \text{for} \quad t=0\,, \end{array}\right. \tag{74}$$

where

$$\tilde{\tilde f}:=(1-\eta)f-2(\partial_z(1-\eta))\partial_z u-(\partial_z^2(1-\eta))u+(\partial_z(1-\eta))pe_z,\qquad \tilde{\tilde\rho}:=(\partial_z(1-\eta))u^z\,. \tag{75}$$

Again by Proposition 2 we have

$$\begin{aligned} & ||(\partial_t-\partial_z^2)\tilde{\tilde u}'||_{(-\infty,1)}+||\nabla'\nabla\tilde{\tilde u}'||_{(-\infty,1)}+||\partial_t\tilde{\tilde u}^z||_{(-\infty,1)}+||\nabla^2\tilde{\tilde u}^z||_{(-\infty,1)}+||\nabla\tilde{\tilde p}||_{(-\infty,1)}\\ \lesssim\ & ||\tilde{\tilde f}||_{(-\infty,1)}+||(-\Delta')^{-\frac12}\partial_t\tilde{\tilde\rho}||_{(-\infty,1)}+||(-\Delta')^{-\frac12}\partial_z^2\tilde{\tilde\rho}||_{(-\infty,1)}+||\nabla\tilde{\tilde\rho}||_{(-\infty,1)}, \end{aligned}$$

where $||\cdot||_{(-\infty,1)}$ is the analogue of (49) (see Section (1.4) for notations). Since $u=\tilde u+\tilde{\tilde u}$ in the strip $[0,L)^{d-1}\times(0,1)$, by the triangle inequality and using the maximal regularity estimates above, we get

$$\begin{aligned} & ||(\partial_t-\partial_z^2)u'||_{(0,1)}+||\nabla'\nabla u'||_{(0,1)}+||\partial_t u^z||_{(0,1)}+||\nabla^2 u^z||_{(0,1)}+||\nabla p||_{(0,1)}\\ \lesssim\ & ||(\partial_t-\partial_z^2)\tilde u'||_{(0,\infty)}+||(\partial_t-\partial_z^2)\tilde{\tilde u}'||_{(-\infty,1)}+||\nabla'\nabla\tilde u'||_{(0,\infty)}+||\nabla'\nabla\tilde{\tilde u}'||_{(-\infty,1)}\\ +\ & ||\partial_t\tilde u^z||_{(0,\infty)}+||\partial_t\tilde{\tilde u}^z||_{(-\infty,1)}+||\nabla^2\tilde u^z||_{(0,\infty)}+||\nabla^2\tilde{\tilde u}^z||_{(-\infty,1)}\\ +\ & ||\nabla\tilde p||_{(0,\infty)}+||\nabla\tilde{\tilde p}||_{(-\infty,1)}\\ \lesssim\ & ||\tilde f||_{(0,\infty)}+||\tilde{\tilde f}||_{(-\infty,1)}+||(-\Delta')^{-\frac12}\partial_t\tilde\rho||_{(0,\infty)}+||(-\Delta')^{-\frac12}\partial_t\tilde{\tilde\rho}||_{(-\infty,1)}\\ +\ & ||(-\Delta')^{-\frac12}\partial_z^2\tilde\rho||_{(0,\infty)}+||(-\Delta')^{-\frac12}\partial_z^2\tilde{\tilde\rho}||_{(-\infty,1)}+||\nabla\tilde\rho||_{(0,\infty)}+||\nabla\tilde{\tilde\rho}||_{(-\infty,1)}\,. \end{aligned}$$

By the definitions of $\tilde f$ and $\tilde{\tilde f}$ we get

$$||\tilde f||_{(0,\infty)}+||\tilde{\tilde f}||_{(-\infty,1)}\lesssim ||f||_{(0,1)}+||\partial_z u||_{(0,1)}+||u||_{(0,1)}+||p||_{(0,1)}$$

and similarly for $\tilde{\rho}$ and $\tilde{\tilde{\rho}}$ we have

$$||\nabla\tilde{\rho}||_{(0,\infty)} + ||\nabla\tilde{\tilde{\rho}}||_{(-\infty,1)} \lesssim ||\nabla u||_{(0,1)} + ||u||_{(0,1)}$$

$$||(-\Delta')^{-\frac{1}{2}}\partial_t\tilde{\rho}||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\tilde{\tilde{\rho}}||_{(-\infty,1)} \lesssim ||(-\Delta')^{-\frac{1}{2}}\partial_t u||_{(0,1)}$$

and

$$\begin{aligned} & ||(-\Delta')^{-\frac{1}{2}}\partial_z^2\tilde{\rho}||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z^2\tilde{\tilde{\rho}}||_{(-\infty,1)} \\ \lesssim \ & ||(-\Delta')^{-\frac{1}{2}}u^z||_{(0,1)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z u^z||_{(0,1)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z^2 u^z||_{(0,1)}\,. \end{aligned}$$

Therefore, collecting the estimates, we have

$$\begin{aligned} & ||(\partial_t - \partial_z^2)u'||_{(0,1)} + ||\nabla'\nabla u'||_{(0,1)} + ||\partial_t u^z||_{(0,1)} + ||\nabla^2 u^z||_{(0,1)} + ||\nabla p||_{(0,1)} \\ \lesssim \ & ||f||_{(0,1)} + ||p||_{(0,1)} + ||\nabla u||_{(0,1)} + ||u||_{(0,1)} \\ + \ & ||(-\Delta')^{-\frac{1}{2}}\partial_t u||_{(0,1)} + ||(-\Delta')^{-\frac{1}{2}}u^z||_{(0,1)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z u^z||_{(0,1)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z^2 u^z||_{(0,1)}\,. \end{aligned}$$

Incorporating the horizontal bandedness assumption we find

$$\begin{aligned} ||\partial_z u||_{(0,1)} &\leq R||\nabla'\partial_z u||_{(0,1)}\,, \\ ||u||_{(0,1)} &\leq R^2||(\nabla')^2 u||_{(0,1)}\,, \\ ||p||_{(0,1)} &\leq R||\nabla' p||_{(0,1)}, \\ ||\nabla u||_{(0,1)} &\leq R||\nabla'\nabla u||_{(0,1)}, \\ ||(-\Delta')^{-\frac{1}{2}}\partial_t u||_{(0,1)} &\leq R||\partial_t u||_{(0,1)}\,, \\ ||(-\Delta')^{-\frac{1}{2}}u^z||_{(0,1)} &\leq R^3||\nabla'^2 u^z||_{(0,1)}\,, \\ ||(-\Delta')^{-\frac{1}{2}}\partial_z u^z||_{(0,1)} &\leq R^2||\nabla'\partial_z u^z||_{(0,1)}\,, \\ ||(-\Delta')^{-\frac{1}{2}}\partial_z^2 u^z||_{(0,1)} &\leq R||\partial_z^2 u^z||_{(0,1)}\,. \end{aligned}$$

Thus, for $R < R_0$ where R_0 is sufficiently small, all the terms in the right hand side, except f can be absorbed into the left hand side and the conclusion follows.

4 Proof of main technical lemmas

Remark 2. *In the proof of Lemma 1, Lemma 2 and Lemma 3 we will derive inequalities between quantities where t is integrated between 0 and ∞. From the proof it is clear that the same inequalities are true with t integrated between 0 and t_0 with constants that are not depending on t_0. Therefore dividing by t_0 and taking $\limsup_{t_0\to\infty}$ (see (32)) we shall obtain the desired estimates in terms of the interpolation norm (49).*

4.1 Proof of Lemma 1

Proof of Lemma 1.
In order to simplify the notations, in what follows we will omit the dependency of the functions from the time variable. It is enough to show

$$||\nabla' u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)},$$

since, by equation (60) $\partial_z u = (-\Delta')^{\frac{1}{2}} u + f$. We claim that, in order to prove (4.1), it is enough to show

$$\sup_z \langle|\nabla' u|\rangle' \lesssim \sup_z \langle|f|\rangle' \tag{76}$$

and

$$||\nabla' u||_{(0,\infty)} \lesssim \int \langle|f|\rangle' \frac{dz}{z}\,. \tag{77}$$

Indeed, by definition of the norm $||\cdot||_{(0,\infty)}$ (see (49)) if we select an arbitrary decomposition $\nabla' u = \nabla' u_1 + \nabla' u_2$, where u_1 and u_2 are solutions of the problem (60) with right hand sides f_1 and f_2 respectively, we have

$$\begin{aligned} ||\nabla' u||_{(0,\infty)} &\leq ||\nabla' u_1||_{(0,\infty)} + \sup_z \langle|\nabla' u_2|\rangle' \\ &\leq \int \langle|f_1|\rangle' \frac{dz}{z} + \sup_z \langle|f_2|\rangle'\,. \end{aligned}$$

Passing to the infimum over all the decompositions of f we obtain

$$||\nabla' u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}.$$

We recall that by Duhamel's principle we have the following representation

$$u(x',z) = \int_z^\infty u_{x',z_0}(z)dz_0, \tag{78}$$

where u_{z_0} is the harmonic extension of $f(\cdot, z_0)$ onto $\{z < z_0\}$, i.e it solves the boundary value problem

$$\left\{ \begin{array}{rcll} (\partial_z - (-\Delta')^{\frac{1}{2}})u_{z_0} &=& 0 & \text{for} \quad z < z_0\,, \\ u_{z_0} &=& f & \text{for} \quad z = z_0\,. \end{array} \right. \tag{79}$$

Argument for (76):
Using the representation of the solution of (79) via the Poisson kernel, i.e

$$u_{z_0}(x',z)=\int\frac{z_0-z}{(|x'-y'|^2+(z_0-z)^2)^{\frac{d}{2}}}f(x',z_0)dy'$$

we obtain the following bounds

$$\langle|\nabla' u_{z_0}(\cdot,z)|\rangle'\lesssim\begin{cases} & \langle|\nabla' f(\cdot,z_0)|\rangle', \\ \frac{1}{(z_0-z)} & \langle|f(\cdot,z_0)|\rangle', \\ \frac{1}{(z_0-z)^2} & \langle|\nabla'(-\Delta')^{-1}f(\cdot,z_0)|\rangle'. \end{cases} \tag{80}$$

By using the bandedness assumption in the form of (117) and (119), we have

$$\langle|\nabla' u_{z_0}(\cdot,z)|\rangle'\lesssim\min\left\{\frac{1}{R},\frac{R}{(z_0-z)^2}\right\}\langle|f(\cdot,z_0)|\rangle',$$

hence

$$\begin{aligned}\langle|\nabla' u(\cdot,z)|\rangle' &\lesssim \int_z^\infty\min\left\{\frac{1}{R},\frac{R}{(z_0-z)^2}\right\}\langle|f(\cdot,z_0)|\rangle'dz_0\\ &\lesssim \sup_{z_0\in(0,\infty)}\langle|f(\cdot,z_0)|\rangle'\int_z^\infty\min\left\{\frac{1}{R},\frac{R}{(z_0-z)^2}\right\}dz_0\\ &\lesssim \sup_{z_0\in(0,\infty)}\langle|f(\cdot,z_0)|\rangle',\end{aligned}$$

which, passing to the supremum in z, implies (76).
From the above and applying Fubini's rule, we also have

$$\begin{aligned}\int_0^\infty\langle|\nabla' u(\cdot,z)|\rangle'dz &\le \int_0^\infty\int_z^\infty\min\left\{\frac{1}{R},\frac{R}{(z_0-z)^2}\right\}\langle|f(\cdot,z_0)|\rangle'dz_0dz \qquad (81)\\ &\le \int_0^\infty\int_0^{z_0}\min\left\{\frac{1}{R},\frac{R}{(z_0-z)^2}\right\}dz\langle|f(\cdot,z_0)|\rangle'dz_0\\ &\lesssim \int_0^\infty\langle|f(\cdot,z)|\rangle'dz\,.\end{aligned}$$

Argument for (77):
Let us consider $\chi_{2H\le z\le 4H}f$ and let u_H be the solution to

$$(\partial_z-(-\Delta')^{\frac{1}{2}})u_H=\chi_{2H\le z\le 4H}f.$$

We claim

$$\sup_{z\le H}\langle|\nabla' u_H|\rangle' \le \int_0^\infty \langle|\chi_{2H\le z\le 4H} f|\rangle' \frac{dz}{z} \tag{82}$$

and

$$\int_H^\infty \langle|\nabla' u_H|\rangle' \frac{dz}{z} \le \int_0^\infty \langle|\chi_{2H\le z\le 4H} f|\rangle' \frac{dz}{z}\,. \tag{83}$$

From estimate (82) and (83) the statement (77) easily follow. Indeed, choosing $H = 2^{n-1}$ and summing up over the dyadic intervals, we have

$$\begin{aligned}
||\nabla' u|| &\le \sum_{n\in\mathbb{Z}} ||\nabla' u_{2^{n-1}}||_{(0,\infty)} \\
&\le \sup_{z\le 2^{n-1}}\langle|\nabla' u_{2^{n-1}}|\rangle' + \int_{2^{n-1}}^\infty \langle|\nabla' u_{2^{n-1}}|\rangle' \frac{dz}{z} \\
&\le \sum_{n\in\mathbb{Z}} \int_0^\infty \langle|\chi_{2^n\le z\le 2^{n+1}} f|\rangle' \frac{dz}{z} \\
&= \int_0^\infty \langle|f|\rangle' \frac{dz}{z}\,.
\end{aligned}$$

Argument for (82): Fix $z \le H$. Then, we have

$$\begin{aligned}
\langle|\nabla' u_H|\rangle' &\overset{(80)}{\le} \int_z^\infty \frac{1}{(z_0 - z)} \langle|\chi_{2H\le z\le 4H} f(\cdot, z_0)|\rangle' dz_0 \\
&\lesssim \int_{2H}^{4H} \frac{1}{(z_0 - z)} \langle|\chi_{2H\le z\le 4H} f(\cdot, z_0)|\rangle' dz_0 \\
&\lesssim \frac{1}{H}\int_{2H}^{4H} \langle|\chi_{2H\le z\le 4H} f(\cdot, z_0)|\rangle' dz_0 \\
&\le \int_{2H}^\infty \langle|\chi_{2H\le z\le 4H} f(\cdot, z_0)|\rangle' \frac{dz_0}{z_0} \\
&\le \int_0^\infty \langle|\chi_{2H\le z\le 4H} f(\cdot, z_0)|\rangle' \frac{dz_0}{z_0}\,.
\end{aligned}$$

Taking the supremum over all z proves (82).
Argument for (83): For $z \geq H$ we have

$$\begin{aligned}\int_H^\infty \langle|\nabla' u_H|\rangle' \frac{dz}{z} &\lesssim \frac{1}{H}\int_0^\infty \langle|\nabla' u_H|\rangle' dz \\ &\overset{(81)}{\lesssim} \frac{1}{H}\int_0^\infty \langle|\chi_{2H\leq z\leq 4H} f|\rangle' dz \\ &= \frac{1}{H}\int_{2H}^{4H} \langle|\chi_{2H\leq z\leq 4H} f|\rangle' dz \\ &\lesssim \int_0^\infty \langle|\chi_{2H\leq z\leq 4H} f|\rangle' \frac{dz}{z}\,.\end{aligned}$$

□

4.2 Proof of Lemma 2

Proof of Lemma 2.
Let us first assume $g = 0$. It is enough to show

$$\sup_z \langle|\nabla' u|\rangle' \lesssim \sup_z \langle|f|\rangle' \tag{84}$$

and

$$\int_0^\infty \langle|\nabla' u|\rangle' \frac{dz}{z} \lesssim \int_0^\infty \langle|f|\rangle' \frac{dz}{z}\,. \tag{85}$$

Recall that by Duhamel's principle we have the following representation

$$u(z) = \int_0^z u_{z_0}(\cdot, z) dz_0, \tag{86}$$

where u_{z_0} is the harmonic extension of $f(z_0)$ onto $\{z > z_0\}$, i.e it solves the boundary value problem

$$\left\{\begin{array}{rcll} (\partial_z + (-\Delta')^{\frac{1}{2}})u_{z_0} &=& 0 & \text{for} \quad z > z_0\,, \\ u_{z_0} &=& f & \text{for} \quad z = z_0\,. \end{array}\right. \tag{87}$$

From the Poisson's kernel representation we learn that

$$\langle|\nabla' u_{z_0}(\cdot, z)|\rangle' \lesssim \begin{cases} \langle|\nabla' f(\cdot, z_0)|\rangle'\,, \\ \frac{1}{(z-z_0)^2}\langle|\nabla'(-\Delta')^{-1} f(\cdot, z_0)|\rangle'\,. \end{cases}$$

Using the bandedness assumption in the form of (117) and (119)

$$\langle|\nabla' u_{z_0}(\cdot,z)|\rangle' \lesssim \min\{\frac{1}{R},\frac{R}{(z-z_0)^2}\}\langle|f(\cdot,z_0)|\rangle'$$

and observing (86), we obtain

$$\begin{array}{rcl}\langle|\nabla' u(\cdot,z)|\rangle' & \lesssim & \int_0^z \min\left\{\frac{1}{R},\frac{R}{(z-z_0)^2}\right\}\langle|f(\cdot,z_0)|\rangle' dz_0 \\ & \le & \sup_{z_0}\langle|f(\cdot,z_0)|\rangle' \int_0^z \min\left\{\frac{1}{R},\frac{R}{(z-z_0)^2}\right\} dz_0 \\ & \lesssim & \sup_{z_0}\langle|f(\cdot,z_0)|\rangle' .\end{array} \tag{88}$$

Estimate (84) follows from (88) by passing to the the supremum in z.
From the above (88), multiplying by the weight $\frac{1}{z}$ and observing that $z > z_0$ we have

$$\langle|\nabla' u(\cdot,z)|\rangle' \frac{1}{z} \lesssim \int_0^z \min\left\{\frac{1}{R},\frac{R}{(z-z_0)^2}\right\}\langle|f(\cdot,z_0)|\rangle' \frac{dz_0}{z_0}. \tag{89}$$

After integrating in $z \in (0,\infty)$ and applying Young's estimate we get (85).

Let's assume now the general case, with $g \neq 0$. We want to prove (63). Recall that by definition $\tilde g(x',z) := g(x')$ and consider $u-\tilde g$. By construction it satisfies

$$\left\{\begin{array}{rcll}(\partial_z + (-\Delta')^{-\frac{1}{2}})(u-\tilde g) & = & f - (-\Delta')^{-\frac{1}{2}} g & \text{for} \quad z>0\,, \\ u-\tilde g & = & 0 & \text{for} \quad z=0\,.\end{array}\right.$$

Using the first part of the proof of (64) and triangle inequality, we have

$$||\nabla u||_{(0,\infty)} \lesssim ||\nabla \tilde g||_{(0,\infty)} + ||f||_{(0,\infty)} + ||(-\Delta')^{\frac{1}{2}}\tilde g||_{(0,\infty)}\,.$$

Therefore by the bandedness assumption in the form of (121) we can conclude (63). □

4.3 Proof of Lemma 3

Proof of Lemma 3.
We will show that, for the non-homogeneous heat equation with Dirichlet

boundary condition

$$\left\{\begin{array}{rcll} (\partial_t-\Delta)u &=& f & \text{for}\quad z>0\,,\\ u &=& 0 & \text{for}\quad z=0\,,\\ u &=& 0 & \text{for}\quad t=0\,, \end{array}\right. \tag{90}$$

we have the following estimates

$$\int \left(\langle|(\partial_t-\partial_z^2)u(\cdot,z,\cdot)|\rangle+\langle|\nabla'^2u(\cdot,z,\cdot)|\rangle\right)\frac{dz}{z}\lesssim\int\langle|f(\cdot,z,\cdot)|\rangle\frac{dz}{z}\,, \tag{91}$$

$$\langle|\nabla'\partial_z u(\cdot,z,\cdot)|_{z=0}\rangle\lesssim\int\langle|f(\cdot,z,\cdot)|\rangle\frac{dz}{z}\,, \tag{92}$$

$$\sup_z\langle|\nabla'^2u(\cdot,z,\cdot)|\rangle\lesssim\sup_z\langle|f(\cdot,z,\cdot)|\rangle\,, \tag{93}$$

$$\sup_z\langle|\nabla'\partial_z u(\cdot,z,\cdot)|\rangle\lesssim\sup_z\langle|f(\cdot,z,\cdot)|\rangle\,. \tag{94}$$

In order to bound the off-diagonal components of the Hessian, we consider the decomposition

$$u=u_N+u_C, \tag{95}$$

where u_N solves

$$\left\{\begin{array}{rcll} (\partial_t-\Delta)u_N &=& f & \text{for}\quad z>0\,,\\ \partial_z u_N &=& 0 & \text{for}\quad z=0\,,\\ u_N &=& 0 & \text{for}\quad t=0\,, \end{array}\right. \tag{96}$$

and u_C solves

$$\left\{\begin{array}{rcll} (\partial_t-\Delta)u_C &=& 0 & \text{for}\quad z>0\,,\\ \partial_z u_C &=& \partial_z u & \text{for}\quad z=0\,,\\ u_C &=& 0 & \text{for}\quad t=0\,. \end{array}\right. \tag{97}$$

The splitting (95) is valid by the uniqueness of the Neumann problem. For the auxiliary problems (96) and (97) we have the following bounds

$$\int\langle|\nabla'\partial_z u_N(\cdot,z,\cdot)|\rangle\frac{dz}{z}\lesssim\int\langle|f(\cdot,z,\cdot)|\rangle\frac{dz}{z}\,, \tag{98}$$

$$\sup_z\langle|\nabla'\partial_z u_C(\cdot,z,\cdot)|\rangle\lesssim\langle|\nabla'\partial_z u(\cdot,z,\cdot)|_{z=0}\rangle\,. \tag{99}$$

We claim that estimates (91), (92),(93), (94), (98) and (99) yield (66).
Let us first consider the bound for ∇'^2. Consider $u = u_1 + u_2$, where u_1 and u_2 satisfy (90) with right hand side f_1 and f_2 respectively. We have

$$\begin{aligned}||\nabla'^2 u||_{(0,\infty)} &\lesssim \sup_z \langle|\nabla'^2 u_1|\rangle + \int \langle|\nabla'^2 u_2|\rangle \frac{dz}{z}\\ &\overset{(91)\&(93)}{\lesssim} \sup_z \langle|f_1|\rangle + \int \langle|f_2|\rangle \frac{dz}{z},\end{aligned}$$

which implies, upon taking infimum over all decompositions $f = f_1 + f_2$

$$||\nabla'^2 u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}. \tag{100}$$

We now consider a further decomposition of u_2 , i.e $u_2 = u_{2C} + u_{2N}$ where u_{2C} satisfies (97) and u_{2N} satisfies (96). Therefore $u = u_1 + u_{2C} + u_{2N}$ and we can bound the off-diagonal components of the Hessian

$$\begin{aligned}||\nabla'\partial_z u||_{(0,\infty)} &\lesssim \sup_z \langle|\nabla'\partial_z u_1|\rangle + \sup_z \langle|\nabla'\partial_z u_{2C}|\rangle + \int \langle|\nabla'\partial_z u_{2N}|\rangle \frac{dz}{z}\\ &\overset{(92),(99),(98)\&(94)}{\lesssim} \sup_z \langle|f_1|\rangle + \int \langle|f_2|\rangle \frac{dz}{z}.\end{aligned}$$

From the last inequality, passing to the infimum over all the possible decompositions of f we get

$$||\nabla'\partial_z u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}. \tag{101}$$

On one hand estimate (100) and (101) imply

$$||\nabla\nabla' u||_{(0,\infty)} \lesssim ||\nabla'^2 u||_{(0,\infty)} + ||\nabla'\partial_z u||_{(0,\infty)}\,,$$

on the other hand equation (65) and estimate (100) yield

$$||(\partial_t - \partial_z^2)u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}\,.$$

<u>Argument for 91</u>
Let u be a solution of problem of (90). Keeping in mind Remark (2) it is enough to show

$$\int_0^\infty \int_0^\infty \langle|\nabla'^2 u|\rangle' \frac{dz}{z} dt \lesssim \int_0^\infty \int_0^\infty \langle|f|\rangle' \frac{dz}{z} dt\,.$$

By the Duhamel's principle we have

$$u(x',z,t) = \int_{s=0}^{t} u_s(x',z,t)ds, \tag{102}$$

where u_s is the solution to the homogeneous, initial value problem

$$\begin{cases} (\partial_t - \Delta)u_s &= 0 \quad \text{for} \quad z>0, t>s\,, \\ u_s &= 0 \quad \text{for} \quad z=0, t>s\,, \\ u_s &= f \quad \text{for} \quad z>0, t=s\,. \end{cases} \tag{103}$$

Extending u and f to the whole space by odd reflection [8], we are left to study the problem

$$\begin{cases} (\partial_t - \Delta)u_s &= 0 \quad \text{for} \quad z\in\mathbb{R}, t>s\,, \\ u_s &= f \quad \text{for} \quad z\in\mathbb{R}, t=s\,, \end{cases}$$

the solution of which can be represented via heat kernel as

$$\begin{aligned} u_s(x',z,t) &= \int_{\mathbb{R}} \Gamma(\cdot, z-\tilde{z}, t-s) *_{x'} f(\cdot,\tilde{z},s)d\tilde{z} \\ &= \int_0^\infty [\Gamma(\cdot, z-\tilde{z}, t-s) - \Gamma(\cdot, z+\tilde{z}, t-s)] *_{x'} f(\cdot,\tilde{z},s)d\tilde{z}\,. \end{aligned} \tag{104}$$

The application of ∇'^2 to the representation above yields

$$\begin{aligned} &\nabla'^2 u_s(x',z,t) \\ = &\begin{cases} \int_0^\infty \int_{\mathbb{R}^{d-1}} \nabla'\Gamma_{d-1}(x'-\tilde{x}', t-s)\left(\Gamma_1(z-\tilde{z},t-s) - \Gamma_1(z+\tilde{z},t-s)\right)\nabla' f(\tilde{x}',\tilde{z},s)d\tilde{x}'d\tilde{z}\,, \\ \int_0^\infty \int_{\mathbb{R}^{d-1}} \nabla'^3\Gamma_{d-1}(x'-\tilde{x}', t-s)\left(\Gamma_1(z-\tilde{z},t-s) - \Gamma_1(z+\tilde{z},t-s)\right)(-\Delta')^{-1}\nabla' f(\tilde{x}',\tilde{z},s)d\tilde{x}'d\tilde{z}\,. \end{cases} \end{aligned}$$

Averaging in the horizontal direction we obtain, on the one hand

$$\begin{aligned} &\langle|\nabla'^2 u_s(\cdot,z,t)|\rangle' \\ \lesssim\ & \int_0^\infty \langle|\nabla'\Gamma_{d-1}(\cdot,t-s)|\rangle' |\Gamma_1(z-\tilde{z},t-s) - \Gamma_1(z+\tilde{z},t-s)| \langle|\nabla' f(\cdot,\tilde{z},s)|\rangle' d\tilde{z} \\ \overset{(125)\&(119)}{\lesssim}\ & \int_0^\infty \frac{1}{(t-s)^{\frac{1}{2}}} |\Gamma_1(z-\tilde{z},t-s) - \Gamma_1(z+\tilde{z},t-s)| \frac{1}{R} \langle|f(\cdot,\tilde{z},s)|\rangle' d\tilde{z} \end{aligned}$$

and, on the other hand

$$\begin{aligned} &\langle|\nabla'^2 u_s(\cdot,z,t)|\rangle' \\ \lesssim\ & \int_0^\infty \langle|\nabla'^3\Gamma_{d-1}(\cdot,t-s)\rangle' |\Gamma_1(z-\tilde{z},t-s) - \Gamma_1(z+\tilde{z},t-s)| \langle|(-\Delta')^{-1}\nabla' f(\cdot,\tilde{z},s)|\rangle' d\tilde{z} \\ \overset{(125)\&(117)}{\lesssim}\ & \int_0^\infty |\Gamma_1(z-\tilde{z},t-s) - \Gamma_1(z+\tilde{z},t-s)| \frac{1}{(t-s)^{\frac{3}{2}}} R \langle|f(\cdot,\tilde{z},s)|\rangle' d\tilde{z}\,. \end{aligned}$$

[8] with abuse of notation we will call again u and f these extensions.

Multiplying by the weight $\frac{1}{z}$ and integrating in $z \in (0,\infty)$ we get

$$\int_0^\infty \langle|\nabla'^2 u_s(\cdot,t)|\rangle' \frac{dz}{z} \lesssim \left(\sup_{\tilde z}\int_0^\infty K_{t-s}(z,\tilde z)dz\right)\begin{cases} \frac{1}{(t-s)^{\frac12}}\frac{1}{R}\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle' \frac{d\tilde z}{\tilde z}\,, \\ \frac{R}{(t-s)^{\frac32}}\int_0^\infty \langle|f(x',\tilde z,s)|\rangle' \frac{d\tilde z}{\tilde z}\,, \end{cases}$$

where we called $K_{t-s}(z,\tilde z) = \frac{\tilde z}{z}|\Gamma_1(z-\tilde z,t-s) - \Gamma_1(z+\tilde z,t-s)|$.
From Lemma 5 we infer

$$\sup_{\tilde z}\int_0^\infty K_{t-s}(z,\tilde z)dz \overset{(124)}{\lesssim} \int_{\mathbb{R}} |\Gamma_1(z,t-s)|dz + \sup_{z\in\mathbb{R}}(z^2|\partial_z\Gamma_1(z,t-s)|) \overset{(126)\&(129)}{\lesssim} 1$$

and therefore we have

$$\int_0^\infty \langle|\nabla'^2 u_s(\cdot,z,t)|\rangle' \frac{dz}{z} \lesssim \begin{cases} \frac{1}{(t-s)^{\frac12}}\frac{1}{R}\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle' \frac{d\tilde z}{\tilde z}\,, \\ \frac{1}{(t-s)^{\frac32}}R\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle' \frac{d\tilde z}{\tilde z}\,. \end{cases}$$

Finally, inserting the previous estimate into the Duhamel formula (102)and integrating in time we get

$$\begin{aligned}
&\int_0^\infty \langle|\nabla'^2 u(\cdot,z,t)|\rangle' \frac{dz}{z}dt \\
\overset{102}{\lesssim}\;& \int_0^\infty\int_0^t \langle|\nabla'^2 u_s(\cdot,z,t)|\rangle' \frac{dz}{z}dsdt \\
\lesssim\;& \int_0^\infty\int_s^\infty \min\{\frac{1}{R(t-s)^{\frac12}},\frac{R}{(t-s)^{\frac32}}\}\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle' \frac{d\tilde z}{\tilde z}dtds \\
\lesssim\;& \int_0^\infty\int_s^\infty \min\{\frac{1}{R(t-s)^{\frac12}},\frac{R}{(t-s)^{\frac32}}\}dt\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle' \frac{d\tilde z}{\tilde z}ds \quad (105) \\
\lesssim\;& \int_0^\infty\int_0^\infty \min\{\frac{1}{R\tau^{\frac12}},\frac{R}{\tau^{\frac32}}\}d\tau\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle' \frac{d\tilde z}{\tilde z}ds, \quad (106) \\
\lesssim\;& \int_0^\infty\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle' \frac{d\tilde z}{\tilde z}ds,
\end{aligned}$$

where in the second to last inequality we used

$$\int_0^\infty \min\left\{\frac{1}{R\tau^{\frac12}},\frac{R}{\tau^{\frac32}}\right\}d\tau \lesssim 1\,. \quad (107)$$

<u>Argument for 92:</u>
Let u be a solution of problem of (90). Recall that we need to prove

$$\int_0^\infty \langle|\nabla'\partial_z u|_{z=0}(\cdot,z,t)|\rangle' dt \lesssim \int_0^\infty\int_0^\infty \langle|f(\cdot,z,t)|\rangle dt\frac{dz}{z}\,. \qquad (108)$$

The solution of the equation (103) extended to the whole space by odd reflection can be represented by (104) (see argument for (91)). Therefore

$$\begin{aligned}&\nabla'\partial_z u_s(x',z,t)|_{z=0}\\ =\;&\begin{cases}-2\int_{\mathbb{R}^{d-1}}\int_0^\infty \Gamma_{d-1}(x'-\tilde{x'},t-s)\partial_z\Gamma_1(\tilde z,t-s)\nabla' f(\tilde{x'},\tilde z,s)d\tilde{x'}d\tilde z\,,\\ -2\int_{\mathbb{R}^{d-1}}\int_0^\infty \nabla'\Gamma_{d-1}(x'-\tilde{x'},t-s)\partial_z\Gamma_1(\tilde z,t-s)\nabla'(-\Delta')^{-1}\nabla' f(\tilde{x'},\tilde z,s)d\tilde{x'}d\tilde z\,.\end{cases}\end{aligned}$$

Taking the horizontal average we get, on the one hand

$$\begin{aligned}&\langle|\nabla'\partial_z u_s(\cdot,z,t)|_{z=0}|\rangle'\\ \lesssim\;&\int_0^\infty \langle|\Gamma_{d-1}(\cdot,t-s)|\rangle'|\partial_z\Gamma_1(\tilde z,t-s)|\langle|\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z\\ \overset{(125)}{\lesssim}\;&\int_0^\infty |\partial_z\Gamma_1(\tilde z,t-s)|\langle|\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z\\ \overset{(119)}{\lesssim}\;&\frac{1}{R}\int_0^\infty |\partial_z\Gamma_1(\tilde z,t-s)|\langle|f(\cdot,\tilde z,s)|\rangle' d\tilde z\\ \lesssim\;&\frac{1}{R}\sup_{\tilde z}|\tilde z\partial_z\Gamma_1(\tilde z,t-s)|\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}\end{aligned}$$

and on the other hand

$$\begin{aligned}&\langle|\nabla'\partial_z u_s(\cdot,z,t)|_{z=0}|\rangle'\\ \lesssim\;&\int_0^\infty \langle|(\nabla')^2\Gamma_{d-1}(\cdot,t-s)|\rangle'|\partial_z\Gamma_1(\tilde z,t-s)|\langle|(-\Delta')^{-1}\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z\\ \overset{(125)}{\lesssim}\;&\frac{1}{(t-s)}\int_0^\infty |\partial_z\Gamma_1(\tilde z,t-s)|\langle|(-\Delta')^{-1}\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z\\ \overset{(117)}{\lesssim}\;&\frac{R}{(t-s)}\int_0^\infty |\partial_z\Gamma_1(\tilde z,t-s)|\langle|f(\cdot,\tilde z,s)|\rangle' d\tilde z\\ \lesssim\;&\frac{R}{(t-s)}\sup_{\tilde z}|\tilde z\partial_z\Gamma_1(\tilde z,t-s)|\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}\,.\end{aligned}$$

Using the estimate (128) we get

$$\langle|\nabla'\partial_z u_s(x',z,t)|_{z=0}|\rangle' \lesssim \begin{cases}\frac{1}{(t-s)^{1/2}R}\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}\,,\\ \frac{R}{(t-s)^{3/2}}\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}\,.\end{cases}$$

Finally, inserting into Duhamel's formula and integrating in time we have

$$\begin{aligned}
&\int_0^\infty \langle|\nabla'\partial_z u(\cdot,z,t)|_{z=0}\rangle' dt \\
\overset{(102)}{\lesssim}\quad & \int_0^\infty\int_0^t \langle|\nabla'\partial_z u_s(\cdot,z,t)|_{z=0}\rangle' ds dt \\
\lesssim\quad & \int_0^\infty\int_s^\infty \min\{\frac{1}{R(t-s)^{\frac12}},\frac{R}{(t-s)^{\frac32}}\}\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z} dt ds \\
\overset{(105)\&(106)}{\lesssim}\quad & \int_0^\infty\int_0^\infty \langle|f(x',z,s)|\rangle'\frac{d\tilde z}{\tilde z} ds.
\end{aligned}$$

Argument for (93):

Let u be the solution of problem (90). We recall that we want to prove

$$\sup_z \int_0^\infty \langle|\nabla'^2 u(\cdot,z,t)|\rangle' dt \lesssim \sup_z \int_0^\infty \langle|f(\cdot,z,t)|\rangle' dt\,. \tag{109}$$

The solution of equation (103) extended to the whole space can be represented by (104) (see argument for (91)). Therefore applying ∇'^2 to (104)and considering the horizontal average we have, on the one hand

$$\begin{aligned}
&\langle|\nabla'^2 u_s(\cdot,z,t)|\rangle' \\
\lesssim\quad & \int_{\mathbb{R}} \langle|\nabla'\Gamma_{d-1}(\cdot,t-s)|\rangle' |\Gamma_1(z-\tilde z,t-s)| \langle|\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z \\
\overset{(125)\&(119)}{\lesssim}\quad & \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac12}} |\Gamma_1(z-\tilde z,t-s)| \frac{1}{R} \langle|f(\cdot,\tilde z,s)|\rangle' d\tilde z
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&\langle|\nabla'^2 u_s(\cdot,z,t)|\rangle' \\
\lesssim\quad & \int_{\mathbb{R}} \langle|\nabla'^3\Gamma_{d-1}(\cdot,t-s)|\rangle' |\Gamma_1(z-\tilde z,t-s)| \langle|(-\Delta')^{-1}\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z \\
\overset{(125)\&(117)}{\lesssim}\quad & \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac32}} |\Gamma_1(z-\tilde z,t-s)| R \langle|f(\cdot,\tilde z,s)|\rangle' d\tilde z\,.
\end{aligned}$$

Inserting the above estimates in the Duhamel's formula (102), we have

$$
\begin{aligned}
& \int_0^\infty \int_0^t \langle |\nabla'^2 u_s(z,\cdot)|\rangle' ds dt \\
\lesssim \quad & \int_0^\infty \int_s^\infty \min\left\{ \frac{1}{R(t-s)^{\frac{1}{2}}}, \frac{R}{(t-s)^{\frac{3}{2}}} \right\} \int_{\mathbb{R}} |\Gamma_1(z-\tilde z, t-s)| \langle |f(\cdot,\tilde z,s)|\rangle' d\tilde z ds dt \\
\lesssim \quad & \int_{\mathbb{R}} \left(\int_0^\infty \min\left\{ \frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}} \right\} |\Gamma_1(z-\tilde z,\tau)| d\tau \right) \int_0^\infty \langle |f(\cdot,\tilde z,s)|\rangle' ds d\tilde z \\
\lesssim \quad & \sup_{\tilde z} \int_0^\infty \langle |f(\cdot,\tilde z,s)|\rangle' ds \int_{\mathbb{R}} \int_0^\infty \min\left\{ \frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}} \right\} |\Gamma_1(z-\tilde z,\tau)| d\tau d\tilde z \\
\stackrel{(126)}{\lesssim} \quad & \sup_{\tilde z} \int_0^\infty \langle |f(\cdot,\tilde z,s)|\rangle' ds \int_0^\infty \min\left\{ \frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}} \right\} d\tau \int_{\mathbb{R}} |\Gamma_1(z-\tilde z,\tau)| d\tilde z \\
\stackrel{(107)}{\lesssim} \quad & \sup_{\tilde z} \int_0^\infty \langle |f(\cdot,\tilde z,s)|\rangle' ds \,.
\end{aligned}
$$

Taking the supremum in z we obtain the desired estimate.

Argument for (94):

Let u be the solution of problem (90). We claim

$$
\sup_z \int_0^\infty \langle |\nabla' \partial_z u|\rangle' dt \lesssim \sup_z \int_0^\infty \langle |f|\rangle' dt \,. \tag{110}
$$

The solution of the equation (103) extended to the whole space can be represented by (see argument for (91))

$$
u_s(x',z,t) = \int_{\mathbb{R}} \Gamma(\cdot, z-\tilde z, t-s) *_{x'} f(\cdot,\tilde z,s) d\tilde z \,.
$$

Applying $\nabla'\partial_z$ and considering the horizontal average we obtain, on the one hand

$$
\begin{aligned}
& \langle |\nabla'\partial_z u_s(\cdot,z,t)|\rangle' \\
\lesssim \quad & \int_{\mathbb{R}} \langle |\Gamma_{d-1}(\cdot,t-s)|\rangle' |\partial_z \Gamma_1(z-\tilde z,t-s)| \langle |\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z \\
\stackrel{119}{\lesssim} \quad & \int_{\mathbb{R}} |\partial_z \Gamma_1(z-\tilde z,t-s)| \frac{1}{R} \langle |f(\cdot,\tilde z,s)|\rangle' d\tilde z
\end{aligned}
$$

and, on the other hand

$$\begin{aligned}
&\langle|\nabla'\partial_z u_s(\cdot,z,t)|\rangle' \\
\lesssim\ & \int_{\mathbb{R}}\langle|\nabla'^2\Gamma_{d-1}(\cdot,t-s)|\rangle'|\partial_z\Gamma_1(z-\tilde{z},t-s)|\langle|(-\Delta')^{-1}\nabla' f(\cdot,\tilde{z},s)|\rangle' d\tilde{z} \\
\overset{117}{\lesssim}\ & \int_{\mathbb{R}}\frac{1}{(t-s)}|\partial_z\Gamma_1(z-\tilde{z},t-s)|R\langle|f(\cdot,\tilde{z},s)|\rangle' d\tilde{z}\,.
\end{aligned}$$

Inserting the above estimates in the Duhamel's formula (102), we have

$$\begin{aligned}
&\int_0^\infty\int_0^t\langle|\nabla'\partial_z u_s(z,\cdot)|\rangle' dsdt \\
\lesssim\ & \int_0^\infty\int_s^\infty \min\left\{\frac{1}{R},\frac{R}{(t-s)}\right\}\int_{\mathbb{R}}|\partial_z\Gamma_1(z-\tilde{z},t-s)|\langle|f(\cdot,\tilde{z},s)|\rangle' d\tilde{z}dtds \\
\lesssim\ & \int_{\mathbb{R}}\left(\int_0^\infty \min\left\{\frac{1}{R},\frac{R}{\tau}\right\}|\partial_z\Gamma_1(z-\tilde{z},\tau)|d\tau\right)\int_0^\infty\langle|f(\cdot,\tilde{z},s)|\rangle' dsd\tilde{z} \\
\lesssim\ & \sup_{\tilde{z}}\int_0^\infty\langle|f(\cdot,\tilde{z},s)|\rangle' ds\int_{\mathbb{R}}\int_0^\infty \min\left\{\frac{1}{R},\frac{R}{\tau}\right\}|\partial_z\Gamma_1(z-\tilde{z},\tau)|d\tau d\tilde{z} \\
\overset{(126)}{\lesssim}\ & \sup_{\tilde{z}}\int_0^\infty\langle|f(\cdot,\tilde{z},s)|\rangle' ds\int_0^\infty \min\left\{\frac{1}{R\tau^{\frac{1}{2}}},\frac{R}{\tau^{\frac{3}{2}}}\right\}d\tau \\
\overset{(107)}{\lesssim}\ & \sup_{\tilde{z}}\int_0^\infty\langle|f(\cdot,\tilde{z},s)|\rangle' ds\,.
\end{aligned}$$

Taking the supremum in z we obtain the desired estimate.

Argument for (98)

We recall that we want to show

$$\int_0^\infty\int_0^\infty\langle|\nabla'\partial_z u_N|\rangle'\frac{dz}{z}dt\lesssim\int_0^\infty\int_0^\infty\langle|f|\rangle'\frac{dz}{z}dt,$$

where u_N be the solution to the non-homogeneous heat equation with Neumann boundary conditions (96). By the Duhamel's principle we have

$$u_N(x',z,t)=\int_{s=0}^t u_{N_s}(x',z,t)ds,$$

where u_{N_s} is solution to

$$\begin{cases}
(\partial_t-\Delta)u_{N_s} = 0 & \text{for } z>0, t>s\,,\\
\partial_z u_{N_s} = 0 & \text{for } z=0, t>s\,,\\
u_{N_s} = f & \text{for } z>0, t=s\,,
\end{cases}$$

is the solution of problem (90). Extending this equation to the whole space by even reflection [9], we are left to study the problem

$$\begin{cases} (\partial_t - \Delta)u_{N_s} &= 0 \qquad \text{for} \quad z\in\mathbb{R}, t>s\,,\\ u_{N_s} &= f \qquad \text{for} \quad t=s\,,\end{cases}$$

the solution of which can be represented via heat kernel as

$$\begin{aligned} u_{N_s}(x',z,t) &= \int_{\mathbb{R}} \Gamma(\cdot, z-\tilde z, t-s) *_{x'} f(\cdot,\tilde z,s)d\tilde z\\ &= \int_0^\infty [\Gamma(\cdot,\tilde z+z,t-s)+\Gamma(\cdot,\tilde z-z,t-s)] *_{x'} f(\cdot,\tilde z,s)d\tilde z\,.\end{aligned}$$

Applying $\nabla'\partial_z$ to the representation above

$$\begin{aligned} &\nabla'\partial_z u_{N_s}(x',z,t)\\ =& \begin{cases} \int_0^\infty\int_{\mathbb{R}^{d-1}}\Gamma_{d-1}(x'-\tilde{x}',t-s)\,(\partial_z\Gamma_1(\tilde z+z,t-s)-\partial_z\Gamma_1(\tilde z-z,t-s))\,\nabla' f(\tilde{x}',\tilde z,s)d\tilde{x}'d\tilde z\,,\\ \int_0^\infty\int_{\mathbb{R}^{d-1}}\nabla'^2\Gamma_{d-1}(x'-\tilde{x}',t-s)\,(\partial_z\Gamma_1(\tilde z+z,t-s)-\partial_z\Gamma_1(\tilde z-z,t-s))\,(-\Delta')^{-1}\nabla' f(\tilde{x}',\tilde z,s)d\tilde{x}'d\tilde z\end{cases}\end{aligned}$$

and averaging in the horizontal direction we obtain, on the one hand

$$\begin{aligned} &\langle|\nabla'\partial_z u_{N_s}(\cdot,z,t)|\rangle'\\ \lesssim\;& \int_0^\infty \langle|\Gamma_{d-1}(\cdot,t-s)|\rangle'|\partial_z\Gamma_1(\tilde z+z,t-s)-\partial_z\Gamma_1(\tilde z-z,t-s)|\langle|\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z\\ \overset{(125)\&(119)}{\lesssim}\;& \frac{1}{R}\int_0^\infty |\partial_z\Gamma_1(\tilde z+z,t-s)-\partial_z\Gamma_1(\tilde z-z,t-s)|\langle|f(\cdot,\tilde z,s)|\rangle' d\tilde z\end{aligned}$$

and, on the other hand

$$\begin{aligned} &\langle|\nabla'\partial_z u_{N_s}(\cdot,z,t)|\rangle'\\ \lesssim\;& \int_0^\infty \langle|\nabla'^2\Gamma_{d-1}(\cdot,t-s)|\rangle'|\partial_z\Gamma_1(\tilde z+z,t-s)-\partial_z\Gamma_1(\tilde z-z,t-s)|\langle|(-\Delta')^{-1}\nabla' f(\cdot,\tilde z,s)\rangle' d\tilde z\\ \overset{(125)\&(117)}{\lesssim}\;& \frac{R}{(t-s)}\int_0^\infty |\partial_z\Gamma_1(\tilde z+z,t-s)-\partial_z\Gamma_1(\tilde z-z,t-s)|\langle|f(\cdot,\tilde z,s)|\rangle' d\tilde z\,.\end{aligned}$$

Multiplying by the weight $\frac{1}{z}$ and integrating in $z\in(0,\infty)$ we get

$$\int_0^\infty \langle|\nabla'\partial_z u_{N_s}(\cdot,z,t)|\rangle'\frac{dz}{z} \lesssim \sup_{\tilde z}\int_0^\infty K_{t-s}(z,\tilde z)dz \begin{cases}\frac{1}{R}\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}\,,\\ \frac{1}{(t-s)}R\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}\,,\end{cases}$$

[9] With abuse of notation we will denote with u_{N_s} and f their even reflection

where we called $K_{t-s}(z,\tilde z) = \frac{\tilde z}{z}|\partial_z\Gamma_1(\tilde z - z, t-s) - \partial_z\Gamma_1(z+\tilde z, t-s)|$.
Recalling

$$\sup_{\tilde z}\int_0^\infty K_{t-s}(z,\tilde z)dz \overset{(124)}{\lesssim} \int_{\mathbb{R}} |\partial_z\Gamma_1(z,t-s)|dz + \sup_{z\in\mathbb{R}}(z^2|\partial_z^2\Gamma_1(z,t-s)|)$$

and observing that, in this case

$$\int_{\mathbb{R}} |\partial_z\Gamma_1(z,t-s)|dz + \sup_{z\in\mathbb{R}}(z^2|\partial_z\Gamma_1(z,t-s)|) \overset{(126)\&(129)}{\lesssim} \frac{1}{(t-s)^{\frac12}},$$

we can conclude that

$$\int_0^\infty \langle|\nabla'\partial_z u_{N_s}(\cdot,t)|\rangle'\frac{dz}{z} \lesssim \begin{cases} \frac{1}{(t-s)^{\frac12}}\frac{1}{R}\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z} \\ \frac{1}{(t-s)^{\frac32}}R\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}. \end{cases}$$

Finally, inserting (102) and integrating in time we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \langle|\nabla'\partial_z u_{N_s}(\cdot,z,t)|\rangle'\frac{dz}{z}dt \\
\overset{(102)}{\lesssim}\quad &\int_0^\infty\int_0^\infty\int_0^t \langle|\nabla'\partial_z u_{N_s}(\cdot,\tilde z,t)|\rangle'\frac{dz}{z}dsdt \\
\lesssim\quad &\int_s^\infty\int_0^\infty \min\{\frac{1}{R(t-s)^{\frac12}},\frac{R}{(t-s)^{\frac32}}\}\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}dsdt \\
\overset{(105)\&(106)}{\lesssim}\quad &\int_0^\infty\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}ds\,.
\end{aligned}$$

<u>Argument for (99):</u>
Recall that we need to prove

$$\sup_z\int_0^\infty|\nabla'\partial_z u_C|dt \lesssim \langle|\nabla'\partial_z u|_{z=0}\rangle'\,.$$

By equation (97), the even extension $\overline{u_C}$ satisfies

$$(\partial_t-\Delta)\overline{u_C} = -[\partial_z\overline{u_C}]\delta_{z=0} = -2\partial_z u_C\delta_{z=0} = -2\partial_z u|_{z=0}\delta_{z=0} \tag{111}$$

and therefore we study the following problem on the whole space

$$\begin{cases} (\partial_t-\Delta)\overline{u_C} = -2\partial_z u|_{z=0}\delta & \text{for}\quad z\in\mathbb{R}, t>0\,, \\ \overline{u_C} = 0 & \text{for}\quad t=0\,. \end{cases} \tag{112}$$

By Duhamel's principle

$$\overline{u_C}(x',z,t) = \int_{s=0}^{t} \overline{u_{C_s}}(x',z,t)ds, \tag{113}$$

where $\overline{u_{C_s}}$ solves the initial value problem

$$\begin{cases} (\partial_t - \Delta)\overline{u_{C_s}} &= 0 \qquad\qquad \text{for} \quad z \in \mathbb{R}, t > s\,, \\ \overline{u_{C_s}} &= -2\partial_z u|_{z=0}\delta \qquad \text{for} \quad z \in \mathbb{R}, t = s\,. \end{cases} \tag{114}$$

The solution of problem (114) can be represented via the heat kernel as

$$\begin{aligned} \overline{u_{C_s}}(x',z,t) &= \int \Gamma(z-\tilde{z}, t-s) *_{x'} (-2\partial_z u|_{z=0}\delta)(\tilde{z},s)d\tilde{z}, \\ &= -2\Gamma(z,t-s) *_{x'} \partial_z u(z,s)|_{z=0}\,. \end{aligned}$$

We apply $\nabla'\partial_z$ to the representation above

$$\nabla'\partial_z\overline{u_{C_s}}(x',z,t) = \int_{\mathbb{R}^{d-1}} -2\Gamma_{d-1}(x'-\tilde{x'}, t-s)\partial_z\Gamma_1(z,t-s)\nabla'\partial_z u(\cdot,z,s)|_{z=0}d\tilde{x'}$$

and then average in the horizontal direction,

$$\begin{aligned} & \langle|\nabla'\partial_z\overline{u_{C_s}}(x',z,t)|\rangle' \\ \lesssim\; & \langle|\Gamma_{d-1}(x',t-s)|\rangle'|\partial_z\Gamma_1(z,t-s)|\langle|\nabla'\partial_z u(\cdot,z,s)|_{z=0}|\rangle' \\ \overset{125}{\lesssim}\; & |\partial_z\Gamma_1(z,t-s)|\langle|\nabla'\partial_z u(\tilde{x'},z,s)|_{z=0}|\rangle'\,. \end{aligned}$$

Inserting the previous estimate in the Duhamel formula 113 and integrating in time we get

$$\begin{aligned} & \int_0^\infty \langle|\nabla'\partial_z\overline{u_C}(x',z,t)|\rangle' dt \\ \leq\; & \int_0^\infty \int_0^t \langle|\nabla'\partial_z\overline{u_{C_s}}(x',z,t)|\rangle' ds dt \\ \lesssim\; & \int_0^\infty \int_s^\infty |\partial_z\Gamma_1(z,t-s)|dt\langle|\nabla'\partial_z u(\tilde{x'},z,s)|_{z=0}|\rangle' ds \\ \overset{127}{\lesssim}\; & \int_0^\infty \langle|\nabla'\partial_z u(\tilde{x'},z,s)|_{z=0}|\rangle' ds\,. \end{aligned} \tag{115}$$

The estimate (99) follows immediately after passing to the supremum in (115). □

5 Appendix

5.1 Preliminaries

We start this section by proving some elementary bounds and equivalences, coming directly from the definition of horizontal bandedness. These will turn to be crucial in the proof of the main result.

Lemma 4.

a) If

$$\mathcal{F}'r(k', z, t) = 0 \quad \text{unless} \quad R|k'| \geq 4 \tag{116}$$

then

$$\langle |r(\cdot, z, t)| \rangle' \leq R \langle |\nabla' r(\cdot, z, t)| \rangle' . \tag{117}$$

In particular

$$||r||_{(0,\infty)} \leq R ||\nabla' r||_{(0,\infty)} .$$

b) If

$$\mathcal{F}'r(k', z, t) = 0 \quad \text{unless} \quad R|k'| \leq 1 \tag{118}$$

then

$$\langle |\nabla' r(\cdot, z, t)| \rangle' \leq \frac{1}{R} \langle |r(\cdot, z, t)| \rangle' . \tag{119}$$

In particular

$$||\nabla' r||_{(0,\infty)} \leq R ||r||_{(0,\infty)} .$$

c) If

$$\mathcal{F}'r(k', z, t) = 0 \quad \text{unless} \quad 1 \leq R|k'| \leq 4$$

then

$$||\nabla' (-\Delta')^{-\frac{1}{2}} r||_{(0,\infty)} \sim ||r||_{(0,\infty)} , \tag{120}$$

and

$$||(-\Delta')^{\frac{1}{2}} r||_{(0,\infty)} \sim ||\nabla' r||_{(0,\infty)} . \tag{121}$$

Remark 3. *All the results stated in Lemma 4 are valid with the norm* $||\cdot||_{(0,\infty)}$ *replaced with* $||\cdot||_{(0,1)}$.

Remark 4. *Notice that from (120) and (121), it follows*

$$||\nabla'(-\Delta')^{-1}\nabla' \cdot r||_{(0,\infty)} \lesssim ||r||_{(0,\infty)} . \qquad (122)$$

Proof.

a) By rescaling we may assume $R = 1$.
Let $\phi \in \mathcal{S}(\mathbb{R}^{d-1})$ be a Schwartz function such that

$$\mathcal{F}'\phi(k') = \begin{cases} 0 & \text{for } |k'| \geq 1 \\ 1 & \text{for } |k'| \leq 1 \end{cases}$$

and such that $\int_{\mathbb{R}^{d-1}} \phi(x')dx' = 1$.
We claim that, under assumption (116), there exists $\psi \in L^1(\mathbb{R}^{d-1})$ such that

$$(\mathrm{Id} - \phi*')\mathrm{r} = \psi *' \nabla \mathrm{r} . \qquad (123)$$

Since $r = r - \phi * r$, if we assume (123) the conclusion follows from Young's inequality

$$\int_{\mathbb{R}^{d-1}} |r(x',z)|dx' \leq \int_{\mathbb{R}^{d-1}} |\psi(x')|dx' \int_{\mathbb{R}^{d-1}} |\nabla r(x',z)|dx' .$$

Argument for (123):
Using the assumptions on ϕ and performing suitable change of vari-

ables, we find

$$
\begin{aligned}
& r(x',z) - \int \phi(x'-y')r(y',z)dy' \\
= & \int \phi(x'-y')(r(x',z)-r(y',z))dy' \\
= & \int_{\mathbb{R}^{d-1}} \phi(x'-y') \int_0^1 (x'-y')\nabla' r(tx'+(t-1)(x'-y'),z)dy'dt \\
= & \int_0^1 \int_{\mathbb{R}^{d-1}} \phi(\xi)\nabla' r(x'+(t-1)\xi,z)\cdot \xi d\xi dt \\
= & \int_0^1 \int_{\mathbb{R}^{d-1}} \phi\left(\frac{\hat{y}'-x'}{t}\right) \nabla r(\hat{y}',z)\cdot \frac{\hat{y}'-x'}{t} dt \frac{1}{t^{d-1}} d\hat{y}' \\
= & \int_{\mathbb{R}^{d-1}} \nabla' r(\hat{y}',z)\cdot \left(\int_0^1 \phi\left(\frac{\hat{y}'-x'}{t}\right)\frac{\hat{y}'-x'}{t^d}dt\right) d\hat{y}' \\
= & \int_{\mathbb{R}^{d-1}} \nabla' r(\hat{y}',z)\psi\left(\frac{\hat{y}'-x'}{t}\right) d\hat{y}',
\end{aligned}
$$

where

$$\psi(x') = \int_0^1 \phi\left(\frac{-x'}{t}\right)\frac{x'}{t^d}dt\,.$$

We notice that $\psi \in L^1(\mathbb{R}^{d-1})$, in fact

$$\int_{\mathbb{R}^{d-1}} |\psi(x')|dx' \le \int_0^1 \int_{\mathbb{R}^{d-1}} |\phi(x'/t)\frac{x'}{t^d}|dx'dt = \int_{\mathbb{R}^{d-1}} |\phi(\xi)\xi|d\xi\,.$$

b) In Fourier space we have

$$\mathcal{F}'\nabla' r(k',z) = ik'\mathcal{F}' r(k',z) = R^{-1}\mathcal{F}'G(Rk')\mathcal{F}' r(k',z) = R^{-1}\mathcal{F}'G_R(k')\mathcal{F}' r(k',z),$$

where G is a Schwartz function and $G_R(x') = R^{-d}\mathcal{F}'G(x'/R)$. Since $\int |G_R|dx' = \int |G|dx'$ is independent of R, we may conclude by Young

$$\int |\nabla' r|dx' \le \frac{1}{R}\int |G_R|dx' \int |r|dx' \lesssim \frac{1}{R}\int |r|dx'\,.$$

□

Here we prove an elementary estimate that will be applied in the argument for (91) and (98), Lemma 3

Lemma 5.
Let $K = K(z)$ be a real function and define

$$\overline{K}(z,\tilde{z}) = \frac{\tilde{z}}{z}|K(\tilde{z}-z) - K(z+\tilde{z})|\,.$$

Then

$$\sup_{\tilde{z}} \int_0^\infty \overline{K}(z,\tilde{z})dz \lesssim \int_{\mathbb{R}} |K(z)|dz + \sup_{z\in\mathbb{R}}(z^2|\partial_z K(z)|)\,. \qquad (124)$$

Proof. Let us distinguish two regions: $\frac{1}{2}\left|\frac{\tilde{z}}{z}\right| < 1$ and $\frac{1}{2}\left|\frac{\tilde{z}}{z}\right| > 1$.
For $|z| \geq \frac{1}{2}|\tilde{z}|$ we have

$$\begin{aligned}
& \sup_{\tilde{z}} \int_{|z|\geq\frac{1}{2}|\tilde{z}|} |\overline{K}(z,\tilde{z})|dz \\
\leq\ & \max_{\tilde{z}} \int_{|z|\geq\frac{1}{2}|\tilde{z}|} |K(\tilde{z}-z) - K(z+\tilde{z})|dz \lesssim \int |K(z)|dz\,.
\end{aligned}$$

While for the region $|z| \leq \frac{1}{2}|\tilde{z}|$ we have,

$$\begin{aligned}
& \max_{\tilde{z}} |\tilde{z}| \int_{|z|\leq\frac{1}{2}|\tilde{z}|} \frac{1}{|z|}|K(\tilde{z}-z) - K(z+\tilde{z})|dz \\
=\ & \max_{\tilde{z}} |\tilde{z}| \int_{|z|\leq\frac{1}{2}|\tilde{z}|} \frac{1}{|z|}\left|\int_{-1}^{1} K'(\tilde{z}+tz)zdt\right| dz \\
\leq\ & \max_{\tilde{z}} |\tilde{z}| \int_{-1}^{1} \frac{1}{t} \int_{|z|\leq\frac{t}{2}|\tilde{z}|} |K'(\tilde{z}+z)|dzdt \\
\overset{\frac{1}{2}|\tilde{z}|\leq|\tilde{z}+z|}{\leq}\ & \max_{\tilde{z}} \int_{-1}^{1} \frac{1}{t} \int_{|z|\leq\frac{t}{2}|\tilde{z}|} 2|\tilde{z}+z||K'(\tilde{z}+z)|dtdz \\
\leq\ & \max_{\tilde{z}} \int_{-1}^{1} \frac{2}{t} \max_{|z|\leq\frac{t}{2}|\tilde{z}|} \{|\tilde{z}+z||K'(\tilde{z}+z)|\} \left(\int_{|z|\leq\frac{t}{2}|\tilde{z}|} dz\right) dt \\
=\ & \max_{\tilde{z}} \int_{-1}^{1} \frac{1}{t} \max_{|z|\leq\frac{t}{2}|\tilde{z}|} \{|\tilde{z}+z||K'(\tilde{z}+z)|\} t|\tilde{z}|dt \\
=\ & 2\max_{\tilde{z}} |\tilde{z}| \max_{|z|\leq\frac{t}{2}|\tilde{z}|} \{|\tilde{z}+z||K'(\tilde{z}+z)|\} \\
\overset{\frac{1}{2}|\tilde{z}|\leq|\tilde{z}+z|}{\leq}\ & 4\max_{\tilde{z}} \max_{|z|\leq\frac{t}{2}|\tilde{z}|} \{|z+\tilde{z}|^2|K'(\tilde{z}+z)|\}\,.
\end{aligned}$$

In conclusion we have

$$\max_z \int |\bar{K}(z,\tilde{z})|dz \lesssim \int |K(z)|dz + \max_z |z|^2|K'(z)|\,.$$

□

5.2 Heat kernel: elementary estimates

In this section we recall the definition of the heat kernel and some properties and estimates that we will use throughout the paper.
The function $\Gamma : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is defined as

$$\Gamma(x,t) = \frac{1}{t^{d/2}} \exp(-\frac{|x|^2}{4t})$$

and we can rewrite it as

$$\Gamma(x,t) = \Gamma_1(z,t)\Gamma_{d-1}(x',t) \qquad x' \in \mathbb{R}^{d-1}, z \in \mathbb{R},$$

where

$$\Gamma_1(z,t) = \frac{1}{t^{1/2}} \exp(-\frac{z^2}{4t})$$

and

$$\Gamma_{d-1}(z,t) = \frac{1}{t^{(d-1)/2}} \exp(-\frac{|x'|^2}{4t})\,.$$

Here we list the bounds on the derivatives of Γ that are used in Section 5, Lemma 3:

1. $$\langle |(\nabla')^n \Gamma_{d-1}|\rangle' \approx \frac{1}{t^{\frac{n}{2}}}\,. \tag{125}$$

2. $$\int_{\mathbb{R}} |\partial_z^n \Gamma_1| dz \lesssim \frac{1}{t^{\frac{n}{2}}}\,. \tag{126}$$

3. $$\int_0^\infty |\partial_z \Gamma_1(z,t)|dt = \int_0^\infty \left| \frac{1}{\hat{t}^{3/2}} \exp\left(-\frac{1}{4\hat{t}}\right) \right| d\tilde{t} \lesssim 1\,, \tag{127}$$

   where we have used the change of variable $\hat{t} = \frac{t}{z^2}$.

4.

$$\sup_{z\in\mathbb{R}} \left(z|\partial_z \Gamma_1(z,t)|\right) = \sup_{\xi} \left| \frac{1}{t^{\frac{1}{2}}} \xi^2 \exp^{-\xi^2} \right| \lesssim \frac{1}{t^{\frac{1}{2}}}\,, \tag{128}$$

where we have used the change of variable $\xi = \frac{z}{t^{\frac{1}{2}}}$.

5.

$$\sup_{z\in\mathbb{R}} \left(z^2|\partial_z \Gamma_1(z,t)|\right) = \sup_{\xi} \left| \xi^3 \exp^{-\xi^2} \right| \lesssim 1\,, \tag{129}$$

where we have used the change of variable $\xi = \frac{z}{t^{\frac{1}{2}}}$.

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