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Harmonic functions on metric measure spaces

by

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ABSTRACT. In this paper, we study harmonic functions on metric measure spaces with Riemannian Ricci curvature bounded from below, which were introduced by Ambrosio-Gigli-Savaré. We prove a Cheng-Yau type local gradient estimate for harmonic functions on these spaces. Furthermore, we derive various optimal dimension estimates for spaces of polynomial growth harmonic functions on metric measure spaces with nonnegative Riemannian Ricci curvature.

1. INTRODUCTION

In [BE85] Bakry and Émery introduced the so-called Γ-calculus and a purely analytical *Curvature-Dimension condition* $BE(K, N), N \in [1, \infty]$ for Riemannian manifolds, which are applicable to the general setting of Dirichlet forms and the associated Markov semigroups. Some years later Lott-Villani [LV09] and Sturm [Stu06a, Stu06b] introduced independently another *Curvature-Dimension condition* $CD(K, N), N \in [1, \infty]$ for general metric measure spaces (mms for short) coming from a better understanding of gradient flows on associated Wasserstein spaces. To overcome the lack of a local-to-global property for $CD(K, N)$ with finite N, Bacher-Sturm [BS10] introduced a weaker notion called *reduced Curvature-Dimension condition* $CD^*(K, N), N \in [1, \infty)$. The two notions $BE(K, N)$ and $CD(K, N)$ are both equivalent to the condition that for weighted Riemannian manifolds the weighted N-Ricci curvature has lower bound K .

Most recently, Ambrosio-Gigli-Savaré made a breakthrough in series of fundamental papers [AGS11, AGS13, AGMR12], by showing that the two notions $CD(K, \infty)$ and $BE(K, \infty)$ are equivalent for *infinitesimal Hilbertian* mms. They gave a new notion $RCD(K, \infty)$ to indicate such spaces and called them *mms with Riemannian Ricci curvature bounded from below*. Actually these spaces exclude Finsler manifolds since the infinitesimal Hilbertian property implies that the heat flow is linear. Later Erbar-Kuwada-Sturm [EKS13] introduced for finite dimensional constant N the class $RCD^*(K, N)$ and they established the equivalence between $CD^*(K, N)$ and $BE(K, N)$ for infinitesimal Hilbertian mms. This was also independently discovered in an unpublished paper by Ambrosio-Mondino-Savaré [AMS13].

The main goal of this paper is to study harmonic functions on $RCD^*(K, N)$ mms for *finite N* (see Definition 2.1). We refer to [EKS13] for various equivalent definitions of $RCD^*(K, N)$. With the calculus developed in recent years, one may expect that many results in smooth Riemannian manifolds and Alexandrov spaces (see [BGP92, BBI01] for definitions) can be extended to $RCD^*(K, N)$ mms. We

will show in this paper that a local calculus can be developed in order to prove local gradient estimates for harmonic functions, as well as dimension estimates for spaces of polynomial growth harmonic functions.

Let us start with a brief introduction of the framework. Throughout the paper, we assume

(X, d, m) is a metric measure space,
where (X, d) is a complete and separable metric space,
and m is a nonnegative σ-finite Borel measure.

Note that we do not require X to be compact. We make the setup for σ-finite Borel measures in order to include important geometric objects such as noncompact finite dimensional Riemannian manifolds with Ricci lower bounds equipped with the volume measure, the measured Gromov-Hausdorff limit spaces of Riemannian manifolds with a uniform Ricci lower bound and a uniform dimension upper bound equipped with a natural Radon measure, see Cheeger-Colding [CC97, CC00a, CC00b] and finite dimensional Alexandrov spaces with Ricci lower bounds, see Zhang-Zhu [ZZ10] and Petrunin [Pet11].

The first part of the paper is devoted to a quantitative gradient estimate of harmonic functions. In 1975, Yau [Yau75] proved a Liouville type theorem for harmonic functions on Riemannian manifolds with nonnegative Ricci curvature. Then Cheng-Yau [CY75] used Bochner's technique to derive a local gradient estimate for harmonic functions, which is now a fundamental result in geometric analysis. On general metric spaces, Bochner's technique fails due to the lack of higher differentiability of the metric. By the semigroup approach, Garofalo-Mondino [GM13] proved Li-Yau type global gradient estimate for solutions of heat equations on $RCD^*(K, N)$ mms equipped with a probability measure. Their arguments heavily rely on the probability measure assumption. A generalization to the σ-finite measure case would meet essential difficulties. In addition, their results are global from which one cannot easily derive the local version.

Our concern is to obtain a local gradient estimate for harmonic functions on $RCD^*(K, N)$ mms with σ-finite measures. For this purpose, we shall first need a local version of Bochner inequality for $RCD^*(K, N)$ mms. The global version was proven by Erbar-Kuwada-Sturm [EKS13]. Here "global" means the statement of their Bochner inequality is only valid for global $W^{1,2}$-functions. Note that Zhang-Zhu [ZZ12] proved a similar Bochner inequality on Alexandrov spaces with Ricci curvature bounded below. A delicate local structure, so-called DC-differential structure (see e.g. [Per]), of Alexandrov spaces plays an essential role in the proof of [ZZ12]. This rules out the possibility of this strategy in our setting. Nevertheless, we can choose nice cut-off functions and apply the global Bochner inequality proven by Erbar-Kuwada-Sturm [EKS13] to derive the local one. This is the novelty of our approach. An important ingredient we need is the local Lipschitz regularity for $W_{loc}^{1,2}$ functions with Laplacian in L^p, which was obtained independently by Jiang [Jia13] and the second author [Kel13] using the method initiated in [KRS03, Jia11], see Lemma 3.2 below.

Theorem 1.1 (Local Bochner inequality). *Let (X, d, m) be an $RCD^*(K, N)$ mms. Let u be a function in $\mathcal{D}_{loc}(\Delta)$ with $\Delta u \in W_{loc}^{1,2} \cap L_{loc}^p(X, d, m)$ for $p > N$. Then*

$$|\nabla u|^2 \in W_{loc}^{1,2}(X, d, m)$$

and the Bochner inequality holds in the weak sense of measures

$$\mathcal{L}_{|\nabla u|^2} \geq 2\left(\frac{(\Delta u)^2}{N}dm + \langle \nabla u, \nabla(\Delta u)\rangle dm + K|\nabla u|^2 dm\right), \tag{1}$$

that is, for all $\varphi \in W^{1,2}(X,d,m)$ with compact support we have

$$\int \langle \nabla\varphi, \nabla|\nabla u|^2\rangle dm \quad \geq \quad 2\Big(\int \varphi\frac{(\Delta u)^2}{N}dm + \int \varphi\langle \nabla u, \nabla(\Delta u)\rangle dm$$
$$+ K\int \varphi|\nabla u|^2 dm\Big).$$

One of our main result is Cheng-Yau type local gradient estimate for harmonic functions on $RCD^*(K,N)$-mms. A function u is called harmonic (subharmonic resp.) on an open set $\Omega \subset X$ if $u \in W^{1,2}_{\text{loc}}(\Omega)$ and

$$\int_\Omega \langle \nabla u, \nabla\varphi\rangle dm = 0 \ (\leq 0 \text{ resp.})$$

for any $0 \leq \varphi \in \text{Lip}(\Omega)$ with compact support. This is equivalent to say that $\mathcal{L}_u = 0 \ (\geq 0 \text{ resp.})$ (see Section 2).

Theorem 1.2 (Cheng-Yau type gradient estimate). *Let (X,d,m) be an $RCD^*(K,N)$ mms for $K \leq 0$. Then there exists a constant $C = C(N)$ such that every positive harmonic function u on geodesic ball $B_{2R} \subset X$ satisfies*

$$\frac{|\nabla u|}{u} \leq C\frac{1+\sqrt{-K}R}{R} \quad \text{in } B_R.$$

Since Bochner's technique using the maximum principle on Riemannian manifolds is not available on metric spaces, we adopt the Moser iteration to prove the local gradient estimate, following the idea of Zhang-Zhu [ZZ12]. Note that for the case $K < 0$, one shall carry out a more delicate Moser iteration as done by the first and third authors in [HX13, Xia13]. In order to carry out the Moser iteration, we need the regularity result, $|\nabla u|^2 \in W^{1,2}_{\text{loc}}(X)$, for a harmonic function u. This follows from a general result by Savaré using Dirichlet form calculation, see [Sav13, Lemma 3.2] or Lemma 3.1 below. For a different proof of this result on Alexandrov spaces with Ricci lower bounds, we refer to [ZZ12, Theorem 1.2]. This will also be crucial in order to prove Theorem 1.5, where we essentially use the fact that $|\nabla u|^2 \in W^{1,2}_{\text{loc}}(X)$ is subharmonic for every harmonic function u on $RCD^*(0,N)$ mms.

Theorem 1.2 immediately yields Cheng's Liouville theorem for sublinear growth harmonic functions on $RCD^*(0,N)$ spaces.

Corollary 1.3 (Cheng's Liouville theorem). *On an $RCD^*(0,N)$ mms, there are no nonconstant harmonic functions of sublinear growth, i.e. if u is harmonic and*

$$\limsup_{R\to\infty} \frac{1}{R}\sup_{B_R}|u| = 0$$

then it is constant.

The second part of the paper is on dimension estimates for spaces of polynomial growth harmonic functions on $RCD^*(K,N)$ mms. The history leading to these results started in the study of Riemannian geometry. Cheng-Yau's gradient estimate

[CY75] implies that sublinear growth harmonic functions on Riemannian manifolds with nonnegative Ricci curvature are constant. Yau further conjectured in [Yau87, Yau93] that the space of polynomial growth harmonic functions on such manifolds with growth rate less than or equal to d should be of finite dimension. Colding-Minicozzi [CM97b, CM97a, CM98b] gave an affirmative answer to Yau's conjecture in a very general framework of weighted Riemannian manifolds utilizing volume doubling property and Poincaré inequality which are even adaptable to general metric spaces. A simplified argument by the mean value inequality can be found in [CM98a, Li97] where the dimension estimates are nearly optimal. This inspired many generalizations on manifolds [Tam98, LW99, LW00, STW00, KL00, Lee04, CW07]. The crucial ingredients of these proofs are the volume growth property and the Poincaré inequality (or mean value inequality).

Let $H^d(X) := \{u \in W^{1,2}_{\text{loc}}(X) : \mathcal{L}_u = 0, |u(x)| \leq C(1 + d(x,p))^d\}$ denote the space of polynomial growth harmonic functions on X with growth rate less than or equal to d for some (hence all) $p \in X$. Before stating the theorem, we shall point out a main difference between harmonic functions on Riemannian manifolds and those on other metric spaces. The unique continuation property for harmonic functions on mms is unknown, leaving us with the problem of verifying the inner product property of the following bilinear form

$$\langle u, v \rangle_R = \int_{B_R} uvdm, \quad u, v \in L^2(X, m),$$

where B_R is a geodesic ball with radius R. We circumvent this difficulty by a lemma in [Hua11] (see Lemma 5.1 below). By using the Bishop-Gromov volume comparison (see Theorem 2.7) and the Poincaré inequality (see Theorem 2.8) on $RCD^*(0, N)$ spaces, we obtain the following optimal dimension estimate for $H^d(X)$.

Theorem 1.4 (Polynomial growth harmonic functions). *Let (X, d, m) be an $RCD^*(0, N)$ mms. Then there exists some constant $C = C(N)$ such that*

$$\dim H^d(X) \leq Cd^{N-1}.$$

For the space of linear growth harmonic functions, we can give more precise estimate.

Theorem 1.5 (Linear growth harmonic functions). *Let (X, d, m) be an $RCD^*(0, N)$ mms and $p \in X$. Suppose the volume growth of (X, d, m) satisfies*

$$\limsup_{R \to \infty} \frac{m(B_R(p))}{R^n} < \infty \tag{2}$$

for some $n \leq N$, then

$$\dim H^1(X) \leq n + 1.$$

On Riemannian manifolds, Theorem 1.5 was studied by Li-Tam [LT89] and the equality case was characterized by Cheeger-Colding-Minicozzi [CCM95]. For the investigation of linear growth harmonic functions, see also Wang [Wan95], Li [Li95] on Kähler manifolds and Munteanu-Wang [MW11] on weighted Riemannian manifolds with nonnegative Ricci curvature.

One of the key ingredients of the proof is the following so-called mean value theorem at infinity for nonnegative subharmonic functions (see Theorem 5.4),

$$\lim_{R \to \infty} \frac{1}{m(B_R)} \int_{B_R} u \, dm = \operatorname*{ess\,sup}_X u.$$

The original proof for this mean value theorem by Li [Li86] (see also [Li12, Lemma 16.4]) used heat kernel estimates. It can also be proved by a tricky monotonicity formula involving the mean value of harmonic functions on geodesic spheres, see [MW11, Theorem 3.3] and [ZZ12, Corollary 6.6]. However, these methods seem hard to be extended to general metric spaces. For our purpose, we present a new proof only using the weak Harnack inequality for superharmonic functions, which is a consequence of Moser iteration, see Theorem 4.1 below. This will be the main ingredient to prove the optimal dimensional bound of the space of linear growth harmonic functions.

We remark that since the proof of above theorem involves the Bochner inequality, Theorem 1.5 is the first result on the dimension estimate of linear growth harmonic functions on nonsmooth metric spaces, even on Alexandrov spaces [Hua09, Hua11, Jia12].

To summarize, we prove a local Bochner inequality on $RCD^*(K, N)$ mms. Then we adopt a delicate Moser iteration to show Cheng-Yau type local gradient estimate of harmonic functions. By using the Bishop-Gromov's volume comparison, the Poincaré inequality and the Bochner inequality, we extend various optimal dimension estimates of the spaces of polynomial growth harmonic functions on Riemannian manifolds to a large class of nonsmooth mms satisfying the $RCD^*(0, N)$ condition. To this extent, we provide a relatively complete picture of global properties of harmonic functions on mms with nonnegative Riemannian Ricci lower bound.

The paper is organized as follows: In Section 2 we collect the basics of the analysis on mms with Riemannian Ricci curvature bounds. Section 3 is devoted to the proof of the local Bochner inequality, Theorem 1.1. In Section 4 we prove the Cheng-Yau type gradient estimate on RCD spaces, Theorem 1.2. In the last section, we prove the optimal dimension estimates of the spaces of polynomial growth harmonic functions and linear growth harmonic functions, Theorem 1.4 and Theorem 1.5 respectively.

2. Preliminaries

We will only introduce some necessary notations and refer to [Gig12, AGS13] for proofs of the statements and further references.

Throughout the paper we assume (X, d, m) is a metric measure spaces and the measure m is σ-finite and satisfying a maximum growth bound, i.e. for some $C > 0$

$$|B_r(x)| \le C \cdot e^{Cr^2},$$

where $|B_r(x)|$ is an abbreviation for $m(B_r(x))$. Additionally assume that (X, d) is a locally compact length space. Both assumptions simplify the following statements. Since any RCD mms will satisfy them they are in no way restrictive.

For the subset of $L^2(X, m)$ containing all Lipschitz functions with compact support one can define a *weak upper gradient* $|\nabla f|_w$ (see e.g. [AGS13]). The *Cheeger*

energy Ch is defined by

$$\mathrm{Ch}(f) = \int |\nabla f|_w^2 \, dm.$$

The subset of L^2-functions with finite Cheeger energy will be denoted by $W^{1,2}(X, m)$. Equipped with the norm

$$\|u\|_{W^{1,2}} := \|u\|_{L^2} + \mathrm{Ch}(u),$$

$W^{1,2}(X, m)$ is a Banach space. It can be shown that all Lipschitz functions with compact support have weak upper gradients in $L^\infty(X, m)$ and are contained in $W^{1,2}(X, m)$. Because the weak upper gradient is a local object, there is also a well-defined notation of $W^{1,2}_{\mathrm{loc}}(X, m)$: For any open set $\Omega \subset X$, $W^{1,2}(\Omega)$ is the set of functions whose weak upper gradient restricting to Ω have finite $L^2(\Omega)$ norm; The set $W^{1,2}_{\mathrm{loc}}(\Omega)$ is defined as the set of functions which belongs to $W^{1,2}(\Omega')$ for any precompact open set $\Omega' \subset \Omega$. We denote by $\mathrm{Lip}_c(\Omega)$ and $W^{1,2}_c(\Omega)$ the set of functions in $\mathrm{Lip}(\Omega)$ and $W^{1,2}(\Omega)$ with compact support in Ω respectively.

It can be shown that the Cheeger energy Ch is convex and lower semicontinuous. Thus the natural gradient flow $P_t : L^2 \to L^2$, called heat flow, can be defined. By the calculus developed in [AGS13] one can define a natural Laplace operator Δ on a dense subset of L^2 as subdifferential of Ch. Furthermore, the following holds

$$P_t \text{ is linear} \iff \mathrm{Ch} \text{ is a quadratic form} \iff \Delta \text{ is linear},$$

An mms whose Cheeger energy is quadratic will be called *infinitesimal Hilbertian*.

Our main focus will be the following subset of infinitesimal Hilbertian spaces

Definition 2.1 ($RCD^*(K, N)$ mms). We say an infinitesimal Hilbertian metric measure spaces is a (finite-dimensional) $RCD^*(K, N)$ mms or satisfies the $RCD^*(K, N)$ condition for some $K \in \mathbb{R}$ and $N > 0$, if for any $f \in W^{1,2}(X, m)$ and we have m-a.e. in X,

$$|\nabla P_t f|_w^2 + \frac{4Kt^2}{N(e^{2Kt} - 1)} |\Delta P_t f|^2 \le e^{-2Kt} P_t(|\nabla f|_w^2).$$

Remark 2.2. By [EKS13, Theorem 7] this is equivalent to the more classical $CD^*(K, N)$ condition defined via Wasserstein geodesics or the Bochner inequality (6) defined below.

Because Ch is a quadratic form in $W^{1,2}(X, m)$, there is an associated Dirichlet form $\mathcal{E}$, i.e., $\mathcal{E} : W^{1,2}(X, m) \times W^{1,2}(X, m) \to \mathbb{R}$ is the unique bilinear symmetric form satisfying

$$\mathcal{E}(f, f) = \mathrm{Ch}(f).$$

It was proven (see Section 4.3 in [AGS11]) that the Dirichlet form can be written as

$$\mathcal{E}(u, v) = \frac{1}{2} \int \langle \nabla u, \nabla v \rangle \, dm, \quad u, v \in W^{1,2}(X, m)$$

where

$$\langle \nabla u, \nabla v \rangle(x) = \lim_{\epsilon \to 0} \frac{|\nabla(v + \epsilon u)|_w^2 - |\nabla v|_w^2}{2\epsilon}.$$

The notion $\langle \nabla u, \nabla v \rangle$ should be understood as a bilinear and symmetric map from $W^{1,2} \times W^{1,2} \to L^1$. We remark that $|\nabla u|^2 = \langle \nabla u, \nabla u \rangle$ and $\langle \nabla u, \nabla v \rangle \in L^1$ are a well-defined objects whereas ∇u is not.

Gigli [Gig12] also showed that each $u \in W^{1,2}_{\text{loc}}(X,m)$ admits a measure valued Laplacian $\mathcal{L}_u$ such that

$$\mathcal{E}(v,u) = -\frac{1}{2}\int v d\mathcal{L}_u$$

for all $v \in W^{1,2}_c(X,m)$. If $\mathcal{L}_u$ has local L^2-density w.r.t. m we just write Δu. The subset of such functions will be denoted by $\mathcal{D}_{\text{loc}}(\Delta)$ (resp. $\mathcal{D}(\Delta)$ if $u \in W^{1,2}(X,m)$ and $\Delta u \in L^2(X,m)$).

Definition 2.3 (Harmonic, subharmonic and superharmonic functions). A function u is called harmonic (subharmonic, superharmonic resp.) on the domain Ω if $u \in W^{1,2}_{\text{loc}}(\Omega)$ and

$$-\int_\Omega \langle \nabla u, \nabla \varphi \rangle dm = 0 \ (\geq 0, \leq 0 \text{ resp.})$$

for any $0 \leq \varphi \in \text{Lip}_c(\Omega)$

Remark 2.4. It turns out [Gig12, Prop. 4.12] that this is equivalent to

$$\mathcal{L}_u = 0 \ (\geq 0, \leq 0 \text{ resp.}).$$

By the definition of harmonic (subharmonic) functions and integration by parts with cut-off functions, we obtain the following Caccioppoli inequality. We omit the proof here.

Lemma 2.5 (Caccioppoli inequality). *Let (X,d,m) satisfy $RCD^*(K,N)$ condition. Then for any nonnegative subharmonic function u on B_{2R}, we have*

$$\int_{B_R} |\nabla u|^2 dm \leq \frac{C}{R^2}\int_{B_{2R}} u^2 dm,$$

where $C = C(N)$.

We summarize the local calculus of the weak upper gradient and the Laplace operator as follows.

Theorem 2.6 ([AGS13, Gig12]). *Assume (X,d,m) is an infinitesimal Hilbertian mms. Assume $u,v,w \in W^{1,2}_{\text{loc}}(X,m)$ Then the following holds:*

(1) *$|\nabla u|_w = |\nabla \tilde{u}|_w \quad m$-a.e. on $\{u = \tilde{u}\}$ and $|\nabla u|_w = 0$ m-a.e. on $\{u = c\}$ for $c \in \mathbb{R}$.*

(2) *$u \in W^{1,2}_{\text{loc}}(X,m)$ iff $u \cdot \chi \in W^{1,2}(X,m)$ for all $\chi \in \text{Lip}_c$, moreover, if u has compact support then it is in $W^{1,2}(X,m)$.*

(3) *Assume $u \in \mathcal{D}_{\text{loc}}(\Delta)$. The Laplace operator is a local object, i.e. if $\Omega \subset X$ is open and $\{\Omega_i\}_{i \in I}$ an open covering of Ω then*

    (a) *$\mathcal{L}_{u|\Omega} = \mu$ iff for all $v \in W^{1,2}(X,m)$ with $\text{supp}\, v \subset \Omega$*

$$\mathcal{E}(v,u) = -\frac{1}{2}\int v d\mathcal{L}_u,$$

    (b) *set $\mathcal{L}_{u|\Omega_i} = \mu_i$; if $\mu_{i|\Omega_i \cap \Omega_j} = \mu_{j|\Omega_i \cap \Omega_j}$ whenever $\Omega_i \cap \Omega_j \neq \varnothing$ then*

$$(\mathcal{L}_{u|\Omega})_{\Omega \cap \Omega_i} = \mu_i.$$

(4) *$|\nabla \cdot|_w$ and Δ satisfy the chain rule, i.e. if $\varphi : \mathbb{R} \to \mathbb{R}$ is Lipschitz then*

$$|\nabla \phi(u)|_w = |\phi'(u)||\nabla u|_w \quad m\text{-a.e.}$$

and if, in addition, u is Lipschitz then

$$\Delta \phi(f) = \phi'(f)\Delta f + \phi''(f)|\nabla f|^2_w \quad m\text{-a.e.}$$

(5) The inner product $\langle\nabla\cdot,\nabla\cdot\rangle$ and Δ satisfy the product rule, i.e. if $u, v \in W^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(X, m)$ then

$$\langle\nabla(u \cdot v), \nabla w\rangle = v\langle\nabla u, \nabla w\rangle + u\langle\nabla v, \nabla w\rangle \quad m\text{-a.e.}$$

and

$$\Delta(u \cdot v) = v \cdot \Delta u + u \cdot \Delta v + 2\langle\nabla u, \nabla v\rangle \quad m\text{-a.e.}$$

(6) The space $W^{1,2}_{\mathrm{loc}}(X, m) \cap L^\infty_{\mathrm{loc}}(X, m)$ and $\mathcal{D}_{\mathrm{loc}}(\Delta) \cap L^\infty_{\mathrm{loc}}(X, m)$ are algebras, i.e. if $u, v \in W^{1,2}_{\mathrm{loc}}(X, m) \cap L^\infty(X, m)$ then $u \cdot v \in W^{1,2}_{\mathrm{loc}}(X, m) \cap L^\infty(X, m)$ and similar for $\mathcal{D}_{\mathrm{loc}}(\Delta) \cap L^\infty_{\mathrm{loc}}(X, m)$.

(7) The Cauchy-Schwarz inequality holds,

$$|\langle\nabla u, \nabla v\rangle| \leq |\nabla u||\nabla v|.$$

Next, we recall some basic geometric properties of $RCD^*(K, N)$ mms. The following volume comparison theorem is well-known.

Theorem 2.7 (Bishop-Gromov volume comparison)**.** *Let (X, d, m) be an $RCD^*(K, N)$ space. Then for any $0 < r < R < \infty$,*

$$\frac{|B_R(x)|}{|B_r(x)|} \leq \frac{V_K^N(R)}{V_K^N(r)}, \tag{3}$$

where $V_K^N(R)$ is the volume of the ball of radius R in a complete simply-connected N-dimensional manifold of constant sectional curvature $\frac{K}{N-1}$. In particular, if $K = 0$, then

$$\frac{|B_R(x)|}{|B_r(x)|} \leq \left(\frac{R}{r}\right)^N. \tag{4}$$

In order to prove Cheng-Yau's local gradient estimate for the space satisfying $RCD^*(K, N)$ condition with $K < 0$, we need the Poincaré and Sobolev inequalities with precise dependence on K and radii of the balls.

The Poincaré inequality was proved by several authors, see Lott-Villani [LV07], von Renesse [vR08], Rajala [Theorem 1.1] [Raj12a], [Raj12b] for mms satisfying $CD(K, N)$.

Theorem 2.8 (Poincaré inequalty)**.** *Let (X, d, m) be mms satisfying $CD(K, N)$ for $K \leq 0$. For any $1 \leq p < \infty$, there exists a constant $C = C(N, p)$ such that for all $u \in W^{1,2}(B_{2R})$,*

$$\int_{B_R} |u - u_{B_R}|^p dm \leq CR^p e^{C\sqrt{-K}R} \int_{B_{2R}} |\nabla u|^p dm, \tag{5}$$

where $u_{B_R} = \frac{1}{|B_R|} \int_{B_R} u\,dm$.

Remark 2.9. For a precise statement of the $CD(K, N)$ condition see [LV09, Stu06b, Raj12a]. In infinitesimal Hilbertian setting this is equivalent to the $RCD^*(K, N)$ condition stated above.

3. A LOCAL BOCHNER INEQUALITY

In the rest of the paper, we assume throughout that (X, d, m) is an $RCD^*(K, N)$ mms for $K \leq 0$.

The following lemma was proven in [EKS13]. We will include a more technical statement of Savaré [Sav13].

Lemma 3.1 (Bochner formula, [EKS13, Theorem 5], [Sav13, Lemma 3.2]). *Let (X, d, m) be an $RCD^*(K, N)$ mms. For any $u \in D(\Delta)$ with $\Delta u \in W^{1,2}(X, d, m)$ and all bounded and nonnegative $g \in D(\Delta)$ with $\Delta g \in L^\infty(X, m)$ we have*

$$\frac{1}{2} \int \Delta g |\nabla u|^2 dm \geq \frac{1}{N} \int g(\Delta u)^2 dm + \int g \langle \nabla u, \nabla(\Delta u) \rangle dm + K \int g |\nabla u|^2 dm. \quad (6)$$

Furthermore, if $u \in \mathrm{Lip}(X) \cap L^\infty(X)$ then $|\nabla u|^2 \in W^{1,2}(X, d, m) \cap L^\infty(X, m)$ and

$$\mathcal{L}_{|\nabla u|^2} \geq 2 \left(\frac{(\Delta u)^2}{N} dm + \langle \nabla u, \nabla(\Delta u) \rangle dm + K |\nabla u|^2 dm \right).$$

Before proving the local version of Bochner inequality, Theorem 1.1, we point out that every function with Laplacian in $L^p(X, m)$ for $p > N$ is locally Lipschitz continuous. This was first proven for Ahlfors regular spaces in [KRS03, Jia11] but also holds in our setting (see [Jia13] for the L^∞ case and more general spaces and [Kel13] for necessary adjustments for the case $p > N$ in our setting).

Lemma 3.2 (Lipschitz regularity [KRS03, Jia11],[Jia13, Kel13]). *Let (X, d, m) be an $RCD^*(K, N)$ mms. Any $u \in W^{1,2}_{\mathrm{loc}}(X, d, m)$ with $\Delta u \in L^p_{\mathrm{loc}}(X, d, m)$ and $p > N$ is locally Lipschitz continuous.*

One of the main tool to prove Theorem 1.1 is the construction of the following nice cut-off functions.

Lemma 3.3. *Let (X, d, m) be an $RCD^*(K, N)$ space and x_0 any point in X. There is a Lipschitz function $\Psi = \Psi_{x_0} : X \to \mathbb{R}$ such that $\Psi, \Delta\Psi \in W^{1,2}_c(X, d, m) \cap L^\infty(X, m)$ and Ψ is constant in a neighborhood of x_0.*

Proof. Note that by the $RCD^*(K, N)$ assumption P_t is hypercontractive,([AGS11, Remark 6.4]) i.e.

$$P_t : L^2(X, m) \to \mathcal{D}(\Delta) \cap L^\infty(X, m).$$

Furthermore, if p_t is the heat kernel of P_t then

$$p_t(x_0, x) = P_{\frac{t}{2}}(p_{\frac{t}{2}}(x_0, \cdot))(x).$$

In addition, note that $P_s \Delta u = \Delta P_s u$ for any $u \in \mathcal{D}(\Delta)$.

Now let $\psi(x) := p_t(x_0, x)$. Then $\psi \in \mathcal{D}(\Delta) \cap L^\infty(X, d, m)$ with $\Delta\psi \in \mathcal{D}(\Delta) \cap L^\infty(X, d, m)$. Because ψ is Lipschitz, $|\nabla\psi|^2 \in W^{1,2}(X, d, m) \cap L^\infty(X, d, m)$ by Lemma 3.1.

If we choose t sufficiently small than there are two constants $0 < l < L < \infty$ such that $\psi(x) > L$ in a neighborhood U of x_0 and $\psi(x) < l$ outside a sufficiently large neighborhood Ω. Now define a smooth real-valued cut-off function $h : \mathbb{R} \to \mathbb{R}$ such that $h(r) = 1$ for $r \geq L$ and $h(r) = 0$ for $r \leq l$. By chain rule we have for $\Psi := h \circ \psi$

$$\Delta\Psi = h' \circ \psi \Delta\psi + h'' \circ \psi |\nabla\psi|^2.$$

Because h is smooth and ψ is bounded, Ψ is Lipschitz and

$$\Psi, \Delta\Psi \in W^{1,2}(X, d, m) \cap L^\infty(X, d, m).$$

In particular, Ψ and $\Delta\Psi$ have compact support and are constant in a neighborhood of x_0. $\qquad\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. It suffices to show that the Bochner inequality holds locally, that is for all $\varphi \in W_c^{1,2}(X, d, m)$ we have

$$
\begin{aligned}
\int \langle \nabla\varphi, \nabla|\nabla u|^2 \rangle dm &= \int \varphi d\mathcal{L}_{|\nabla u|^2} \\
&\geq 2\Big(\int \varphi \frac{(\Delta u)^2}{N} dm + \int \varphi \langle \nabla u, \nabla(\Delta u) \rangle dm \\
&\quad + K \int \varphi |\nabla u|^2 dm \Big).
\end{aligned}
$$

Furthermore, we can assume without loss of generality that for each $x_0 \in X$ there is a neighborhood U' such that the Bochner inequality above holds for all $\phi \in W^{1,2}(X, d, m)$ with support in U'.

Let Ψ be the cut-off function of the previous lemma and let Ω be open and bounded containing the support of Ψ. In addition, let U be a neighborhood of x_0 such that Ψ is constant in U. Then

$$\Delta(u \cdot \Psi)_{|U} \equiv \Delta u_{|U}.$$

By Leibniz rule

$$\Delta(u \cdot \Psi) = u\Delta\Psi + \Psi\Delta u + 2\langle \nabla u, \nabla\Psi \rangle.$$

Also note by bilinearity of $\langle \nabla\cdot, \nabla\cdot \rangle$

$$
2\langle \nabla u, \nabla\Psi \rangle(x) =
\begin{cases}
|\nabla(u+\Psi)|^2(x) - (|\nabla u|^2(x) + |\nabla\Psi|^2(x)) & x \in \Omega \\
0 & x \notin \Omega
\end{cases},
$$

i.e. $\langle \nabla u, \nabla\Psi \rangle$ is a sum of local $W^{1,2}$-functions and thus itself in $W_{\mathrm{loc}}^{1,2}(X, d, m)$. Because it has compact support, it is in $W^{1,2}(X, m)$. So that we conclude $\Delta(u\cdot\Psi) \in W^{1,2}(X, d, m)$.

Because $\Delta u \in L_{\mathrm{loc}}^p(X, m)$ we know by Lemma 3.2 that u is locally Lipschitz. Therefore, $u \cdot \Psi$ is Lipschitz continuous and has compact support. Thus we can apply the Bochner inequality (6) to $u\cdot\Psi$, in particular, the inequality holds for test functions with support in U. Because u and $u \cdot \Psi$ agree on U we have for all open $U' \subset\subset U$

$$\int_{U'} \varphi d\mathcal{L}_{|\nabla u|^2} \geq 2\left(\int_{U'} \varphi \frac{(\Delta u)^2}{N} dm + \int_{U'} \varphi \langle \nabla u, \nabla(\Delta u) \rangle dm + K \int_{U'} \varphi |\nabla u|^2 dm \right).$$

$\qquad\square$

4. Cheng-Yau type local gradient estimate

In this section, we shall prove Cheng-Yau type local gradient estimate for harmonic functions on $RCD^*(K, N)$ mms. We denote by $B_R = B_R(x_0)$ the open ball in X centered at some point $x_0 \in X$ with radius R.

For $RCD^*(0, N)$ mms, the standard Moser iteration using the volume doubling property (implied by the Bishop-Gromov volume comparison (3)) and the Poincaré inequality (5) yield the following Harnack inequality, see e.g. Han-Lin [HL97].

Theorem 4.1. *Let (X, d, m) be an $RCD^*(0, N)$ mms. Then*

(a) (mean value inequality) there exists a constant $C = C(N)$ such that for any subharmonic function u on B_{2R}

$$\operatorname*{ess\,sup}_{B_R} |u| \le C \fint_{B_{2R}} |u| dm, \tag{7}$$

where $\fint_{B_{2R}} |u| dm = \frac{1}{|B_{2R}|} \int_{B_{2R}} |u| dm$.

(b) (weak Harnack inequality) there exists a constant $C = C(N)$ such that for any nonnegative superharmonic function u on B_{2R}

$$\operatorname*{ess\,inf}_{B_R} u \ge C \fint_{B_{2R}} u dm.$$

(c) (Harnack inequality) there exists a constant $C = C(N)$ such that for any nonnegative harmonic function u on B_{2R}

$$\sup_{B_R} u \le C \inf_{B_R} u.$$

In order to prove Theorem 1.2, we need to adopt a delicate Moser iteration based on the local Bochner inequality. By the Bishop-Gromov volume comparison and the Poincaré inequality for $RCD^*(K, N)$ mms above, we can prove the following Sobolev inequality, see e.g. [MW11, Lemma 3.2].

Theorem 4.2 (local uniform Sobolev inequality)**.** *Let (X, d, m) satisfy $RCD^*(K, N)$, with $K \le 0$. Then there exist two constants $\nu > 2$ and C, both depending only on N, such that for $B_R \subset X$ and $u \in W^{1,2}_{\mathrm{loc}}(B_R)$,*

$$\left(\fint_{B_R} (u - u_{B_R})^{\frac{2\nu}{\nu-2}} dm \right)^{\frac{\nu-2}{\nu}} \le e^{C(1+\sqrt{-K}R)} R^2 |B_R|^{-\frac{2}{\nu}} \int_{B_R} |\nabla u|^2 dm, \tag{8}$$

where $u_{B_R} = \fint_{B_R} u dm$. In particular,

$$\left(\fint_{B_R} u^{\frac{2\nu}{\nu-2}} dm \right)^{\frac{\nu-2}{\nu}} \le e^{C(1+\sqrt{-K}R)} R^2 |B_R|^{-\frac{2}{\nu}} \int_{B_R} (|\nabla u|^2 + R^{-2}u^2) dm. \tag{9}$$

By the Sobolev inequality above, we may adopt a delicate Moser iteration as in Hua-Xia [HX13] to prove Cheng-Yau's local gradient estimate, Theorem 1.2. For the completeness, we include the proof here.

Proof of Theorem 1.2. Without loss of generality, we may assume that u is a positive harmonic function on B_{4R}. Lemma 3.2 and Theorem 1.1 yield that u is locally Lipschitz continuous in B_{4R} and $|\nabla u|^2 \in W^{1,2}_{\mathrm{loc}}(B_{4R})$. Set $v := \log u$. One can easily verify that

$$\mathcal{L}_v = -|\nabla v|^2 \cdot dm. \tag{10}$$

Since $v \in \mathrm{Lip}(B_{2R})$, by setting $f = |\nabla v|^2$, it follows from the Bochner inequality (1) in Theorem 1.1 that for any $0 \leq \eta \in \mathrm{Lip}_c(B_{2R})$,

$$\int_{B_{2R}} \langle \nabla \eta, \nabla f \rangle dm \leq \int_{B_{2R}} \eta \left(2 \langle \nabla v, \nabla f \rangle - 2Kf - \frac{2}{N} f^2 \right) dm. \qquad (11)$$

In fact, by an approximation argument, (11) holds for any $0 \leq \eta \in W_c^{1,2}(B_{2R}) \cap L^\infty(B_{2R})$. Let $\eta = \phi^2 f^\beta$, with $\phi \in \mathrm{Lip}_c(B_{2R})$, $0 \leq \phi \leq 1$ and $\beta \geq 1$. Then η is an admissible test function for (11). Hence we have from (11) that

$$\int_{B_{2R}} \left(\beta \phi^2 f^{\beta-1} |\nabla f|^2 + 2\phi f^\beta \langle \nabla f, \nabla \phi \rangle \right) dm$$
$$\leq \int_{B_{2R}} \phi^2 f^\beta \left(2 \langle \nabla v, \nabla f \rangle - 2Kf - \frac{2}{N} f^2 \right) dm.$$

It follows from the Cauchy-Schwarz inequality ((7) in Theorem 2.6) that

$$\frac{4\beta}{(\beta+1)^2} \int_{B_{2R}} \phi^2 |\nabla f^{\frac{\beta+1}{2}}|^2 dm \quad \leq \quad \frac{4}{\beta+1} \int_{B_{2R}} \phi f^{\frac{\beta+1}{2}} |\nabla \phi| |\nabla f^{\frac{\beta+1}{2}}| dm$$
$$+ \frac{4}{\beta+1} \int_{B_{2R}} \phi^2 f^{\frac{\beta+2}{2}} |\nabla f^{\frac{\beta+1}{2}}| dm$$
$$- \int_{B_{2R}} \frac{2}{N} \phi^2 f^{\beta+2} dm - \int_{B_{2R}} 2K \phi^2 f^{\beta+1} dm.$$

Using the Hölder inequality, we obtain

$$\int_{B_{2R}} \phi^2 |\nabla f^{\frac{\beta+1}{2}}|^2 dm \quad \leq \quad C \int_{B_{2R}} |\nabla \phi|^2 f^{\beta+1} dm + C \int_{B_{2R}} \phi^2 f^{\beta+2} dm$$
$$- C\beta \int_{B_{2R}} \phi^2 f^{\beta+2} dm - C\beta K \int_{B_{2R}} \phi^2 f^{\beta+1} dm.$$

We remark that from now on, C denotes various constants depend only on N.

For β sufficiently large, we can absorb the second term on the right hand side and get

$$\int_{B_{2R}} |\nabla(\phi f^{\frac{\beta+1}{2}})|^2 dm + C\beta \int_{B_{2R}} \phi^2 f^{\beta+2} dm$$
$$\leq \quad 2C \int_{B_{2R}} |\nabla \phi|^2 f^{\beta+1} dm - C\beta K \int_{B_{2R}} \phi^2 f^{\beta+1} dm. \qquad (12)$$

Using the Sobolev inequality (9), we obtain

$$\left(\int_{B_{2R}} \phi^{2\chi} f^{(\beta+1)\chi} dm \right)^{\frac{1}{\chi}} \leq e^{C(1+\sqrt{-K}R)} R^2 |B_{2R}|^{-\frac{2}{\nu}} \left(C \int_{B_{2R}} |\nabla \phi|^2 f^{\beta+1} dm \right.$$
$$\left. + (CR^{-2} - C\beta K) \int_{B_{2R}} \phi^2 f^{\beta+1} dm - \beta \int_{B_{2R}} \phi^2 f^{\beta+2} dm \right), \qquad (13)$$

where $\chi = \nu/(\nu - 2)$.

We first use (13) to prove the following:

Lemma 4.3. *There exists two large positive constants C_0 and C such that for $\beta_0 = C_0(1 + \sqrt{-K}R)$ and $\beta_1 = (\beta_0 + 1)\chi$, we have $f \in L^{\beta_1}(B_{\frac{3}{2}R})$ and*

$$\|f\|_{L^{\beta_1}\left(B_{\frac{3}{2}R}\right)} \leq C\frac{(1 + \sqrt{-K}R)^2}{R^2}|B_{2R}|^{\frac{1}{\beta_1}}. \tag{14}$$

Proof. Let C_0 be large enough such that $\beta_0 = C_0(1 + \sqrt{-K}R)$ satisfies (12) and (13). We rewrite (13) for $\beta = \beta_0$ as

$$\left(\int_{B_{2R}} \phi^{2\chi} f^{(\beta_0+1)\chi} dm\right)^{\frac{1}{\chi}} \leq e^{C\beta_0}|B_{2R}|^{-\frac{2}{\nu}}\left(CR^2\int_{B_{2R}} |\nabla\phi|^2 f^{\beta_0+1} dm \tag{15}\right.$$
$$\left. +C_1\beta_0^3\int_{B_{2R}} \phi^2 f^{\beta_0+1} dm - \beta_0 R^2\int_{B_{2R}} \phi^2 f^{\beta_0+2} dm\right).$$

We estimate the second term on the right-hand side of (15) as follows:

$$C_1\beta_0^3\int_{B_{2R}} \phi^2 f^{\beta_0+1} dm = C_1\beta_0^3\left(\int_{\{f\geq 2C_1\beta_0^2 R^{-2}\}} \phi^2 f^{\beta_0+1} dm + \int_{\{f<2C_1\beta_0^2 R^{-2}\}} \phi^2 f^{\beta_0+1} dm\right)$$
$$\leq \frac{1}{2}\beta_0 R^2\int_{B_{2R}} \phi^2 f^{\beta_0+2} dm + C^{\beta_0+1}\beta_0^3\left(\frac{\beta_0}{R}\right)^{2(\beta_0+1)}|B_{2R}|. \tag{16}$$

Set $\phi = \psi^{\beta_0+2}$ with $\psi \in \text{Lip}_0(B_{2R})$ satisfying

$$0 \leq \psi \leq 1, \quad \psi \equiv 1 \text{ in } B_{\frac{3}{2}R}, \ |\nabla\psi| \leq \frac{C}{R}.$$

Then

$$R^2|\nabla\phi|^2 \leq C\beta_0^2\phi^{\frac{2(\beta_0+1)}{\beta_0+2}}.$$

By the Hölder inequality and the Young inequality, the first term in the right-hand side of (15) can be estimated as follows:

$$CR^2\int_{B_{2R}} |\nabla\phi|^2 f^{\beta_0+1} dm \leq C\beta_0^2\int_{B_{2R}} \phi^{\frac{2(\beta_0+1)}{\beta_0+2}} f^{\beta_0+1} dm$$
$$\leq C\beta_0^2\left(\int_{B_{2R}} \phi^2 f^{\beta_0+2} dm\right)^{\frac{\beta_0+1}{\beta_0+2}}|B_{2R}|^{\frac{1}{\beta_0+2}}$$
$$\leq \frac{1}{2}\beta_0 R^2\int_{B_{2R}} \phi^2 f^{\beta_0+2} dm + C\beta_0^{\beta_0+3} R^{-2(\beta_0+1)}|B_{2R}|. \tag{17}$$

Substituting the estimates (16) and (17) into (15), we obtain

$$\left(\int_{B_{2R}} \phi^{2\chi} f^{(\beta_0+1)\chi} dm\right)^{\frac{1}{\chi}} \leq 2e^{C\beta_0} C^{\beta_0+1}\beta_0^3\left(\frac{\beta_0}{R}\right)^{2(\beta_0+1)}|B_{2R}|^{1-\frac{2}{\nu}}.$$

Taking the $(\beta_0 + 1)$-st root on both sides, we get

$$\|f\|_{L^{\beta_1}(B_{\frac{3}{2}R})} \leq C\left(\frac{\beta_0}{R}\right)^2 |B_{2R}|^{\frac{1}{\beta_1}}.$$

$\square$

Now we start from (13) and use Moser's iteration to prove Theorem 1.2.

Let $R_k = R + R/2^k$ and $\phi_k \in \mathrm{Lip}_0(B_{R_k})$ satisfy

$$0 \le \phi_k \le 1, \quad \phi_k \equiv 1 \text{ in } B_{R_{k+1}}, \quad |\nabla \phi_k| \le C\frac{2^{k+1}}{R}.$$

Let β_0, β_1 be the numbers in Lemma 4.3 and $\beta_{k+1} = \beta_k \chi$ for $k \ge 1$. One can deduce from (13) with $\beta + 1 = \beta_k$ and $\phi = \phi_k$ that (we have dropped the last term on the right-hand side of (13) since it is negative)

$$\|f\|_{L^{\beta_{k+1}}(B_{R_{k+1}})} \le e^{C\frac{\beta_0}{\beta_k}} |B_{2R}|^{-\frac{2}{\nu}\frac{1}{\beta_k}} (4^k + \beta_0^2 \beta_k)^{\frac{1}{\beta_k}} \|f\|_{L^{\beta_k}(B_{R_k})}.$$

Hence by iteration we get

$$\|f\|_{L^\infty(B_R)} \le e^{C\beta_0 \sum_k \frac{1}{\beta_k}} |B_{2R}|^{-\frac{2}{\nu}\sum_k \frac{1}{\beta_k}} \prod_k (4^k + 2\beta_0^3 \chi^k)^{\frac{1}{\beta_k}} \|f\|_{L^{\beta_1}(B_{\frac{3}{2}R})}.$$

Since $\sum_k \frac{1}{\beta_k} = \frac{\nu}{2}\frac{1}{\beta_1}$ and $\sum_k \frac{k}{\beta_k}$ converges, we have

$$\begin{aligned}
\|f\|_{L^\infty(B_R)} &\le & Ce^{C\frac{\beta_0}{\beta_1}} \beta_0^{\frac{3\nu}{2}\frac{1}{\beta_1}} |B_{2R}|^{-\frac{1}{\beta_1}} \|f\|_{L^{\beta_1}(B_{\frac{3}{2}R})} \\
&\le & C|B_{2R}|^{-\frac{1}{\beta_1}} \|f\|_{L^{\beta_1}(B_{\frac{3}{2}R})}.
\end{aligned}$$

Using Lemma 4.3, we conclude

$$\|f\|_{L^\infty(B_R)} \le C(N)\frac{(1 + \sqrt{-K}R)^2}{R^2},$$

which implies

$$\|\nabla \log u\|_{L^\infty(B_R)} \le C(N)\frac{1 + \sqrt{-K}R}{R}.$$

This proves Theorem 1.2.

$\square$

5. POLYNOMIAL GROWTH HARMONIC FUNCTIONS

Since the Laplace operator on RCD mms is linear, we may study the dimension of the space of polynomial growth harmonic functions as Colding-Minicozzi did [CM97a, CM98a, CM98b]. In this section, we will prove our main results on dimension estimates, Theorem 1.4 and Theorem 1.5.

Fix some $p \in X$. For any $d > 0$, let

$$H^d(X) := \{u \in W^{1,2}_{\mathrm{loc}}(X) : \mathcal{L}_u = 0, |u(x)| \le C(1 + d(x,p))^d\}$$

denote the space of polynomial growth harmonic functions on X with growth rate less than or equal to d.

To estimate the dimension of $H^d(X)$, we need the Bishop-Gromov volume comparison (3) and the Poincaré inequality (5), see [CM98b, Li97, Hua11]. In the following, we shall prove Theorem 1.4 for which we do not need the Bochner inequality. The first lemma follows easily from a contradiction argument.

Lemma 5.1 ([Hua11, Lemma 3.4]). *For any finite dimensional subspace $S \subset H^d(X)$, there exists a constant $R_0(S)$ depending on S, such that for $\forall\ R \geq R_0$*

$$\langle u, v \rangle_R = \int_{B_R} uv\, dm$$

is an inner product on S.

The next lemma follows verbatim from [Li12, Lemma 28.3] or [Hua11, Lemma 3.7].

Lemma 5.2. *Let (X, d, m) be an $RCD^*(0, N)$ mms and S be a k-dimensional subspace of $H^d(X)$. For any $p \in X, \beta > 1, \delta > 0, R_0 > 0$, there exists $R > R_0$ such that if $\{u_i\}_{i=1}^k$ is an orthonormal basis of S with respect to the inner product $< u, v >_{\beta R} := \int_{B_{\beta R}(p)} uv\, dm$, then*

$$\sum_{i=1}^k \int_{B_R(p)} u_i^2 dm \geq k\beta^{-(2d+N+\delta)}.$$

The following lemma can be derived from the mean value inequality for subharmonic functions, see Theorem 4.1 (a).

Lemma 5.3. *Let (X, d, m) be an $RCD^*(0, N)$ mms and S be a k-dimensional subspace of $H^d(X)$. Then there exists a constant $C(N)$ such that for any basis of S, $\{u_i\}_{i=1}^k$, $\forall\ p \in X, R > 0, 0 < \epsilon < \frac{1}{2}$ we have*

$$\sum_{i=1}^k \int_{B_R(p)} u_i^2 dm \leq C(N)\epsilon^{-(N-1)} \sup_{u \in \langle A, U \rangle} \int_{B_{(1+\epsilon)R}(p)} u^2 dm,$$

where $\langle A, U \rangle = \{v = \sum_i a_i u_i, \sum_i a_i^2 = 1\}$.

Proof. For fixed $x \in B_R(p)$, we set $S_x = \{u \in S \mid u(x) = 0\}$. The subspace $S_x \subset S$ is of at most codimension 1, since for any $v, w \notin S_x$, $v - \frac{v(x)}{w(x)}w \in S_x$. Then there exists an orthogonal transformation mapping $\{u_i\}_{i=1}^k$ to $\{v_i\}_{i=1}^k$ such that $v_i \in S_x$, $i \geq 2$. By the mean value inequality (7), we have

$$\begin{aligned}
\sum_{i=1}^k u_i^2(x) &= \sum_{i=1}^k v_i^2(x) = v_1^2(x) \leq C \fint_{B_{(1+\epsilon)R-r(x)}(x)} v_1^2 dm \\
&\leq C|B_{(1+\epsilon)R-r(x)}(x)|^{-1} \sup_{u \in \langle A, U \rangle} \int_{B_{(1+\epsilon)R}(p)} u^2 dm, \qquad (18)
\end{aligned}$$

where $r(x) = d(p, x)$. For simplicity, we denote $V_x(t) = |B_t(x)|$ and $A_x(t) = \frac{d}{dt}V_x(t)$ for a.e. $t \in [0, \infty)$.

By the Bishop-Gromov volume comparison (3), we have

$$V_x((1+\epsilon)R - r(x)) \geq \left(\frac{(1+\epsilon)R - r(x)}{2R}\right)^N V_x(2R) \geq \left(\frac{(1+\epsilon)R - r(x)}{2R}\right)^N V_p(R).$$

Hence, substituting it into (18) and integrating over $B_R(p)$, we have

$$\sum_{i=1}^k \int_{B_R(p)} u_i^2 dm \leq \frac{C2^n}{V_p(R)} \sup_{u \in \langle A, U \rangle} \int_{B_{(1+\epsilon)R}(p)} u^2 dm \int_{B_R(p)} (1+\epsilon-R^{-1}r(x))^{-N} dm(x) \tag{19}$$

Define $f(t) = (1 + \epsilon - R^{-1}t)^{-N}$, then $f'(t) = \frac{N}{R}(1 + \epsilon - R^{-1}t)^{-(N+1)} \geq 0$. Since $|\nabla r|(x) = 1$ for m-a.e. $x \in X$ (see [Gig12, Theorem 5.3] and the proof of [Gig12, Corollary 5.15]), the coarea formula implies that

$$\int_{B_R(p)} f(r(x))dm(x) \leq \int_0^R f(t)A_p(t)dt.$$

Since $A_p(t) = V'_p(t)$ a.e., integrating by parts we obtain

$$\int_0^R f(t)A_p(t)dt = f(t)V_p(t) \mid_0^R - \int_0^R V_p(t)f'(t)dt.$$

Noting that $f'(t) \geq 0$ and the Bishop-Gromov volume comparison (3), we have

$$\int_0^R V_p(t)f'(t)dt \quad \geq \quad \frac{V_p(R)}{R^N} \int_0^R t^N f'(t)dt$$

$$= \quad \frac{V_p(R)}{R^N}\{t^N f(t) \mid_0^R -N \int_0^R t^{(N-1)}f(t)dt\}$$

Therefore

$$\int_{B_R(p)} f(r(x))dm(x) \leq \frac{NV_p(R)}{R^N} \int_0^R t^{(N-1)}f(t)dt \leq \frac{N}{N-1}V_p(R)\epsilon^{-(N-1)}.$$

Combining this with (19), we prove the lemma. $\qquad\square$

By using the previous two lemmas, we are able to prove the optimal dimension estimate for the space of polynomial growth harmonic functions.

Proof of Theorem 1.4. For any k-dimensional subspace $S \subset H^d(X)$, we set $\beta = 1 + \epsilon$. Let $\{u_i\}_{i=1}^k$ be an orthonormal basis of S with respect to the inner product $< \cdot, \cdot >_{\beta R}$. By Lemma 5.2, we have

$$\sum_{i=1}^k \int_{B_R(p)} u_i^2 dm \geq k(1 + \epsilon)^{-(2d+N+\delta)}.$$

Lemma 5.3 implies

$$\sum_{i=1}^k \int_{B_R(p)} u_i^2 dm \leq C(N)\epsilon^{-(N-1)}.$$

Setting $\epsilon = \frac{1}{2d}$ and letting δ tend to 0, we have

$$k \leq C(N) \left(\frac{1}{2d}\right)^{-(N-1)} \left(1 + \frac{1}{2d}\right)^{(2d+N+\delta)} \leq Cd^{N-1}. \qquad (20)$$

Noting that (20) holds for arbitrary subspace S of $H^d(X)$, we prove the theorem.
$\qquad\square$

By Corollary 1.3, we know that $H^\alpha(X) = 1$ for any $\alpha < 1$. The Bochner inequality can be used to obtain the following dimension estimate for the space of harmonic functions of linear growth. This estimate for the dimension of the space of linear growth harmonic functions is more tricky, see [Li12].

The following auxiliary theorem on the behavior of subharmonic functions on $RCD^*(0, N)$ mms appearing in the proof of Theorem 1.5 is interesting in its own right.

Theorem 5.4 (Mean value theorem at infinity)**.** *Let (X, d, m) be an $RCD^*(0, N)$ mms. Then for any bounded nonnegative subharmonic function u,*

$$\lim_{R \to \infty} \fint_{B_R} u \, dm = \operatorname*{ess\,sup}_X u,$$

where $\fint_{B_R} u \, dm = \frac{1}{|B_R|} \int_{B_R} u \, dm$.

Proof. Set $w = \operatorname*{ess\,sup}_X u - u$. It suffices to show that

$$\lim_{R \to \infty} \fint_{B_R} w \, dm = 0.$$

Since $\operatorname*{ess\,inf}_X w = 0$, for any $\epsilon > 0$ there exists an $R_\epsilon > 0$ such that

$$\operatorname*{ess\,inf}_{B_{R_\epsilon}} w < \epsilon.$$

Note that w is a bounded nonnegative superharmonic function on X. The weak Harnack inequality, Theorem 4.1 (b), implies that for any $R \geq 2R_\epsilon$

$$\fint_{B_R} w \, dm \leq C \operatorname*{ess\,inf}_{B_{\frac{1}{2}R}} w \leq \operatorname*{ess\,inf}_{B_{R_\epsilon}} w < \epsilon.$$

This proves the lemma. □

Now we prove the dimension estimate for linear growth harmonic functions on $RCD^*(0, N)$ mms using only the weak Harnack inequality for superharmonic functions.

Proof of Theorem 1.5. We claim that for any $f \in H^1(X)$, $|\nabla f|^2$ is a bounded subharmonic function on X. By the Bochner inequality in Theorem 1.1, $|\nabla f|^2$ is a subharmonic function. Using the Caccioppoli inequality, Theorem 2.5, and the mean value inequality, Theorem 4.1 (a), we have

$$\begin{aligned}
\operatorname*{ess\,sup}_{B_R} |\nabla f|^2 \quad &\leq \quad C \fint_{B_{2R}} |\nabla f|^2 dm \leq \frac{C}{R^2} \fint_{B_{4R}} f^2 dm \\
&\leq \quad C \frac{(\sup_{B_{4R}} f)^2}{R^2} \leq C,
\end{aligned}$$

where we used the linear growth property of f in the last inequality. This is true for any $R > 0$, hence $\operatorname*{ess\,sup}_X |\nabla f|^2 \leq C$. This proves the claim. Hence, Theorem 5.4 yields that

$$\lim_{R \to \infty} \fint_{B_R} |\nabla f|^2 dm = \operatorname*{ess\,sup}_X |\nabla f|^2. \tag{21}$$

For a fixed point $p \in X$, we define a subspace of $H^1(X)$ by $H' = \{f \in H^1(X) | f(p) = 0\}$, and a bilinear form D on H' by

$$D(f, g) = \lim_{R \to \infty} \fint_{B_R(p)} \langle \nabla f, \nabla g \rangle dm.$$

It is easy to see that H' is of at most codimension one in $H^1(X)$. In addition, D is an inner product on H' by the mean value theorem at infinity, Theorem 5.4. Given any finite dimensional subspace H'' in H' with $\dim H'' = k$, let $\{f_1, f_2, \cdots, f_k\}$ be an orthonormal basis of H'' with respect to the inner product D. Set $F^2(x) := \sum_{i=1}^k f_i^2(x)$ and $F^\delta(x) := \sqrt{\sum_{i=1}^k f_i^2(x) + \delta}$, $(\delta > 0)$. Since $\{f_i\}_{i=1}^k$ are Lipschitz, the weak upper gradient $|\nabla f_i|_w(x)$ is well defined for m-a.e. $x \in X$, that is, there

exists a measurable subset $Y \subset X$ with $m(X \setminus Y) = 0$ such that $|\nabla f_i|_w(y)$ is well defined for all $y \in Y$. For any $y \in Y$, there is an orthogonal transformation $T_y : \mathbb{R}^k \to \mathbb{R}^k$ such that $T_y(f_1(y), f_2(y), \cdots, f_k(y)) = (\sqrt{\sum_{i=1}^{k} f_i^2(y)}, 0, \cdots, 0)$. We denote $g_i(z) := \sum_{j=1}^{k} T_{y,ij} f_j(z)$ for any $z \in X$. Clearly, $\{g_i\}_{i=1}^{k}$ is an orthonormal basis of H'' with respect to D and $(F^\delta)^2(z) = \sum_{i=1}^{k} g_i^2(z) + \delta$ for all $z \in X$. Since T_y is a constant matrix, $|\nabla g_i|_w(z)$ is well defined for all $z \in Y$.

By the product rule (5) in Theorem 2.6,

$$
\begin{aligned}
|\nabla F^\delta|_w(y) F^\delta(y) &= \sum_{i=1}^{k} g_i(y)|\nabla g_i|_w(y) \\
&= g_1(y)|\nabla g_1|_w(y), \quad (\text{by } g_i(y) = 0, \ i \geq 2).
\end{aligned}
$$

Since $\{g_i\}_{i=1}^{k}$ is the orthonormal basis of H'', the equation (21) implies that $\operatorname{ess\,sup}_X |\nabla g_i| \leq 1$ for any $1 \leq i \leq k$. Then $\{g_i\}_{i=1}^{k}$ are Lipschitz functions of Lipschitz constant at most 1. Hence the chain rule yields for all $y \in Y$,

$$
|\nabla F^\delta|(y) \leq \frac{g_1(y)}{\sqrt{g_1^2(y) + \delta}} \leq 1.
$$

Hence for any $\delta > 0$, F^δ is a Lipschitz function of Lipschitz constant at most 1. Since $F^\delta(x) \to F(x)$ for any $x \in X$ as $\delta \to 0$, F is a Lipschitz function with $|\nabla F| \leq 1$. Note that $F(p) = 0$, integrating along a geodesic we have

$$
F(x) \leq d(x, p).
$$

For any $R > 0$, $\epsilon > 0$, we define a cut-off function as

$$
\chi_\epsilon(x) := \frac{(R + \epsilon - d(x, p))_+}{\epsilon} \wedge 1.
$$

Then χ_ϵ is a Lipschitz function supported in $B_{R+\epsilon}$ with $\chi_\epsilon|_{B_R} \equiv 1$ and $|\nabla \chi_\epsilon| \leq \frac{1}{\epsilon}$. Since $\{f_i\}_{i=1}^{k}$ are harmonic, F^2 is subharmonic, i.e. $\mathcal{L}_{F^2} \geq 0$. Then

$$
\begin{aligned}
2 \sum_{i=1}^{k} \int_{B_R} |\nabla f_i|^2 dm &= \int_{B_R} \mathcal{L}_{F^2} \leq \mathcal{L}_{F^2}(\chi_\epsilon) \\
&= -\int_{B_{R+\epsilon}} \langle \nabla F^2, \nabla \chi_\epsilon \rangle dm \\
&\leq \frac{2}{\epsilon} \int_{B_{R+\epsilon} \setminus B_R} F |\nabla F| dm \\
&\leq \frac{2(R+\epsilon)}{\epsilon} |B_{R+\epsilon} \setminus B_R|.
\end{aligned}
$$

Let $\epsilon \to 0$, we have for a.e. $R > 0$

$$
\sum_{i=1}^{k} \fint_{B_R} |\nabla f_i|^2 dm \leq \frac{R |A_R|}{|B_R|}. \tag{22}
$$

The fact that $\{f_i\}_{i=1}^{k}$ is the orthonormal basis for the inner product D implies that for any $\epsilon_1 > 0$, there exists R_{ϵ_1} such that for any $R \geq R_{\epsilon_1}$ we have

$$
\sum_{i=1}^{k} \fint_{B_R} |\nabla f_i|^2 dm \geq k - \epsilon_1.
$$

Combining this with (22), we obtain for any $R \geq R_{\epsilon_1}$,

$$\frac{k - \epsilon_1}{R} \leq \frac{|A_R|}{|B_R|}.$$

Integrating this inequality from R_{ϵ_1} to R, we have for any $R \geq R_{\epsilon_1}$

$$\left(\frac{R}{R_{\epsilon_1}}\right)^{k - \epsilon_1} \leq \frac{|B_R|}{|B_{R_{\epsilon_1}}|}.$$

Hence the assumption (2) on the volume growth of X yields $k - \epsilon_1 \leq n$. Let $\epsilon_1 \to 0$, we prove the theorem.

$\square$

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