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A Regularity Result for Quasilinear Stochastic Partial Differential Equations of Parabolic Type

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A REGULARITY RESULT FOR QUASILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE

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Abstract. We consider a quasilinear parabolic stochastic partial differential equation driven by a multiplicative noise and study regularity properties of its weak solution satisfying classical *a priori* estimates. In particular, we determine conditions on coefficients and initial data under which the weak solution is Hölder continuous in time and possesses spatial regularity that is only limited by the regularity of the given data. Our proof is based on an efficient method of increasing regularity: the solution is rewritten as the sum of two processes, one solves a linear parabolic SPDE with the same noise term as the original model problem whereas the other solves a linear parabolic PDE with random coefficients. This way, the required regularity can be achieved by repeatedly making use of known techniques for stochastic convolutions and deterministic PDEs.

1. Introduction

In this paper, we are interested in the regularity of weak solutions of quasilinear parabolic stochastic partial differential equation driven by a multiplicative noise. Let $D \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, let $T > 0$ and set $D_T = (0,T)\times D$, $S_T = (0,T]\times\partial D$. We study the following problem

(1.1)
$$\begin{cases} \mathrm{d}u = \mathrm{div}(B(u))\,\mathrm{d}t + \mathrm{div}\,(A(u)\nabla u)\,\mathrm{d}t + F(u)\,\mathrm{d}t + H(u)\,\mathrm{d}W & \text{in } D_T, \\ \quad u = 0 & \text{in } S_T, \\ u(0) = u_0 & \text{in } D. \end{cases}$$

where W a cylindrical Wiener process on some Hilbert space K and H a mapping with values in the space of the γ-radonifying operators from K to certain Sobolev spaces. The precise description of the problem setting will be given in the next section.

It is a well known fact in the field of PDEs and SPDEs that many equations do not, in general, have classical or strong solutions and can be solved only in some weaker sense. Unlike deterministic problems, in the case of stochastic equations we can only ask whether the solution is smooth in the space variable since the time regularity is limited by the regularity of the

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stochastic integral. Thus, the aim of the present work is to determine conditions on coefficients and initial data under which there exists a spatially smooth solution to (1.1).

Such a regularity result is fundamental and interesting by itself. Equations of the form (1.1) appear in many sciences. Regularity of solutions is an important property when one wants to study qualitative behaviour. It is also a preliminary step when studying numerical approximations. Our original motivation is that such models arise as limit of random kinetic equations. An example of such equations is treated in [8]. The problem is linear there and the limit is a limit stochastic parabolic equation. But we wish to treat more general kinetic equations and expect limit equations of the form (1.1). The rigourous justification of this limit requires the results obtained in this article.

The issue of existence of a classical solution to deterministic parabolic problems is well understood, among the main references stands the extensive book [12] which is mainly concerned with the solvability of initial-boundary value problems and the Cauchy problem to the basic linear and quasilinear second order PDEs of parabolic type. A special attention is paid to the connection between the smoothness of solutions and the smoothness of known data entering into the problem (initial condition and coefficients), nevertheless, due to technical complexity of the proofs a direct generalization to the stochastic case is not obvious.

In the case of linear parabolic problems, let us mention the classical Schauder theory (see e.g. [13]) that provides *a priori* estimates relating the norms of solutions of initial- boundary value problems, namely the parabolic Hölder norms, to the norms of the known quantities in the problems. These results are usually employed in order to deal with quasilinear equations: the application of the Schauder fixed point theorem leads easily to the existence of a smooth solution under very weak hypotheses on the coefficients. In our proof, we make use of the Schauder theory as well, yet in an entirely different approach.

Regularity of parabolic problems in the stochastic setting was also studied in several works. In the previous work of the third author [11], semilinear parabolic SPDEs (i.e. the diffusion matrix A independent of the solution) were studied and a regularity result established by using semigroup arguments. In [9], a maximum principle is obtained for a SPDE similar to (1.1) but with a more general diffusion H, it may depend on the gradient of u. Hölder continuity of solutions to nonlinear parabolic systems under suitable structure conditions was proved in [3] by energy methods. In comparison to this work, the quasilinear case considered in the present paper is more delicate and different techniques need to be applied.

The transposition of the deterministic method exposed in [12] seems to be quite difficult. Fortunately, we have found a trick to avoid this. We introduce a new method that is based on a very simple idea: a weak solution to (1.1) that satisfies *a priori* estimates is decomposed into two parts $u = y + z$

where z is a solution to a linear parabolic SPDE with the same noise term as (1.1) and y solves a linear parabolic PDE with random coefficients. As a consequence, the problem of regularity of u is reduced to showing regularity of z and regularity of y which can be handled by known techniques for stochastic convolutions and deterministic PDEs. It is rather surprising that this classical idea used to treat semilinear equations can be applied also for quasilinear problems.

Let us explain this method more precisely. As the main difficulties come from the second order and the stochastic term, for simplicity of the introduction we assume $B = F = 0$ and consider periodic boundary conditions, i.e. $D = \mathbb{T}^N$ is the N-dimensional torus. Let u be a weak solution to

$$\begin{cases} \mathrm{d}u = \operatorname{div}(A(u)\nabla u)\,\mathrm{d}t + H(u)\,\mathrm{d}W, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

and let z be a solution to

$$\begin{cases} \mathrm{d}z = \Delta z\,\mathrm{d}t + H(u)\,\mathrm{d}W, \\ z(0) = 0. \end{cases}$$

Then z is given by the stochastic convolution with the semigroup generated by the Laplacian, denoted by $(S(t))_{t\geq 0}$, i.e.

$$z(t) = \int_0^t S(t-s)H(u)\,\mathrm{d}W(s)$$

and regularization properties are known. Setting $y = u - z$ it follows immediately that y solves

$$\begin{cases} \partial_t y = \operatorname{div}(A(u)\nabla y) + \operatorname{div}((A(u) - \mathrm{I})\nabla z), \\ y(0) = u_0, \end{cases} \tag{1.3}$$

which is a (pathwise) deterministic linear parabolic PDE. According to *a priori* estimates for (1.2), it holds

$$u \in L^p(\Omega; L^\infty(0,T; L^p(\mathbb{T}^N))) \cap L^2(\Omega; L^2(0,T; W^{1,2}(\mathbb{T}^N))), \quad \forall p \in [2,\infty),$$

and making use of the factorization method it is possible to show that z possesses enough regularity so that ∇z is a function with good integrability properties. Now, a classical result for deterministic linear parabolic PDEs with discontinuous coefficients (see [12]) yields Hölder continuity of y (in time and space) and consequently also Hölder continuity of u itself. Having this in hand, the regularity of z can be increased to a level where the Schauder theory for linear parabolic PDEs with Hölder continuous coefficients applies to (1.3) (see [13]) and higher regularity of y is obtained. Repeating this approach then allows us to conclude that u is λ-Hölder continuous in time with $\lambda < 1/2$ and possesses as much regularity in space as allowed by the regularity of the coefficients and the initial data.

The paper is organized as follows. In Section 2, we introduce the basic setting and state our regularity results, Theorem 2.5, Theorem 2.6. Section

3 gives a preliminary result concerning the stochastic convolution. The remainder of the paper is devoted to the proof of Theorem 2.5 and Theorem 2.6 that is divided into several parts. In Section 4, we establish our first regularity result, Theorem 2.5, that gives some Hölder continuity in time and space of a weak solution to (1.1). The regularity is then inductively improved in the final Section 5 and Theorem 2.6 is proved.

2. Notations, hypotheses and the main result

2.1. **Notations.** In this paper, we adopt the following conventions. For $r \in [1,\infty]$, the Lebesgue spaces $L^r(D)$ are denoted by L^r and the corresponding norm by $\|\cdot\|_r$. In order to measure higher regularity of functions we make use of the Bessel potential spaces $H^{a,r}(D)$, $a\in\mathbb{R}$ and $r\in(1,\infty)$, we also shorten the notation to $H^{a,r}$ with the norm $\|\cdot\|_{a,r}$. The choice of this scale of function spaces is more natural for our method than the Sobolev-Slobodeckij spaces $W^{a,r}$, namely, the spaces $H^{a,r}_0$ coincide with the domains of fractional powers of the Laplace operator with null Dirichlet boundary conditions, which is an important ingredient for proving regularity of the stochastic convolution. For the reader's convenience we include a reminder of the basic properties of these spaces in Section 3.

Another important scale of function spaces which is used throughout the paper are the Hölder spaces. In particular, if X and Y are two Banach spaces and $\alpha\in(0,1)$, $C^\alpha(X;Y)$ denotes the space of bounded Hölder continuous functions with values in Y equipped with the norm

$$\|f\|_{C^\alpha(X;Y)} = \sup_{x\in X}\|f(x)\|_Y + \sup_{x,x'\in X, x\neq x'}\frac{\|f(x)-f(x')\|_Y}{\|x-x'\|_X^\alpha}.$$

In the sequel, we consider the spaces $C^\alpha(\overline{D}) = C^\alpha(\overline{D};\mathbb{R})$, $C^\alpha([0,T];X)$ where $X=H^{a,r}$ or $X=C^\beta(\overline{D})$ and $C^\alpha([0,T]\times\overline{D}) = C^\alpha([0,T]\times\overline{D};\mathbb{R})$. Besides, we employ Hölder spaces with different regularity in time and space, i.e. $C^{\alpha,\beta}([0,T]\times\overline{D})$ equipped with the norm

$$\|f\|_{C^{\alpha,\beta}} = \sup_{(t,x)}|f(t,x)| + \sup_{(t,x)\neq(s,y)}\frac{|f(t,x)-f(s,y)|}{|t-s|^\alpha+|x-y|^\beta}.$$

With usual modifications we can also consider $\alpha,\beta\geq 1$. Note that it holds $C^\alpha([0,T];C^\beta(\overline{D})) \subsetneqq C^{\alpha,\beta}([0,T]\times\overline{D})$ and therefore we have to distinguish these two spaces (see [14]).

2.2. **Hypotheses.** Let us now introduce the precise setting of (1.1). We work on a finite-time interval $[0,T]$, $T>0$, and on a bounded domain D in $\mathbb{R}^N$ with smooth boundary. We denote by D_T the cylinder $(0,T)\times D$ and by S_T the lateral surface of D_T, that is $S_T=(0,T]\times\partial D$. Concerning the coefficients A, B, F, H, we only state here the basic assumptions that guarantee the existence of a weak solution and are valid throughout the paper. Further regularity hypotheses are necessary in order to obtain better regularity of the weak solution and will be specified later. We assume that the flux function

$$B = (B_1, \dots, B_N) : \mathbb{R} \longrightarrow \mathbb{R}^N$$

is continuous with linear growth. The diffusion matrix

$$A = (A_{ij})_{i,j=1}^N : \mathbb{R} \longrightarrow \mathbb{R}^{N \times N}$$

is supposed to be continuous, symmetric, positive definite and bounded. In particular, there exist constants ν, $\mu > 0$ such that for all $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$,

$$\nu|\xi|^2 \le A(u)\xi \cdot \xi \le \mu|\xi|^2. \tag{2.1}$$

The drift coefficient $F : \mathbb{R} \to \mathbb{R}$ is continuous with linear growth.

Regarding the stochastic term, let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. The driving process W is a cylindrical Wiener process: $W(t) = \sum_{k\ge1} \beta_k(t)e_k$ with $(\beta_k)_{k\ge1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)_{t\ge0}$ and $(e_k)_{k\ge1}$ a complete orthonormal system in a separable Hilbert space K. For each $u \in L^2(D)$ we consider a mapping $H(u) : K \to L^2(D)$ defined by $H(u)\, e_k = H_k(\cdot, u(\cdot))$. In particular, we suppose that $H_k \in C(D \times \mathbb{R})$ and the following linear growth condition holds true

$$\sum_{k\ge1} |H_k(x, \xi)|^2 \le C\big(1 + |\xi|^2\big), \qquad \forall x \in D,\, \xi \in \mathbb{R}. \tag{2.2}$$

This assumption implies in particular that H maps $L^2(D)$ to $L_2(K; L^2(D))$ where $L_2(K; L^2(D))$ denotes the collection of Hilbert-Schmidt operators from K to $L^2(D)$. Thus, given a predictable process u that belongs to $L^2(\Omega; L^2(0, T; L^2(D)))$, the stochastic integral $t \mapsto \int_0^t H(u)\mathrm{d}W$ is a well defined process taking values in $L^2(D)$ (see [6] for a thorough exposition).

Later on we are going to estimate the weak solution of (1.1) in certain Bessel potential spaces $H^{a,r}$ with $a \ge 0$ and $r \in [2, \infty)$ and therefore we need to ensure the existence of the stochastic integral in (1.1) as an $H^{a,r}$-valued process. We recall that the Bessel potential spaces $H^{a,r}$ with $a \ge 0$ and $r \in [2, \infty)$ belong to the class of 2-smooth Banach spaces since they are isomorphic to $L^r(0, 1)$ according to [16, Theorem 4.9.3] and hence they are well suited for the stochastic Itô integration (see [4], [5] for the precise construction of the stochastic integral). So, let us denote by $\gamma(K, X)$ the space of the γ-radonifying operators from K to a 2-smooth Banach space X. We recall that $\Psi \in \gamma(K, X)$ if the series

$$\sum_{k\ge0} \gamma_k \Psi(e_k)$$

converges in $L^2(\widetilde{\Omega}, X)$, for any sequence $(\gamma_k)_{k\ge0}$ of independent Gaussian real-valued random variables on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and any orthonormal basis $(e_k)_{k\ge0}$ of K. Then, the space $\gamma(K, X)$ is endowed with the norm

$$\|\Psi\|_{\gamma(K,X)} := \left(\widetilde{\mathbb{E}}\left|\sum_{k\geq 0}\gamma_k\Psi(e_k)\right|_X^2\right)^{1/2}$$

(which does not depend on $(\gamma_k)_{k\geq 0}$, nor on $(e_k)_{k\geq 0}$) and is a Banach space. Now, if $a\geq 0$ and $r\in[2,\infty)$ we denote by ($\mathrm{H}_{a,r}$) the following hypothesis

$$(\mathrm{H}_{a,r})\qquad \|H(u)\|_{\gamma(K,H^{a,r})} \leq \begin{cases} C\big(1+\|u\|_{a,r}\big), & a\in[0,1],\\ C\big(1+\|u\|_{a,r}+\|u\|_{1,ar}^a\big), & a>1,\end{cases}$$

i.e. H maps $H^{a,r}$ to $\gamma(K,H^{a,r})$ provided $a\in[0,1]$ and it maps $H^{a,r}\cap H^{1,ar}$ to $\gamma(K,H^{a,r})$ provided $a>1$. The precise values of parameters a and r will be given later in each of our regularity results.

Remark 2.1. We point out that, thanks to the linear growth hypothesis (2.2) on the functions $(H_k)_{k\geq 1}$, one can easily verify that, for all $r\in[2,\infty)$, the bound ($\mathrm{H}_{0,r}$) holds true.

In order to clarify the assumption ($\mathrm{H}_{a,r}$), let us present the main examples we have in mind.

Example 2.2. Let W be a d-dimensional $(\mathscr{F}_t)$-Wiener process, that is $W(t)=\sum_{k=1}^d W_k(t)\,e_k$, where W_k, $k=1,\dots,d$, are independent standard $(\mathscr{F}_t)$-Wiener processes and $(e_k)_{k=1}^d$ is an orthonormal basis of $K=\mathbb{R}^d$. Then the hypothesis ($\mathrm{H}_{a,r}$) is satisfied for $a\geq 0$, $r\in[2,\infty)$ provided the functions $H_1,\dots,H_d$ are sufficiently smooth (for more details we refer the reader to [15]). Note that in this example it is necessary to restrict ourselves to the subspace $H^{a,r}\cap H^{1,ar}$ of $H^{a,r}$ so that the corresponding Nemytskij operators $u\mapsto H_k(\cdot,u(\cdot))$ take values in $H^{a,r}$. In fact, if $1+1/r\leq a\leq N/r$, $r\in(1,\infty)$, then only linear operators map $H^{a,r}$ to itself (see [15]).

Example 2.3. In the case of linear operator H we are able to deal with an infinite dimensional noise. Namely, let W be a $(\mathscr{F}_t)$-cylindrical Wiener process on $K=L^2(D)$, that is $W(t)=\sum_{k\geq 1}W_k(t)\,e_k$, where W_k, $k\geq 1$, are independent standard $(\mathscr{F}_t)$-Wiener processes and $(e_k)_{k\geq 1}$ an orthonormal basis of K. We assume that H is linear of the form $H(u)e_k := u\,Qe_k$, $k\geq 1$, where Q denotes a linear operator from K to K. Then, one can verify that the hypothesis ($\mathrm{H}_{a,r}$) is satisfied for $a\geq 0$, $r\in[2,\infty)$ provided we assume the following regularity property: $\sum_{k\geq 1}\|Qe_k\|_{a,\infty}^2<\infty$. We point out that, in this example, H maps $H^{a,r}$ to $\gamma(K,H^{a,r})$ for any $a\geq 0$ and $r\in[2,\infty)$.

As we are interested in proving the regularity up to the boundary for weak solutions of (1.1), it is necessary to impose certain compatibility conditions upon the initial data and the null Dirichlet boundary condition. To be more precise, since u_0 can be random in general, let us assume that $u_0\in L^0(\Omega;C(\overline{D}))$ with $u_0=0$ on ∂D. Further integrability and regularity assumptions on u_0 will be specified later.

Note that other boudary conditions could be studied with similar arguments.

2.3. **Existence of weak solutions.** Let us only give a short comment here as the existence of weak solutions is not our main concern and we will only make use of *a priori* estimates for parabolic equations of the form (1.1). In the recent work [7], the authors gave a well-posedness result for degenerate parabolic SPDEs (with periodic boundary conditions) of the form

$$\begin{cases} \mathrm{d}u = \operatorname{div}(B(u))\,\mathrm{d}t + \operatorname{div}(A(u)\nabla u)\,\mathrm{d}t + H(u)\,\mathrm{d}W, \\ u(0) = u_0, \end{cases}$$

where the diffusion matrix was supposed to be positive semidefinite. One can easily verify that the Dirichlet boundary conditions and the drift term $F(u)$ in (1.1) do not cause any additional difficulties in the existence part of the proofs and therefore the corresponding results in [7], namely Section 4 (with the exception of Subsection 4.3) and Proposition 5.1, are still valid in the case of (1.1). In particular, we have the following.

Theorem 2.4. *There exists* $\big((\tilde\Omega,\tilde{\mathscr{F}},(\tilde{\mathscr{F}}_t),\tilde{\mathbb{P}}),\tilde W,\tilde u\big)$ *which is a weak martingale solution to* (1.1) *and, for all* $p\in[2,\infty)$,

$$\tilde u\in L^2(\tilde\Omega;C([0,T];L^2))\cap L^p(\tilde\Omega;L^\infty(0,T;L^p))\cap L^2(\tilde\Omega;L^2(0,T;W^{1,2})).$$

In the sequel, we assume the existence of a weak solution on the original probability space $(\Omega,\mathscr{F},\mathbb{P})$ and show that it possesses as much regularity as we want provided the coefficients and initial data are sufficiently regular. We point out that this assumption is taken without loss of generality since pathwise uniqueness can be proved once we have sufficient regularity in hand and hence existence of a pathwise solution can be then obtained by usual methods (cf. [7, Subsection 4.3]).

A similar result can be obtained in the case of null Dirichlet boundary conditions as well.

2.4. **The main result.** To conclude this section let us state our main results to be proved precisely.

Theorem 2.5. *Let u be a weak solution to* (1.1) *such that, for all* $p\in[2,\infty)$,

$$u\in L^2(\Omega;C([0,T];L^2))\cap L^p(\Omega;L^\infty(0,T;L^p))\cap L^2(\Omega;L^2(0,T;W^{1,2})).$$

Assume that

(i) $u_0\in L^m(\Omega;C^\iota(\overline{D}))$ *for some* $\iota>0$ *and all* $m\in[2,\infty)$,
(ii) $(\mathrm{H}_{1,2})$ *is fulfilled.*

Then there exists $\eta>0$ *such that, for all* $m\in[2,\infty)$, *the weak solution u belongs to* $L^m(\Omega;C^\eta(\overline{D_T}))$.

Theorem 2.6. *Let* $k\in\mathbb{N}$. *Let u be a weak solution to* (1.1) *such that, for all* $p\in[2,\infty)$,

$$u\in L^2(\Omega;C([0,T];L^2))\cap L^p(\Omega;L^\infty(0,T;L^p))\cap L^2(\Omega;L^2(0,T;W^{1,2})).$$

Assume that

(i) $u_0\in L^m(\Omega;C^{k+\iota}(\overline{D}))$ *for some* $\iota>0$ *and all* $m\in[2,\infty)$,

(ii) $A,\, B \in C_b^k$ *and* $F \in C_b^{k-1}$,
(iii) ($\mathrm{H}_{a,r}$) *is fulfilled for all* $a < k+1$ *and* $r \in [2,\infty)$.

Then for all $\lambda \in (0,1/2)$ *there exists* $\beta > 0$ *such that, for all* $m \in [2,\infty)$, *the weak solution* u *belongs to* $L^m(\Omega; C^{\lambda,k+\beta}(\overline{D_T}))$.

3. Regularity of the stochastic convolution

Our proof of Theorem 2.5 and Theorem 2.6 is based on a regularity result that concerns mild solutions to linear SPDEs of the form

$$\begin{cases} \mathrm{d}Z = \Delta_D Z\,\mathrm{d}t + \Psi(t)\,\mathrm{d}W_t, \\ Z(0) = 0, \end{cases} \tag{3.1}$$

where Δ_D is the Laplacian on D with null Dirichlet boundary conditions acting on various Bessel potential spaces.

In order to motivate the use of these spaces let us recall their basic properties (for a thorough exposition we refer the reader to the books of Triebel [16], [17]). In the case of $\mathbb{R}^N$ (or $\mathbb{T}^N$) the Bessel potential spaces are defined in terms of Fourier transform of tempered distributions: let $a \in \mathbb{R}$, $r \in (1,\infty)$ then

$$H^{a,r}(\mathbb{R}^N) = \big\{ f \in \mathcal{S}'(\mathbb{R}^N);\ \|f\|_{H^{a,r}} := \big\|\mathcal{F}^{-1}(1+|\xi|^2)^{a/2}\mathcal{F}f\big\|_{L^r} < \infty \big\}$$

and they belong to the Triebel-Lizorkin scale $F^a_{r,s}(\mathbb{R}^N)$ in the sense that $H^{a,r}(\mathbb{R}^N) = F^a_{r,2}(\mathbb{R}^N)$. As a consequence, they are generally different from the Sobolev-Slobodeckij spaces $W^{a,r}(\mathbb{R}^N)$ which belong to the Besov scale $B^a_{r,s}(\mathbb{R}^N)$ in the sense that $W^{a,r}(\mathbb{R}^N) = B^a_{r,r}(\mathbb{R}^N)$ if $a > 0$, $a \notin \mathbb{N}$. Nevertheless, we have the following two relations which link the two scales of function spaces together

$$W^{a,r}(\mathbb{R}^N) = H^{a,r}(\mathbb{R}^N) \quad \text{if} \quad a \in \mathbb{N}_0,\, r \in (1,\infty) \quad \text{or} \quad a \geq 0,\, r = 2,$$

and

$$H^{a+\varepsilon,r}(\mathbb{R}^N) \hookrightarrow W^{a,r}(\mathbb{R}^N) \hookrightarrow H^{a-\varepsilon,r}(\mathbb{R}^N) \qquad a \in \mathbb{R},\, r \in (1,\infty),\, \varepsilon > 0.$$

The Bessel potential spaces $H^{a,r}(\mathbb{R}^N)$ behave well under the complex interpolation, i.e. for $a_0, a_1 \in \mathbb{R}$ and $r_0, r_1 \in (1,\infty)$ it holds that

$$[H^{a_0,r_0}(\mathbb{R}^N), H^{a_1,r_1}(\mathbb{R}^N)]_\theta = H^{a,r}(\mathbb{R}^N), \tag{3.2}$$

where $\theta \in (0,1)$ and $a = (1-\theta)a_0 + \theta a_1$, $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, which makes them more suitable for studying regularity for linear elliptic and parabolic problems. Indeed, under the assumption of bounded imaginary powers of a positive operator $\mathcal{A}$ on a Banach space X, the domains of fractional powers of $\mathcal{A}$ are given by the complex interpolation as well: let $0 \leq \alpha < \beta < \infty$, $\theta \in (0,1)$ then

$$[D(\mathcal{A}^\alpha), D(\mathcal{A}^\beta)]_\theta = D(\mathcal{A}^{(1-\theta)\alpha+\theta\beta}).$$

Furthermore, the expression (3.2) suggests how the spaces $H^{a,r}(D)$ may be defined for a general domain D: if $s \geq 0$ and $m \in \mathbb{N}$ such that $s \leq m < s+1$ then we define

$$H^{a,r}(D) := [W^{m,r}(D), L^r(D)]_{(m-a)/m}.$$

If D is sufficiently regular then $H^{a,r}(D)$ coincides with the space of restrictions to D of functions in $H^{a,r}(\mathbb{R}^N)$ and the Sobolev embedding theorem holds true. The spaces $H_0^{a,r}(D)$, $a \geq 0$, $r \in (1,\infty)$, are then defined as the closure of $C_c^\infty(D)$ in $H^{a,r}(D)$. Note, that $H_0^{a,r}(D) = H^{a,r}(D)$ if $a \leq 1/r$ and $H_0^{a,r}(D)$ is strictly contained in $H^{a,r}$ if $a > 1/r$. Besides, an interpolation result similar to (3.2) holds for these spaces as well

$$[H_0^{a_0,r_0}(D), H_0^{a_1,r_1}(D)]_\theta = H_0^{a,r}(D).$$

Let us now take a closer look at the Dirichlet Laplacian Δ_D. Considered as an operator on L^r, its domain is $H_0^{2,r}$ and it is the infinitesimal generator of an analytic semigroup denoted by $S = (S(t))_{t\geq 0}$. Moreover, it follows from the above considerations that the domains of its fractional powers coincide with the Bessel potential spaces, that is $D((-\Delta_D)^\alpha) = H_0^{2\alpha,r}$, $\alpha \geq 0$. Therefore, one can build a fractional power scale (or a Sobolev tower, see [2], [10]) generated by $(L^r, -\Delta_D)$ to get

$$\big[\big(H_0^{2\alpha,r}, -\Delta_{D,2\alpha,r}\big);\, \alpha \geq 0\big], \tag{3.3}$$

where $-\Delta_{D,2\alpha,r}$ is the $H_0^{2\alpha,r}$-realization of $-\Delta_D$. Having this in hand, an important result [2, Theorem V.2.1.3] describes the behavior of the semigroup S in this scale. More precisely, the operator $\Delta_{D,2\alpha,r}$ generates an analytic semigroup $S_{2\alpha,r}$ on $H_0^{2\alpha,r}$ which is naturally obtained from S by restriction, i.e. $S_{2\alpha,r}(t)$ is the $H_0^{2\alpha,r}$-realization of $S(t)$, $t \geq 0$, and we have the following regularization property: for any $\delta > 0$ and $t > 0$, $S_{2\alpha,r}(t)$ maps $H_0^{2\alpha,r}$ into $H_0^{2\alpha+\delta,r}$ with

$$\big\|S_{2\alpha,r}(t)\big\|_{\mathcal{L}(H_0^{2\alpha,r},H_0^{2\alpha+\delta,r})} \leq \frac{C}{t^{\delta/2}}. \tag{3.4}$$

For notational simplicity of the sequel we do not directly specify the spaces where the operators Δ_D and $S(t)$, $t \geq 0$, are acting since this is always clear from the context.

The solution to (3.1) is given by the stochastic convolution, that is

$$Z(t) = \int_0^t S(t-s)\Psi(s)\,\mathrm{d}W_s, \quad t \in [0,T].$$

In order to describe the connection between its regularity and the regularity of Ψ, we recall the following proposition.

Proposition 3.1. *Let $a \geq 0$ and $r \in [2,\infty)$ and let Ψ be a progressively measurable process in $L^p(\Omega; L^p(0,T; \gamma(K, H^{a,r})))$.*

(i) *Let* $p \in (2,\infty)$ *and* $\delta \in (0, 1-2/p)$. *Then, for any* $\gamma \in [0, 1/2 - 1/p - \delta/2)$, $Z \in L^p(\Omega; C^\gamma(0,T;H^{a+\delta,r}))$ *and*

$$\mathbb{E}\|Z\|^p_{C^\gamma(0,T;H^{a+\delta,r})} \leq C\,\mathbb{E}\|\Psi\|^p_{L^p(0,T;\gamma(K,H^{a,r}))}.$$

(ii) *Let* $p \in [2,\infty)$ *and* $\delta \in (0,1)$. *Then* $Z \in L^p(\Omega; L^p(0,T;H^{a+\delta,r}))$ *and*

$$\mathbb{E}\|Z\|^p_{L^p(0,T;H^{a+\delta,r})} \leq C\,\mathbb{E}\|\Psi\|^p_{L^p(0,T;\gamma(K,H^{a,r}))}.$$

Proof. Having established the behavior of the Dirichlet Laplacian and the corresponding semigroup along the fractional power scale (3.3), the proof of (i) is an application of the factorization method and can be found in [4, Corollary 3.5] whereas the point (ii) follows from the Burkholder-Davis-Gundy inequality and regularization properties (3.4) of the semigroup. □

4. First step in the regularity problem

In this section, we show the first step towards regularity of the weak solution u to (1.1). We consider the following auxiliary problem

$$\begin{cases} \mathrm{d}z = \Delta z\,\mathrm{d}t + H(u)\,\mathrm{d}W_t & \text{in } D_T, \\ z = 0 & \text{in } S_T, \\ z(0) = 0 & \text{in } D. \end{cases} \tag{4.1}$$

It can be rewritten in the abstract form

$$\begin{cases} \mathrm{d}z = \Delta_D z\,\mathrm{d}t + H(u)\,\mathrm{d}W_t, \\ z(0) = 0 \end{cases}$$

and hence its solution is given by the stochastic convolution

$$z(t) = \int_0^t S(t-s)H(u_s)\,\mathrm{d}W_s, \quad t \in [0,T]. \tag{4.2}$$

Next, we define the process $y := u - z$. It follows immediately that y solves the following linear parabolic PDE with random coefficients

$$\begin{cases} \partial_t y = \operatorname{div}(A(u)\nabla y) + \operatorname{div}(B(u)) + F(u) + \operatorname{div}((A(u)-\mathrm{I})\nabla z) & \text{in } D_T, \\ y = 0 & \text{in } S_T, \\ y(0) = u_0 & \text{in } D. \end{cases} \tag{4.3}$$

This way, we have split u into two parts, i.e. y and z, that are much more convenient in order to study regularity. Our first regularity result reads as follows.

Proposition 4.1. *Let* $u_0 \in L^m(\Omega; C^\iota(\overline{D}))$ *for some* $\iota > 0$ *and all* $m \in [2,\infty)$. *We assume that* ($\mathrm{H}_{1,2}$) *is fulfilled. Then, there exists* $\eta > 0$ *such that, for all* $m \in [2,\infty)$, *the weak solution* u *to* (1.1) *belongs to* $L^m(\Omega; C^\eta(\overline{D_T}))$.

Proof. Step 1: Regularity of z. According to Theorem 2.4, the weak solution u to (1.1) belongs to $L^2(\Omega; L^2(0,T; H^{1,2}))$ so that, thanks to the hypothesis ($\mathrm{H}_{1,2}$), we have that $H(u)$ belongs to $L^2(\Omega; L^2(0,T; \gamma(K, H^{1,2})))$. As a result, with Proposition 3.1 - (ii) and the bound ($\mathrm{H}_{1,2}$), we have that for any $a \in (0,2)$, $z \in L^2(\Omega; L^2(0,T; H^{a,2}))$ with

$$\mathbb{E}\|z\|^2_{L^2(0,T;H^{a,2})} \leq C\Big(1 + \mathbb{E}\|u\|^2_{L^2(0,T;H^{1,2})}\Big).$$

Besides, since for all $p \in [2,\infty)$, the weak solution u to (1.1) belongs to $L^p(\Omega; L^p(0,T; L^p))$, we obtain, with the hypothesis ($\mathrm{H}_{0,p}$) (see Remark 2.1), that $H(u)$ belongs to $L^p(\Omega; L^p(0,T; \gamma(K, L^p)))$. As a consequence, with Proposition 3.1 - (ii) and ($\mathrm{H}_{0,p}$), we have that for any $b \in (0,1)$, $z \in L^p(\Omega; L^p(0,T; H^{b,p}))$ with

$$\mathbb{E}\|z\|^p_{L^p(0,T;H^{b,p})} \leq C\Big(1 + \mathbb{E}\|u\|^p_{L^p(0,T;L^p)}\Big).$$

Since for any $a \in (0,2)$ and $b \in (0,1)$, we have $z \in L^2(\Omega; L^2(0,T; H^{a,2}))$ and $z \in L^p(\Omega; L^p(0,T; H^{b,p}))$, we can interpolate to obtain that (see [1])

$$z \in L^r(\Omega; L^r(0,T; H^{c,r})),$$

where, for $\theta \in (0,1)$,

$$\begin{cases} \dfrac{1}{r} = \dfrac{\theta}{2} + \dfrac{1-\theta}{p}, \\ c = a\theta + b(1-\theta), \end{cases}$$

with the bound

$$\mathbb{E}\|z\|^r_{L^r(0,T;H^{c,r})} \leq \Big(\mathbb{E}\|z\|^2_{L^2(0,T;H^{a,2})}\Big)^{r\theta/2} \Big(\mathbb{E}\|z\|^p_{L^p(0,T;H^{b,p})}\Big)^{r(1-\theta)/p} < \infty. \tag{4.4}$$

Note that by choosing $\theta \in (0,1)$ and $p \in [2,\infty)$ appropriately, r can be arbitrary in $[2,\infty)$. Furthermore, when $\theta \in (0,1)$ is fixed, it is always possible to take $(a,b) \in (0,2) \times (0,1)$ such that $c > 1$. As a result, for all $r \in [2,\infty)$, there exists $c_r > 1$ such that

$$z \in L^r(\Omega; L^r(0,T; H^{c_r,r})).$$

This gives, for all $r \in [2,\infty)$,

$$\nabla z \in L^r(\Omega; L^r(0,T; L^r)),$$

and, due to the boundedness of the mapping A,

$$(A(u) - \mathrm{I})\nabla z \in L^r(\Omega; L^r(0,T; L^r)),$$

with, thanks to (4.4),

$$\mathbb{E}\|(A(u) - \mathrm{I})\nabla z\|^r_{L^r(0,T;L^r)} \leq C\mathbb{E}\|z\|^r_{L^r(0,T;H^{c,r})} < \infty, \tag{4.5}$$

where $C > 0$ depends on $\|A\|_\infty$. Note that, thanks to the linear growth property of the coefficients B and F, we obviously have, for all $r \in [2,\infty)$,

$$\mathbb{E}\|B(u)\|^r_{L^r(0,T;L^r)} + \mathbb{E}\|F(u)\|^r_{L^r(0,T;L^r)} \leq C(1 + \mathbb{E}\|u\|^r_{L^r(0,T;L^r)}) < \infty. \tag{4.6}$$

Step 2: Regularity of y. From now on, we consider that $r \geq r_0$ where r_0 is fixed such that for all $r \geq r_0$,

$$\frac{2+N}{r} < \frac{1}{2}. \tag{4.7}$$

Concerning the regularity of y, we intend to apply the regularization result given in the second part of [12, Theorem 10.1, Ch. III] to deduce that y has in fact α-Hölder continuous paths in $\overline{D_T}$ for some $\alpha > 0$. Precisely, we set $\Gamma' = S_T$ and

$$a_i = b_i = a = 0, \quad f_i = B_i(u) + ((A(u) - \mathrm{I})\nabla z)_i, \quad f = F(u),$$

and observe that the conditions (1.2), (7.1) and (7.2) in [12, Ch. III] are satisfied thanks to (2.1) and the bounds (4.5)−(4.6) coupled with (4.7). Note also that [12, Theorem 7.1, Ch. III] applies and gives $y \in L^\infty(D_T)$ a.s. Thus we can now employ the second part of [12, Theorem 10.1, Ch. III] which yields $y \in C^{\alpha/2,\alpha}(\overline{D_T})$ where $\alpha \in (0, \iota]$ is determined by N, ν, μ and r_0. In particular, we point out that α is deterministic. Furthermore, studying the proofs of [12, Theorem 7.1, Theorem 10.1, Ch. III] in detail, we have the following bound

$$\begin{aligned} \|y\|_{C^{\alpha/2,\alpha}(\overline{D_T})} &\leq C(1 + \|u_0\|_{C^\iota(\overline{D})}) \\ &\quad \times (1 + \|B(u) + (A(u) - \mathrm{I})\nabla z\|^{2N+1}_{L^r(0,T;L^r)} + \|F(u)\|^{2N+1}_{L^r(0,T;L^r)}) \end{aligned} \tag{4.8}$$

for some deterministic constant $C > 0$ depending on the constants of the problem and on r_0. Therefore, if $2(2N+1)m < r$, we obtain due to (4.5)−(4.6), the hypothesis made on u_0 and the Cauchy-Schwarz inequality

$$\begin{aligned} &\mathbb{E}\|y\|^m_{C^{\alpha/2,\alpha}} \leq C\,(1 + \mathbb{E}\|u_0\|^{2m}_{C^\iota(\overline{D})}) \times \\ &(1 + \mathbb{E}\|B(u) + (A(u) - \mathrm{I})\nabla z\|^r_{L^r(0,T;L^r)} + \mathbb{E}\|F(u)\|^r_{L^r(0,T;L^r)}) < \infty. \end{aligned} \tag{4.9}$$

Since r is arbitrary in $[r_0, \infty)$, the result holds for all $m \in [2, \infty)$.

Step 3: Hölder regularity of z. In order to complete the proof it is necessary to improve the regularity of z. Recall that for all $m \in [2, \infty)$, the solution u to (1.1) belongs to $L^m(\Omega; L^m(0,T; L^m))$ and that $H(u)$ belongs to $L^m(\Omega; L^m(0,T; \gamma(K, L^m)))$. We now apply Proposition 3.1 - (i) and ($\mathrm{H}_{0,m}$) to obtain that for $m \in (2, \infty)$, $\delta \in (0, 1 - 2/m)$ and $\gamma \in [0, 1/2 - 1/m - \delta/2)$, $z \in L^m(\Omega; C^\gamma([0,T]; H^{\delta,m}))$ with

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];H^{\delta,m})} \leq C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;L^m)}\right).$$

Note that we can choose δ and γ to be independent of m. For instance, let us suppose in the sequel that $m \geq 3$; then $\delta = 1/6$ and $\gamma = 1/12$ satisfies the conditions above for any $m \geq 3$. Furthermore, from now on, we also suppose that $m \geq 7N := m_0$. This implies that $m \geq 3$ and $\delta m > N$, so that the following Sobolev embedding holds true

$$H^{\delta,m} \hookrightarrow C^\lambda, \quad \lambda := \delta - N/m_0.$$

We conclude that, for all $m \geq m_0$,

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];C^\lambda)} \leq C\left(1+\mathbb{E}\|u\|^m_{L^m(0,T;L^m)}\right) < \infty. \tag{4.10}$$

Note that for $m \in [2, m_0)$, we can write with the Hölder inequality

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];C^\lambda)} \leq \left(\mathbb{E}\|z\|^{m_0}_{C^\gamma([0,T];C^\lambda)}\right)^{m/m_0} < \infty. \tag{4.11}$$

Step 4: Conclusion. Finally, we set $\eta := \min(\alpha/2, \gamma, \lambda) > 0$ and we recall that $u = y + z$ so that the conclusion follows from (4.9), (4.10), (4.11) due to the fact that $C^\eta([0,T]; C^\eta(\overline{D})) \subset C^\eta([0,T] \times \overline{D})$. □

5. Increasing the regularity

In this final section, we complete the proof of Theorem 2.6. Having Proposition 4.1 in hand, it is quite straightforward to significantly increase the regularity of u using the same auxiliary problems (4.1) and (4.3) together with the Schauder theory for deterministic parabolic PDEs with Hölder continuous coefficients.

Proposition 5.1. *Let $u_0 \in L^m(\Omega; C^{1+\iota}(\overline{D}))$ for some $\iota > 0$ and all $m \in [2,\infty)$. Suppose that A, $B \in C^1_b$. If ($\mathrm{H}_{a,r}$) is fulfilled for all $a < 2$ and $r \in [2,\infty)$, then for all $\lambda \in (0,1/2)$ there exists $\beta > 0$ such that for all $m \in [2,\infty)$ the weak solution u to (1.1) belongs to $L^m(\Omega; C^{\lambda,1+\beta}(\overline{D_T}))$.*

Proof. The proof is divided in two parts: we first increase the regularity in space and then in time.

Spatial regularity. Step 1: Regularity of z. First, we improve the regularity of z that was defined in (4.2). According to Proposition 4.1, there exists $\eta > 0$ such that for all $m \in [2,\infty)$, $u \in L^m(\Omega; C^\eta(\overline{D_T}))$. In particular, this implies that $u \in L^m(\Omega; L^m(0,T; H^{\kappa,m}))$ provided $\kappa < \eta$. With ($\mathrm{H}_{\kappa,m}$), we deduce that $H(u) \in L^m(\Omega; L^m(0,T; \gamma(K, H^{\kappa,m})))$. Application of Proposition 3.1 yields that $z \in L^m(\Omega; C^\gamma([0,T]; H^{\kappa+\delta,m}))$ for every $m \in (2,\infty)$ with

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];H^{\kappa+\delta,m})} \leq C\left(1+\mathbb{E}\|u\|^m_{L^m(0,T;H^{\kappa,m})}\right),$$

where $\delta \in (0, 1-2/m)$ and $\gamma \in [0, 1/2 - 1/m - \delta/2)$. In the sequel, we assume that $m \geq (N+4)/\kappa := m_0$. Then $\delta = 1 - 3/m_0$ and $\gamma = 1/(4m_0)$ satisfies the conditions above uniformly in $m \geq m_0$. Furthermore, observe that $(\kappa+\delta)m > \kappa m \geq \kappa m_0 \geq N$ so that the following Sobolev embedding holds true

$$H^{\kappa+\delta,m} \hookrightarrow C^\sigma, \quad \sigma = \kappa + \delta - N/m_0.$$

Besides, by definition of m_0, $\sigma = \kappa + 1 - (N+3)/m_0 > 1$. Finally, we deduce that for all $m \geq m_0$, $z \in L^m(\Omega; C^\gamma([0,T]; C^\sigma(\overline{D})))$ with

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];C^\sigma)} \leq C\left(1+\mathbb{E}\|u\|^m_{L^m(0,T;H^{\kappa,m})}\right). \tag{5.1}$$

Step 2: Regularity of y. Next, we improve the regularity of y that is given by (4.3). Namely, we intend to make use of the classical Schauder theory for deterministic parabolic PDEs, see e.g. [13, Theorem 6.48]. As a consequence of Proposition 4.1 and (5.1), we obtain due to the assumptions upon A, B and F that, for all $m\in[2,\infty)$

$$\begin{aligned} A(u)&\in L^m(\Omega;C^{\alpha/2,\alpha}(\overline{D_T}),\\ B(u)+(A(u)-\mathrm{I})\nabla z&\in L^m(\Omega;C^{\alpha/2,\alpha}(\overline{D_T}),\\ F(u)&\in L^m(\Omega;L^m(0,T;L^m)),\\ u_0&\in L^m(\Omega;C^{1+\alpha}(\overline{D})),\end{aligned}$$

where $\alpha:=\min(\iota,\eta,\sigma-1,\gamma)>0$. Thus the hypotheses of [13, Theorem 4.8, Theorem 6.48] are fulfilled and we obtain the following (pathwise) estimate

$$\begin{aligned}\|y\|_{C^{(1+\alpha)/2,1+\alpha}}\leq C\Big(&\|u_0\|_{C^{1+\alpha}}+\|B(u)+(A(u)-\mathrm{I})\nabla z\|_{C^{\alpha/2,\alpha}}\\ &+\|F(u)\|_{L^r(0,T;L^r)}\Big),\end{aligned}$$

where $r\in[2,\infty)$ is large enough. We conclude that, for all $m\in[2,\infty)$,

$$y\in L^m(\Omega;C^{(1+\alpha)/2,1+\alpha}(\overline{D_T})) \tag{5.2}$$

which together with (5.1) yields $u\in L^m(\Omega;C^{\gamma,1+\alpha}(\overline{D_T}))$.

Time regularity. Having in hand the improved regularity of u, we consider again the stochastic convolution z, repeat the approach from the first step of this proof and obtain due to Proposition 4.1 (with $\delta=0$) and ($\mathrm{H}_{1+\kappa,m}$)

$$\begin{aligned}&\mathbb{E}\|z\|^m_{C^\lambda([0,T];H^{1+\kappa,m})}\\ &\quad\leq C\left(1+\mathbb{E}\|u\|^m_{L^m(0,T;H^{1+\kappa,m})}+\mathbb{E}\|u\|^{(1+\kappa)m}_{L^{(1+\kappa)m}(0,T;H^{1,(1+\kappa)m})}\right)<\infty,\end{aligned} \tag{5.3}$$

where $\kappa<\alpha$ and $\lambda\in(0,1/2-1/m)$. Therefore for any $\lambda\in(0,1/2)$ there exists m_0 large enough so that (5.3) holds true for any $m\geq m_0$ and the Sobolev embedding then implies that $z\in L^m(\Omega;C^\lambda([0,T];C^{1+\beta}(\overline{D})))$ for $\beta<\kappa$. Since we already have (5.2) the proof is complete. □

Due to the properties of the stochastic convolution it is not possible to increase the time regularity of u. Nevertheless, it is possible to continue in the same manner as before and increase its space regularity.

Proposition 5.2. *Let $u_0\in L^m(\Omega;C^{2+\iota}(\overline{D}))$ for some $\iota>0$ and all $m\in[2,\infty)$. Suppose that A, $B\in C^2_b$ and that $F\in C^1_b$. If ($\mathrm{H}_{a,r}$) is fulfilled for all $a<3$ and $r\in[2,\infty)$, then for all $\lambda\in(0,1/2)$ there exists $\beta>0$ such that for all $m\in[2,\infty)$ the weak solution u to (1.1) belongs to $L^m(\Omega;C^{\lambda,2+\beta}(\overline{D_T}))$.*

First, we give the proof of Proposition 5.2 in the periodic case where $D=\mathbb{T}^N$. We point out that in this simpler setting the proof can exactly be reproduced in order to establish Proposition 5.3 below which achieves higher regularity of u. Then, we give the proof of Proposition 5.2 in the general case of a bounded domain D of $\mathbb{R}^N$ with smooth boundary. Unlike the periodic setting, this proof does not directly extend to the proof of Proposition 5.3.

Thus, the technique requires some improvements which are detailed in the proof of Proposition 5.3.

Proof. The periodic case. From now on, let $D=\mathbb{T}^N$. The proof follows similar ideas as in Proposition 5.1 only with some modifications in the second step.

Spatial regularity. Step 1: Regularity of z. As in Proposition 5.1, we first increase the regularity of z. With Proposition 5.1, for any $\lambda\in(0,1/2)$, there exists $\beta>0$ such that for all $m\in[2,\infty)$, $u\in L^m(\Omega;C^{\lambda,1+\beta}(\overline{D_T}))$. Then we deduce

$$\begin{aligned}\mathbb{E}\|z\|^m_{C^\gamma([0,T];H^{1+\kappa+\delta,m})}\\ &\leq C\left(1+\mathbb{E}\|u\|^m_{L^m(0,T;H^{1+\kappa,m})}+\mathbb{E}\|u\|^{(1+\kappa)m}_{L^{(1+\kappa)m}(0,T;H^{1,(1+\kappa)m})}\right)<\infty,\end{aligned}$$

where $\kappa<\beta$, $\delta\in(0,1-2/m)$ and $\gamma\in[0,1/2-1/m-\delta/2)$. By a similar reasoning as above we obtain due to the Sobolev embedding that $z\in L^m(\Omega;C^\gamma([0,T];C^\sigma(\overline{D})))$ where $m\in[2,\infty)$ and $\sigma>2$.

Step 2: Regularity of y. In order to improve the space regularity of y we derive the equation that is satisfied by ∂y where the operator ∂ can stand for any partial derivative with respect to space variable x: $\partial=\partial_{x_i}$, $i=1,\dots,N$. We obtain

$$\begin{cases}\partial_t(\partial y)=\operatorname{div}(A(u)\nabla(\partial y))+\operatorname{div}(\partial A(u)\nabla u)+\operatorname{div}(\partial B(u))\\ \qquad\qquad\qquad+\partial F(u)+\operatorname{div}((A(u)-\mathrm{I})\nabla(\partial z)) \qquad \text{in } D_T,\\ \partial y(0)=\partial u_0.\end{cases}$$

The above is again a (pathwise) linear parabolic PDE hence we need to show that its coefficients satisfy the hypotheses of [13, Theorem 6.48]. In particular, according to what was already proved, we have

$$\begin{aligned}A(u)&\in L^m(\Omega;C^{\alpha/2,\alpha}(\overline{D_T})),\\ \partial A(u)\nabla u+\partial B(u)+(A(u)-\mathrm{I})\nabla(\partial z)&\in L^m(\Omega;C^{\alpha/2,\alpha}(\overline{D_T})),\\ \partial F(u)&\in L^m(\Omega;L^m(0,T;L^m)),\\ \partial u_0&\in L^m(\Omega;C^{1+\alpha}(\overline{D})),\end{aligned}$$

for some $\alpha\in(0,\sigma-2]$ and all $m\in[2,\infty)$ provided A, $B\in C^1_b$, $F\in C^1_b$. Therefore [13, Theorem 6.48] applies and we deduce

$$\partial y\in L^m(\Omega;C^{(1+\alpha)/2,1+\alpha}(\overline{D_T})).$$

As a consequence, we see that

$$y\in L^m(\Omega;C^{(1+\alpha)/2,2+\alpha}(\overline{D_T}))$$

hence

$$u\in L^m(\Omega;C^{\gamma,2+\alpha}(\overline{D_T})).$$

Time regularity. Finally, we improve the time regularity of u by considering the stochastic convolution again as in Proposition 5.1. We obtain that for any $\lambda \in (0, 1/2)$ there exists m_0 large enough so that

$$\mathbb{E}\|z\|^m_{C^\lambda([0,T];H^{2+\kappa,m})} \le C\Big(1 + \mathbb{E}\|u\|^m_{L^m(0,T;H^{2+\kappa,m})} + \mathbb{E}\|u\|^{(2+\kappa)m}_{L^{(2+\kappa)m}(0,T;H^{1,(2+\kappa)m})}\Big),$$

holds true for any $m \ge m_0$ and the Sobolev embedding then implies that $z \in L^m(\Omega; C^\lambda([0,T]; C^{2+\beta}(\overline{D})))$ for $\beta < \kappa$ which completes the proof. □

Let us now prove Proposition 5.2 in the general case. In the sequel D is again a bounded domain in $\mathbb{R}^N$ with smooth boundary.

Proof. The general case. The proof follows the same scheme as in the periodic case except for the *Step 2: Regularity of y*. Let us now detail the proof of this step.

Step 2: Regularity of y. In order to improve the space regularity of y we make use of [12, Theorem 5.2, Ch. IV]. In particular, we set

$$a_{ij} = A_{ij}(u), \quad a_j = \nabla u \cdot A'_{\cdot j}(u), \quad a = 0, \quad f = \operatorname{div}\big(B(u) + (A(u) - \mathrm{I})\nabla z\big) + F(u).$$

According to what was already proved, we have

$$a_{ij},\, a_j,\, a,\, f \in L^m(\Omega; C^{\alpha/2,\alpha}(\overline{D_T})), \qquad u_0 \in L^m(\Omega; C^{2+\alpha}(\overline{D})), \tag{5.4}$$

for some $\alpha \in (0, \sigma - 2]$ and all $m \in [2, \infty)$ provided $A, B \in C^2_b$, $F \in C^1_b$. Therefore [12, Theorem 5.2, Ch. IV] applies and we deduce

$$y \in L^m(\Omega; C^{1+\alpha/2,2+\alpha}(\overline{D_T})),$$

hence

$$u \in L^m(\Omega; C^{\gamma,2+\alpha}(\overline{D_T})).$$

This completes the proof. □

Finally, we achieve even higher regularity of u provided the coefficients are smooth enough. We obtain the following result.

Proposition 5.3. *Let $k \in \{3, 4, \dots\}$. Let $u_0 \in L^m(\Omega; C^{k+\iota}(\overline{D}))$ for some $\iota > 0$ and all $m \in [2, \infty)$. Suppose that $A, B \in C^k_b$ and $F \in C^{k-1}_b$. If ($\mathrm{H}_{a,r}$) is fulfilled for all $a < k + 1$ and $r \in [2, \infty)$, then for all $\lambda \in (0, 1/2)$ there exists $\beta > 0$ such that for all $m \in [2, \infty)$ the weak solution u to (1.1) belongs to $L^m(\Omega; C^{\lambda,k+\beta}(\overline{D_T}))$.*

As previously mentioned, the proof of Proposition 5.2 in the periodic case can exactly be reproduced here so that the result of Proposition 5.3 is proved in the setting of periodic boundary conditions.

Nevertheless, the proof of Proposition 5.2 made in the general case does not apply here any more. Indeed, the problem arises from the fact that the regularization result given by [12, Theorem 5.2, Ch. IV] is stated under the condition that the regularity of the coefficients and the source term is in the parabolic scaling, that is, the space regularity is exactly twice the time regularity. In our case, since the time regularity is limited to $\frac{1}{2}^-$, we are limited to 1^- for the space regularity of the coefficients and the source term if we want to fit in the setting of [12, Theorem 5.2, Ch. IV]. As a consequence, we wouldn't obtain a better space regularity of our solution u than 3^-. To handle this issue, we prove an alternative version of the result [12, Theorem 5.2, Ch. IV] where we avoid the hypothesis of the parabolic regularity of the coefficients and initial data. The result is the following.

Theorem 5.4. *Let $\mathscr{L}$ denote the linear parabolic differential operator given by* [12, (5.1), Ch. IV]

$$\mathscr{L}u = \partial_t u - \sum_{i,j=1}^{N} a_{ij}\partial^2_{x_i,x_j}u + \sum_{i=1}^{N} a_i\partial_{x_i}u + au,$$

and u the solution to the null Dirichlet problem [12, (5.3), Ch. IV]

$$\begin{cases} \mathscr{L}u = f & \text{in } D_T, \\ u = 0 & \text{in } S_T, \\ u(0) = u_0 & \text{in } D. \end{cases}$$

Let α, $\beta \geq 0$ such that $2\alpha \leq \beta$. Assume that the coefficients of $\mathscr{L}$ and the source f belong to $C^{\alpha,\beta}(\overline{D_T})$ and that u_0 belongs to $C^{\beta}(\overline{D})$. Then, for all $\varepsilon > 0$, u is $C^{\alpha+1-\varepsilon,\beta+2-\varepsilon}(\overline{D_T})$ with

$$\|u\|_{C^{\alpha+1-\varepsilon,\beta+2-\varepsilon}} \leq C(\|f\|_{C^{\alpha,\beta}} + \|u_0\|_{C^{\beta}}).$$

Proof of Proposition 5.3. For the time being, let us suppose that this result holds true. The proof of Proposition 5.3 is then exactly the same as in Proposition 5.2 in the general case except that (5.4) is replaced by

$$\begin{aligned} a_{ij},\, a_j,\, a,\, f &\in L^m(\Omega; C^{\gamma,(k-2)+\alpha}(\overline{D_T})), \\ u_0 &\in L^m(\Omega; C^{k+\alpha}(\overline{D})), \end{aligned} \tag{5.5}$$

for any $\gamma < 1/2$ and some $\alpha \in (0, \sigma - k]$ where $\sigma > k$ and that we then apply Theorem 5.4 instead of [12, Theorem 5.2, Ch. IV]. □

Thus it only remains to prove Theorem 5.4.

Proof of Theorem 5.4. The proof of [12, Theorem 5.2, Ch. IV] is divided into two steps. The first one is to prove the desired result on the whole space and on the half-space in the case where a_{ij} are constant coefficients and $a_i = a = 0$; the results are the bounds (6.4) and (6.5) in [12, Theorem 6.1, Ch. IV] (the bound (6.6) deals with the case of Neumann boundary conditions). The second one is to freeze the coefficients, to use a localization technique and to handle the lower order terms of $\mathscr{L}$ by some compactness argument and finally to prove [12, Theorem 5.2, Ch. IV] using (6.4) and (6.5) of [12, Theorem 6.1, Ch. IV]; this second step is achieved in [12, Section 7, Ch. IV]. As a result, we only need to prove that the bounds (6.4) and (6.5) of [12, Theorem 6.1, Ch. IV] hold true whenever the regularity of the source term is not in the parabolic scaling. Furthermore, as explained in the proof of [12, Theorem 6.1, Ch. IV], it is sufficient to deal with the case $a_{ij} = \delta_{ij}$.

To sum up, let $f \in C^{\alpha,\beta}([0,T] \times \mathbb{R}^N)$, $g \in C^{\alpha,\beta}([0,T] \times \overline{\mathbb{R}^N_+})$, and w, v the solutions of

$$\begin{cases} \partial_t w - \Delta w = f \text{ in } (0,T) \times \mathbb{R}^N, \\ w(0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t v - \Delta v = g \text{ in } (0,T) \times \mathbb{R}^N_+, \\ v|_{x_N=0} = 0, \\ v(0) = 0, \end{cases}$$

where $\mathbb{R}^N_+$ denotes the half-space $\{(x_1, ..., x_N) \in \mathbb{R}^N,\ x_N > 0\}$, it remains to prove that, for all $\varepsilon > 0$,

$$\|w\|_{C^{\alpha+1-\varepsilon,\beta+2-\varepsilon}([0,T]\times\mathbb{R}^N)} \leq C\|f\|_{C^{\alpha,\beta}([0,T]\times\mathbb{R}^N)}, \tag{5.6}$$

$$\|v\|_{C^{\alpha+1-\varepsilon,\beta+2-\varepsilon}([0,T]\times\overline{\mathbb{R}^N_+})} \leq C\|g\|_{C^{\alpha,\beta}([0,T]\times\overline{\mathbb{R}^N_+})}. \tag{5.7}$$

The bound (5.6) can be justified exactly as in the case of the parabolic scaling, see the proof of [12, (2.1), Ch. IV]. It gives the bound (5.6) where we can take $\varepsilon = 0$, that is

$$\|w\|_{C^{\alpha+1,\beta+2}([0,T]\times\mathbb{R}^N)} \leq C\|f\|_{C^{\alpha,\beta}([0,T]\times\mathbb{R}^N)}.$$

Unfortunately, the proof made in [12] in the case of the half-space does not work any more when we are not in the parabolic scaling. So, let us define $(S(t))_{t\geq 0}$ the semigroup of the Dirichlet Laplacian on the half-space $\mathbb{R}^N_+$. Precisely, $\psi = S(t)h$ satisfies

$$\begin{cases} \partial_t \psi - \Delta \psi = 0 \text{ in } (0,\infty) \times \mathbb{R}^N_+, \\ \psi|_{x_N=0} = 0, \\ \psi(0) = h. \end{cases} \tag{P^+_h}$$

It is classical that $S(1)$ maps $C^\gamma(\overline{\mathbb{R}^N_+})$ to $C^\infty(\overline{\mathbb{R}^N_+})$ so that we can deduce the following bound, for any $h \in C^\gamma(\overline{\mathbb{R}^N_+})$ and $\delta > 0$,

$$\|S(1)h\|_{C^{\gamma+\delta}(\overline{\mathbb{R}^N_+})} \leq C\|h\|_{C^\gamma(\overline{\mathbb{R}^N_+})}. \tag{5.8}$$

Now, let $t > 0$ and $h \in C^\gamma(\overline{\mathbb{R}^N_+})$. We define $\widetilde{h}(x) := h(xt^{\frac{1}{2}})$ and consider the solution ψ to the problem ($\mathrm{P}^+_{\widetilde{h}}$). Finally, we set $\varphi(s,x) := \psi(st^{-1}, xt^{-\frac{1}{2}})$ which is well defined in the half-space and satisfies (P^+_h). As a result, $\varphi(s,x) = S(s)h$. Thus observe that we have $S(t)h = \varphi(t,x) = \psi(1, xt^{-\frac{1}{2}}) = S(1)\widetilde{h}(xt^{-\frac{1}{2}})$ so that we deduce, with (5.8),

$$\|S(t)h\|_{C^{\gamma+\delta}(\overline{\mathbb{R}^N_+})} = \|S(1)\widetilde{h}(\cdot\ t^{-\frac{1}{2}})\|_{C^{\gamma+\delta}(\overline{\mathbb{R}^N_+})} \leq Ct^{-(\gamma+\delta)/2}\|\widetilde{h}\|_{C^\gamma(\overline{\mathbb{R}^N_+})}.$$

As a result, since $\|\widetilde{h}\|_{C^\gamma(\overline{\mathbb{R}^N_+})} \leq t^{\gamma/2}\|h\|_{C^\gamma(\overline{\mathbb{R}^N_+})}$, we are led to

$$\|S(t)h\|_{C^{\gamma+\delta}(\overline{\mathbb{R}^N_+})} \leq Ct^{-\delta/2}\|h\|_{C^\gamma(\overline{\mathbb{R}^N_+})}. \tag{5.9}$$

Finally, let us conclude the proof of the bound (5.7). The solution v is given by

$$v(t) = \int_0^t S(t-s)g(s)\,\mathrm{d}s,$$

so that with (5.9) we deduce

$$\|v\|_{C^{0,\gamma+\delta}([0,T]\times\overline{\mathbb{R}^N_+})} \leq C\|g\|_{C^{0,\gamma}([0,T]\times\overline{\mathbb{R}^N_+})}, \tag{5.10}$$

provided $\delta < 2$. Besides, thanks to the result [12, (6.5), Ch. IV] in the parabolic scaling, we have the bound

$$\|v\|_{C^{\sigma/2+1,\sigma+2}([0,T]\times\overline{\mathbb{R}^N_+})} \leq C\|g\|_{C^{\sigma/2,\sigma}([0,T]\times\overline{\mathbb{R}^N_+})}. \tag{5.11}$$

Since the bounds (5.10) and (5.11) holds true for any γ, $\sigma \geq 0$ and $\delta < 2$, we deduce, by interpolation, that for any $\varepsilon > 0$,

$$\|v\|_{C^{\alpha+1-\varepsilon,\beta+2-\varepsilon}([0,T]\times\overline{\mathbb{R}^N_+})} \leq C\|g\|_{C^{\alpha,\beta}([0,T]\times\overline{\mathbb{R}^N_+})},$$

which concludes the proof. □

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