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Dynamical torsion in view of a distinguished class of Dirac operators

by

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DYNAMICAL TORSION IN VIEW OF A DISTINGUISHED CLASS OF DIRAC OPERATORS

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Abstract. In this paper we discuss geometric torsion in terms of a distinguished class of Dirac operators. We demonstrate that from this class of Dirac operators a variational problem for torsion can be derived similar to that of Yang-Mills gauge theory. As a consequence, one ends up with a propagating torsion even in vacuum as opposed to the usual variational problem encountered in ordinary Einstein-Cartan theory.

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1. Introduction

A connection on the frame bundle of any smooth manifold M is known to yield the two independent geometrical concepts of *curvature* and *torsion*. There are various (but equivalent) approaches to the torsion of a connection, depending on the geometrical setup. For instance, the torsion of a connection $\nabla \equiv \nabla^{\rm TM}$ on the tangent bundle of M may be defined by

$$\tau_\nabla := d_\nabla \mathfrak{Id} \in \Omega^2(M, TM)\,. \tag{1}$$

Here, d_∇ denotes the exterior covariant derivative with respect to $\nabla^{\rm TM}$. The canonical one-form $\mathfrak{Id} \in \Omega^1(M, TM)$ is defined as $\mathfrak{Id}_x(v) := v$ for all tangent vectors $v \in T_xM$ and $x \in M$. This canonical one-form corresponds to the soldering form on the frame bundle of M.

In general relativity a field equation for the torsion is obtained by the so-called "Palatini formalism", where the metric and the connection on the tangent bundle are

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regarded as being independent from each other (c.f. [11]). The resulting field equation for torsion in Einstein's theory of gravity is known to be given by

$$\tau_\nabla = \lambda_{\rm grav}\, \mathrm{j}_{\rm spin}\,, \tag{2}$$

where the real coupling constant $\lambda_{\rm grav}$ is proportional to the gravitational constant. The so-called "spin-current" $\mathrm{j}_{\rm spin} \in \Omega^2(M, TM)$ is obtained by the variation of the action functional that dynamically describes matter fields with respect to the metric connection. Usually, ordinary bosonic matter does not depend on the metric connection as, for instance, described by the Standard Model. According to (2) the connection is thus provided by the Levi-Civita connection. This holds true, in particular, when matter is disregarded. When matter is defined in terms of spinor fields, as in the case of the Dirac action, then the right-hand side of (2) may be non-vanishing. This is usually rephrased by the statement

"*Spin is the source of torsion*".

However, even in this case torsion is not propagating in space-time since (2) is purely algebraic relation between torsion and matter. Furthermore, the ordinary Dirac action is real only if it is defined in terms of a torsion-free connection (c.f. [4]).

In this work, we discuss torsion within the framework of general relativity which is different from the ordinary Palatini formalism. We obtain field equations for the torsion from a functional that looks similar to the Einstein-Hilbert-Yang-Mills-Dirac action. This functional is derived from a certain class of Dirac operators. The geometrical background of these Dirac operators is basically dictated by the reality condition imposed on the action including torsion. Furthermore, this class of Dirac operators fits well with those giving rise to Einstein's theory of gravity, ordinary Yang-Mills-Dirac theory and non-linear $\sigma-$models, as discussed in [15].

From a physics point of view torsion provides an additional degree of freedom to Einstein's theory of gravity. In the latter the action of a gravitational field is described in terms of the curvature of a smooth four dimensional manifold M. Even more, this curvature is assumed to be uniquely determined by the Levi-Civita connection on the tangent bundle of M with fiber metric $g_{\rm M}$. In other words, the geometrical model behind Einstein's theory of gravity is known to be given by a smooth (orientable) Lorentzian four-manifold $(M, g_{\rm M})$ of signature $s = \pm 2$ (resp. a diffeomorphism class thereof). For a "space-time" $(M, g_{\rm M})$ to be physically admissible the metric field $g_{\rm M}$ has to fulfill the Einstein equation of gravity (and maybe topological restriction on M, like global hyperbolicity), whereby the source of gravity is given by the energy-momentum current of matter. The latter is either phenomenologically described by a mass density or in terms of matter fields (i.e. sections of certain vector bundles over space-time). This holds true, especially, if the spin of matter is taken into account. In this case, matter is geometrically described by (Dirac) spinor fields. According to the above mentioned statement about the relation between spin and torsion a huge variety of generalizations of Einstein's theory of gravity including torsion has been proposed over the last decades, going under the name "Einstein-Cartan theory", "Poincare gauge gravity", "teleparallel gravity", etc. (see, for instance, [7], as well as the more recent essay [8] and the references cited therein). For higher order gravity with (propagating) torsion discussed within the realm of Connes' non-commutative geometry we refer to [6], [12] and the references sited therein.

In some of the above mentioned approaches torsion does not propagate, whereas other approaches propose only torsion but no curvature. In any case, spin is considered to be the source of torsion. Since spin is fundamental torsion plays also a prominent role in (super-)string theory (see, for instance, [2] and [9]). For an overview about the role of torsion in theoretical physics we refer to [5]. Also, we refer to [16] as a reasonable source of references to the issue. From a physics perspective it is speculated that torsion might contribute to dark energy, whose existence seems experimentally confirmed by the observed acceleration of the universe.

The approach to a dynamical torsion presented in this work is different, for it starts out with Rarita-Schwinger fermions to which torsion minimally couples. These fermions are geometrically modeled by sections of a twisted spinor bundle, where the "inner degrees of freedom" are generated by the (co-)tangent bundle of the underlying manifold. As a consequence, the resulting coupling to torsion completely parallels that of spinor-electrodynamics. Since the known matter is geometrically described by Dirac spinors, the Rarita-Schwinger fermions may serve as a geometrical model to physically describe dark matter (or parts thereof). Accordingly, the energy momentum of torsion (the underlying gauge field) may contribute to dark energy. The coupling constant between the Rarita-Schwinger and the torsion field, however, is a free parameter like in ordinary gauge theory, although the underlying action is derived from "first (geometrical) principles". Moreover, this gauge coupling constant is an additional free parameter as opposed to ordinary Einstein-Cartan theory.

This paper is organized as follows: We start out with a summary of the necessary geometrical background of Dirac operators in terms of general Clifford module bundles. Afterwards, we discuss torsion in the context of a distinguished class of Dirac operators which give rise to field equations similar to Dirac-Yang-Mills equations.

2. Geometrical background

The geometrical setup presented fits well with that already discussed in [15] for non-linear $\sigma-$models and Yang-Mills theory. For the convenience of the reader we briefly summarize the basic geometrical background. In particular, we present the basic features of Dirac operators of simple type. This class of Dirac operators will play a fundamental role in our discussion. For details we refer to [14] (or [15]) and [1], as well as to [3] which serves as a kind of "standard reference" for what follows.

In the sequel, $(M, g_{\rm M})$ always denotes a smooth orientable (semi-)Riemannian manifold of finite dimension $n \equiv p + q$. The index of the (semi-)Riemannian metric $g_{\rm M}$ is $s \equiv p - q \not\equiv 1 \bmod 4$. The *bundle of exterior forms* of degree $k \geq 0$ is denoted by $\Lambda^k T^*M \to M$ with its canonical projection. Accordingly, the *Grassmann bundle* is given by $\Lambda T^*M \equiv \bigoplus_{k\geq 0} \Lambda^k T^*M \to M$. It naturally inherits a metric denoted by $g_{\Lambda \rm M}$, such that the direct sum is orthogonal and the restriction of $g_{\Lambda \rm M}$ to degree one equals to the fiber metric $g^*_{\rm M}$ of the cotangent bundle $T^*M \to M$.

The mutually inverse *musical isomorphisms* in terms of $g_{\rm M}$ (resp. $g^*_{\rm M}$) are denoted by $\;^{\flat/\sharp} : TM \simeq T^*M$, such that, for instance, $g_{\rm M}(u, v) = g^*_{\rm M}(u^\flat, v^\flat)$ for all $u, v \in TM$.

The *Clifford bundle* of $(M, g_{\rm M})$ is denoted by $\pi_{\rm Cl} : Cl_{\rm M} = Cl^+_{\rm M} \oplus Cl^-_{\rm M} \to M$. Its canonical "even/odd" grading involution is $\tau_{\rm Cl} \in {\rm End}(Cl_{\rm M})$. The hermitian structure $\langle \cdot, \cdot \rangle_{\rm Cl}$ is the one induced by the metric $g_{\rm M}$ due to the canonical linear isomorphism between the Clifford and the Grassmann bundle (c.f. (8), below). We always assume the Clifford and the Grassmann bundle to be generated by the cotangent bundle of M.

Throughout the present work we always identify on $(M, g_{\rm M})$ the vector bundle $\pi_{\Lambda \rm T^*M}|_{\Lambda^2 \rm T^*M} : \Lambda^2 T^*M \to M$ with the (Lie algebra) bundle $\pi_{\rm so} : so(TM) \to M$ of the $g_{\rm M}$ skew-symmetric endomorphisms on the tangent bundle $\pi_{\rm TM} : TM \to M$ due to the canonical linear (bundle) isomorphism (over the identity on M)

$$\begin{aligned} \Lambda^2 T^*M &\stackrel{\simeq}{\longrightarrow} so(TM) \subset \mathrm{End}(TM) \\ \omega &\mapsto \Omega\,, \end{aligned} \tag{3}$$

where for all $u, v \in TM$: $g_{\rm M}(u, \Omega(v)) := \omega(u, v)$. Accordingly, we always take advantage of the induced isomorphism

$$\begin{aligned} \Lambda^2 T^*M \otimes TM &\stackrel{\simeq}{\longrightarrow} T^*M \otimes so(TM) \\ \omega \otimes v &\mapsto v^\flat \otimes \Omega\,. \end{aligned} \tag{4}$$

A smooth complex vector bundle $\pi_{\mathcal{E}} : \mathcal{E} \to M$ is called a *Clifford module bundle*, provided there is a *Clifford map*. That is, there is a smooth linear (bundle) map (over the identity on M)

$$\begin{aligned} \gamma_{\mathcal{E}} : T^*M &\longrightarrow \mathrm{End}(\mathcal{E}) \\ \alpha &\mapsto \gamma_{\mathcal{E}}(\alpha)\,, \end{aligned} \tag{5}$$

satisfying $\gamma_{\mathcal{E}}(\alpha)^2 = \epsilon g^*_{\rm M}(\alpha,\alpha)\mathrm{Id}_{\mathcal{E}}$. Here, $\epsilon \in \{\pm 1\}$ depends on how the *Clifford product* is defined. That is, $\alpha^2 := \pm g^*_{\rm M}(\alpha,\alpha)1_{\rm Cl} \in Cl_{\rm M}$, for all $\alpha \in T^*M \subset Cl_{\rm M}$ and $1_{\rm Cl} \in Cl_{\rm M}$ denotes the unit element.

To emphasize the module structure we write

$$\pi_{\mathcal{E}} : (\mathcal{E}, \gamma_{\mathcal{E}}) \longrightarrow (M, g_{\rm M})\,. \tag{6}$$

The bundle (6) is called an *odd hermitian Clifford module bundle*, provided it is $\mathbb{Z}_2-$graded, with grading involution $\tau_{\mathcal{E}}$, and endowed with an hermitian structure $\langle\cdot,\cdot\rangle_{\mathcal{E}}$, such that $\gamma_{\mathcal{E}} \circ \tau\varepsilon = -\tau_{\mathcal{E}} \circ \gamma_{\mathcal{E}}$ and both the grading involution and Clifford action are either hermitian or skew-hermitian. In what follows, (6) always means an odd hermitian Clifford module.

The linear map

$$\begin{aligned} \delta_\gamma : \Omega(M, \mathrm{End}(\mathcal{E})) &\longrightarrow \Omega^0(M, \mathrm{End}(\mathcal{E})) \\ \omega \equiv \alpha \otimes \mathfrak{B} &\mapsto \not\omega \equiv \gamma_{\mathcal{E}}\big(\sigma^{-1}_{\rm Ch}(\alpha)\big) \circ \mathfrak{B} \end{aligned} \tag{7}$$

is called the *"quantization map"*. It is determined by the linear isomorphism called *symbol map*:

$$\begin{aligned} \sigma_{\rm Ch} : Cl_{\rm M} &\stackrel{\simeq}{\longrightarrow} \Lambda T^*M \\ \mathfrak{a} &\mapsto \Gamma_{\rm Ch}(\mathfrak{a})1_\Lambda\,. \end{aligned} \tag{8}$$

Here, $1_\Lambda \in \Lambda T^*M$ is the unit element. The homomorphism $\Gamma_{\rm Ch} : Cl_{\rm M} \to \mathrm{End}(\Lambda T^*M)$ is given by the canonical Clifford map:

$$\begin{aligned} \gamma_{\rm Cl} : T^*M &\longrightarrow \mathrm{End}(\Lambda T^*M) \\ v &\mapsto \left\{ \begin{array}{ccc} \Lambda T^*M & \longrightarrow & \Lambda T^*M \\ \omega & \mapsto & \epsilon\, int(v)\omega + ext(v^\flat)\omega\,, \end{array} \right. \end{aligned} \tag{9}$$

where, respectively, "*int*" and "*ext*" indicate "interior" and "exterior" multiplication.

When restricted to $\Omega^1(M, \mathrm{End}(\mathcal{E}))$ the quantization map (7) has a canonical right-inverse given by

$$\begin{aligned} ext_\Theta : \Omega^0(M, \mathrm{End}(\mathcal{E})) &\longrightarrow \Omega^1(M, \mathrm{End}(\mathcal{E})) \\ \Phi &\mapsto \Theta\Phi\,, \end{aligned} \tag{10}$$

where the *canonical one-form* $\Theta \in \Omega^1(M, \mathrm{End}(\mathcal{E}))$ is given by $\Theta(v) := \frac{\varepsilon}{n}\gamma_{\mathcal{E}}(v^\flat)$, for all $v \in TM$. The associated projection operators are $\mathfrak{p} \equiv ext_\Theta \circ \delta_\gamma|_{\Omega^1}$ and $\mathfrak{q} := \mathrm{Id}_{\Omega^1} - \mathfrak{p}$, such that

$$\Omega^1(M, \mathrm{End}(\mathcal{E})) = \mathfrak{p}\big(\Omega^1(M, \mathrm{End}(\mathcal{E}))\big) \oplus \mathfrak{q}\big(\Omega^1(M, \mathrm{End}(\mathcal{E}))\big)\,. \tag{11}$$

Notice that for any connection on a Clifford module bundle the first order operator $\mathcal{T}_\nabla \equiv \mathfrak{q}(\nabla^{\mathcal{E}}) : \Omega^0(M, \mathcal{E}) \to \Omega^1(M, \mathcal{E})$, $\psi \mapsto \nabla^{\mathcal{E}}\psi - \Theta\big(\nabla\!\!\!/^{\mathcal{E}}\psi\big)$ is the associated *twister operator*. Here, $\nabla\!\!\!/^{\mathcal{E}} \equiv \delta_\gamma(\nabla^{\mathcal{E}})$ denotes the Dirac operator associated to the connection (see below).

A (linear) connection on a Clifford module bundle is called a *Clifford connection* if the corresponding covariant derivative $\nabla^{\mathcal{E}}$ "commutes" with the Clifford map $\gamma_{\mathcal{E}}$ in the following sense:

$$[\nabla^{\mathcal{E}}_X, \gamma_{\mathcal{E}}(\alpha)] = \gamma_{\mathcal{E}}\big(\nabla^{T^*M}_X\alpha\big) \qquad \big(X \in \mathfrak{Sec}(M, TM),\ \alpha \in \mathfrak{Sec}(M, T^*M)\big)\,. \tag{12}$$

Here, ∇^{T^*M} is the Levi-Civita connection on the co-tangent bundle with respect to g^*_{M}.

Equivalently, a connection on a Clifford module bundle is a Clifford connection if and only if it fulfills:

$$\nabla^{T^*M\otimes \mathrm{End}(\mathcal{E})}_X \Theta = 0 \qquad \big(X \in \mathfrak{Sec}(M, TM)\big)\,. \tag{13}$$

Apparently, Clifford connections provide a distinguished class of connections on any Clifford module bundle.

We denote Clifford connections by ∂_{A}. This notation is used because Clifford connections are parametrized by a family of locally defined one-forms $A \in \Omega^1(U, \mathrm{End}_\gamma(\mathcal{E}))$. Here, $\mathrm{End}_\gamma(\mathcal{E}) \subset \mathrm{End}(\mathcal{E})$ denotes the total space of the algebra bundle of endomorphisms which commute with the Clifford action that is provided by the Clifford map $\gamma_{\mathcal{E}}$.

We call in mind that a *Dirac operator* $D\!\!\!\!/$ on a Clifford module bundle is a first order differential operator acting on sections $\psi \in \mathfrak{Sec}(M, \mathcal{E})$, such that $[D\!\!\!\!/, df]\psi = \gamma_{\mathcal{E}}(df)\psi$ for all smooth functions $f \in \mathcal{C}^\infty(M)$. The set of all Dirac operators on a given Clifford module bundle is denoted by $\mathfrak{Dir}(\mathcal{E}, \gamma_{\mathcal{E}})$. It is an affine set over the vector space $\Omega^0(M, \mathrm{End}(\mathcal{E}))$. Moreover, Dirac operators are *odd* operators on odd (hermitian) Clifford module bundles: $D\!\!\!\!/\tau_{\mathcal{E}} = -\tau_{\mathcal{E}}D\!\!\!\!/$. In this case, the underlying vector space reduces to $\Omega^0(M, \mathrm{End}^-(\mathcal{E}))$.

We call the Dirac operator $\nabla\!\!\!/^{\mathcal{E}} \equiv \delta_\gamma(\nabla^{\mathcal{E}})$ the "quantization" of a connection $\nabla^{\mathcal{E}}$ on a Clifford module bundle. Let $e_1, \ldots, e_n \in \mathfrak{Sec}(U, TM)$ be a local frame and $e^1, \ldots, e^n \in \mathfrak{Sec}(U, T^*M)$ its dual frame. For $\psi \in \mathfrak{Sec}(M, \mathcal{E})$ one has

$$\nabla\!\!\!/^{\mathcal{E}}\psi := \sum_{k=1}^n \delta_\gamma(e^k)\nabla^{\mathcal{E}}_{e_k}\psi = \sum_{k=1}^n \gamma_{\mathcal{E}}(e^k)\nabla^{\mathcal{E}}_{e_k}\psi\,, \tag{14}$$

where the canonical embedding $\Omega(M) \hookrightarrow \Omega(M, \mathrm{End}(\mathcal{E}))$, $\omega \mapsto \omega \equiv \omega \otimes \mathrm{Id}_{\mathcal{E}}$ is taken into account.

Every Dirac operator has a canonical first-order decomposition:

$$D\!\!\!\!/ = \partial\!\!\!/_{\mathrm{B}} + \Phi_{\mathrm{D}}\,. \tag{15}$$

Here, ∂_{B} denotes the (covariant derivative of the) *Bochner connection* that is defined by $D\!\!\!\!/$ as

$$2ev_g\big(df, \partial_{\mathrm{B}}\psi\big) := \epsilon\big([D\!\!\!\!/^2, f] - \delta_g df\big)\psi \qquad \big(\psi \in \mathfrak{Sec}(M, \mathcal{E})\big)\,, \tag{16}$$

with ev_g" being the evaluation map with respect to g_{M} and δ_g the dual of the exterior derivative (see [3]).

The zero-order section $\Phi_{\mathrm{D}} := \not{D} - \not{\partial}_{\mathrm{B}} \in \mathfrak{Sec}(M, \mathrm{End}(\mathcal{E}))$ is thus also uniquely determined by $\not{D}$. We call the Dirac operator $\not{\partial}_{\mathrm{B}}$ the *"quantized Bochner connection"*.

Since the set $\mathfrak{Dir}(\mathcal{E}, \gamma_{\mathcal{E}})$ is an affine space, every Dirac operator can be written as

$$\not{D} = \not{\partial}_{\mathrm{A}} + \Phi\,. \tag{17}$$

However, this decomposition is far from being unique. The section $\Phi \in \mathfrak{Sec}(M, \mathrm{End}(\mathcal{E}))$ depends on the chosen Clifford connection ∂_{A}. In general, a Dirac operator does not uniquely determine a Clifford connection.

Definition 2.1. *A Dirac operator is said to be of "simple type" provided that Φ_D anti-commutes with the Clifford action:*

$$\Phi_D \gamma_{\mathcal{E}}(\alpha) = -\gamma_{\mathcal{E}}(\alpha)\Phi_D \qquad \big(\alpha \in T^*M\big)\,. \tag{18}$$

It follows that a Dirac operator of simple type uniquely determines a Clifford connection ∂_{A} together with a zero-order operator $\phi_{\mathrm{D}} \in \mathfrak{Sec}(M, \mathrm{End}_\gamma(\mathcal{E}))$, such that (c.f. [14])

$$\not{D} = \not{\partial}_{\mathrm{A}} + \tau_{\mathcal{E}}\phi_{\mathrm{D}}\,. \tag{19}$$

These Dirac operators play a basic role in the geometrical description of the Standard Model (c.f. [14]). They are also used in the context of the family index theorem (see, for instance, [3]). Apparently, Dirac operators of simple type provide a natural generalization of quantized Clifford connections. Indeed, they build the biggest class of Dirac operators such that the corresponding Bochner connections are also Clifford connections.

Every Dirac operator is known to have a unique *second order decomposition*

$$\not{D}^2 = \triangle_{\mathrm{B}} + V_{\mathrm{D}}\,, \tag{20}$$

where the *Bochner-Laplacian* (or "trace Laplacian") is given in terms of the Bochner connection as $\triangle_{\mathrm{B}} := \epsilon ev_g\big(\partial_{\mathrm{B}}^{T^*M\otimes\mathcal{E}} \circ \partial_{\mathrm{B}}\big)$. The trace of the zero-order operator $V_{\mathrm{D}} \in \mathfrak{Sec}(M, \mathrm{End}(\mathcal{E}))$ explicitly reads (c.f. [14]):

$$tr_{\mathcal{E}} V_{\mathrm{D}} = tr_\gamma\big(curv(\not{D}) - \varepsilon\,\mathrm{ev}_g(\omega_{\mathrm{D}}^2)\big) - \varepsilon\delta_g\big(tr_{\mathcal{E}}\omega_{\mathrm{D}}\big)\,, \tag{21}$$

where $curv(\not{D}) \in \Omega^2(M, \mathrm{End}(\mathcal{E}))$ denotes the curvature of the Dirac connection of $\not{D} \in \mathfrak{Dir}(\mathcal{E}, \gamma_{\mathcal{E}})$ and $tr_\gamma := tr_{\mathcal{E}} \circ \delta_\gamma$ the "quantized trace". The Dirac connection of $\not{D}$ is defined by $\partial_{\mathrm{D}} := \partial_{\mathrm{B}} + \omega_{\mathrm{D}}$, where $\omega_{\mathrm{D}} \equiv ext_\Theta\Phi_{\mathrm{D}} \in \Omega^1(M, \mathrm{End}^+(\mathcal{E}))$. The Dirac connection has the property that it is uniquely determined by $\not{D}$ and $\not{\partial}_{\mathrm{D}} = \not{D}$.

Let M be closed compact. We call the functional

$$\begin{aligned} \mathcal{I}_{\mathrm{D}} : \mathfrak{Dir}(\mathcal{E}, \gamma_{\mathcal{E}}) &\to \mathbb{C} \\ \not{D} &\mapsto \int_M *tr_{\mathcal{E}} V_{\mathrm{D}} \end{aligned} \tag{22}$$

the *"universal Dirac action"* and

$$\begin{aligned} \mathcal{I}_{\mathrm{D,tot}} : \mathfrak{Dir}(\mathcal{E}, \gamma_{\mathcal{E}}) \times \mathfrak{Sec}(M, \mathcal{E}) &\to \mathbb{C} \\ (\not{D}, \psi) &\mapsto \int_M *\big(\langle\psi, \not{D}\psi\rangle_{\mathcal{E}} + tr_{\mathcal{E}} V_{\mathrm{D}}\big) \end{aligned} \tag{23}$$

the *"total Dirac action"*. Here, "$*$" is the Hodge map with respect to g_{M} and a chosen orientation of M.

If the Dirac connection of $\not{D}$ is a Clifford connection, then also $\partial_{\rm D} = \partial_{\rm B}$. In this case, the universal Dirac action (22) reduces to the usual Einstein-Hilbert functional.

In contrast, for Dirac operators of simple type the universal Dirac action becomes

$$\mathcal{I}_{\rm D}\big(\not{\partial}_{\rm A} + \tau_{\mathcal{E}}\phi_{\rm D}\big) = \int_M *\big(- \epsilon \tfrac{rk(\mathcal{E})}{4} scal(g_{\rm M}) + tr_{\mathcal{E}}\phi_{\rm D}^2\big)\,, \tag{24}$$

with $rk(\mathcal{E}) \geq 1$ being the rank of the underlying Clifford module bundle and the smooth function $scal(g_{\rm M})$ is the *scalar curvature* of the Levi-Civita connection of $g_{\rm M}$. The explicit formula (24) is a direct consequence of Lemma 4.1 and the Corollary 4.1 of Ref. [14] (see also Sec. 6 in loc. site). The restriction of the universal Dirac action (22) to Dirac operators of simple type (19) therefore corresponds to the Einstein-Hilbert action with a cosmological constant Λ, where (up to numerical factors)

$$\Lambda = tr_{\mathcal{E}}\phi_{\rm D}^2 \equiv \pm\|\phi_{\rm D}\|^2\,. \tag{25}$$

3. Dirac operators with torsion

To this end let $(M, g_{\rm M})$ be a (semi-)Riemannian *spin*-manifold of even dimension $n = p + q$ and signature $p - q \not\equiv 1 \mod 4$. Let $\pi_{\rm S} : S = S^+ \oplus S^- \to M$ be a (complexified) spinor bundle with grading involution $\tau_{\rm S} \in {\rm End}(S)$. The hermitian structure is denoted by $\langle\cdot,\cdot\rangle_{\rm S}$. The Clifford action is provided by the canonical Clifford map $\gamma_{\rm S} : T^*M^{\mathbb{C}} \to {\rm End}_{\mathbb{C}}(S) \simeq Cl_{\rm M}^{\mathbb{C}}$. The induced Clifford action is supposed to be anti-hermitian. The Clifford action also anti-commutes with the grading involution. The grading involution is assumed to be either hermitian or anti-hermitian.

We consider the twisted spinor bundle

$$\pi_{\mathcal{E}_1} : \mathcal{E}_1 := S \otimes_{\rm M} TM \longrightarrow M \tag{26}$$

with the grading involution $\tau_{\mathcal{E}_1} := \tau_{\rm S} \otimes {\rm Id}_{\rm TM}$ and Clifford action $\gamma_{\mathcal{E}_1} := \gamma_{\rm S} \otimes {\rm Id}_{\rm TM}$. The hermitian structure reads: $\langle\cdot,\cdot\rangle_{\mathcal{E}_1} := \langle\cdot,\cdot\rangle_{\rm S}\, g_{\rm M}$.

The *Clifford extension* of (26) is denoted by (c.f. [15])

$$\pi_{\mathcal{E}} : \mathcal{E} := \mathcal{E}_1 \otimes_{\rm M} Cl_{\rm M} \longrightarrow M\,. \tag{27}$$

Here, the grading involution and Clifford action, respectively, are given by $\tau_{\mathcal{E}} := \tau_{\mathcal{E}_1} \otimes \tau_{\rm Cl}$ and $\gamma_{\mathcal{E}} := \gamma_{\mathcal{E}_1} \otimes {\rm Id}_{\rm Cl}$. The hermitian structure is $\langle\cdot,\cdot\rangle_{\mathcal{E}} := \langle\cdot,\cdot\rangle_{\mathcal{E}_1} \langle\cdot,\cdot\rangle_{\rm Cl}$.

In what follows all vector bundles are regarded as complex vector bundles, though we do not explicitly indicate their complexifications.

We denote the covariant derivative of the spin connection by $\nabla^{\rm S}$. The corresponding spin-Dirac operator is $\not{\nabla}^{\rm S}$.

For $A \in \Omega^1(M, \Lambda^2 T^*M)$, the most general metric connection on the tangent bundle is known to be given by the covariant derivative

$$\nabla^g := \nabla^{\rm LC} + A\,. \tag{28}$$

Here, $\nabla^{\rm LC}$ is the covariant derivative of the Levi-Civita connection on the tangent bundle with respect to $g_{\rm M}$.

Accordingly, the torsion $\tau_{\nabla^g} \in \Omega^2(M, TM)$ of a metric connection (28) can be expressed by the *torsion form* (also called "torsion tensor")

$$\tau_{\rm A}(u, v) \equiv A(u)v - A(v)u \qquad (u, v \in TM)\,. \tag{29}$$

Indeed, the definitions (28) and (29) imply that for all smooth tangent vector fields $X, Y \in \mathfrak{Sec}(M, TM)$:

$$\begin{aligned} \tau_{\mathrm{A}}(X,Y) &= \nabla^g_X Y - \nabla^{\mathrm{LC}}_X Y - \nabla^g_Y X + \nabla^{\mathrm{LC}}_Y X \\ &= \nabla^g_X Y - \nabla^g_Y X - [X,Y] \\ &= d_{\nabla^g}\mathfrak{Id}(X,Y)\,. \end{aligned} \tag{30}$$

Hence, we do not make a distinction between the torsion $\tau_{\nabla^g} \in \Omega^2(M, TM)$ of a metric connection ∇^g and the torsion form $\tau_{\mathrm{A}} \in \Omega^2(M, TM)$ of $A \in \Omega^1(M, \Lambda^2 T^*M)$.

Definition 3.1. *Let ∇^g be the covariant derivative of a metric connection on the tangent bundle of (M, g_M). We call the section*

$$A := \nabla^g - \nabla^{LC} \in \Omega^1(M, \Lambda^2 T^*M) \tag{31}$$

the "torsion potential" of $\tau_A = d_{\nabla^g}\mathfrak{Id}$.

Consider the following hermitian *Clifford connection* on the Clifford module bundle (26) that is provided by the following covariant derivative:

$$\begin{aligned} \partial_{\mathrm{A}} &:= \nabla^{\mathrm{S}} \otimes \mathrm{Id}_{\mathrm{TM}} + \mathrm{Id}_{\mathrm{S}} \otimes \nabla^g \\ &= \nabla^{\mathrm{S}} \otimes \mathrm{Id}_{\mathrm{TM}} + \mathrm{Id}_{\mathrm{S}} \otimes \nabla^{\mathrm{LC}} + \mathrm{Id}_{\mathrm{S}} \otimes A \\ &\equiv \nabla^{\mathcal{E}_1} + \mathrm{Id}_{\mathrm{S}} \otimes A\,. \end{aligned} \tag{32}$$

Clearly, (32) is but the gauge covariant derivative of a twisted spin connection on (26) that is defined by the lift of (28).

Of course, every metrical connection (28) on (M, g_{M}) can be lifted to the spinor bundle $\pi_{\mathrm{S}} : S \to M$. However, in this case the resulting spin connection is neither a Clifford connection, nor is the ordinary Dirac action real-valued.

With respect to an oriented orthonormal frame $e_1, \ldots, e_n \in \mathfrak{Sec}(U, TM)$, with the dual frame being denoted by $e^1, \ldots, e^n \in \mathfrak{Sec}(U, T^*M)$, the corresponding *twisted spin-Dirac operator* reads:

$$\begin{aligned} \not{\partial}_{\mathrm{A}} &= \not{\nabla}^{\mathrm{S}} \otimes \mathrm{Id}_{\mathrm{TM}} + \sum_{k=1}^n \gamma_{\mathrm{S}}(e^k) \otimes \nabla^g_{e_k} \\ &= \not{\nabla}^{\mathrm{S}} \otimes \mathrm{Id}_{\mathrm{TM}} + \sum_{k=1}^n \gamma_{\mathrm{S}}(e^k) \otimes \nabla^{\mathrm{LC}}_{e_k} + \sum_{k=1}^n \gamma_{\mathrm{S}}(e^k) \otimes A(e_k) \\ &\equiv \not{\nabla}^{\mathcal{E}_1} + \sum_{k=1}^n \gamma_{\mathrm{S}}(e^k) \otimes A(e_k) \equiv \not{\nabla}^{\mathcal{E}_1} + \not{A}\,. \end{aligned} \tag{33}$$

It looks similar to the usual gauge covariant Dirac operator encountered in ordinary electrodynamics on Minkowski space-time.

Accordingly, on the Clifford extension (27) we consider the Clifford connection

$$\begin{aligned} \tilde{\nabla}^{\mathcal{E}} &:= \partial_{\mathrm{A}} \otimes \mathrm{Id}_{\mathrm{Cl}} + \mathrm{Id}_{\mathcal{E}_1} \otimes \nabla^{\mathrm{Cl}} \\ &\equiv \nabla^{\mathcal{E}} + \mathrm{Id}_{\mathrm{S}} \otimes A \otimes \mathrm{Id}_{\mathrm{Cl}}\,, \end{aligned} \tag{34}$$

with ∇^{Cl} being the induced Levi-Civita connection on the Clifford bundle.

With regard to the canonical embedding $\pi_{\mathcal{E}}|_{\mathrm{S}} : \mathcal{E}_1 \hookrightarrow \mathcal{E} \to M,\ z \mapsto z \equiv z \otimes 1$ one obtains for $\psi \equiv \psi \otimes 1 \in \mathfrak{Sec}(M, \mathcal{E})$ the equality

$$\tilde{\nabla}^{\mathcal{E}}\psi = \partial_{\mathrm{A}}\psi \otimes 1\,. \tag{35}$$

Definition 3.2. *Let $d_{\nabla^{\rm LC}}$ be the exterior covariant derivative induced by the Levi-Civita connection with curvature $F_{\nabla^{\rm LC}}$. Also, let $F_{\nabla^g} \in \Omega^2(M, \Lambda^2 T^*M)$ be the curvature of ∇^g. We call the relative curvature*

$$\begin{aligned} F_A &:= F_{\nabla^g} - F_{\nabla^{\rm LC}} = d_{\nabla^{\rm LC}} A + A \wedge A \\ &= d_{\nabla^{\rm LC}} A + \tfrac{1}{2}[A, A] \in \Omega^2(M, \Lambda^2 T^*M) \end{aligned} \tag{36}$$

the "torsion field strength" associated to the torsion potential $A = \nabla^g - \nabla^{\rm LC}$.

On a metrical flat manifold $(M, g_{\rm M})$ the torsion field strength fulfills a Bianchi identity and therefore becomes a true curvature that is defined by torsion. In some approaches to torsion, this curvature is used to geometrical describe gravity on metrical flat space-time manifolds.

Let again $e_1, \ldots, e_n \in \mathfrak{Sec}(U, TM)$ be a local (oriented orthonormal) frame with the dual frame being denoted by $e^1, \ldots, e^n \in \mathfrak{Sec}(U, T^*M)$. We consider the following one-form:

$$\begin{aligned} \Sigma &:= \sum_{b=1}^{n} e^b \otimes \Sigma_b \in \Omega^1(M, {\rm End}^-_\gamma(\mathcal{E}))\,, \\ \Sigma_b &:= \sum_{a=1}^{n} {\rm Id}_{\rm S} \otimes F_{\rm A}(e_b, e_a) \otimes e^a \in \mathcal{C}^\infty(U, {\rm End}^-_\gamma(\mathcal{E}))\,. \end{aligned} \tag{37}$$

Again, ${\rm End}^-_\gamma(\mathcal{E}) \subset {\rm End}(\mathcal{E})$ denotes the sub-algebra of the (odd) endomorphisms which commute with the Clifford action provided by $\gamma_{\mathcal{E}}$.

We consider the Dirac operator of simple type

$$\not{D} := \tilde{\not{\nabla}}^{\mathcal{E}'} + \tau_{\mathcal{E}'} \phi_{\rm D} \tag{38}$$

on the *Clifford twist* (see, again, [15])

$$\pi_{\mathcal{E}'} : \mathcal{E}' := \mathcal{E} \otimes_{\rm M} Cl_{\rm M} \to M \tag{39}$$

of (27). The grading involution, the Clifford action and the hermitian product are defined by

$$\tau_{\mathcal{E}'} := \tau_{\mathcal{E}} \otimes {\rm Id}_{\rm Cl}\,, \quad \gamma_{\mathcal{E}'} := \gamma_{\mathcal{E}} \otimes {\rm Id}_{\rm Cl}\,, \quad \langle\cdot,\cdot\rangle_{\mathcal{E}'} := \langle\cdot,\cdot\rangle_{\mathcal{E}} \langle\cdot,\cdot\rangle_{\rm Cl}\,. \tag{40}$$

Also,

$$\tilde{\nabla}^{\mathcal{E}'} := \tilde{\nabla}^{\mathcal{E}} \otimes {\rm Id}_{\rm Cl} + {\rm Id}_{\mathcal{E}} \otimes \nabla^{\rm Cl} \tag{41}$$

is the covariant derivative of the induced Clifford connection on the Clifford twist of (27) and

$$\begin{aligned} \phi_{\rm D} &:= -\sum_{b=1}^{n} \Sigma_b \otimes e^b \\ &= \sum_{a,b=1}^{n} {\rm Id}_{\rm S} \otimes F_{\rm A}(e_a, e_b) \otimes e^a \otimes e^b \in \mathfrak{Sec}(M, {\rm End}^-_\gamma(\mathcal{E}'))\,. \end{aligned} \tag{42}$$

Notice that the Dirac operator of simple type (38) is fully determined by the Clifford connection (32). In contrast, the quantization of the lift of ∇^g to the spinor bundle is neither a quantized Clifford connection, nor a Dirac operator of simple type.

Theorem 3.1. *Let M be closed compact. When restricted to the class of simple type Dirac operators* (38) *and to the sections $\psi \in \mathfrak{S}ec(M,\mathcal{E}_1) \subset \mathfrak{S}ec(M,\mathcal{E}')$, the total Dirac action decomposes as*

$$\mathcal{I}_{D,tot}(\not{D},\psi) = \int_M *\Big(-\epsilon\tfrac{rk(\mathcal{E}')}{4} scal(g_M) + \langle\psi,\, \not{\partial}_A\psi\rangle_{\mathcal{E}_1} - 2^{2n} rk(S)\|F_A\|^2\Big)\,, \tag{43}$$

where $\|F_A\|^2 \equiv -g^*_M(e^a,e^c)\, g^*_M(e^b,e^d)\, tr\big(F_A(e_a,e_b)F_A(e_c,e_d)\big) \equiv -trF_{ab}F^{ab} \in \mathcal{C}^\infty(M)$.

In particular, the variation of the total Dirac action (43) *with respect to the global torsion potential $A \in \Omega^1(M,\Lambda^2T^*M)$ yields a Yang-Mills like equation for the torsion field strength:*

$$\delta_{\nabla^{\rm LC}} F_A = -\lambda_0 Re\langle\psi,\Theta\psi\rangle_S\,. \tag{44}$$

Here, $\lambda_0 \equiv 2^{-2(n+1)}\varepsilon n/rk(S)$ and $\delta_{\nabla^{\rm LC}} \equiv (-1)^{n(k+1)+q+1} * d_{\nabla^{\rm LC}} *$ is the formal adjoint of the exterior covariant derivative $d_{\nabla^{\rm LC}}$.

With respect to an oriented orthonormal frame the right-hand side of (44) *explicitly reads:*

$$\begin{aligned} Re\langle\psi,\Theta\psi\rangle_S &:= \frac{\epsilon}{n}\sum_{i,j,k,l,m=1}^{n} g_M(e_i,e_j)g_M(e_l,e_k)Re\langle\psi^k,\gamma_S(e^j)\psi^m\rangle_S\, e^i\otimes e^l\otimes e_m \\ &\equiv \frac{\epsilon}{n}\sum_{i,j,k=1}^{n} Re\langle\psi_i,\gamma_k\psi^j\rangle_S\, e^k\otimes e^i\otimes e_j \\ &\equiv \frac{\epsilon}{n}\sum_{i,j,k=1}^{n} \langle\psi_i,\gamma_k\psi_j\rangle_S\, e^k\otimes e^i\wedge e^j \in \Omega^1(M,\Lambda^2T^*M)\,, \end{aligned} \tag{45}$$

whereby $\psi =: \sum_{k=1}^n \psi^k\otimes e_k \in \mathfrak{S}ec(M,\mathcal{E}_1)$ and $\psi_i \equiv \sum_{j=1}^n g_M(e_i,e_j)\psi^j$.

Before proving the theorem (3.1) it might be worthwhile adding some comments first: Clearly, the functional (43) looks much like the usual Dirac-Yang-Mills action including gravity. This holds true, especially, when $(M,g_{\rm M})$ is supposed to be flat. Hence, the Euler-Lagrange equation (44) may not come as a surprise. In fact, for flat $(M,g_{\rm M})$, (44) formally coincides with the inhomogeneous Yang-Mills equation, where the electric current is proportional to the Dirac current. A crucial distinction to ordinary Yang-Mills theory arises since the torsion field strength not only depends on the torsion potential but also on the Levi-Civita connection (determined by the Einstein equation). As a consequence, the energy-momentum current not only depends on the metric but also on its first derivative. This additional dependence, however, can be always (point-wise) eliminated by the choice of *normal coordinates* (nc.), such that at $x \in M$:

$$F_{\rm A}|_x \overset{\rm nc.}{=} \big(dA + \tfrac{1}{2}[A,A]\big)|_x\,. \tag{46}$$

Such a choice of local trivialization of the frame bundle does not affect the torsion potential (in contrast to the action of diffeomorphisms on M).

In order to end up with the field equation (44) the sections $\psi \in \mathfrak{S}ec(M,\mathcal{E}_1)$ are twisted fermions of spin 3/2, as opposed to ordinary Dirac-Yang-Mills theory. In the presented approach to torsion the additional spin-one degrees of freedom of matter are regarded as "internal gauge degrees" that couple to torsion.

We stress that the functional (43) is real-valued, indeed. This is because, the Dirac operator $\not{\partial}_{\rm A}$ is symmetric and the twisted spin-connection provided by $\nabla^{\mathcal{E}_1}$ is torsion-free. If the spinor bundle were not twisted with the tangent bundle in (26), then one has to use (28) instead of $\nabla^{\mathcal{E}_1}$ to define the ordinary Dirac action. In this case, however, the functional (43) would be necessarily complex, as mentioned already. The demand to derive a *real action including torsion* from Dirac operators of simple type basically dictates the geometrical setup presented that eventually leads to (43).

Proof. The statement of (3.1) is a special case of the Proposition 6.2 in [15]. To explicitly prove the statement we consider the $\mathbb{Z}_2-$graded hermitian vector bundle $\pi_{\rm E} : E := TM \otimes_{\rm M} Cl_{\rm M} \otimes_{\rm M} Cl_{\rm M} \to M$. The grading involution and hermitian structure are given, respectively, by $\tau_{\rm E} := {\rm Id}_{\rm TM} \otimes \tau_{\rm Cl} \otimes {\rm Id}_{\rm Cl}$ and $\langle \cdot, \cdot \rangle_{\rm E} := g_{\rm M} \langle \cdot, \cdot \rangle_{\rm Cl} \langle \cdot, \cdot \rangle_{\rm Cl}$. Hence, $\pi_{\mathcal{E}'} : \mathcal{E}' = S \otimes_{\rm M} E \to M$ is an odd twisted hermitian spinor bundle, with the grading involution $\tau_{\mathcal{E}'} := \tau_{\rm S} \otimes \tau_{\rm E}$ and the Clifford action provided by $\gamma_{\mathcal{E}'} := \gamma_{\rm S} \otimes {\rm Id}_{\rm E}$. The hermitian structure is $\langle \cdot, \cdot \rangle_{\mathcal{E}'} := \langle \cdot, \cdot \rangle_{\rm S} \langle \cdot, \cdot \rangle_{\rm E}$. Furthermore, the twisted spinor bundle carries the canonical Clifford connection that is provided by $\nabla^{\mathcal{E}'} = \nabla^{S\otimes E}$, where $\nabla^{\rm E} := \nabla^{\rm TM\otimes Cl \otimes Cl}$ and $\nabla^{\rm TM} \equiv \nabla^{\rm LC}$.

From the general statement concerning the universal Dirac action restricted to Dirac operators of simple type it follows that

$$\mathcal{I}_{\rm D}(\not{D}) = \int_M *tr_\gamma\big(curv(\not{\nabla}^{\mathcal{E}'}) + tr_{\mathcal{E}'}\phi_{\rm D}^2\big)\,. \tag{47}$$

One infers from the ordinary Lichnerowicz-Schrödinger formula of twisted spin-Dirac operators (c.f. [10] and [13]) that

$$tr_\gamma curv(\not{\nabla}^{\mathcal{E}'}) = -\epsilon\tfrac{rk(\mathcal{E}')}{4} scal(g_{\rm M})\,. \tag{48}$$

Furthermore, it is straightforward to check that

$$tr_{\mathcal{E}'}\phi_{\rm D}^2 \sim \|F_{\rm A}\|^2\,. \tag{49}$$

Finally, when restricting to sections $\psi \in \mathfrak{Sec}(M, \mathcal{E}_1) \subset \mathfrak{Sec}(M, \mathcal{E}')$ one obtains

$$\begin{aligned} \langle \psi, \not{D}\psi\rangle_{\mathcal{E}'} &= \langle \psi, \not{\partial}_{\rm A}\psi\rangle_{\mathcal{E}_1} \\ &= \langle \psi, \not{\nabla}^{\mathcal{E}_1}\psi\rangle_{\mathcal{E}_1} + \langle \psi, \not{A}\psi\rangle_{\mathcal{E}_1}\,. \end{aligned} \tag{50}$$

The first equality holds because

$$\begin{aligned} \langle \psi, \tau_{\mathcal{E}'}\phi_{\rm D}\psi\rangle_{\mathcal{E}'} &= -\sum_{a,b=1}^{n} \langle \psi, ({\rm Id}_{\rm S} \otimes F_{\rm A}(e_a, e_b))\psi\rangle_{\mathcal{E}_1} \langle e^a, 1\rangle_{\rm Cl}\langle e^b, 1\rangle_{\rm Cl} \\ &= 0\,. \end{aligned} \tag{51}$$

This proves (43). To also prove (44) we remark that (50) formally coincides with the gauge covariant Dirac-Lagrangian and

$$*\,\|F_{\rm A}\|^2 \sim tr\big(F_{\rm A} \wedge *F_{\rm A}\big) \tag{52}$$

formally coincides with the usual Yang-Mills-Lagrangian. The basic difference is that the torsion potential $A \in \Omega^1(M, \Lambda^2T^*M)$ itself does not define a connection, in general. As already mentioned, if $(M, g_{\rm M})$ is flat, then the torsion field strength $F_{\rm A}$ has the geometrical meaning of the curvature of a general metric connection. In general, however, the torsion field strength determines the torsion of a general metric connection. □

We also mention that by an appropriate re-definition of the sections ϕ and ψ one may always recast the functional (43) into

$$\mathcal{I}_{\rm D,tot}(\not{D}, \psi) \sim -\tfrac{\varepsilon}{\lambda_{\rm grav}}\int_M *scal(g_{\rm M}) + \int_M *\langle\psi, \not{\partial}_{\rm A}\psi\rangle_{\mathcal{E}_1} - \frac{1}{2g^2}\int_M tr\big(F_{\rm A} \wedge *F_{\rm A}\big)\,, \tag{53}$$

with $g > 0$ being an arbitrary positive (coupling) constant like in ordinary non-abelian Yang-Mills theory. Furthermore, by re-scaling the torsion potential A, which is admissible since the torsion potential belongs to a vector space as opposed to gauge potentials, the field equation (44) changes to

$$\delta_{\nabla^{\rm LC}} F_{\rm A} = -\lambda_{\rm tor} Re\big\langle \psi, \Theta\psi \big\rangle_{\rm S} \,. \tag{54}$$

The dimensionless constant $\lambda_{\rm tor} \equiv \varepsilon n g > 0$ is a free parameter analogous to ordinary Dirac-Yang-Mills theory. Accordingly, the Dirac equation becomes

$$\not{\partial}_{\rm A}\psi = 0 \quad \Leftrightarrow \quad \not{\nabla}^{\mathcal{E}_1}\psi = -g\not{A}\psi \,. \tag{55}$$

As in ordinary general relativity the Levi-Civita connection is determined by the Einstein equation with the energy-momentum current similarly defined to the usual Dirac-Yang-Mills theory. As already mentioned, when (53) is varied with respect to the metric one also has to take into account that the torsion field strength itself depends on $g_{\rm M}$ (and its first derivative).

The field equation (54) for torsion should be contrasted with the field equation (2) of ordinary Einstein-Cartan theory. The coupling constant g determines the coupling strength of the *inner degrees* of freedom of the fermions to torsion similar to ordinary Dirac-Yang-Mills theory. This coupling constant is *dimensionless* and independent of the gravitational constant as opposed to the coupling constant obtained by the Palatini formalism of general relativity. Therefore, the assumption $0 < g << 1$ allows to treat all the field equations perturbatively. In particular, one also obtains non-trivial solutions of (54), even if the coupling to the fermions is omitted (e.g. in matter free regions). This is also in strong contrast to (2).

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