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LIOUVILLE THEOREMS FOR f-HARMONIC MAPS INTO HADAMARD SPACES

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Abstract. In this paper, we study harmonic functions on weighted manifolds and harmonic maps from weighted manifolds into Hadamard spaces introduced by Korevaar and Schoen. We prove several Liouville type theorems for these harmonic maps.

1. Introduction

Weighted Riemannian manifolds, also called manifolds with density or smooth metric measure spaces in the literature, are Riemannian manifolds equipped with weighted measures. Appearing naturally in the study of self-shrinkers, Ricci solitons, harmonic heat flows and many others, the weighted manifolds are proved to be nontrivial generalizations of Riemannian manifolds. There are many geometric investigations of weighted manifolds, see for instance Morgan [33] and Wei-Wylie [47]. In this paper, we investigate various Liouville type theorems for harmonic functions on weighted manifolds as well as harmonic maps from weighted manifolds into Hadamard spaces, i.e. global nonpositively curved spaces in the sense of Alexandrov (also called CAT(0) spaces), see [17] and [2].

A weighted Riemannian manifold is a triple $(M, g, e^{-f}dV_g)$, where (M, g) is an n-dimensional Riemannian manifold, dV_g is the Riemannian volume element and f is a smooth positive function on M. The f-Laplacian

$$\Delta_f = \Delta - \nabla f \cdot \nabla$$

is a natural generalization of Laplace-Beltrami operator as it is self-adjoint with respect to the weighted measure $e^{-f}dV_g$, i.e.

$$\int_M \Delta_f uve^{-f}dV_g = \int_M u\Delta_f ve^{-f}dV_g \text{ for } u, v \in C_0^\infty(M).$$

A function $u \in W^{1,2}_{\mathrm{loc}}(M)$ is called f-harmonic (f-subharmonic, f-superharmonic) if it satisfies $\Delta_f u = 0(\geq 0, \leq 0)$ in the weak sense, i.e.

$$\int_M \langle \nabla u, \nabla \varphi \rangle e^{-f}dV_g = 0(\leq 0, \geq 0) \text{ for any } 0 \leq \varphi \in C_0^\infty(M).$$

The Dirichlet f-energy of u is defined by $D^f(u) = \int_M |\nabla u|^2 e^{-f}dV_g$.

On the other hand, f-harmonic maps from weighted manifolds $(M, g, e^{-f}dV_g)$ to (singular) metric spaces (Y, d) have wide geometric applications. Harmonic maps into metric spaces were initiated by Gromov-Schoen [14] and then investigated independently by Korevaar-Schoen [20] and Jost [15]. In particular, when the domain

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is a Riemannian manifold, Korevaar-Schoen gave a complete exposition in [20, 21]. In this paper we call a map $u : M \to Y$ f-harmonic if u locally minimizes the f-energy functional E^f in the sense of Korevaar-Schoen. For a detailed definition and its properties, we refer to [20] or Section 4 below. For the special case, f-harmonic maps from the Gaussian spaces, $(\mathbb{R}^n, |\cdot|, e^{-|x|^2/4}dx)$, to Riemannian manifolds are called quasi-harmonic spheres which emerge in the blow-up analysis of harmonic heat flow [31, 23]. In this paper, we study Liouville theorems for f-harmonic maps into metric spaces, which generalize the previous results for harmonic maps in both aspects of domain manifolds and target spaces.

Analysis on weighted manifolds and corresponding f-Laplacian have been extensively studied recently. We refer to [34, 35, 3, 29, 30] for the f-harmonic functions on weighted manifolds, to [24, 49, 26, 25] for f-harmonic functions on the Gaussian spaces, to [12, 13] for heat kernel estimates, and to [31, 46, 4, 37, 41, 42] for f-harmonic maps.

In the first part of this paper we concern with Liouville theorems for f-harmonic functions on weighted manifolds. There are various Liouville theorems for f-harmonic functions on the Gaussian spaces, also called quasi-harmonic functions [49, 24]. The main techniques there are gradient estimates and separation of variables coupled with ODE results. In this paper, we propose another approach, which seems to be overlooked in the literature, to reprove almost all previous results on Liouville theorems for quasi-harmonic functions. Since this method is adapted to the more general setting, we obtain Liouville theorems for f-harmonic functions on general weighted manifolds.

Our observation is that the weighted version of L^p Liouville theorem for weighted manifolds can be used to derive various Liouville theorems concerning the growth of f-harmonic functions. Yau [48] first proved the L^p-Liouville theorem $(1 < p < \infty)$ for harmonic functions on any Riemannian manifold. Later, Karp [18] obtained a quantitative version of this result. Li-Schoen [27] proved other L^p-Liouville theorems (e.g. $0 < p < 1$) under the curvature assumption of manifolds. Karp's version of L^p-Liouville theorem has been generalized by Sturm [43] to the setting of strongly local regular Dirichlet forms. In particular, our f-harmonic functions lie in this setting. By applying Sturm's L^p-Liouville theorem to f-harmonic functions, we immediately obtain several consequences which generalize previous results of [49, 24, 26, 25]. Although the proof of L^p-Liouville theorem is quite general which only involves the integration by parts and the Caccioppoli inequality (thus it holds for all reasonable spaces), it is surprisingly powerful to obtain various Liouville theorems for f-harmonic functions for weighted manifolds with slow volume growth, especially for the Gaussian spaces, see Corollary 2.2 and 2.3 in Section 2. This does provide another approach to derive Liouville theorems without using gradient estimates.

The second part of this paper is devoted to the study of Liouville type theorems for harmonic maps from weighted manifolds to Hadamard spaces. For applications of f-harmonic maps with singular targets we refer to Gromov-Schoen [14]. Our first result is an analogue to Kendall's theorem [19, Theorem 3.2]. The essence of Kendall's theorem is that validity of Liouville theorem for f-harmonic maps into Hadamard spaces, a priori a nonlinear problem, is reduced to that of Liouville theorem of f-harmonic functions, a linear problem. Kendall [19] proved this theorem for harmonic maps between Riemannian manifolds, by using probabilistic methods and potential theory. Kuwae-Sturm [22] generalized Kendall's method to a class of harmonic maps between general metric spaces in the framework of Markov processes. However harmonic maps they considered are different from that of Korevaar-Schoen when targets are singular metric spaces. For harmonic maps into Hadamard spaces in the sense of Korevaar-Schoen, we adopt a geometric proof due to Li-Wang [28], in which the local compactness of the targets is crucial, to show the following Kendall-type theorem. Recall that a geodesic space (Y, d) is called locally compact if every closed geodesic ball is compact.

Theorem 1.1. *Let $(M, g, e^{-f}dV_g)$ be a complete weighted Riemannian manifold admitting no nonconstant bounded f-harmonic function and (Y, d) be a locally compact Hadamard space. Then any f-harmonic map from M to Y having bounded image must be constant.*

In the same spirit of Kendall's theorem, Cheng-Tam-Wan [5] proved a Liouville-type theorem for harmonic maps with finite energy. Our second result is a generalization of Cheng-Tam-Wan's theorem to f-harmonic maps into Hadamard spaces.

Theorem 1.2. *Let $(M, g, e^{-f}dV_g)$ be a complete noncompact weighted Riemannian manifold satisfying that any f-harmonic function with finite Dirichlet f-energy is bounded. Let (Y, d) be an Hadamard space. Then any f-harmonic map from M to Y with finite f-energy must have bounded image.*

We will follow the line of Cheng-Tam-Wan's reasoning, but using the techniques in potential theory, especially the theory of Royden-Nakai decomposition and classification of Riemannian manifolds [38, 36, 39]. This possible approach of potential theory was implicitly suggested by Lyons in [5, pp. 278]. We figure out the detailed arguments of this insight and apply them to Liouville theorems of f-harmonic maps. Royden-Nakai decomposition theorem and Virtanen theorem play important roles in the classification theory of Riemannian manifolds developed by Royden, Nakai, Sario etc. more than half a century ago. We shall dwell on these theories in the framework of weighted manifolds in Section 5 and adopt them to prove Theorem 1.2.

The following theorem is, more or less, a consequence of the combination of Theorem 1.1 and 1.2. This theorem has important applications.

Theorem 1.3. *Let $(M, g, e^{-f}dV_g)$ be a complete noncompact weighted Riemannian manifold admitting no nonconstant bounded f-harmonic functions and (Y, d) be a locally compact Hadamard space. Then any f-harmonic map from M to Y with finite f-energy must be constant.*

Bakry-Émery [1] introduced weighted Ricci curvature for weighted manifolds. Particularly, the ∞-Bakry-Émery Ricci curvature

$$Ric_f := Ric + \nabla^2 f$$

turns out to be a suitable and important curvature quantity for weighted manifolds. The nonnegativity of Ric_f corresponds to the curvature-dimension condition

$CD(0,\infty)$ on metric measure spaces via optimal transport, in the sense of Lott-Villani [32] and Sturm [44, 45].

By a theorem of Brighton [3], see also Li [30], the weighted manifold $(M, g, e^{-f}dV_g)$ satisfying $Ric_f \geq 0$ admits no nonconstant bounded f-harmonic functions. Hence by Theorem 1.3 we immediately have

Theorem 1.4. *Let $(M, g, e^{-f}dV_g)$ be a complete noncompact weighted Riemannian manifold satisfying $Ric_f \geq 0$ and (Y, d) be a locally compact Hadamard space. Then any f-harmonic map from M to Y with finite f-energy must be constant.*

When the target is a Riemannian manifold of nonpositive curvature, Theorem 1.4 was already proved by Wang-Xu [46] and Rimoldi-Veronelli [37] with additional assumption $\int_M e^{-f}dV_g = \infty$ (in [37] some other assumptions were also considered). We remark that unlike the unweighted case, $Ric_f \geq 0$ and noncompactness do not imply $\int_M e^{-f}dV_g = \infty$ in general. We will give a short proof in Section 3 for the case $\int_M e^{-f}dV_g < \infty$ when the target is further assumed to be a Cartan-Hadamard manifold. Note that one cannot drop both the assumptions that $\int_M e^{-f}dV_g = \infty$ and the target is Cartan-Hadamard simultaneously in view of a counterexample constructed in [37].

For harmonic maps into singular Hadamard spaces, the arguments in [46] and [37], both following Schoen-Yau [40], do not work any more since we cannot apply Bochner techniques as in [46] and [37] due to the singularity of targets. Although a weak Bochner formula can be derived as in [20], it is insufficient for these arguments. Fortunately, we can circumvent these technical problems by proving Theorem 1.3 and applying the Royden-Nakai theory. This does provide another approach to Liouville theorems for f-harmonic maps without using Bochner techniques. This is one of the main points of the paper.

The rest of the paper is organized as follows. In Section 2, we study L^p Liouville theorem for f-harmonic functions and give some applications. In Section 3, we prove Theorem 1.3 in the case that Y is a Cartan-Hadamard manifold. In Section 4, we define f-harmonic maps from weighted Riemannian manifolds into Hadamard spaces and prove Theorem 1.1. In Section 5, we dwell on the Royden-Nakai theory and prove Theorem 1.2 and 1.3.

2. f-harmonic functions

In this section, we collect a well-known L^p Liouville theorem for f-harmonic functions and its applications. We will show that L^p Liouville theorem is quite powerful for weighted manifolds with finite volume.

The L^p Liouville theorem, $1 < p < \infty$, for harmonic functions (or nonnegative subharmonic functions) was initiated by Yau [48] on complete Riemannian manifolds. Karp [18] obtained a quantitative version of this Liouville theorem. Later, Sturm [43] proved L^p Liouville theorem for strongly local regular Dirichlet forms. The following theorem is a special case of Sturm's result for f-harmonic functions. We denote by $B_r := B_r(x_0)$ the closed geodesic ball of radius r centered at a fixed point $x_0 \in M$.

Theorem 2.1 ([43], Theorem 1)**.** *Let $(M, g, e^{-f}dV_g)$ be a complete weighted Riemannian manifold and u be a nonnegative f-subharmonic function (or an f-harmonic* *function). For $1 < p < \infty$, set $v(r) := \int_{B_r} |u|^p e^{-f} dV_g$. Then either*

$$\inf_{a>0} \int_a^\infty \frac{r}{v(r)} dr < \infty,$$

or u is a constant.

We state several consequences of Theorem 2.1.

A quite useful consequence is about f-parabolicity of M. Recall that a weighted manifold $(M, g, e^{-f} dV_g)$ is called *f-parabolic* if there is no nonconstant nonnegative f-superharmonic functions on M. For a compact set $K \subset M$, the *f-capacity* of K is defined as

$$\mathrm{cap}^f(K) := \inf_{\substack{\phi \in \mathrm{Lip}_0(M) \\ \phi|_K = 1}} \int_M |\nabla \phi|^2 e^{-f} dV_g,$$

where $\mathrm{Lip}_0(M)$ is the space of compactly supported Lipschitz functions on M.

Proposition 2.1 (f-parabolicity)**.** *Let $(M, g, e^{-f} dV_g)$ be a complete weighted manifold. Then the following are equivalent:*

(i) M is f-parabolic;
(ii) $\mathrm{cap}^f(K) = 0$ for some (then any) compact set $K \subset M$;
(iii) any bounded f-superharmonic function on M is constant.

Proof. $(i) \Leftrightarrow (ii)$. This follows from [9, Proposition 3], see [11, Proposition 2.1].

$(i) \Leftrightarrow (iii)$. This follows from the fact that any nonnegative f-superharmonic function u can be approximated by bounded f-superharmonic functions $u_n = \min\{u, n\}$, $n \in \mathbb{N}$. □

We say a weighted manifold $(M, g, e^{-f} dV_g)$ has the *moderate volume growth property* if

$$\int_1^\infty \frac{r}{V_f(B_r)} dr = \infty, \tag{1}$$

where $V_f(B_r) := \int_{B_r} e^{-f} dV_g$.

Corollary 2.1. *Let $(M, g, e^{-f} dV_g)$ be a complete weighted Riemannian manifold satisfying the moderate volume growth property. Then M is f-parabolic.*

Proof. Let u be a bounded f-superharmonic function on M. Then for any $a > 0$,

$$\int_a^\infty \frac{r}{v(r)} dr \geq C \int_a^\infty \frac{r}{V_f(B_r)} dr = \infty.$$

Theorem 2.1 yields that u is a constant. This proves the corollary. □

Remark 2.1. Corollary 2.1 slightly generalizes [46, Theorem 1.4]. In particular, this corollary implies [49, Theorem 2].

We can also derive several Liouville type theorems for f-harmonic functions from Theorem 2.1.

Corollary 2.2. *Let $(M, g, e^{-f} dV_g)$ be a complete weighted Riemannian manifold and u be a nonnegative f-subharmonic function (or f-harmonic function). Assume one of the following holds:*

(i) $u = O(w^\alpha)$ for some nonnegative function w satisfying $\int_M w d^{-2}(\cdot, x_0) e^{-f} dV_g < \infty$ and some $\alpha \in (0, 1)$;

(ii) $\int_M d^k(\cdot, x_0)e^{-f}dV_g < \infty$ *for some* $k > -2$ *and* $u = O(d^\beta(\cdot, x_0))$ *for some* $\beta \in (0, k+2)$*;*

(iii) $\int_M e^{-f}dV_g < \infty$ *and* $u = O(d^\beta(\cdot, x_0))$ *for* $\beta \in (0, 2)$*;*

(iv) $f \geq Cd(\cdot, x_0)^\gamma$ *for some* $C > 0, \gamma > 0$ *and* $\int_M e^{-\delta f}dV_g < \infty$ *for some* $0 < \delta < 1$ *and* u *has polynomial growth;*

(v) $f \geq Cd(\cdot, x_0)^\gamma$ *for some* $C > 0, \gamma > 0$ *and* dV_g *has polynomial volume growth and* $u = O(e^{\alpha C d(\cdot, x_0)^\gamma})$*,* $\alpha \in (0, 1)$*.*

Then u *is a constant.*

Proof. For (i), we see that there exists $p \in (1, \infty)$ such that $|u|^p = O(w)$. Hence

$$\begin{aligned} \frac{1}{r^2 \log r} v(r) &= \frac{1}{r^2 \log r}\int_{B_r} |u|^p e^{-f} dV_g \\ &\leq \frac{C}{\log r}\int_{B_r} \frac{w(x)}{d^2(x, x_0)} e^{-f(x)} dV_g(x) = o(1). \end{aligned}$$

It follows from Theorem 2.1 that u is a constant. (ii) follows from (i) by letting $w = d^{k+2}(\cdot, x_0)$. (iii) follows from (ii) by letting $k = 0$.

For (iv), let us observe for any $1 < p < \infty$,

$$\int_M |u|^p e^{-f} dV_g \leq C \int_M d^{sp}(x, x_0) e^{-f(x)} dV_g(x) \leq C \int_M e^{-\delta f} dV_g < \infty,$$

where $s > 0$. Then the statement also follows from Theorem 2.1. (v) can be proved in a similar way. □

The following result is a direct corollary of the above (v).

Corollary 2.3. *Let* u *be an* f*-harmonic function on the Gaussian space, i.e.,*

$$\Delta u - \frac{1}{2}\langle x, \nabla u\rangle = 0.$$

If $u = O(e^{\alpha \frac{|x|^2}{4}})$ *as* $x \to \infty$*, for some* $0 < \alpha < 1$*, then* u *is a constant.*

Remark 2.2. Corollary 2.3 implies that there is no nonconstant polynomial growth f-harmonic functions on the Gaussian space. This improves the result in [24, Theorem 4.2]. By Caccioppoli's inequality, Corollary 2.3 can be also derived from Li-Yang [25, Corollary 1.2] .

In the remaining part of this section, we study the L^p Liouville theorem introduced by Zhu-Wang [49] using a different measure from ours. We shall explain why the critical exponent of their L^p Liouville theorem in [49, Theorem 3] is $p = \frac{n}{n-2}$ $(n \geq 3)$ by applying our result. Let $(M, g, e^{-f}dV_g)$ be an n-dimensional $(n \geq 3)$ complete weighted manifold. In fact, they consider the L^p space with respect to the Riemannian volume in a modified Riemannian manifold $\widetilde{M} = (M, \tilde{g}, dV_{\tilde{g}})$ where $\tilde{g} = e^{-\frac{2f}{n-2}}g$, i.e. $L^p(\widetilde{M}, dV_{\tilde{g}})$. Since this new manifold $\widetilde{M}$ may be incomplete, e.g. for the Gaussian space, Yau's L^p Liouville theorem fails in this setting. In the following, we use the L^p Liouville theorem on weighted manifolds to show the one on the modified Riemannian manifolds.

Theorem 2.2. *Let* $(M, g, e^{-f}dV_g)$ *be an* n*-dimensional* $(n \geq 3)$ *complete weighted manifold,* $\widetilde{M} = (M, \tilde{g}, dV_{\tilde{g}})$ *be the modified Riemannian manifold and* u *be a nonnegative* f*-subharmonic function (or* f*-harmonic function) on* M*. For any* $p > \frac{n}{n-2}$*,* *there exists a constant* $\delta = \delta(p,n) \in (0,1)$ *such that if* $\int_M e^{-\delta f} dV_g < \infty$ *and* $u \in L^p(\widetilde{M}, dV_{\tilde{g}})$*, then* u *is a constant.*

Proof. For any $p > \frac{n}{n-2}$, let $q = \frac{2p}{p+\frac{n}{n-2}} > 1$, $\alpha = \frac{p}{q} > \frac{n}{n-2}$ and $\alpha^* = \frac{\alpha}{\alpha-1} \in (1, \frac{n}{2})$. Set $\delta = \frac{n-2\alpha^*}{n-2} \in (0,1)$. By Hölder inequality, we can verify that

$$\begin{aligned}\int_M u^q e^{-f} dV_g = \int_M u^q e^{\frac{2f}{n-2}} dV_{\tilde{g}} &\leq \left(\int_M u^{q\alpha} dV_{\tilde{g}}\right)^{\frac{1}{\alpha}} \left(\int_M e^{\frac{2\alpha^* f}{n-2}} dV_{\tilde{g}}\right)^{\frac{1}{\alpha^*}} \\ &= \left(\int_M u^p dV_{\tilde{g}}\right)^{\frac{1}{\alpha}} \left(\int_M e^{-\delta f} dV_g\right)^{\frac{1}{\alpha^*}} < \infty.\end{aligned}$$

The statement follows from Theorem 2.1. □

This yields a direct corollary which generalizes [49, Theorem 3], which is restricted to the Gaussian spaces, to general weighted manifolds. Let $A_g(r)$ denote the area induced by the Riemannian metric g of the geodesic sphere $S_r(x_0)$ centered at x_0 of radius r. The Riemannian manifold (M, g, dV_g) is called of subexponential volume growth if $V_g(r) = \exp\{o(d(\cdot, x_0))\}$ for some (then all) $x_0 \in M$.

Corollary 2.4. *Let* $(M, g, e^{-f} dV_g)$ *be an* n*-dimensional* $(n \geq 3)$ *complete weighted manifold satisfying that* $f \geq C d^\gamma(\cdot, x_0)$ *for some* $C > 0, \gamma > 0$ *and* $A_g(r) = \exp\{o(d^\gamma(\cdot, x_0))\}$*. Let* $\widetilde{M} = (M, \tilde{g}, dV_{\tilde{g}})$ *be the modified Riemannian manifold. Then for any* $p > \frac{n}{n-2}$*, the* f*-harmonic function in* $L^p(\widetilde{M}, dV_{\tilde{g}})$ *is constant. In particular, for* $\gamma = 1$*, it suffices to assume* (M, g, dV_g) *has subexponential volume growth.*

3. f-HARMONIC MAPS INTO CARTAN-HADAMARD MANIFOLDS

In this section, we prove Theorem 1.4 in the case that the target $Y = N$ is a Cartan-Hadamard manifold. Recall that a Cartan-Hadamard manifold is a simply connected Riemannian manifold of nonpositive (sectional) curvature.

Theorem 3.1. *Let* $(M, g, e^{-f} dV_g)$ *be a complete weighted Riemannian manifold which is* f*-parabolic and* N *be a Cartan-Hadamard manifold. Then any* f*-harmonic map with finite* f*-energy, i.e.* $E^f(u) := \int_M |\nabla u|^2 e^{-f} dV_g < \infty$*, is a constant map.*

Proof. We use a construction by Rimoldi-Veronelli [37] which associates an f-harmonic map with a (unweighted) harmonic map on some higher dimensional warped product manifold.

Precisely, let $\bar{M} := M \times_{e^{-f}} \mathbb{S}^1$ denote a warped product, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with $Vol(\mathbb{S}^1) = 1$, with the metric on $\bar{M}$ given by $\bar{g}(x,t) = g(x) + e^{-2f(x)} dt^2$. Note that $\bar{M}$ is complete. It follows from [37], Proposition 2.5 and Lemma 2.6 that $\bar{M}$ is parabolic and the map $\bar{u} : \bar{M} \to N$, defined by $\bar{u}(x,t) = u(x)$ is a harmonic map. Moreover, $E_{\bar{M}}(\bar{u}) = E^f_M(u) < \infty$.

Now by applying [5, Proposition 2.1 and Theorem 3.1] to $\bar{u}$ and $\bar{M}$, we know that the image of $\bar{u}$, $\bar{u}(\bar{M}) = u(M)$, is bounded in N. Since N is a Cartan-Hadamard manifold, $d^2(\bar{u}(\cdot), Q)$ is a subharmonic function for any $Q \in N$, which is also bounded. By the parabolicity of $\bar{M}$, we know that $d^2(\bar{u}(\cdot), Q)$ is constant for any $Q \in N$. This proves the theorem. □

Theorem 3.2. *Let $(M, g, e^{-f}dV_g)$ be a complete weighted Riemannian manifold satisfying $Ric_f \geq 0$ and N be a Cartan-Hadamard manifold. Then any f-harmonic map with finite f-energy $E^f(u) < \infty$ is a constant map.*

Proof. We divide the theorem into two cases: (a) $\int_M e^{-f}dV_g = \infty$, (b) $\int_M e^{-f}dV_g < \infty$. For the case (a), it was already proved in [46, Theorem 1.2] or [37, Theorem 3.3] for general Riemannian target of nonpositive curvature (without the assumption of simple connectedness). For the case (b), we observe that M satisfies the moderate volume growth property (1). By Corollary 2.1, M is f-parabolic. Then the statement follows from Theorem 3.1. Hence we finish the proof. □

Remark 3.1. Comparing Theorem 3.2 with [46, Theorem 1.2] or [37, Theorem 3.3], we removed the condition of infinity f-volume for M but added the assumption that N is simply connected. In view of a counterexample constructed by Rimoldi-Veronelli [37, Remark 3.7], we see that the simple connectedness of N in Theorem 3.2 cannot be removed.

4. f-harmonic maps into Hadamard spaces

In this section, we define f-harmonic maps from an n-dimensional complete weighted Riemannian manifold $(M, g, e^{-f}dV_g)$ to a general metric space (Y, d). For that purpose we investigate an f-energy functional E^f whose definition given here follows Korevaar- Schoen [20]. In [20], the authors developed a Sobolev space theory for maps from a Riemannian domain to a metric space (Y, d). And the 2-Sobolev energy functional was further extended to maps from a complete (noncompact) Riemannian manifold (and even more generally, the so-called admissible Riemannian polyhedron with simplexwise smooth Riemannian metric) in Eells-Fuglede [7] (see Chapter 9 therein).

We consider Borel-measurable (equivalently, measurable w.r.t. $e^{-f}dV_g$) maps $u : M \to Y$ (u then has separable range since M is a separable metric space, see Problem 10 of [6, Section 4.2]). The space $L^2_{loc}(M_f, Y)$ is defined as the set of Borel-measurable maps u for which $d(u(\cdot), Q) \in L^2_{loc}(M, e^{-f}dV_g)$ for some point Q (and hence for any Q by triangle inequality) in Y. Since this space is unchanged if we use the unweighted measure dV_g instead of $e^{-f}dV_g$ in its definition, we will write $L^2_{loc}(M, Y)$ for simplicity in the following. When M is compact, $L^2_{loc}(M, Y)$ is a complete metric space, with distance function $\hat{d}$ defined by

$$\hat{d}^2(u, v) := \int_M d^2(u(x), v(x))e^{-f(x)}dV_g(x),$$

provided that (Y, d) is complete.

The approximate energy density for a map $u \in L^2_{loc}(M, Y)$ is defined for $\varepsilon > 0$ as

$$e_\varepsilon(u) := \frac{1}{\omega_n} \int_{S(x,\varepsilon)} \frac{d^2(u(x), u(y))}{\varepsilon^2} \frac{d\sigma_{x,\varepsilon}(y)}{\varepsilon^{n-1}}, \tag{2}$$

where $d\sigma_{x,\varepsilon}(y)$ is the $(n-1)$-dimensional surface measure on the sphere $S(x, \varepsilon)$ of radius ε centered at x induced by the Riemannian metric g, and ω_n is the volume of

n-dimensional unit Euclidean ball. One can check that the function $e_\varepsilon(u) \in L^1_{loc}(M)$ (see [20]). Then we can define the f-energy functional E^f by

$$E^f(u) := \sup_{\substack{\eta \in C_0(M) \\ 0 \leq \eta \leq 1}} \left(\limsup_{\varepsilon \to 0} \int_M \eta e_\varepsilon(u) e^{-f} dV_g \right).$$

We say a map $u \in L^2_{loc}(M, Y)$ is locally of finite energy, denoted by $u \in W^{1,2}_{loc}(M, Y)$, if $E^f(u_{|\Omega}) < \infty$ for any relatively compact domain $\Omega \subset M$.

Theorem 4.1. *If $u \in W^{1,2}_{loc}(M, Y)$, then there exists a function $e(u) \in L^1_{loc}(M)$, such that for any $\eta \in C_0(M)$,*

$$\lim_{\varepsilon \to 0} \int_M \eta e_\varepsilon(u) e^{-f} dV_g = \int_M \eta e(u) e^{-f} dV_g. \tag{3}$$

Proof. By definition, $u \in W^{1,2}_{loc}(M, Y)$ implies that for any connected, open and relatively compact subset $\Omega \subset M$, $u_{|\Omega} \in L^2(\Omega, Y)$ and

$$\sup_{\substack{\eta' \in C_0(\Omega) \\ 0 \leq \eta' \leq 1}} \left(\limsup_{\varepsilon \to 0} \int_\Omega \eta' e_\varepsilon(u_{|\Omega}) dV_g \right) < \infty,$$

that is, $u_{|\Omega} \in W^{1,2}(\Omega, Y)$ in Korevaar-Schoen's notation [20].

Now by [20, Theorem 1.5.1 and Theorem 1.10], we know that there exists a function $e(u_{|\Omega}) \in L^1(\Omega)$ such that

$$\lim_{\varepsilon \to 0} \int_\Omega \eta' e_\varepsilon(u) dV_g = \int_\Omega \eta' e(u_{|\Omega}) dV_g, \;\; \forall \;\, \eta' \in C_0(\Omega). \tag{4}$$

In particular, one has

$$\lim_{\varepsilon \to 0} \int_\Omega \eta e_\varepsilon(u) e^{-f} dV_g = \int_\Omega \eta e(u_{|\Omega}) e^{-f} dV_g, \;\; \forall \;\, \eta \in C_0(\Omega). \tag{5}$$

We then define a function $e(u)$ on M by $e(u)_{|\Omega} := e(u_{|\Omega})$ for any $\Omega \subset M$ with smooth boundary. One can show that $e(u)$ is well defined. For that purpose, one only needs to check $e(u_{|\Omega}) = e(u_{|\Omega_1})$ on $\Omega_1 \subset \Omega$ where both Ω_1 and $\Omega \setminus \Omega_1$ have Lipschitz boundary. This is true since by the trace theory [20, Theorem 1.12.3], one has

$$\int_\Omega e(u_{|\Omega}) dV_g = \int_{\Omega_1} e(u_{|\Omega_1}) dV_g + \int_{\Omega \setminus \Omega_1} e(u_{|\Omega \setminus \Omega_1}) dV_g.$$

Then (3) follows from (5) which proves this theorem. □

Remark 4.1. By the definition of $e(u)$ and (4), we know

$$e(u)(x) = |\nabla u|^2(x),$$

where $|\nabla u|^2(x)$ is the energy density function in [20]. This function is consistent with the usual way of defining $|du|^2$ for maps between Riemannnian manifolds. Therefore we will use $|\nabla u|^2(x)$ instead of $e(u)(x)$ in the following.

Remark 4.2. By a polarization argument, we can check that for any two functions $h_1, h_2 \in W^{1,2}_{loc}(M, e^{-f}dV_g)$,

$$\lim_{\varepsilon\to 0}\int_M \eta(x)\frac{1}{\omega_n}\int_{S(x,\varepsilon)}\frac{(h_1(x)-h_1(y))(h_2(x)-h_2(y))}{\varepsilon^2}\frac{d\sigma_{x,\varepsilon}(y)}{\varepsilon^{n-1}}e^{-f(x)}dV_g(x)$$
$$=\int_M \eta(x)\langle\nabla h_1(x),\nabla h_2(x)\rangle e^{-f(x)}dV_g(x),\ \forall\ \eta\in C_0(M).$$

Remark 4.3. With (3) in hand, by the definition of E^f, we can derive (see [7, Theorem 9.1]),

$$E^f(u)=\int_M |\nabla u|^2 e^{-f}dV_g,\ \forall\ u\in W^{1,2}_{loc}(M,Y).$$

In particular, $E^f(u)=D^f(u)$ when $Y=\mathbb{R}$.

Remark 4.4. As in [20], the definition of E^f is unchanged if we replace $e_\varepsilon(x)$ by ${}_\nu e_\varepsilon(x):=\int_0^2 e_{\lambda\varepsilon}(x)d\nu(\lambda)$, where ν is any Borel measure on the interval $(0,2)$ satisfying $\nu\geq 0,\ \nu((0,2))=1,\ \int_0^2\lambda^{-2}d\nu(\lambda)<\infty$. For example, the approximate energy density function can be chosen as follows.

(1) When $n\geq 3$, for the measure $d\nu_1(\lambda)=n\lambda^{n-1}d\lambda,\ 0<\lambda<1$,

$${}_{\nu_1}e_\varepsilon(x)=\frac{n}{\omega_n}\int_{B(x,\varepsilon)}\frac{d^2(u(x),u(y))}{d^2(x,y)}\frac{dV_g(y)}{\varepsilon^n};$$

(2) For the measure $d\nu_2(\lambda)=(n+2)\lambda^{n+1}d\lambda,\ 0<\lambda<1$,

$${}_{\nu_2}e_\varepsilon(x)=\frac{n+2}{\omega_n}\int_{B(x,\varepsilon)}\frac{d^2(u(x),u(y))}{\varepsilon^2}\frac{dV_g(y)}{\varepsilon^n}.$$

Remark 4.5. For $n\geq 3$, by introducing a conformal change of the metric $\widetilde{M}=(M,\tilde{g},dV_{\tilde{g}})$ where $\tilde{g}=e^{-\frac{2f}{n-2}}g$ and employing the energy density ${}_{\nu_1}e_\varepsilon$, many problems for weighted manifolds can be reduced to those on (possibly incomplete) unweighted manifolds. However, we prefer to write the proofs in a unified way which includes the case $n=2$.

We call a map $u\in W^{1,2}_{loc}(M,Y)$ *f-harmonic* if it is a local minimizer of the energy functional E^f, i.e., for any connected, open and relatively compact domain $\Omega\subset M$, $E^f(u)\leq E^f(v)$ for every map $v\in W^{1,2}_{loc}(M,Y)$ such that $u=v$ in $M\setminus\Omega$.

We now investigate the properties of the function $d(u(\cdot),Q)$ on M, where $u: M\to Y$ is an f-harmonic map and $Q\in Y$. The first observation is that

$$E^f(d(u,Q))\leq E^f(u). \tag{6}$$

This can be derived from the triangle inequality

$$(d(u(x),Q)-d(u(y),Q))^2\leq d^2(u(x),u(y)).$$

Recall that an *Hadamard space* (also called global NPC space) is a complete geodesic space which is globally nonpositively curved in the sense of Alexandrov, i.e., Toponogov's triangle comparison for nonpositive curvature holds for any geodesic triangle. The class of Hadamard spaces, natural generalizations of Cartan-Hadamard manifolds, includes all simply connected local NPC spaces (see e. g. [2]). When the target space (Y,d) is an Hadamard space, we have the following theorem.

Theorem 4.2. *If $u \in W^{1,2}_{loc}(M,Y)$ is an f-harmonic map into an Hadamard space Y, then for any $Q \in Y$,*

$$-\int_M \langle \nabla\eta(x), \nabla d(u(x),Q)\rangle e^{-f} dV_g \geq 0, \ \forall\ 0 \leq \eta \in \mathrm{Lip}_0(M), \tag{7}$$

i.e., $d(u(x),Q) \in W^{1,2}_{loc}(M)$ is an f-subharmonic function.

This theorem is a consequence of Lemma 5 in Jost [16]. The subharmonicity of $d(u(\cdot),Q)$ for harmonic maps from an admissible Riemannian polyhedron with simplexwise smooth Riemannian metric to an Hadamard space was obtained by Eells-Fuglede [7, Lemma 10.2]. Their argument essentially also works in our setting. Using Remark 4.2, Jost's lemma can be reformulated in our setting as follows.

Lemma 4.1 ([16], Lemma 5)**.** *If $u \in W^{1,2}_{loc}(M,Y)$ is an f-harmonic map into an Hadamard space Y, then for any $Q \in Y$ and $\eta \in \mathrm{Lip}_0(M)$, $0 \leq \eta \leq 1$,*

$$-\int_M \langle \nabla\eta(x), \nabla d^2(u(x),Q)\rangle e^{-f(x)} dV_g(x) \geq 2\int_M \eta(x)|\nabla u|^2(x) e^{-f(x)} dV_g(x). \tag{8}$$

In fact, (8) still holds for nonnegative functions $\eta \in W^{1,2}(M)$ with compact support. (When $E^f(u)$ is finite, (8) even holds for $0 \leq \eta \in W^{1,2}_0(M)$.) Now we can prove Theorem 4.2 concerning the f-subharmonicity of $d(u(\cdot),Q)$.

Proof of Theorem 4.2. Denote by $\varphi(x) := \sqrt{x^2+\epsilon}$ for $\epsilon > 0$. For any $0 \leq \eta \in \mathrm{Lip}_0(M)$, we choose a compactly supported function

$$\eta_1(x) := \frac{\eta(x)}{2\varphi(d(u(x),Q))} \in W^{1,2}(M).$$

Then we calculate (we suppress the measure $e^{-f}dV_g$ in the notations)

$$\begin{aligned}
&-\int_M \left\langle \nabla\eta(x), \nabla\sqrt{d^2(u(x),Q)+\epsilon}\right\rangle = -\int_M \left\langle \nabla\eta(x), \frac{\nabla d^2(u(x),Q)}{2\varphi(d(u(x),Q))}\right\rangle \\
&= -\int_M \langle \nabla\eta_1(x), \nabla d^2(u(x),Q)\rangle - \int_M 2\eta_1 \frac{d(u(x),Q)\varphi'(d(u(x),Q))}{\varphi(d(u(x),Q))} |\nabla d(u(x),Q)|^2.
\end{aligned}$$

Note that

$$\frac{d(u(x),Q)\varphi'(d(u(x),Q))}{\varphi(d(u(x),Q))} = \frac{d^2(u(x),Q)}{d^2(u(x),Q)+\epsilon} \leq 1,$$

and by (6),

$$|\nabla d(u(x),Q)|^2 \leq |\nabla u(x)|^2,$$

we obtain

$$-\int_M \left\langle \nabla\eta(x), \nabla\sqrt{d^2(u(x),Q)+\epsilon}\right\rangle \geq -\int_M \langle \nabla\eta_1(x), \nabla d^2(u(x),Q)\rangle - 2\int_M \eta_1 |\nabla u(x)|^2. \tag{9}$$

Applying Lemma 4.1, and letting $\epsilon \to 0$, we complete the proof. ☐

Now we adopt the method of Li-Wang [28], a geometric analysis method, to prove Kendall's theorem when the target is a locally compact Hadamard space.

Proof of Theorem 1.1. By assumption, the space of bounded f-harmonic functions is of dimension one. Then by the arguments of Grigor'yan [10], every two f-massive subsets of M have a non-empty intersection. Here by f-massive subset, we mean an open proper subset of $\Omega \subset M$ on which there is a bounded, nonnegative, nontrivial,

f-subharmonic function h such that $h_{|\partial\Omega} = 0$. Such function h is called an f-potential of the set Ω.

Let $\hat{M}$ be the Stone-Cěch compactification of M. Then every bounded continuous functions on M can be continuously extended to $\hat{M}$. Let Ω be an f-massive subset of M, we then define the set

$$S := \bigcap_{\substack{h: f\text{-potential}\\ \text{functions of } \Omega}} \{\hat{x} \in \hat{M} \mid h(\hat{x}) = \sup h\}.$$

By the maximum principle for f-subharmonic functions, we know $S \subset \hat{M} \setminus M$.

Then, by the same arguments as in [28, Theorem 2.1], we can prove $S \neq \emptyset$. Furthermore, for any bounded f-subharmonic function v, we have $S \subset \{\hat{x} \in \hat{M} \mid v(\hat{x}) = \sup v\}$.

Let us take a point $Q_0 \in \overline{u(M)}$. If $u(M) = \{Q_0\}$, then we complete the proof. Otherwise, we have $u(M) \setminus \{Q_0\} \neq \emptyset$. Since u is an f-harmonic map, by Theorem 4.2, the function $h_1(x) := d(u(x), Q_0)$ is an f-subharmonic function, which is bounded and nonconstant. Hence h_1 attains its maximum at every point of S. For a point $\hat{x} \in S$, there is a sequence $\{x_n\}$ in M converging to $\hat{x}$ in $\hat{M}$. Note that u has bounded image. Thus by local compactness of the target Y, there exists a subsequence of $\{u(x_n)\}$ converging to $Q_1 \in Y$. Now again, if $u(M) = \{Q_1\}$, we complete the proof. Therefore, we can assume $u(M) \setminus \{Q_1\} \neq \emptyset$. By Theorem 4.2, the function $h_2(x) := d(u(x), Q_1)$ is a bounded f-subharmonic function. Thus h_2 achieves its maximum on S, in particular at $\hat{x}$. That is

$$\sup h_2(x) = h_2(\hat{x}) = d(Q_1, Q_1) = 0.$$

This contradicts our assumption. Therefore $u(M) = \{Q_1\}$ is a constant map. □

Remark 4.6. As pointed out to us by K. Kuwae, one can prove Kendall's theorem by combining the methods of Li-Wang [28] and Kuwae-Sturm [22] for harmonic maps into Hadamard spaces on which the weak topology (see [15, Definition 2.7]) coincides with the strong one, i.e. Hadamard spaces (Y, d) satisfying that the distance function $d(x, \cdot)$ is weakly continuous for any $x \in Y$.

5. Liouville type theorems

In this section, we shall prove our main theorem. First of all, let us review the classical classification theory of Riemannian manifolds in the framework of weighted manifolds. For more details we refer to [8] and [39].

We recall some function spaces of $(M, g, e^{-f}dV_g)$. Let $D^f(M)$ be the set of Tonelli functions[1] on M with finite Dirichlet f-energy. The Royden algebra $BD^f(M)$ is the set of bounded functions in $D^f(M)$. Under the norm $\|u\| = \sup_M |u| + \sqrt{D^f(u)}$, $BD^f(M)$ becomes a Banach algebra. For a sequence $\{u_n\}$ in $D^f(M)$, we say $u = C - \lim u_n$ if u_n converges to u uniformly on compact subsets and $u = B - \lim u_n$ if in addition $\{u_n\}$ is uniformly bounded. We say $u = D^f - \lim u_n$ if $\lim D^f(u_n - u) = 0$. We also denote by $u = CD^f - \lim u_n$ or $u = BD^f - \lim u_n$ to indicate two types of convergence.

[1]A Tonelli function is a continuous function with locally L^2-integrable weak derivatives.

Let $C_0^\infty(M)$ be the set of smooth functions with compact support and $D_0^f(M)$ be its closure under the CD^f-topology. We also denote by $HD^f(M)$ and $HBD^f(M)$ the sets of f-harmonic functions in $D^f(M)$ and $BD^f(M)$ respectively.

Proposition 5.1. *Let $(M, g, e^{-f}dV_g)$ be an f-parabolic weighted Riemannian manifold. Then any f-subharmonic function with finite Dirichlet f-energy is constant. In particular, any function in $HD^f(M)$ is constant.*

Proof. Let $u \in D^f(M)$ be f-subharmonic. we may assume $u \geq 0$ since $\max\{u, 0\}$ is also f-subharmonic. Let $\{M_n\}$ be an exhaustion of M and take $w_k \in BD^f(M)$ with $w_k|_{M_0} = 1$, $w_k|_{M\setminus M_k} = 0$ and f-harmonic in $M_k\setminus\overline{M_0}$. It follows from the f-parabolicity of M that $BD^f - \lim w_k = 1$. On the other hand, set $v_k \in BD^f(M)$ with $v_k|_{M_0} = u$, $v_k|_{M\setminus M_k} = 0$ and f-harmonic in $M_k\setminus\overline{M_0}$, one can verify that $v = BD^f - \lim v_k$ exists. Set now $\tilde{u} = u - v$, and $\tilde{u}_m = \min\{\tilde{u}, m\}$. Then $\tilde{u} = D^f - \lim \tilde{u}_m$. Since $\tilde{u}$ is nonnegative and f-subharmonic, we can compute

$$0 \geq -\int_{M_k\setminus M_0} \tilde{u}_m w_k \Delta_f \tilde{u} e^{-f} dV_g = \int_M \langle \nabla(\tilde{u}_m w_k), \nabla\tilde{u}\rangle e^{-f} dV_g. \qquad (10)$$

As $w_k \to 1$ in D^f-topology, we deduce from (10) by letting $k \to \infty$ that

$$\int_M \langle \nabla\tilde{u}_m, \nabla\tilde{u}\rangle e^{-f} dV_g = 0,$$

which yields $D^f(\tilde{u}) = 0$ by letting $m \to \infty$. Since $\tilde{u}|_{M_0} = 0$, we see $u = v$. Finally,

$$D^f(u) = \int_M \langle \nabla u, \nabla v\rangle e^{-f} dV_g = \lim_{k\to\infty} \int_M \langle \nabla u, \nabla v_k\rangle e^{-f} dV_g \leq 0,$$

and hence u is a constant. □

The following are the weighted version of the Royden-Nakai decomposition theorem and Virtanen theorem. The proofs are almost the same as the unweighted case. For the convenience of the readers, we shall give proofs here.

Theorem 5.1 (Royden-Nakai decomposition theorem)**.** *Let $(M, g, e^{-f}dV_g)$ be a non-f-parabolic weighted Riemannian manifold. Then any function $u \in D^f(M)$ has a unique decomposition $u = h + g$, where $h \in HD^f(M)$ and $g \in D_0^f(M)$. Moreover, if u is f-subharmonic, then $u \leq h$.*

Proof. Let $u \in D^f(M)$. Assume first $u \geq 0$. Let $\{M_k\}$ be an exhaustion of M and take $h_k \in HD^f(M_k)$ with $h_k|_{\partial M_k} = u$, and $g_k = u - h_k$. It follows from the maximum principle that $h_k \geq 0$. One can check that $D^f(h_k) = D^f(u) - D^f(g_k)$ and $D^f(h_m) - D^f(h_k) = D^f(h_m - h_k) \geq 0$ for $m \leq k$. Thus $\{h_k\}$ is a D^f-Cauchy sequence. Let $w_k \in BD^f(M)$ with $w_k|_{M_0} = 1$, $w_k|_{M\setminus M_k} = 0$ and harmonic in $M_k\setminus\overline{M_0}$. It follows from the non-f-parabolicity of M that $w = BD^f - \lim w_k$ satisfies $D^f(w) > 0$. We can compute

$$\int_M \langle \nabla g_k, \nabla w_k\rangle e^{-f} dV_g = \int_{\partial M_0} g_k \frac{\partial w_k}{\partial \nu} e^{-f} dA_g,$$

where ν is the unit outward normal of ∂M_0. Combining this with the fact that $\int_{\partial M_0} \frac{\partial w_k}{\partial \nu} e^{-f} dV_g = D^f(w_k)$ and

$$\left| \int_M \langle \nabla g_k, \nabla w_k\rangle e^{-f} dV_g \right| \leq \left[D^f(g_k) D^f(w_k)\right]^{\frac{1}{2}} \leq \left[D^f(u) D^f(w_k)\right]^{\frac{1}{2}},$$

we find

$$\inf_{M_0} h_k \leq \sup u + \left[\frac{D^f(u)}{D^f(w)}\right]^{\frac{1}{2}}.$$

By the Harnack inequality, $\sup_{M_0} h_k$ is bounded. Hence h_k converges to some h in CD^f-topology and $h \in HD^f(M)$. It is easy to check that g_k converges to $g = u - h$ in CD^f-topology and thus $g \in D_0^f(M)$. Furthermore, it follows from the construction of h_k that $h \geq u$.

If u is not nonnegative, we can run the same process for $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$ as before and get the same result.

The uniqueness follows from the fact that any $h \in HD^f(M)$ and $g \in D_0^f(M)$ satisfy $\int_M \langle \nabla h, \nabla g \rangle e^{-f} dV_g = 0$. □

Theorem 5.2 (Virtanen theorem)**.** *For every $u \in HD^f(M)$ there exists a sequence $h_k \in HBD^f(M)$ such that $u = CD^f - \lim h_k$. In particular, M admits no nonconstant f-harmonic function on M with finite Dirichlet f-energy if and only if M admits no nonconstant bounded f-harmonic function on M with finite Dirichlet f-energy.*

Proof. We may assume M is non-f-parabolic, since otherwise, any $u \in HD^f(M)$ is constant, due to Proposition 5.1, whence the statement is trivial. We may also assume $u \geq 0$, since otherwise we do the same analysis on u^+ and u^-. Set for any $k \in \mathbb{N}$, $u_k = \min\{u, k\}$. Then u_k is f-superharmonic and $u = D^f - \lim u_k$. By Royden-Nakai decomposition, $u_k = h_k + g_k$, where $h_k \in HD^f(M)$ and $g_k \in D_0^f(M)$. Moreover, $g_k \geq 0$. One can verify

$$D^f(u - u_k) = D^f(u - h_k) + D^f(g_k).$$

Hence $D^f(u - h_k) \to 0$ and $D^f(g_k) \to 0$. Since $0 \leq g_k \leq u_k \leq u$ is bounded in any compact set of M, we conclude that g_k converges to some constant function c in CD^f-topology. It follows from the non-f-parabolicity of M that $c = 0$. Therefore h_k converges to u in CD^f-topology.

The second assertion follows easily from this approximation. □

The following lemma was first proved by Cheng-Tam-Wan [5, Theorem 1.2].

Lemma 5.1. *Let $(M, g, e^{-f}dV_g)$ be a weighted Riemannian manifold. Then the following two statements are equivalent:*

(i) *any $u \in HD^f(M)$ is bounded;*
(ii) *any nonnegative f-subharmonic function on M with finite Dirichlet f-energy is bounded.*

Proof. (ii)⇒(i). This is quite simple by observing the fact that if $u \in HD^f(M)$, then $\sqrt{u^2 + 1}$ is a nonnegative f-subharmonic function on M with finite Dirichlet f-energy.

(i)⇒(ii). Assume u is a nonnegative f-subharmonic function on M with finite Dirichlet f-energy. If M is f-parabolic, then the two statements are both true by virtue of Proposition 5.1 and hence equivalent. If M is non-f-parabolic, then by Theorem 5.1, $u = h + g$ for $h \in HD^f(M)$ and $g \in D_0^f(M)$. Moreover, since u is f-subharmoic, we know $u \leq h$. By the assumption (i), h is bounded. Thus u is also bounded. This proves the lemma. □

Using Lemma 5.1, we can prove the main Theorem 1.2.

Proof of Theorem 1.2. Let u be an f-harmonic map from M to Y with finite f-energy. It follows from Theorem 4.2 that the function $v : M \to \mathbb{R}$, $v(x) = \sqrt{d^2(u(x), Q) + 1}$ is subharmonic, where $Q \in Y$. Also, the finiteness of the f-energy of u implies the finiteness of the Dirichlet f-energy of v (recall (6)). Using the assumption and the equivalence in Lemma 5.1, we know that any nonnegative f-subharmonic function on M with finite Dirichlet f-energy is bounded. Hence v is bounded, in turn, u has bounded image. This proves the theorem. □

For harmonic maps from f-parabolic weighted manifolds, we don't need the local compactness assumption of the targets to obtain the Liouville theorem.

Corollary 5.1. *Let $(M, g, e^{-f}dV_g)$ be a complete noncompact f-parabolic weighted Riemannian manifold and (Y, d) be an Hadamard space. Then any f-harmonic map from M to Y with finite f-energy must be constant.*

Proof. Let u be an f-harmonic map from M to Y with finite f-energy. By Proposition 5.1 and Theorem 1.2, the image of u is bounded. Hence for any $Q \in Y$, the f-subharmonic function $d(u(x), Q)$ is bounded. By the f-parabolicity of M and Proposition 2.1, the function $d(u(x), Q)$ is constant for any $Q \in Y$. This yields that u is a constant map. The corollary follows. □

Combining Theorem 1.1 and Theorem 1.2, we obtain Theorem 1.3 by the potential theory.

Proof of Theorem 1.3. Since any bounded f-harmonic function on M is constant (see Brighton [3]), by Theorem 5.2, we know that any f-harmonic function on M with finite Dirichlet f-energy is constant. Using Theorem 1.2, we see that any f-harmonic map from M to Y with finite f-energy must have bounded image.

On the other hand, by Theorem 1.1, we know that any f-harmonic map from M to Y having bounded image is constant. Hence any f-harmonic map from M to Y with finite f-energy must be constant. This proves the theorem. □

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