Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Isoperimetric type problems and Alexandrov-Fenchel type inequalities in the hyperbolic space

by

Guofang Wang and Chao Xia

Preprint no.: 21 2014

ISOPERIMETRIC TYPE PROBLEMS AND ALEXANDROV-FENCHEL TYPE INEQUALITIES IN THE HYPERBOLIC SPACE

GUOFANG WANG AND CHAO XIA

Abstract. In this paper, we solve various isoperimetric problems for the quermassintegrals and the curvature integrals in the hyperbolic space $\mathbb{H}^n$, by using quermassintegral preserving curvature flows. As a byproduct, we obtain hyperbolic Alexandrov-Fenchel inequalities.

1. Introduction

Isoperimetric type problems play an important role in mathematics. The classical isoperimetric theorem in the Euclidean space says that among all bounded domains in $\mathbb{R}^n$ with given volume, the minimum of the area of the boundary is achieved precisely by the round balls. This can be formulated as an optimal inequality

$$\mathrm{Area}(\partial K) \geq n^{\frac{n-1}{n}} \omega_{n-1}^{\frac{1}{n}} \mathrm{Vol}(K)^{\frac{n-1}{n}}, \tag{1.1}$$

for any bounded domain $K \subset \mathbb{R}^n$, and equality holds if and only if K is a geodesic ball. Here and throughout this paper, ω_k denotes the k-th dimensional Hausdorff measure of the k-dimensional unit sphere $\mathbb{S}^k$, and by a bounded domain we mean a compact set with non-empty interior. When $n = 2$, inequality (1.1) is

$$L^2 \geq 4\pi A, \tag{1.2}$$

where L is the length of a closed curve γ in $\mathbb{R}^2$ and A is the area of the enclosed domain by γ. Inequalities (1.1) and (1.2) are the classical isoperimetric inequalities. Their general forms are the Alexandrov-Fenchel quermassintegral inequalities. A special, but interesting class of the Alexandrov-Fenchel quermassintegral establishes the relationship between the quermassintegrals or the curvature integrals:

$$\int_{\partial K} H_k d\mu \geq \omega_{n-1}^{\frac{k-l}{n-1-l}} \left(\int_{\partial K} H_l d\mu \right)^{\frac{n-1-k}{n-1-l}}, \quad 0 \leq l < k \leq n-1, \tag{1.3}$$

for any convex bounded domain $K \subset \mathbb{R}^n$ with C^2 boundary, where H_k is the *(normalized) k-th mean curvature* of ∂K as an embedding in $\mathbb{R}^n$. These inequalities have been intensively studied by many mathematicians and have many applications in differential geometry and integral geometry. See the excellent books of Burago-Zalgaller [7], Santalo

2010 *Mathematics Subject Classification.* 52A40, 53C65, 53C44.

Key words and phrases. quermassintegral, curvature integral, isoperimetric problem, Alexandrov-Fenchel inequality.

GW is partly supported by SFB/TR71 "Geometric partial differential equations" of DFG.

CX is supported by funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no. 267087. Part of this work was done while CX was visiting the mathematical institute of Albert-Ludwigs-Universität Freiburg. He would like to thank the institute for its hospitality.

[37] and Schneider [39]. Recently, the Alexandrov-Fenchel quermassintegral inequalities in $\mathbb{R}^n$ have been extended to certain classes of non-convex domains. See for example [11, 25, 29].

All these above inequalities solve the problem if one geometric quantity attains its minimum or maximum at geodesic balls among a class of (smooth) bounded domains in $\mathbb{R}^n$ with another given geometric quantity. We call such problems *isoperimetric type problems*.

It is a very natural question to ask if such isoperimetric type problems also hold in the hyperbolic space $\mathbb{H}^n$. We remark that in this paper $\mathbb{H}^n$ denotes the hyperbolic space with the sectional curvature -1. One of the main motivations to study this problem comes naturally from integral geometry in $\mathbb{H}^n$. Another main motivation comes from the recent study of ADM mass, Gauss-Bonnet-Chern mass and quasi-local mass in asymptotically hyperbolic manifolds, see [21]. The isoperimetric problem between volume and area in $\mathbb{H}^n$ was already solved by Schmidt [38] 70 years ago. Due to its complication, a simple explicit inequality like (1.1) is in general not available. When $n = 2$, there is an explicit form, namely the hyperbolic isoperimetric inequality in this case is

$$L^2 \geq 4\pi A + A^2, \tag{1.4}$$

where L is the length of a closed curve γ in $\mathbb{H}^2$ and A is the area of the enclosed domain by γ. Moreover, equality holds if and only if γ is a circle. Comparing to (1.2), inequality (1.4) has an extra term. This is a well-known phenomenon, which indicates that the isoperimetric type problems in $\mathbb{H}^n$ are more complicated than the ones in $\mathbb{R}^n$.

Till now, the Alexandrov-Fenchel type inequalities or the isoperimetric type problems in the hyperbolic space are quite open except some special cases. See for example [5, 16, 17]. In [16], Gallego-Solanes proved the following interesting inequality for convex domain in $\mathbb{H}^n$

$$\int_\Sigma H_k d\mu > c|\Sigma|,$$

where $c = 1$ if $k > 1$ and $c = (n-2)/(n-1)$ if $k = 1$ and $|\Sigma|$ is the area of Σ. Their method depends heavily on the integral interpretation of the quermassintegrals. However, the results obtained there are far away from being optimal. Here we say that a geometric inequality for bounded domains is optimal, if equality holds if and only if the domain is a geodesic ball. In other words, only geodesic balls solve the corresponding isoperimetric problem. More recently, several interesting works have appeared in this research field, see [4, 14, 19, 20, 33]. In [19, 20, 33], the authors solve some special cases of the isoperimetric type problems by establishing the following inequalities as the Alexandrov-Fenchel inequalities (1.3) for the curvature integrals: for $1 \leq k \leq \frac{n-1}{2}$,

$$\int_{\partial K} H_{2k} d\mu \geq \omega_{n-1} \left\{ \left(\frac{|\partial K|}{\omega_{n-1}}\right)^{\frac{1}{k}} + \left(\frac{|\partial K|}{\omega_{n-1}}\right)^{\frac{1}{k}\frac{n-1-2k}{n-1}} \right\}^k, \tag{1.5}$$

for any horospherical convex domain $K \subset \mathbb{H}^n$. Here $|\partial K|$ is the area of ∂K. This is optimal, in the sense that equality holds if and only if K is a geodesic ball in $\mathbb{H}^n$. When $k = 1$, inequality (1.5) was proved in [33] under a weaker condition that ∂K is star-shaped and 2-convex.

In order to state our results we give more precise definitions about quermassintegrals and curvature integrals.

Let us first recall two different kinds of convexity in $\mathbb{H}^n$. A domain $K \subset \mathbb{H}^n$ is said to be *(geodesically) convex* if for every point $p \in \partial K$, K lies on one side of some totally geodesic sphere through p. A domain $K \subset \mathbb{H}^n$ is said to be *horospherical convex*, or *h-convex*, or have *h-convex boundary*, if for every point $p \in \partial K$, K lies on the convex side of some horosphere $S_h(p)$ through p. Recall that a horosphere in $\mathbb{H}^n$ is a hypersurface obtained as the limit of a geodesic sphere of $\mathbb{H}^n$ when its center goes to the infinity along a fixed geodesic ray. It is well-known (see e.g. [15]) that a horosphere in $\mathbb{H}^n$ has all its principal curvatures being equal to 1 and the h-convexity of $K \subset \mathbb{H}^n$ is equivalent to that all the principal curvatures of its boundary ∂K are bounded below by 1. We say a domain $K \subset \mathbb{H}^n$ is strictly h-convex if all the principal curvatures of its boundary ∂K are strictly bigger than 1. The geodesic balls in $\mathbb{H}^n$ are all strictly h-convex. An h-convex domain must be convex, but the converse is not true. In some sense, the horospherical convexity is more natural geometric concept than the convexity in $\mathbb{H}^n$, see e.g. [15]. The horospherical convexity plays a crucial role in the proof of (1.5) in [19, 20] for $k \geq 2$. It is also crucial for this paper.

For a (geodesically) convex domain $K \subset \mathbb{H}^n$, the *quermassintegrals* are defined by

$$W_k(K) := \frac{(n-k)\omega_{k-1}\cdots\omega_0}{n\omega_{n-2}\cdots\omega_{n-k-1}} \int_{\mathcal{L}_k} \chi(L_k \cap K) dL_k, \quad k = 1, \cdots, n-1;$$

where $\mathcal{L}_k$ is the space of k-dimensional totally geodesic subspaces L_k in $\mathbb{H}^n$ and dL_k is the natural (invariant) measure on $\mathcal{L}_k$, see Section 2 for more details. The function χ is given by $\chi(K) = 1$ if $K \neq \emptyset$ and $\chi(\emptyset) = 0$. For simplicity, we also use the convention

$$W_0(K) = \mathrm{Vol}(K), \quad W_n(K) = \frac{\omega_{n-1}}{n}.$$

We remark that the Cauchy-Crofon formula (See [37] or [42] Proposition 2.2.1) tells that

$$W_1(K) = \frac{1}{n}|\partial K|.$$

When the boundary ∂K is C^2-differentiable, one can define the *curvature integrals* by

$$V_{n-1-k}(K) = \int_{\partial K} H_k d\mu, \quad k = 0, \cdots, n-1,$$

where H_k are the (normalized) k-th mean curvature of ∂K as an embedding in $\mathbb{H}^n$ and $d\mu$ is the area element on ∂K induced from $\mathbb{H}^n$.

From the viewpoint of integral geometry, the quermassintegrals seem to be more important and play a central role. Nevertheless, the curvature integrals are also very important geometric quantities not only in integral geometry, but also in the theory of submanifolds. In $\mathbb{R}^n$, the quermassintegrals coincide with the curvature integrals, up to a constant multiple. However, the quermassintgrals and the curvature integrals in $\mathbb{H}^n$ do not coincide. Nevertheless they are closely related (see e.g. [41], Proposition 7):

$$\begin{aligned} V_{n-1-k}(K) &= n\left(W_{k+1}(K) + \frac{k}{n-k+1}W_{k-1}(K)\right), \quad k = 1, \cdots, n-1, \\ V_{n-1}(K) &= nW_1(K) = |\partial K|. \end{aligned}$$

In this paper, we will solve a large class of isoperimetric type problems in $\mathbb{H}^n$ involving the quermassintegrals and the curvature integrals for h-convex bounded domains with smooth boundary.

The first main result of this paper is the following Alexandrov-Fenchel type inequalities for the quermassintegrals.

Theorem 1.1. *Let $\mathcal{K}$ be the space of h-convex bounded domains in $\mathbb{H}^n$ with smooth boundary and $K \in \mathcal{K}$. For $0 \le l < k \le n-1$, we have*

$$W_k(K) \ge f_k \circ f_l^{-1}(W_l(K)).$$

Equality holds if and only if K is a geodesic ball. Here $f_k : [0, \infty) \to \mathbb{R}_+$ is a monotone function defined by $f_k(r) = W_k(B_r)$, the k-th quermassintegral for the geodesic ball of radius r, and f_l^{-1} is the inverse function of f_l. In other words, the minimum of W_k among the domains in $\mathcal{K}$ with given W_l is achieved precisely by geodesic balls.

Moreover, from Theorem 1.1 we solve the following isoperimetric type problems.

Theorem 1.2. *Let $\mathcal{K}$ be the space of h-convex bounded domains in $\mathbb{H}^n$ with smooth boundary. Then the following holds:*

(i) *For $0 \le l < k \le n-1$, V_{n-1-k} attains its minimum at a geodesic ball among the domains in $\mathcal{K}$ with given W_l;*
(ii) *For $0 \le k \le n-1$, V_{n-1-k} attains its minimum at a geodesic ball among the domains in $\mathcal{K}$ with given volume $W_0 = Vol$;*
(iii) *For $1 \le k \le n-1$, V_{n-1-k} attains its minimum at a geodesic ball among the domains in $\mathcal{K}$ with given area $|\partial K| = nW_1 = V_{n-1}$ of the boundary ∂K;*
(iv) *For $0 \le l < k \le n-1$ and $k - l = 2m$ for some $m \in \mathbb{N}$, V_{n-1-k} attains its minimum at a geodesic ball among the domains in $\mathcal{K}$ with given V_{n-1-l}.*

Theorem 1.1 and 1.2 give an affirmative answer to the question posed by Gao-Hug-Schneider in [17] for $\mathbb{H}^n$ (in the case of h-convex bounded domains with smooth boundary).

Unlike in $\mathbb{R}^n$, most of above results for quermassintegrals and the curvature integrals in $\mathbb{H}^n$ have no explicit (inequality) form. As mentioned above, even the classical isoperimetric problem between volume and area in $\mathbb{H}^n$ solved in [38] has in general no explicit from. Here we are able to formulate Statement (iii) in Theorem 1.2 in an optimal inequality.

Theorem 1.3. *Let $1 \le k \le n-1$. Any h-convex bounded domain K in $\mathbb{H}^n$ with smooth boundary satisfies*

$$\int_{\partial K} H_k d\mu \ge \omega_{n-1} \left\{ \left(\frac{|\partial K|}{\omega_{n-1}} \right)^{\frac{2}{k}} + \left(\frac{|\partial K|}{\omega_{n-1}} \right)^{\frac{2}{k}\frac{(n-k-1)}{n-1}} \right\}^{\frac{k}{2}}. \tag{1.6}$$

Equality holds if and only if K is a geodesic ball.

Inequality (1.6) was called as a hyperbolic Alexandrov-Fenchel inequality in [19]. As mentioned above, (1.6) was proved in [33] for $k = 2$ under a weaker condition, in [19] for $k = 4$ and in [20] for general even k. For general odd integer k inequality (1.6) was conjectured in [20] after the authors showed (1.6) for $k = 1$ with a help of a result of Cheng and Xu [12]. For the related works about the result of Cheng and Xu [12], see also [13], [18] and [22].

Recently Theorem 1.3 (for k odd) was used in [21] to prove a Penrose type inequality for a higher order mass on asymptotically hyperbolic manifolds.

The approaches used in [19, 20, 33], and also in [4, 14], are finding a suitable geometric quantity, which is monotone under a suitable inverse curvature flow studied by Gerhardt [23], and managing to compute the limit of the geometric quantity. However in this paper we will not use an inverse curvature flow. Instead we will use a (normalized) generalized mean curvature flow to prove Theorem 1.1. The crucial points of this paper are: (i) the choice of the quermassintegrals W_k as this suitable geometric quantity, (ii) the use of the quermassintegral preserving curvature flows, along which one quermassintegral is preserved and the other is monotone. The flow we consider is

$$\frac{\partial X}{\partial t}(x,t) = \left\{ c(t) - \left(\frac{H_k}{H_l}\right)^{\frac{1}{k-l}}(x,t) \right\} \nu(x,t), \tag{1.7}$$

where $\nu(\cdot,t)$ is the outer normal of the evolved hypersurface and $c(t)$ is defined by

$$c(t) := c_{k,l}(t) := \frac{\int_{\Sigma_t} H_k^{\frac{1}{k-l}} H_l^{1-\frac{1}{k-l}} d\mu_t}{\int_{\Sigma_t} H_l d\mu_t}.$$

We will show that this flow converges exponentially to a geodesic sphere, provided that the initial hypersurface is h-convex. The study of this flow is motivated by the work of [8, 26, 34, 35], especially the work of Makowski [34], who considered the mixed volume (in our words, the curvature integrals) preserving curvature flows in $\mathbb{R}^n$ and $\mathbb{H}^n$ respectively. In [34] the isoperimeteric result of Schmidt mentioned above was reproved by a flow method. The method of using geometric flows to prove geometric inequalities seems to be powerful. Various flows have been employed to prove geometric inequalities, see for instance [2, 4, 14, 19, 20, 22, 25, 26, 28, 33, 34, 35, 40].

The rest of this paper is organized as follows. In Section 2, we present basic concepts and facts about integral geometry in the hyperbolic space. In Section 3, we study the quermassintegral preserving curvature flows and prove a rigidity result. In Section 4, we choose a special flow to prove our main theorems.

2. Curvature integrals and Quermassintegrals

In this section, we recall some basic concepts in integral geometry in the hyperbolic space, we refer to Santaló's book [37], Part IV, and Solanes' thesis [42] for more details.

For a (geodesically) convex domain $K \subset \mathbb{H}^n$, the *quermassintegrals* are defined by

$$W_k(K) := \frac{(n-k)\omega_{k-1}\cdots\omega_0}{n\omega_{n-2}\cdots\omega_{n-k-1}} \int_{\mathcal{L}_k} \chi(L_k \cap K) dL_k, \quad k = 1, \cdots, n-1; \tag{2.1}$$

where $\mathcal{L}_k$ is the space of k-dimensional totally geodesic subspaces L_k in $\mathbb{H}^n$ and dL_k is the natural (invariant) measure on $\mathcal{L}_k$. Since dL_k is unique up to a constant factor, we use an easy interpretation given in [16] to illustrate the normalization or the choice of dL_k. Each totally geodesic subspace L_k is determined by its orthogonal subspace $L_{n-k}(0)$, which is through the origin, and the intersection point $x = L_k \cap L_{n-k}(0)$. Hence one can consider

$\mathcal{L}_k$ as a bundle over the Grassmannian manifolds $G(n-k,n)$ which consists of all subspaces $L_{n-k}(0)$. Then dL_k is given by

$$dL_k = \cosh\left(d_{\mathbb{H}^n}(x,0)\right)^k dx dV_{n-k},$$

where $d_{\mathbb{H}^n}(x,0)$ is the distance function between x and the origin, dx is the volume element on $L_{n-k}(0)$ and dV_{n-k} is the volume element on $G(n-k,n)$.

The function χ is given by $\chi(K)=1$ if $K\neq\emptyset$ and $\chi(\emptyset)=0$. For simplicity, we also use the notation

$$W_0(K)=\mathrm{Vol}(K),\quad W_n(K)=\frac{\omega_{n-1}}{n}.$$

It is clear from definition (2.1) that the quermassintegrals W_k, $k=0,1,\cdots,n-1$, are strictly increasing under set inclusion, i.e.,

$$\text{if } K_1 \subsetneqq K_2, \text{ then } W_k(K_1)<W_k(K_2). \tag{2.2}$$

This simple fact plays a role in the proof of the convergence of curvature flows considered below.

Let σ_k be the k-th elementary symmetric function $\sigma_k:\mathbb{R}^{n-1}\to\mathbb{R}$ defined by

$$\sigma_k(\Lambda)=\sum_{i_1<\cdots<i_k}\lambda_{i_1}\cdots\lambda_{i_k}\quad \text{for } \Lambda=(\lambda_1,\cdots,\lambda_{n-1})\in\mathbb{R}^{n-1}.$$

As convention, we take $\sigma_0=1$. The definition of σ_k can be easily extended to the set of all symmetric matrices. The Garding cone Γ_k^+ is defined as

$$\Gamma_k^+=\{\Lambda\in\mathbb{R}^{n-1}|\sigma_j(\Lambda)>0,\quad \forall j\le k\}.$$

We denote by $\overline{\Gamma_k^+}$ the closure of Γ_k^+.

Let

$$H_k=H_k(\Lambda)=\frac{\sigma_k(\Lambda)}{\binom{n}{k}}$$

be the normalized symmetric functions. We have the following Newton-MacLaurin inequalities. For the proof we refer to a survey of Guan [24].

Proposition 2.1. *For $1\le l<k\le n-1$ and $\Lambda\in\overline{\Gamma_k^+}$, the following inequalities hold:*

$$H_{k-1}H_l\ge H_kH_{l-1}. \tag{2.3}$$

$$H_l\ge H_k^{\frac{l}{k}}. \tag{2.4}$$

Equalities hold in (2.3) *or* (2.4) *if and only if $\lambda_i=\lambda_j$ for all $1\le i,j\le n-1$.*

For a domain $K\subset\mathbb{H}^n$, if the boundary ∂K is C^2-differentiable, the *(normalized) k-th mean curvatures* are

$$H_k(x)=H_k(\kappa(x))\quad \text{for } x\in\partial K,\quad k=0,\cdots,n-1,$$

where $\kappa=(\kappa_1,\cdots,\kappa_{n-1})$ is the set of the principal curvatures of ∂K as an embedding in $\mathbb{H}^n$. The *curvature integrals* are defined by

$$V_{n-1-k}(K)=\int_{\partial K}H_k d\mu,\quad k=0,\cdots,n-1,$$

where $d\mu$ is the area element on ∂K induced from $\mathbb{H}^n$.

The curvature integrals have a similar meaning of the mixed volume in the Euclidean space, in view of the Steiner formula (see [37], IV.18.4) which says that for a smooth convex domain K and some positive number $\rho \in \mathbb{R}$, its parallel set $K[\rho] := \{x \in \mathbb{H}^n | d_{\mathbb{H}^n}(x, K) \leq \rho\}$ has the volume

$$\mathrm{Vol}(K[\rho]) = \mathrm{Vol}(K) + \sum_{k=0}^{n-1} \binom{n}{k} V_k(K) \int_0^\rho \cosh^k(s) \sinh^{n-1-k}(s) ds.$$

Recall that the quermassintegrals and the curvature integrals are related (see e.g. [41], Proposition 7) by

$$\begin{aligned} V_{n-1-k}(K) &= n\left(W_{k+1}(K) + \frac{k}{n-k+1} W_{k-1}(K)\right), \quad k = 1, \cdots, n-1, \\ V_{n-1}(K) &= nW_1(K) = |\partial K|. \end{aligned} \tag{2.5}$$

From (2.5) it is easy to express W_k as a linear combination of several curvature integrals (see e.g. [37], IV.17.4 or [41], Corollary 8):

- for $1 \leq k \leq n-1$ and k is even,

$$\begin{aligned} W_k(K) &= \frac{1}{n} \sum_{i=0}^{\frac{k}{2}-1} (-1)^i \frac{(k-1)!!(n-k)!!}{(k-1-2i)!!(n-k+2i)!!} \int_{\partial K} H_{k-1-2i} d\mu \\ &\quad + (-1)^{\frac{k}{2}} \frac{(k-1)!!(n-k)!!}{n!!} \mathrm{Vol}(K); \end{aligned} \tag{2.6}$$

- for $1 \leq k \leq n-1$ and k is odd,

$$W_k(K) = \frac{1}{n} \sum_{i=0}^{\frac{k-1}{2}} (-1)^i \frac{(k-1)!!(n-k)!!}{(k-1-2i)!!(n-k+2i)!!} \int_{\partial K} H_{k-1-2i} d\mu. \tag{2.7}$$

Here the notation $k!!$ means the product of all odd (even) integers up to odd (even) k. For $k = n$, the formulas (2.6) and (2.7) can be viewed as the Gauss-Bonnet-Chern theorem for domains in the hyperbolic space.

From (2.6) and (2.7), one can see the difference between quermassintegrals W_{2k} and W_{2k+1}. In fact, W_{2k} is extrinsic and W_{2k+1} is intrinsic, namely it depends only on the induced metric g on ∂K. The latter follows from the fact that H_{2k} can be expressed in terms of intrinsic geometric quantities, the Gauss-Bonnet curvatures. For the proof see [20].

3. Quermassintegral preserving curvature flows

Let $K_0 \in \mathcal{K}$ be an h-convex bounded domain in $\mathbb{H}^n$ with smooth boundary $\Sigma_0 = \partial K_0$. We consider the following curvature evolution equation

$$\frac{\partial X}{\partial t}(x,t) = (c(t) - F(\mathcal{W}(x,t)))\nu(x,t), \tag{3.1}$$

where $X(\cdot, t) : M^{n-1} \to \mathbb{H}^n$ are parametrizations of a family of hypersurfaces $\Sigma_t \subset \mathbb{H}^n$ which encloses K_t, $\nu(\cdot, t)$ is the unit outward normal to Σ_t, F is a smooth curvature

function evaluated at the matrix of the Weingarten map $\mathcal{W}$ of Σ_t. The time dependent term $c(t)$ will be explained later.

The function F should have the following properties (P):

- $F(A) = f(\lambda(A))$, where $\lambda(A) = (\lambda_1, \cdots, \lambda_{n-1})$ are the eigenvalues of the matrix A and f is a smooth, symmetric function defined on the positive cone
$$\Gamma^+ = \{\lambda \in \mathbb{R}^{n-1} | \lambda_i > 0, \forall i = 1, \cdots, n-1\};$$
- f is positively homogeneous of degree 1: $f(t\lambda) = tf(\lambda)$ for any $t > 0$;
- f is strictly increasing in each argument: $\frac{\partial f}{\partial \lambda_i} > 0$;
- f is normalized by setting $f(1, \cdots, 1) = 1$;
- f is concave and inverse concave, i.e., $f^*(\lambda) := -f(\lambda_1^{-1}, \cdots, \lambda_{n-1}^{-1})$ is concave.

We use the notation $\dot{f}^i = \frac{\partial f}{\partial \lambda_i}$, $\ddot{f}^{ij} = \frac{\partial f}{\partial \lambda_i \partial \lambda_j}$, $F^{ij} = \frac{\partial F}{\partial A_{ij}}$ and $F^{ij,rs} = \frac{\partial^2 F}{\partial A_{ij} \partial A_{rs}}$. Also we use "$\nabla$" or ";" to denote the covariant derivative on hypersurfaces. Unless stated otherwise, the summation convention is used throughout this paper. For our purpose, F is viewed as a function on $h_i^j = g^{jk} h_{ik}$, i.e., $F = F(h_i^j) = F(g^{jk} h_{ik}) = f(\kappa)$, where g_{ij} and h_{ij} the first and second fundamental form respectively and $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ is the set of the principal curvatures.

We have the evolution equations for the quermassintegrals and the curvature integrals associated with K_t under flow (3.1).

Proposition 3.1. *Along flow* (3.1)*, we have*

$$\frac{d}{dt} Vol(K_t) = \int_{\Sigma_t} (c(t) - F)\, d\mu_t; \tag{3.2}$$

$$\frac{d}{dt} |\Sigma_t| = \int_{\Sigma_t} (n-1) H_1 (c(t) - F)\, d\mu_t; \tag{3.3}$$

$$\frac{d}{dt} \int_{\Sigma_t} H_k d\mu_t = \int_{\Sigma_t} \{(n-1-k) H_{k+1} + k H_{k-1}\} (c(t) - F)\, d\mu_t, \quad k = 1, \cdots, n-1; \tag{3.4}$$

$$\frac{d}{dt} W_k(K_t) = \frac{n-k}{n} \int_{\Sigma_t} H_k (c(t) - F)\, d\mu_t, \quad k = 0, \cdots, n-1. \tag{3.5}$$

Proof. (3.2)–(3.4) are now well-known and were proved in [36]. We now prove (3.5) by induction. In view of (3.2) and (3.3), it is true for $k = 0, 1$. Assume it is true for $k-1$, we can compute by using (2.5), (3.4) and the inductive assumption that

$$\begin{aligned}
\frac{d}{dt} W_{k+1}(K_t) &= \frac{1}{n} \frac{d}{dt} \int_{\Sigma_t} H_k d\mu_t - \frac{k}{n-k+1} \frac{d}{dt} W_{k-1}(K_t) \\
&= \frac{1}{n} \int_{\Sigma_t} ((n-1-k) H_{k+1} + k H_{k-1}) (c(t) - F)\, d\mu_t \\
&\quad - \frac{k}{n-k+1} \frac{n-k+1}{n} \int_{\Sigma_t} H_{k-1} (c(t) - F)\, d\mu_t \\
&= \frac{n-k-1}{n} \int_{\Sigma_t} H_{k+1} (c(t) - F)\, d\mu_t.
\end{aligned}$$

□

The choice of $c(t)$ depends on which geometric quantity we want to preserve. In this paper, we will take

$$c(t) = c_l(t) := \frac{\int_{\Sigma_t} H_l F d\mu_t}{\int_{\Sigma_t} H_l d\mu_t} \tag{3.6}$$

so that the flow preserves W_l.

Lemma 3.1. *With the choice of $c(t)$ by (3.6) flow (3.1) preserves the quermassintegral W_l.*

Proof. By (3.5) we have

$$\frac{d}{dt} W_l(K_t) = \frac{n-l}{n} \int_{\Sigma_t} H_l (c_l(t) - F) d\mu_t = 0.$$

□

Under the assumptions (P) on F and the assumption that the initial domain is h-convex, the long time existence and convergence of the flow (3.1) can be proved.

Theorem 3.1. *Let $K_0 \in \mathcal{K}$ be an h-convex domain in $\mathbb{H}^n$ with smooth boundary Σ_0. Let F be a function satisfying the properties (P) and $c(t)$ be defined in (3.6) for some $l \in \{0, \cdots, n-1\}$. Then flow (3.1) has a smooth solution $X(t)$ for $t \in [0, \infty)$. Moreover, $X(t)$ converges exponentially to a geodesic sphere of radius r_0, which has the same quermassintegral W_l as K_0, i.e., there exists some $\delta > 0$ such that $X(t)$ can be written as graphs over $\mathbb{S}^{n-1}$ and the graph function $u(t)$ satisfies*

$$|u(t) - r_0| \leq e^{-\delta t}. \tag{3.7}$$

Before the proof, let us spend a few words to compare Theorem 3.1 with the main one in [34]. On one hand, flow (3.1) is slightly different from that in [34] in the sense that the curvature integral preserving property is replaced by quermassintegral preserving one. Hence we need to check that the same strategy works for our flow. On the other hand, we impose a weaker condition that we start with an h-convex domain, rather than a strictly h-convex domain. In fact, our first step illustrates that the domain becomes immediately strictly h-convex during the flow.

Proof of Theorem 3.1. The proof will be divided into two steps.

Step I. The flow (3.1) exists at least in a short time interval $[0, T^*)$ for some $T^* > 0$ and the evolving hypersurface Σ_t is strictly h-convex for all $t \in (0, T^*)$.

The short time existence is now well-known, since the third condition in (P) ensures that the flow is strictly parabolic. To prove the strict h-convexity, we shall use the following constant rank theorem.

Theorem 3.2. *Let Σ_t be a smooth solution to the flow (3.1) in $[0, T]$ for some $T > 0$ which is h-convex, i.e., the matrix $(S_{ij}) = (h_{ij} - g_{ij}) \geq 0$ for $t \in [0, T]$. Then (S_{ij}) is of constant rank $l(t)$ for each $t \in (0, T]$ and $l(s) \leq l(t)$ for all $0 < s \leq t \leq T$.*

Proof of Theorem 3.2. The proof follows similar arguments as that of the proof of Proposition 5.1 in [3]. For the convenience of the readers, we sketch the proof.

The h-convexity of Σ_0 means that $(S_{ij}) \geq 0$ at $t = 0$. For $\varepsilon > 0$, define a symmetric matrix $W = (S_{ij} + \varepsilon g_{ij})$. Let $l(t)$ be the minimal rank of $(S_{ij}(x,t))$. For a fixed $t_0 \in (0,T]$, let $x_0 \in \Sigma_{t_0}$ such that $(S_{ij}(x,t_0))$ attains its minimal rank at x_0. Set $l := l(t_0)$ and

$$\phi(x,t) = \sigma_{l+1}(W(x,t)) + \frac{\sigma_{l+2}}{\sigma_{l+1}}(W(x,t)).$$

It is proved in Section 2 in [3] that ϕ is in $C^{1,1}$. We will show that there are constants C_1, C_2 and δ, depending on $\|X\|_{C^{3,1}(M\times[0,T^*))}$ but independent of ε and ϕ, such that in some neighborhood $\mathcal{O}$ of x_0 and for $t \in (t_0 - \delta, t_0]$,

$$F^{ij}\phi_{;ij} - \frac{\partial}{\partial t}\phi \leq C_1\phi + C_2|\nabla\phi|. \tag{3.8}$$

First, one verifies the evolution equation for S_{ij} (see e.g. (4.23) in [23] or (3.21) in [34]):

$$\begin{aligned}
\frac{\partial}{\partial t}S_{ij} &= F^{kl}h_{ij;kl} + F^{kl,rs}h_{kl;i}h_{rs;j} + F^{kl}h_{lr}h^r_k h_{ij} - c(t)h^k_i h_{jk} \\
&\quad -(2F - c(t))g_{ij} + F^{kl}g_{kl}h_{ij} - 2(F - c(t))h^k_i S_{jk} \\
&= F^{kl}S_{ij;kl} + F^{kl,rs}S_{kl;i}S_{rs;j} \\
&\quad + F^{kl}h_{lr}h^r_k S_{ij} - c(t)S^k_i S_{jk} - 2c(t)S_{ij} + F^{kl}g_{kl}S_{ij} - 2(F - c(t))h^k_i S_{jk} \\
&\quad + F^{kl}h_{lr}h^r_k g_{ij} + (F^{kl}g_{kl} - 2F)g_{ij}.
\end{aligned} \tag{3.9}$$

The last line of (3.9) can be further computed as

$$\begin{aligned}
& F^{kl}h_{lr}h^r_k g_{ij} + (F^{kl}g_{kl} - 2F)g_{ij} \\
&= \left(F^{kl}S_{lr}S^r_k + 2F^{kl}S_{kl} + F^{kl}g_{kl} + F^{kl}g_{kl} - 2F^{kl}h_{kl}\right)g_{ij} \\
&= F^{kl}S_{lr}S^r_k g_{ij} \geq 0,
\end{aligned} \tag{3.10}$$

since F^{kl} is positive definite and $F^{kl}h_{kl} = F$ due to the 1-homogeneity of F.

As in [3], in $\mathcal{O}\times(t_0-\delta,t_0]$, the index set $\{1,\cdots,n-1\}$ can be divided into two subsets B and G, where for $i \in B$, the eigenvalues $\tilde{\lambda}_i$ of W is small and for $j \in G$, $\tilde{\lambda}_j$ is uniformly positive away from 0. By choosing suitable coordinates, we may assume at each point of computation, $W_{ij}(x,t)$ is diagonal. Let $O(\phi)$ denote a quantity which can be controlled by $C\phi$ for a universal constant C depending on $\|X\|_{C^{3,1}(M\times[0,T^*))}$ but independent of ε and ϕ. Notice that $\varepsilon = O(\phi)$ near (x_0,t_0) (see (3.8) in [3]). With help of this, we can compute by using (3.9) and (3.10) that

$$\begin{aligned}
F^{kl}\phi_{;kl} - \frac{\partial}{\partial t}\phi &= \phi^{ii}(F^{kl}W_{ii;kl} - \frac{\partial}{\partial t}W_{ii}) + F^{kl}\phi^{ij,rs}W_{ij;r}W_{kl;s} \\
&\leq \phi^{ii}\big(-F^{kl,rs}W_{kl;i}W_{rs;i} - F^{kl}h_{lr}h^r_k W_{ii} \\
&\quad + c(t)W^k_i W_{ik} + 2c(t)W_{ii} - F^{kl}g_{kl}W_{ii} + 2(F - c(t))h^k_i W_{ik}\big) \\
&\quad + F^{kl}\phi^{ij,rs}W_{ij;k}W_{kl;s} + O(\phi).
\end{aligned} \tag{3.11}$$

Here we use the notation $\phi^{ij} = \frac{\partial\phi}{\partial W_{ij}}$ and $\phi^{ij,kl} = \frac{\partial^2\phi}{\partial W_{ij}\partial W_{kl}}$.

As in [3], in $\mathcal{O}\times(t_0-\delta,t_0]$, the index set $\{1,\cdots,n-1\}$ can be divided into two subsets B and G, where for $i \in B$, the eigenvalues $\tilde{\lambda}_i$ of W is small and for $j \in G$, $\tilde{\lambda}_j$ is uniformly

positive away from 0. By choosing suitable coordinates, we may assume at each point of computation, W_{ij} is diagonal. Since $\phi^{jj} = O(\phi)$ for $j \in G$ ((3.14) in [3]) and $W_{ii} = O(\phi)$ for $i \in B$, inequality (3.11) can be reduced to the following one:

$$F^{kl}\phi_{;kl} - \frac{\partial}{\partial t}\phi \leq -\phi^{ii}F^{kl,rs}W_{kl;i}W_{rs;i} + F^{kl}\phi^{ij,rs}W_{ij;k}W_{kl;s} + O(\phi) \tag{3.12}$$

One can find that the right hand side of (3.12) are exactly the same as that of (3.13) in [3] (Notice here W_{ij} satisfies Codazzi property and the situation is somehow simpler since F only depends on second order term W_{ij}, but no lower order term). By a routine computation of ϕ^{ii} and $\phi^{ij,rs}$ as Theorem 3.2 in [3], one can obtain

$$\begin{aligned}
& -\phi^{ii}F^{kl,rs}W_{kl;i}W_{rs;i} + F^{kl}\phi^{ij,rs}W_{ij;k}W_{kl;s} \\
= \;& O(\phi + \sum_{i,j\in B}|\nabla W_{ij}|) - \frac{1}{\sigma_1(B)}\sum_{k,l}\sum_{i,j\in B, i\neq j}F^{kl}W_{ij;k}W_{ij;l} \\
& -\frac{1}{\sigma_1^3(B)}\sum_{k,l}\sum_{i\in B}F^{kl}\left(W_{ii;k}\sigma_1(B) - W_{ii}\sum_{j\in B}W_{jj;k}\right)\left(W_{ii;l}\sigma_1(B) - W_{ii}\sum_{j\in B}W_{jj;l}\right) \\
& -\sum_{i\in B}\left[\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}\right]\cdot \\
& \cdot\left[\sum_{k,l,r,s\in G}F^{kl,rs}W_{kl;i}W_{rs;i} + 2\sum_{k,l\in G}F^{kl}\sum_{j\in G}\frac{1}{\tilde{\lambda}_j}W_{ij;k}W_{ij;l}\right].
\end{aligned} \tag{3.13}$$

Here $\sigma_k(B)$ denotes the elementary symmetric functions σ_k on the eigenvalues $\tilde{\lambda}_i$ for $i \in B$ and $\sigma_k(B|i)$ denotes σ_k on $\tilde{\lambda}_j$ for $j \in B, j \neq i$.

The analysis in Theorem 3.2 in [3] shows that the right hand side of above equation can be controlled by $\phi + |\nabla\phi| - C\sum_{i,j\in B}|\nabla W_{ij}|$. This analysis is quite subtle and depends heavily on the concavity and the inverse concavity of F with respect to W_{ij}. We refer to [3] to the details.

Hence by combining (3.12) and (3.13), we arrive at (3.8). Now letting $\varepsilon \to 0$ and by the standard strong maximum principle for parabolic equations, we conclude that

$$\sigma_{l(t_0)+1}(S_{ij}(x,t)) + \frac{\sigma_{l(t_0)+2}(S_{ij}(x,t))}{\sigma_{l(t_0)+1}(S_{ij}(x,t))} \equiv 0 \text{ for } (x,t) \in \mathcal{O}\times(t_0-\delta, t_0]. \tag{3.14}$$

Since $l(t_0)$ is the minimum rank of $S_{ij}(x,t_0)$ among Σ_{t_0} and Σ_{t_0} is connected, we conclude from (3.14) that the matrices $(S_{ij}(x,t_0))$ is of constant rank $l(t_0)$ in Σ_{t_0} and $l(t) \leq l(t_0)$ for $t \in (t_0-\delta, t_0]$. Since $t_0 \in (0,T]$ is arbitrary, we complete the proof. □

Remark 3.1. *The constant rank theorem was initiated by Caffarelli-Friedman* [10] *and Korevaar-Lewis* [32] *for semilinear elliptic equations and developed by Guan-Ma* [27]*, Caffarelli-Guan-Ma* [9] *and Bian-Guan* [3] *for fully nonlinear elliptic equations. The choice of a suitable auxiliary function ϕ is the key point to prove different kinds of constant rank theorems.*

We return to the proof of Theorem 3.1, Step I. We follow closely the argument in [3].

We may approximate Σ_0 by a family of strictly h-convex hypersurfaces Σ_0^ε. By continuity, there is $\delta > 0$ (independent of ε), such that there is a solution Σ_t^ε to (3.1) for $t \in [0,\delta]$.

Then Σ_t^ε must be strictly h-convex for $t \in [0, \delta]$ by Theorem 3.2. Taking $\varepsilon \to 0$, we have that Σ_t is h-convex for $t \in [0, \delta]$. This implies that the set $\{t \in [0, T] | \Sigma_t \text{ is h-convex}\}$ is open. It is obviously closed and non-empty. Therefore, Σ_t is h-convex for $t \in [0, T]$. On the other hand, a standard argument shows that for every closed hypersurface in $\mathbb{H}^n$, there exists at least one point which is strictly h-convex. Therefore by Theorem 3.2 again, Σ_t is strictly h-convex for all $t \in (0, T]$. We finish the proof of Step I.

Step II: Let Σ_{t_0}, $t_0 \in (0, T^*)$ be a strictly h-convex hypersurface evolving by (3.1), then the long time existence and convergence can be proved.

Starting with a strictly h-convex hypersurface, the flow (3.1) is quite similar to that considered by Makowski [34]. The difference is that the flows he considered preserve the curvature integrals and ours preserve the quermassintegrals. However, this difference makes a very big difference in applications to the Alexandrov-Fenchel type inequalities, though the analytic part of both flows is quite similar. For the convenience of the readers, we sketch the proof and refer to [34] for the formal proof.

Let $\mathbb{H}^n = \mathbb{R} \times \mathbb{S}^{n-1}$ with the hyperbolic metric

$$\bar{g} = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}}$$

where $g_{\mathbb{S}^{n-1}}$ is the standard round metric on the $(n-1)$-dimensional unit sphere. Denote by $\langle \cdot, \cdot \rangle$ the metric $\bar{g}$, and by $\bar{\nabla}$ the covariant derivative on $\mathbb{H}^n$.

1. We see from Step 1 that as long as the flow exists, the strict h-convexity is preserved. Moreover by using Andrews' pinching estimates [1], one can prove the pinching of the principal curvatures is also preserved (Lemma 4.4 in [34]), i.e.,

- if $h_{ij} - g_{ij} \geq \varepsilon(H_1 - 1)g_{ij}$ at $t = t_0$ for some $\varepsilon \in (0, \frac{1}{n-1})$, then it holds as well as $h_{ij} - g_{ij} \geq \varepsilon(F - 1)g_{ij}$ holds as long as the flow exists.

More precisely, one verifies the evolution equation for $\tilde{S}_{ij} := h_{ij} - g_{ij} - \varepsilon(H_1 - 1)g_{ij}$:

$$\begin{aligned}
\frac{d}{dt}\tilde{S}_{ij} &= F^{kl}\tilde{S}_{ij;kl} + F^{kl,rs}h_{kl;p}h_{rs;q}\left(\delta_i^p\delta_j^q - \frac{\varepsilon}{n}g^{pq}g_{ij}\right) \\
&\quad +(F^{kl}h_{lr}h_k^r + F^{kl}g_{kl})(\tilde{S}_{ij} + (1-\varepsilon)g_{ij}) + c(t)\left(\frac{\varepsilon}{n}h_k^l h_l^k g_{ij} - h_i^k h_{kj}\right) \\
&\quad -(2F - c(t))(1-\varepsilon)g_{ij} - 2(F - c(t))h_i^k\tilde{S}_{kj} \\
&=: F^{kl}\tilde{S}_{ij;kl} + F^{kl,rs}h_{kl;p}h_{rs;q}\left(\delta_i^p\delta_j^q - \frac{\varepsilon}{n}g^{pq}g_{ij}\right) + N_{ij}.
\end{aligned}$$

For a vector $v = (v^1, \cdots, v^{n-1})$ such that $\tilde{S}_{ij}v^j = 0$, using (3.10) and the fact that $h_k^l h_l^k \geq nH_1^2$, we check

$$\begin{aligned}
N_{ij}v^iv^j &= (F^{kl}h_{lr}h_k^r + F^{kl}g_{kl} - 2F)g_{ij}v^iv^j \\
&\quad +c(t)\left(\frac{\varepsilon}{n}h_k^l h_l^k g_{ij} - h_i^k h_{kj} + (1-\varepsilon)g_{ij}\right)v^iv^j \\
&\geq c(t)\left(\frac{\varepsilon}{n}h_k^l h_l^k - (1 - \varepsilon + \varepsilon H_1)^2 + (1-\varepsilon)\right)g_{ij}v^iv^j \\
&\geq c(t)\left(\varepsilon(1-\varepsilon)H_1^2 - 2\varepsilon(1-\varepsilon)H_1 - (1-\varepsilon)^2 + (1+\varepsilon)\right)g_{ij}v^iv^j \\
&= \varepsilon(1-\varepsilon)(H_1 - 1)^2 g_{ij}v^iv^j \geq 0.
\end{aligned}$$

Therefore Andrews' pinching theorem (Theorem 4.1 and Theorem 3.2 in [1]) applies to show $\tilde{S}_{ij} \geq 0$ is preserved along the flow. Now the preservation of pinching follows from $H_1 \geq F$ for concave function F.

2. In this substep, we will show that as long as the flow exists, the speed function F is bounded by a constant depending only on the initial hypersurface Σ_{t_0}. Consequently, the time-dependent term $c(t)$ is bounded, and $|\frac{\partial X}{\partial t}|$ is bounded. By the pinching estimate in Step II.1, one can easily deduce the upper boundedness of the principal curvatures. In fact, it follows from

$$\lambda_{max} \leq \frac{1}{\varepsilon}(\lambda_{min} - 1) + 1 \leq \frac{1}{\varepsilon}(F - 1) + 1.$$

Here λ_{max} and λ_{min} denotes the maximum and minimum among all the principal curvatures respectively.

The proof of the boundedness of F is more technique. Hence we give more details for this step.

2.1. As long as the flow exists, the inner radius and the outer radius of K_t can be uniformly bounded by some positive constants r_0 and R_0, dependent only on the initial hypersurface Σ_{t_0}, respectively.

In fact, this is the only place where the property of preserving the quermassintegrals is used. We verify this here. Let $r(t)$ and $R(t)$ be the inner radius and outer radius of Σ_t respectively. Let r_{t_0} be the number so that $W_l(K_{t_0}) = W_l(B_{r_{t_0}})$. By virtue of (2.2), we have that

$$W_l(B_{R(t)}) \geq W_l(K_t) = W_l(K_{t_0}) = W_l(B_{r_{t_0}}).$$

Thus $R(t) \geq r_{t_0}$. According to Step I, the h-convexity is preserved. A remarkable feature of the h-convexity is that the inner radius and the outer radius are comparable (see [34], Theorem 5.2 or [6], Theorem 3.1). Namely, there is a constant $C > 1$ such that

$$r(t) \leq R(t) \leq Cr(t).$$

Hence

$$r(t) \geq C^{-1}R(t) \geq C^{-1}r_{t_0} := r_0.$$

Similarly, from the monotonicity of the quermassintegral (2.2), we have

$$W_l(B_{r(t)}) \leq W_l(K_t) = W_l(K_{t_0}) = W_l(B_{r_{t_0}}),$$

which implies $r(t) \leq r_{t_0}$. Hence, we have

$$R(t) \leq Cr(t) \leq Cr_{t_0} := R_0.$$

2.2. Fix a time $t_1 \in [t_0, T^*)$. Since the inner radius of K_t is uniformly bounded, we can assume $B_{r_{t_1}}(p_{t_1}) \subset K_{t_1}$ is an enclosed ball with the center p_{t_1} and the radius $r_{t_1} \geq r_0$, then we can show that $B_{\frac{1}{2}r_{t_1}}(p_{t_1}) \subset K_t$ in some short time interval $t \in [t_1, t_2)$ for t_2 chosen later.

In fact, let $r(x,t)$ be the distance function of Σ_t from p_{t_1}. Set $\rho(x,t) := \cosh r(x,t)$. Let $u := \langle \bar{\nabla}\rho, \nu \rangle$ be the "support function". Define

$$\varphi := e^{(n-1)c_0(t-t_1)}\rho(x,t),$$

where c_0 is the constant in (3.17). Using the fact that $\rho_{;ij} = \rho g_{ij} - uh_{ij}$ and $F = F^{ij}h_{ij}$, one can easily check that

$$\frac{d}{dt}\varphi - F^{ij}\varphi_{;ij} = \varphi\left((n-1)c_0 - F^{ij}g_{ij}\right) + c(t)e^{(n-1)c_0(t-t_1)}u \geq 0.$$

By parabolic maximum principle,

$$\inf_{x\in\Sigma_t}\rho(x,t) \geq e^{-(n-1)c_0(t-t_1)}\inf_{x\in\Sigma_{t_1}}\rho(x,t_1) \geq e^{-(n-1)c_0(t-t_1)}\cosh r_{t_1}.$$

Therefore, in the time interval $[t_1, t_2)$, where $t_2 = \min\{t_1 + \frac{1}{(n-1)c_0}\ln\frac{\cosh r_{t_1}}{\cosh\frac{1}{2}r_{t_1}}, T^*\}$, we have $r(x,t) \geq \frac{1}{2}r_{t_1}$, namely, $B_{\frac{1}{2}r_{t_1}}(p_{t_1}) \subset K_t$.

Moreover, in view of a crucial property of h-convexity, which says $\langle\partial_r, \nu\rangle \geq \tanh r$ (see e.g. [8], Theorem 4), we infer that the "support function" $u = \sinh r\langle\partial_r, \nu\rangle$ is bounded below by a positive constant

$$u \geq u_0 := \sinh\frac{1}{2}r_{t_1}\tanh\frac{1}{2}r_{t_1}$$

in the time interval $[t_1, t_2)$. On the other hand, h-convexity ensures that $r(x,t) \leq r(t) + \ln 2 \leq R_0 + \ln 2$ (see e.g. [8], Theorem 4), which implies that u is also bounded above.

2.3. In the time interval $[t_1, t_2)$, we consider an auxiliary function

$$\Phi := \frac{F}{u - \frac{1}{2}u_0}.$$

One verifies the evolution equation for F and u:

$$\frac{d}{dt}F = F^{ij}F_{;ij} + (F - c(t))F^{ij}h_i^k h_{kj} - (F - c(t))F^{ij}g_{ij};$$

$$\frac{d}{dt}u = F^{ij}u_{;ij} + uF^{ij}h_i^k h_{kj} - (2F - c(t))\cosh r.$$

Here $r = r(x,t)$ still denotes the distance function of Σ_t from p_{t_1}. Therefore the evolution equation for Φ is

$$\begin{aligned}\frac{d}{dt}\Phi &= F^{ij}\Phi_{;ij} + \frac{2F^{ij}u_{;i}\Phi_{;j}}{u - \frac{1}{2}u_0} - \frac{c(t)}{u - \frac{1}{2}u_0}(F^{ij}h_i^k h_{kj} - F^{ij}g_{ij}) \\ &\quad - \frac{\frac{1}{2}u_0}{(u - \frac{1}{2}u_0)^2}F^{ij}h_i^k h_{kj}F + \frac{2F - c(t)}{u - \frac{1}{2}u_0}\cosh r\Phi - \frac{F}{u - \frac{1}{2}u_0}F^{ij}g_{ij}.\end{aligned} \tag{3.15}$$

By the h-convexity of Σ_t, we know $F^{ij}h_i^k h_{kj} - F^{ij}g_{ij} \geq 0$. Also, by pinching estimate and 1-homogeneity of F, we have

$$F^{ij}h_i^k h_{kj} \geq F(\varepsilon(F-1)+1) \geq \varepsilon F^2.$$

Hence at the maximum point of Φ in $M \times [t_1, t_2)$, we deduce from (3.15) that

$$0 \leq \frac{d}{dt}\Phi \leq -\frac{1}{2}u_0(u - \frac{1}{2}u_0)\varepsilon\Phi^3 + 2\cosh r\Phi^2. \tag{3.16}$$

Since $u - \frac{1}{2}u_0 \geq \frac{1}{2}u_0$ and $r \leq R_0 + \ln 2$, it follows from (3.16) that for $t \in [t_1, t_2)$, Φ is bounded above by a constant C depending only on Σ_{t_1}. Consequently, as u has also upper

bound, we get that the speed F is bounded above by C for $t \in [t_1, t_2)$. Since t_1 can be chosen arbitrary in $[t_0, T^*)$, we conclude that F has a uniform bound for $t \in [t_0, T^*)$.

3. The flow exists for $t \in [0, \infty)$ and the flow converges exponentially to a geodesic sphere.

Since F^{ij} is homogeneous of degree zero and the principal curvatures of the evolving hypersurfaces Σ_t satisfy $\lambda \geq 1$, we see that flow (3.1) is always uniformly parabolic, i.e., there exists some constant c_0, depending only on Σ_{t_0}, such that

$$c_0^{-1} g^{ij} \leq F^{ij}\big((h^i_j)(x,t)\big) \leq c_0 g^{ij}, \quad t_0 \leq t < T^*. \tag{3.17}$$

On the other hand, in Step II.2, we have shown that the principal curvatures of $\Sigma_t, t \in [t_0, T^*)$ have a uniform upper bound independent of T^*. Also the principal curvatures have a uniform lower bound 1. Therefore the principal curvatures for the evolving hypersurfaces lie in a compact set of Γ^+.

Exactly as in [35], Section 8, using the above facts, we can derive the higher order estimates for the graph function of the evolving hypersurfaces $\Sigma_t, t \in [t_0, T^*)$, still independent of T^*. These estimates enable us to extend the flow beyond T^*, which gives the long time existence.

To show the exponential convergence to a geodesic sphere, one first applies again Andrews' pinching theorem as in Step II.1 to the evolution equation of

$$\hat{S}_{ij} = h_{ij} - g_{ij} - (1 - e^{-\delta t})(H_1 - 1))g_{ij}$$

in order to prove that the pinching of the principal curvatures λ is improving at an exponential rate, i.e.,

$$\lambda_i - 1 \geq (1 - e^{-\delta t})(H_1 - 1) \tag{3.18}$$

for some $\delta > 0$. Combing with the uniform upper bound for λ, (3.18) implies

$$|h|^2 - nH_1^2 \leq Ce^{-\delta_0 t}$$

for some $\delta_0 > 0$. Then the exponential convergence follows in a standard way. For details see Theorem 3.5 in [40]. □

A direct consequence of Theorem 3.1 is the following Alexandrov type theorem for hypersurfaces in $\mathbb{H}^n$.

Corollary 3.1. *Let $0 \leq l < k \leq n-1$. Let $K \in \mathcal{K}$ be an h-convex bounded domain in $\mathbb{H}^n$ with smooth boundary satisfying that $H_k = cH_l$ for some constant $c \in \mathbb{R}$. Then K must be a geodesic ball.*

This result was proved under a weaker condition that ∂K is k-convex, by Korevaar [31] and later Koh [30] by using respectively the method of Alexandrov's reflection based on the maximum principle and the integral method based on Minkowski type formulae. In this paper, the form in Corollary 3.1 is enough for our application.

Proof of Corollary 3.1. We just let K evolve by (3.1) with $F = \left(\frac{H_k}{H_l}\right)^{\frac{1}{k-l}}$. Then the flow is actually stationary. The convergence of Theorem 3.1 implies that K must be a geodesic ball. □

4. Proof of Theorem 1.1-1.3

Before proving the main theorems, we define some auxiliary functions which will be used below.

First recall that, for $0 \le k \le n-1$,

$$f_k : [0,\infty) \to \mathbb{R}_+, \quad f_k(r) = W_k(B_r).$$

It is easy to see that f is smooth and it follows from (2.2) that f_k is strictly monotone increasing. Hence its inverse function f_k^{-1} exists and is also strictly monotone increasing.

For $2 \le k \le n-1$, define

$$g_k : [0,\infty) \to \mathbb{R}_+, \qquad g_k(s) = n f_k \circ f_{k-2}^{-1}(s) + \frac{n(k-1)}{n-k+2} s.$$

Thanks to the monotonicity of f_k, g_k is also strictly monotone increasing and its inverse function g_k^{-1} exists and is strictly monotone increasing. One can easily check from the strict monotonicity of f_k and g_k that

$$\frac{1}{n} s - \frac{k-1}{n-k+2} g_k^{-1}(s) \ge 0. \tag{4.1}$$

For $3 \le k \le n-1$, define

$$h_k : [0,\infty) \to \mathbb{R}_+, \qquad h_k(s) = g_{k+1}\left(\frac{1}{n} s - \frac{k-2}{n-k+3} g_{k-1}^{-1}(s)\right).$$

We claim that h_k is also strictly monotone increasing. Indeed, it is direct to compute that

$$h_k'(s) = g_{k+1}'\left(\frac{1}{n} s - \frac{k-2}{n-k+3} g_{k-1}^{-1}(s)\right) \cdot \left(\frac{1}{n} - \frac{k-2}{n-k+3} \frac{1}{g_{k-1}'(g_{k-1}^{-1}(s))}\right)$$

Since $g_{k+1}' > 0$ and

$$g_{k-1}' = n(f_{k-1} \circ f_{k-3})' + n\frac{k-2}{n-k+3} > n\frac{k-2}{n-k+3},$$

we have that $h_k' > 0$, namely h_k is strictly monotone increasing.

From the definition of g_k and h_k, one can easily see that

$$g_k(W_{k-2}(B_r)) = V_{n-k}(B_r), \quad h_k(V_{n-k+1}(B_r)) = V_{n-k-1}(B_r).$$

Now we start to prove the main theorems. We first prove Theorem 1.1 by using special forms of flow (3.1).

Proof of Theorem 1.1. Let $K = K_0 \in \mathcal{K}$.

To prove Theorem 1.1, we consider flow (3.1) starting from $\Sigma_0 = \partial K_0$ with

$$F = \left(\frac{H_k}{H_l}\right)^{\frac{1}{k-l}}, \quad c(t) = c_l(t) = \frac{\int_{\Sigma_t} H_k^{\frac{1}{k-l}} H_l^{1-\frac{1}{k-l}} d\mu_t}{\int_{\Sigma_t} H_l d\mu_t}.$$

Let $\Sigma_t, t \in [0,\infty)$ be the solution obtained in Theorem 3.1, which encloses K_t. One verifies from (3.5) that

(4.2)
$$\begin{aligned}\frac{d}{dt}W_k(K_t) &= \frac{n-k}{n}\int_{\Sigma_t} H_k\left(c(t)-F\right)\\ &= \frac{n-k}{n}\frac{1}{\int_{\Sigma_t}H_l}\left(\int_{\Sigma_t}H_k\int_{\Sigma_t}H_k^{\frac{1}{k-l}}H_l^{1-\frac{1}{k-l}}-\int_{\Sigma_t}H_l\int_{\Sigma_t}H_k^{1+\frac{1}{k-l}}H_l^{-\frac{1}{k-l}}\right)\end{aligned}$$

It follows from the Hölder inequality that

$$\int_{\Sigma_t}H_k \le \left(\int_{\Sigma_t}H_k^{1+\frac{1}{k-l}}H_l^{-\frac{1}{k-l}}\right)^{\frac{k-l}{k-l+1}}\left(\int_{\Sigma_t}H_l\right)^{\frac{1}{k-l+1}}, \tag{4.3}$$

$$\int_{\Sigma_t}H_k^{\frac{1}{k-l}}H_l^{1-\frac{1}{k-l}} \le \left(\int_{\Sigma_t}H_k^{1+\frac{1}{k-l}}H_l^{-\frac{1}{k-l}}\right)^{\frac{1}{k-l+1}}\left(\int_{\Sigma_t}H_l\right)^{\frac{k-l}{k-l+1}}. \tag{4.4}$$

Inserting (4.3) and (4.4) into (4.2), we have

$$\frac{d}{dt}W_k(K_t) \le 0. \tag{4.5}$$

Note that the flow preserves W_l. Theorem 3.1 says that the flow converges to some geodesic ball B_r with $W_l(B_r) = W_l(K_0) = W_l(K_t)$. Thus we have

$$W_k(K) \ge W_k(B_r), \quad \text{with } W_l(K) = W_l(B_r) \text{ for some } r > 0, \tag{4.6}$$

which is equivalent to

$$W_k(K) \ge f_k \circ f_l^{-1}(W_l(K)). \tag{4.7}$$

Equality in (4.7) holds iff equalities in (4.3) and (4.4) hold, iff $H_k = cH_l$ for some $c \in \mathbb{R}$, which means by Corollary 3.1 that K is a geodesic ball in $\mathbb{H}^n$. □

Proof of Theorem 1.2. Once we have Theorem 1.1 and especially have (4.7), it is easy to see from (2.5) that

$$\begin{aligned}V_{n-1-k}(K) &= n\left(W_{k+1}(K)+\frac{k}{n-k+1}W_{k-1}(K)\right)\\ &\ge \left(nf_{k+1}\circ f_{k-1}^{-1}+\frac{nk}{n-k+1}Id\right)(W_{k-1}(K))\\ &\ge \left(nf_{k+1}\circ f_l^{-1}+\frac{nk}{n-k+1}f_{k-1}\circ f_l^{-1}\right)(W_l(K)),\end{aligned} \tag{4.8}$$

where $Id : \mathbb{R} \to \mathbb{R}$ is the identity function. This leads to Statement (i) in Theorem 1.2.

Statements (ii) and (iii) in Theorem 1.2 are almost included in Statement (i) except that (a) the area V_{n-1} attains its minimum at a geodesic ball among the domains with given volume W_0, and (b) $\int_{\partial K} H_1 d\mu$ attains its minimum at a geodesic ball among the domains with given area of the boundary $|\partial K|$. However, (a) is just the classical isoperimetric inequality in $\mathbb{H}^n$ and (b) was proved in [20] by using results of Cheng-Zhou [12] and Li-Wei-Xiong [33], which was mentioned in the introduction.

We now prove Statement (iv) of Theorem 1.2. First we consider the simple case $k-l=2$. For $l=0, k=2$, the statement is included in Statement (iii). Hence we assume $k \geq 3$.

First of all, we see from (2.5) and (4.7) that

$$\begin{aligned} \int_{\partial K} H_{k-2} d\mu &= nW_{k-1}(K) + \frac{n(k-2)}{n-k+3} W_{k-3}(K) \\ &\geq \left(n f_{k-1} \circ f_{k-3}^{-1} + \frac{n(k-2)}{n-k+3} Id \right) (W_{k-3}(K)) \\ &= g_{k-1}(W_{k-3}(K)). \end{aligned} \tag{4.9}$$

It follows from (4.9) that

$$W_{k-3}(K) \leq g_{k-1}^{-1} \left(\int_{\partial K} H_{k-2} d\mu \right). \tag{4.10}$$

Next, we use (2.5) and (4.7) again on $\int_{\partial K} H_k d\mu$ to obtain that

$$\begin{aligned} \int_{\partial K} H_k d\mu &\geq g_{k+1}(W_{k-1}(K)) \\ &= g_{k+1} \left(\frac{1}{n} \int_{\partial K} H_{k-2} d\mu - \frac{k-2}{n-k+3} W_{k-3}(K) \right). \end{aligned} \tag{4.11}$$

We deduce from (4.10) that

$$\begin{aligned} & \frac{1}{n} \int_{\partial K} H_{k-2} d\mu - \frac{k-2}{n-k+3} W_{k-3}(K) \\ \geq\ & \frac{1}{n} \int_{\partial K} H_{k-2} d\mu - \frac{k-2}{n-k+3} g_{k-1}^{-1} \left(\int_{\partial K} H_{k-2} d\mu \right) \end{aligned} \tag{4.12}$$

In view of (4.1), both sides of (4.12) are nonnegative. Back to (4.11), using the monotonicity of g_{k+1}, we obtain that

$$\begin{aligned} \int_{\partial K} H_k d\mu &\geq g_{k+1} \left[\frac{1}{n} \int_{\partial K} H_{k-2} d\mu - \frac{k-2}{n-k+3} g_{k-1}^{-1} \left(\int_{\partial K} H_{k-2} d\mu \right) \right] \\ &= h_k \left(\int_{\partial K} H_{k-2} d\mu \right). \end{aligned} \tag{4.13}$$

For $k-l=2m$ for $m \in \mathbb{N}$, due to the monotonicity of h_k, we can inductively utilize (4.13) to deduce that

$$\int_{\partial K} H_k d\mu \geq h_k \circ h_{k-2} \circ \cdots \circ h_{l+2} \left(\int_{\partial K} H_l d\mu \right). \tag{4.14}$$

Notice that the inequalities we have used previously are all optimal in the sense that equalities hold iff K is a geodesic ball. Hence we conclude Statement (iv) in Theorem 1.2.

We complete the proof of Theorem 1.2. □

Proof of Theorem 1.3: it is sufficient to explicitly write out formula (4.8) for $l=1$ and $1 \leq k \leq n-1$. A direct calculation yields that

$$f_1(r) = \frac{1}{n} |\partial B_r| = \frac{1}{n} \omega_{n-1} \sinh^{n-1}(r).$$

Thus

$$f_1^{-1}(s) = \sinh^{-1}\left[\left(\frac{ns}{\omega_{n-1}}\right)^{\frac{1}{n-1}}\right].$$

Since $H_k(B_r) = \coth^k(r)$, it follows from (2.6) and (2.7) that if k is odd,

$$f_k(r) = \frac{1}{n}\sum_{i=0}^{\frac{k-1}{2}}(-1)^i\frac{(k-1)!!(n-k)!!}{(k-1-2i)!!(n-k+2i)!!}\omega_{n-1}\coth^{k-1-2i}(r)\sinh^{n-1}(r),$$

while if k is even,

$$\begin{aligned} f_k(r) &= \frac{1}{n}\sum_{i=0}^{\frac{k}{2}-1}(-1)^i\frac{(k-1)!!(n-k)!!}{(k-1-2i)!!(n-k+2i)!!}\omega_{n-1}\coth^{k-1-2i}(r)\sinh^{n-1}(r) \\ &\quad +(-1)^{\frac{k}{2}}\frac{(k-1)!!(n-k)!!}{n!!}\int_0^r \omega_{n-1}\sinh^{n-1}(t)dt. \end{aligned}$$

Hence, for k odd,

$$f_k\circ f_1^{-1}(s) = \frac{1}{n}\sum_{i=0}^{\frac{k-1}{2}}(-1)^i\frac{(k-1)!!(n-k)!!}{(k-1-2i)!!(n-k+2i)!!}\omega_{n-1}\frac{ns}{\omega_{n-1}}\left[1+\left(\frac{ns}{\omega_{n-1}}\right)^{\frac{-2}{n-1}}\right]^{\frac{k-1}{2}-i},$$

and for k even,

$$\begin{aligned} f_k\circ f_1^{-1}(s) &= \frac{1}{n}\sum_{i=0}^{\frac{k}{2}-1}(-1)^i\frac{(k-1)!!(n-k)!!}{(k-1-2i)!!(n-k+2i)!!}\omega_{n-1}\frac{ns}{\omega_{n-1}}\left[1+\left(\frac{ns}{\omega_{n-1}}\right)^{\frac{-2}{n-1}}\right]^{\frac{k-1-2i}{2}} \\ &\quad +(-1)^{\frac{k}{2}}\frac{(k-1)!!(n-k)!!}{n!!}\int_0^{\sinh^{-1}\left[\left(\frac{ns}{\omega_{n-1}}\right)^{\frac{1}{n-1}}\right]}\omega_{n-1}\sinh^{n-1}(t)dt. \end{aligned}$$

Recall that the case $k=1$ was already proved in [20]. From the previous two formulas we can easily compute that for $k\geq 2$,

$$\begin{aligned} nf_{k+1}\circ f_1^{-1}(s) + \frac{nk}{n-k+1}f_{k-1}\circ f_1^{-1}(s) &= \omega_{n-1}\frac{ns}{\omega_{n-1}}\left[1+\left(\frac{ns}{\omega_{n-1}}\right)^{\frac{-2}{n-1}}\right]^{\frac{k}{2}} \\ &= \omega_{n-1}\left[\left(\frac{ns}{\omega_{n-1}}\right)^{\frac{2}{k}}+\left(\frac{ns}{\omega_{n-1}}\right)^{\frac{2}{k}\frac{(n-k-1)}{n-1}}\right]^{\frac{k}{2}}. \end{aligned} \tag{4.15}$$

Letting $s = W_1(K) = \frac{1}{n}|\partial K|$ in (4.15), we obtain from (4.8) inequality (1.6). □

From the proof, one can see again the difference between the even case and the odd case. However, an interesting cancellation gives the uniform inequality (1.6).

Acknowledgment. We learned from Pengfei Guan that he and Junfang Li obtained the same type results with the same functionals and a different flow, a modified inverse curvature flow, under weaker conditions that K is a k-convex star-shaped domain and satisfies a

technical condition that $|\bar{\nabla}\log\cosh r| \leq C(n)$, where $C(n) \geq 4$ is a dimensional constant. This technical condition is believed removable. We would like to thank Yuxin Ge, Pengfei Guan and Jie Wu for helpful discussions. We also thank the anonymous referees for their valuable comments and suggestions.

References

[1] B. Andrews, Pinching estimates and motion of hypersurfaces by curvature functions, J. reine angew. Math. 608 (2007) 17–31.

[2] B. Andrews, Aleksandrov-Fenchel inequalities and curvature flows, Centre for Mathematics and its Applications, School of Mathematical Sciences, (1993).

[3] B. Bian, P. Guan, A microscopic convexity principle for nonlinear partial differential equations. Invent. Math. 177 (2009), no. 2, 307–335.

[4] S. Brendle, P.-K. Hung, M. -T. Wang, A Minkowski-type inequality for hypersurfaces in the Anti-deSitter-Schwarzschild manifold, **arXiv:1209.0669**.

[5] A. Borisenko, V. Miquel, Total curvatures of convex hypersurfaces in hyperbolic space. Illinois J. Math. 43 (1999), no. 1, 61–78.

[6] A. Borisenko, V. Miquel, Comparison theorems on convex hypersurfaces in Hadamard manifolds. Ann. Global Anal. Geom. 21 (2002), no. 2, 191–202.

[7] Y.D. Burago, V.A. Zalgaller, Geometric Inequalities, Springer, Berlin, (1988).

[8] E. Cabezas-Rivas, V. Miquel, Volume preserving mean curvature flow in the hyperbolic space. Indiana Univ. Math. J. 56 (2007), no. 5, 2061–2086.

[9] L. Caffarelli, P. Guan, X.-N. Ma, A constant rank theorem for solutions of fully nonlinear elliptic equations. Comm. Pure Appl. Math. 60 (2007), no. 12, 1769–1791.

[10] L. A. Caffarelli, A. Friedman, Convexity of solutions of semilinear elliptic equations. Duke Math. J. 52 (1985), no. 2, 431–456.

[11] S.-Y. A. Chang, Y.Wang, On Aleksandrov-Fenchel inequalities for k-convex domains, Milan J. Math., 79 (2011), no. 1, 13–38.

[12] X. Cheng, D. Zhou, Rigidity for closed totally umbilical hypersurfaces in space forms, J. Geom. Anal. (2012) DOI: 10.1007/s12220-012-9375-4.

[13] C. De Lellis, P. Topping, Almost-Schur lemma, Calc. Var. Partial Differential Equations 43 (2012), no. 3-4, 347–354.

[14] L. de Lima, F. Girao An Alexandrov-Fenchel-type inequality in hyperbolic space with an application to a Penrose inequality, **arXiv:1209.0438**.

[15] E. Gallego, A. M. Naveira, G. Solanes, Horospheres and convex bodies in n-dimensional hyperbolic space. Geom. Dedicata 103 (2004), 103–114.

[16] E. Gallego , G. Solanes, Integral geometry and geometric inequalities in hyperbolic space. Diff. Geom. Appl. 22 (2005), no. 3, 315–325.

[17] F. Gao, D. Hug, R. Schneider, Intrinsic volumes and polar sets in spherical space. Homage to Luis Santaló. Vol. 1 (Spanish). Math. Notae 41 (2001/02), 159–176 (2003).

[18] Y. Ge, G. Wang, An almost Schur theorem on 4-dimensional manifolds. Proc. Amer. Math. Soc. 140 (2012), no. 3, 1041–1044.

[19] Y. Ge, G. Wang, J. Wu, Hyperbolic Alexandrov-Fenchel inequalities I, **arXiv:1303.1714**.

[20] Y. Ge, G. Wang, J. Wu, Hyperbolic Alexandrov-Fenchel inequalities II, **arXiv:1304.1417**.

[21] Y. Ge, G. Wang, J. Wu, The GBC mass for asymptotically hyperbolic manifolds, **arXiv:1306.4233**.

[22] Y. Ge, G. Wang, C. Xia, On problems related to an inequality of Andrews, De Lellis, and Topping, Int Math Res Notices 2013 (2013) no. 20, 4798–4818.

[23] C. Gerhardt, Inverse curvature flows in hyperbolic space, J. Diff. Geom. 89 (2011), no. 3, 487–527.

[24] P. Guan, Topics in Geometric Fully Nonlinear Equations, available at "http://www.math.mcgill.ca/guan/notes.html".

[25] P. Guan, J. Li, The quermassintegral inequalities for k-convex starshaped domains. Adv. Math. 221 (2009), no. 5, 1725–1732.

[26] P. Guan, J. Li, A mean curvature type flow in space forms, **arXiv:1309.5099**.

[27] P. Guan, X.-N. Ma, The Christoffel-Minkowski problem. I. Convexity of solutions of a Hessian equation. Invent. Math. 151 (2003), no. 3, 553–577.
[28] P. Guan, G. Wang, Geometric inequalities on locally conformally flat manifolds. Duke Math. J. 124 (2004), no. 1, 177–212.
[29] G.Huisken, in preparation. See also [25].
[30] S.-E. Koh, Sphere theorem by means of the ratio of mean curvature functions. Glasg. Math. J. 42 (2000), no. 1, 91–95.
[31] N. Korevaar, Sphere theorems via Alexandrov for constant Weingarten curvature hypersurfaces. Appendix to a note of A. Ros. J. Diff. Geom., 27 (1988), no. 2, 221–223.
[32] N. Korevaar, J. L. Lewis, Convex solutions of certain elliptic equations have constant rank Hessians. Arch. Rational Mech. Anal. 97 (1987), no. 1, 19–32.
[33] H. Li, Y. Wei, C. Xiong, A geometric inequality on hypersurface in hyperbolic space, **arXiv:1211.4109**.
[34] M. Makowski, Mixed volume preserving curvature flows in hyperbolic space, **arXiv:1208.1898**.
[35] J. A. McCoy, Mixed volume preserving curvature flows. Calc. Var. Partial Differential Equations 24 (2005), no. 2, 131–154.
[36] R. C. Reilly, On the Hessian of a function and the curvatures of its graph. Michigan Math. J. 20 (1973), 373–383.
[37] L. A. Santalo, Integral geometry and geometric probability. Second edition. With a foreword by Mark Kac. Cambridge Mathematical Library. Cambridge University Press, Cambridge, (2004).
[38] E. Schmidt, Die isoperimetrischen Ungleichungen auf der gewöhnlichen Kugel und für Rotationskörper im n-dimensionalen sphärischen Raum. (German) Math. Z. 46, (1940), 743–794.
[39] R. Schneider, Convex bodies: The Brunn-Minkowski theory, Cambridge University, (1993).
[40] F. Schulze, Nonlinear evolution by mean curvature and isoperimetric inequalities, J. Differential Geom. 79 (2008), no. 2, 197–241.
[41] G. Solanes, Integral geometry and the Gauss-Bonnet theorem in constant curvature spaces. Trans. Amer. Math. Soc. 358 (2006), no. 3, 1105–1115 .
[42] G. Solanes, Integrals de curvatura i geometria integral a l'espai hiperbolic, Univ. Aut. Barcelona, PhD Thesis, (2003). English version available at "http://mat.uab.cat/ egallego/gil/tesiEng.pdf".

Albert-Ludwigs-Universität Freiburg, Mathematisches Institut, Eckerstr. 1, 79104 Freiburg, Germany

E-mail address: guofang.wang@math.uni-freiburg.de

Max-Planck-Institut für Mathematik in den Naturwissenschaft, Inselstr. 22, D-04103, Leipzig, Germany

E-mail address: chao.xia@mis.mpg.de