Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Existence and non-existence of area-minimizing hypersurfaces in manifolds of non-negative Ricci curvature

by

Qi Ding, Jürgen Jost, and Yuanlong Xin

EXISTENCE AND NON-EXISTENCE OF AREA-MINIMIZING HYPERSURFACES IN MANIFOLDS OF NON-NEGATIVE RICCI CURVATURE

QI DING, J. JOST, AND Y.L. XIN

ABSTRACT. We study minimal hypersurfaces in manifolds of non-negative Ricci curvature, Euclidean volume growth and quadratic curvature decay at infinity. By comparison with capped spherical cones, we identify a precise borderline for the Ricci curvature decay. Above this value, no complete area-minimizing hypersurfaces exist. Below this value, in contrast, we construct examples.

1. INTRODUCTION

Complete minimal (hyper)surfaces where first considered in Euclidean spaces. In fact, there was one particular problem that inspired much of the spectacular development of the field. This was the Bernstein problem, which was concerned with the question to what extent the classical Bernstein theorem can be generalized. Bernstein's theorem simply says that an entire minimal graph in $\mathbb{R}^3$ has to be a plane. The original proofs were strictly two-dimensional, making essential use of conformal coordinates, but the statement itself certainly is meaningful in any dimension. Partly in order to have mathematical tools with which to approach such questions, the field of geometric measure theory was developed. Higher dimensional generalizations of the Bernstein theorem were achieved by successive efforts of W. Fleming [13], E. De Giorgi [9], F. J. Almgren [1] and J. Simons [28] up to dimension seven within the framework of geometric measure theory. In 1969, Bombieri-De Giorgi-Giusti [4] then constructed a nontrivial entire minimal graph in $\mathbb{R}^{n+1}$ with $n > 7$ whose tangent cone at infinity had been described earlier by Simons.

Clearly, the Bernstein problem can be further generalized. We can not only increase the dimension of the ambient space, but also allow for more general Riemannian geometries than the Euclidean one. In order to see what might happen then, we observe that minimal graphs in Euclidean space are automatically area minimizing. Thus, the Bernstein problem is essentially about the (non-)existence of a particular class of complete area-minimizing hypersurfaces. Therefore, the challenge of the Bernstein problem consists in finding sharp conditions for the existence or non-existence of complete area-minimizing hypersurfaces in curved ambient manifolds.

Let us therefore review the previous results in this direction. Using curvature estimate techniques, Schoen-Simon-Yau [25] obtained L^p-estimates for the squared norm of the second fundamental form for stable minimal hypersurfaces in certain curved ambient

manifolds. As a consequence, they showed that any stable minimal hypersurface with Euclidean volume growth in a flat N^{n+1} with $n \leq 5$ has to be totally geodesic. Later, Fischer-Colbrie and Schoen [12] proved that there are no stable minimal surfaces in 3-dimensional manifolds with positive Ricci curvature. Shen-Zhu [26] proved certain rigidity results for stable minimal hypersurfaces in N^4 or N^5. On the other hand, P. Nabonnand [22] constructed a complete manifold N^{n+1} with positive Ricci curvature which admits area-minimizing hypersurfaces. M. Anderson [3] proved a non-existence result for area-minimizing hypersurfaces in complete non-compact simply connected manifolds N^{n+1} of non-negative sectional curvature with diameter growth conditions. For rotationally symmetric spaces with conical singularities, some explicit results were obtained by F. Morgan in [21]. These results will provide us with important model spaces for the general theory.

In the present paper we will study minimal hypersurfaces in complete Riemannian manifolds that satisfy three conditions:

C1) non-negative Ricci curvature;
C2) Euclidean volume growth;
C3) quadratic decay of the curvature tensor.

Such manifolds can be much more complicated than Euclidean space, but on the other hand, this class of manifolds possesses certain topological and analytical properties [23],[8] that constrain their geometry. They admit tangent cones at infinity over a smooth compact manifold in the Gromov-Hausdorff sense. These cones may be not unique, but they have certain nice properties, proved by Cheeger-Colding [5]. Another important fact is that their Green functions have a well controlled asymptotic behavior. In particular, the Hessian of such a Green function converges to the metric tensor (up to a constant factor 2) point-wisely at infinity, as shown by Colding-Minicozzi [8]. The precise results will be described in section 4.

While our non-existence results are quite general, the existence results that we develop here, mainly for the purpose of showing that our non-existence results are sharp, are more explicit and depend on special constructions. Essentially, for these constructions, we consider ambient manifolds of the form $\Sigma \times \mathbb{R}$ where Σ is an n-dimensional Riemannian manifold with a conformally flat metric whose conformal factor depends only on the radius. This class will include a capped spherical cone with opening angle $2\pi\kappa$, denoted by MCS_κ. Its tangent cone at infinity is the uncapped spherical cone CS_κ, or equivalently, the Euclidean cone over a sphere of radius κ. These cones will be on one hand our main examples for existence results and on the other hand our model spaces for the non-existence results. The border between those two phenomena, existence vs. non-existence, will be sharp. Existence takes place for $\kappa \geq \frac{2}{n}\sqrt{n-1}$, non-existence else. The intuitive geometric reason is simply that for larger values of κ, in order to minimize area, it is most efficient to go through the vertex of the cone, whereas for smaller values of κ, it is better to avoid the vertex and go around the cone. This had already been observed by F. Morgan in [21]. As a by-product we can answer some questions raised by M. Anderson in [3].

Whereas the existence examples are specific, our non-existence results will be general. Essentially, the idea consists in reducing them to the model cases by taking cones at infinity. For this, we need some heavier machinery, including the theory of Gromov-Hausdorff limits [16, 17, 24, 15] and the theory of currents in metric spaces developed by Ambrosio-Kirchheim [2]. In order to apply those tools, we shall analyze the Green function at infinity of the ambient space and minimal hypersurfaces with Euclidean volume growth,

in order to carry the stability inequality for minimal hypersurfaces over to the asymptotic limit. The corresponding results may be of interest in themselves, see Theorem 5.1.

Our main results thus are general non-existence results for stable minimal hypersurfaces in $(n+1)-$manifolds N with conditions C1), C2) and C3) under an additional growth condition on the non-radial Ricci curvature involving a constant κ'. For the capped spherical cones MCS_κ, this constant κ' can be expressed in terms of the constant κ. More precisely, we show that N admits no complete stable minimal hypersurface with at most Euclidean volume growth if the above constant $\kappa' > \frac{(n-2)^2}{4}$, see Theorem 5.5. The existence result of Theorem 3.4 then tells us that our condition on the asymptotic non-radial Ricci curvature is optimal.

2. Preliminaries

Let Σ be an n-dimensional Riemannian manifold with metric $ds^2 = \sigma_{ij}dx_idx_j$ in local coordinates. Let D be the corresponding Levi-Civita connection on Σ. For a subset $\Omega \subset \Sigma$ let M be a graph in the product manifold $\Omega \times \mathbb{R}$ with smooth defining function u on Σ, i.e.,

$$(2.1) \qquad M = \{(x, u(x)) \in \Omega \times \mathbb{R} |\ x \in \Omega\}.$$

Since $N = \Sigma \times \mathbb{R}$ has the product metric $ds^2 = dt^2 + \sigma_{ij}dx_idx_j$, then the induced metric on M is

$$ds^2 = g_{ij}dx_idx_j = (\sigma_{ij} + u_iu_j)dx_idx_j,$$

where $u_i = \frac{\partial u}{\partial x_j}$ and $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ in the sequel. Let (σ^{ij}) be the inverse metric tensor on Σ. Let E_i and E_{n+1} be the dual vectors of dx_i and dt, respectively. Let Γ_{ij}^k be the Christoffel symbols of Σ with respect to the frame E_i, i.e., $D_{E_i}E_j = \sum_k \Gamma_{ij}^k E_k$. Set $u^i = \sigma^{ij}u_j$, $|Du|^2 = \sigma^{ij}u_iu_j$, $D_iD_ju = u_{ij} - \Gamma_{ij}^k u_k$ and $v = \sqrt{1 + |Du|^2}$. If f stands for the immersion (2.1) of Σ in $M \subset N$, then $X_i = f_*E_i = E_i + u_iE_{n+1}$, $i = 1, \cdots, n$, are tangent vectors of M in N. Let ν_M and H be the unit normal vector field and the mean curvature of M in N. Then, direct computation yields

$$\nu_M = \frac{1}{v}(-\sigma^{ij}u_jE_i + E_{n+1}),$$

$$H = \mathrm{div}_\Sigma \left(\frac{Du}{v}\right) = \frac{1}{\sqrt{\det \sigma_{kl}}}\partial_j\left(\sqrt{\det \sigma_{kl}}\frac{\sigma^{ij}u_i}{v}\right).$$

M is a minimal graph in $\Omega \times \mathbb{R}$ if and only if $H \equiv 0$ and u satisfies

$$(2.2) \qquad \mathrm{div}_\Sigma \left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = \frac{1}{\sqrt{\det \sigma_{kl}}}\partial_j\left(\sqrt{\det \sigma_{kl}}\frac{\sigma^{ij}u_i}{\sqrt{1 + |Du|^2}}\right) = 0.$$

This is the Euler-Lagrangian equation of the volume functional of M in N. Moreover, similar to the Euclidean case [30], any minimal graph on Ω is also an area-minimizing hypersurface in $\Omega \times \mathbb{R}$, see Lemma 2.1 below.

We introduce an operator $\mathfrak{L}$ on a domain $\Omega \subset \Sigma$ by

$$(2.3) \qquad \mathfrak{L}F = \left(1 + |DF|^2\right)^{\frac{3}{2}} \mathrm{div}_\Sigma \left(\frac{DF}{\sqrt{1 + |DF|^2}}\right) = \left(1 + |DF|^2\right)\Delta_\Sigma F - F_{,j}F^iF^j,$$

where $F^i = \sigma^{ik}F_k$, and $F_{i,j} = F_{ij} - \Gamma_{ij}^k F_k$ is the covariant derivative. Clearly, $\{(x, F(x)) | \ x \in \Omega\}$ is a minimal graph on Σ if and only if $\mathfrak{L}F = 0$ on Ω. We call F $\mathfrak{L}$-*subharmonic* ($\mathfrak{L}$-*superharmonic*) if $\mathfrak{L}F \geq 0$ ($\mathfrak{L}F \leq 0$).

Lemma 2.1. *Let Ω be a bounded domain in Σ and M be a minimal graph on $\overline{\Omega}$ as in (2.1) with volume element $d\mu_M$. For any hypersurface $W \subset \overline{\Omega} \times \mathbb{R}$ with $\partial M = \partial W$, one has*

$$(2.4) \qquad \int_M d\mu_M \leq \int_W d\mu_W,$$

with equality if and only if $W = M$.

Proof. Let U be the domain in N enclosed by M and W. Let Y be a vector field in M defined by

$$Y = -\sum_{i=1}^n \frac{\sigma^{ij}u_j}{v}E_i + \frac{1}{v}E_{n+1},$$

Viewing u_i and v as functions on Σ and translating Y to W along the E_{n+1} axis, we obtain a vector field in U, denoted by Y, as well. From the minimal surface equation (2.2) we have

$$\overline{\text{div}}(Y) = -\sum_i \frac{1}{\sqrt{\sigma}}\partial_{x_i}\left(\frac{\sqrt{\sigma}\sigma^{ij}u_j}{v}\right) = 0,$$

where $\overline{\text{div}}$ stands for the divergence operator on N. Let ν_M, ν_W be the unit outside normal vectors of M, W respectively. Observe that $Y|_M = \nu_M$. Then by Green's formula,

$$0 = \int_U \overline{\text{div}}(Y) = \int_M \langle Y, \nu_M\rangle d\mu_M - \int_W \langle Y, \nu_W\rangle d\mu_W$$

$$\geq \int_M d\mu_M - \int_W d\mu_W.$$

Obviously, equality holds if and only if $M = W$. $\qquad\square$

The index form from the second variational formula for the volume functional for an oriented minimal hypersurface M in N is (see Chapter 6 of [30])

$$(2.5) \qquad I(\phi, \phi) = \int_M \left(|\nabla\phi|^2 - |\bar{B}|^2\phi^2 - Ric_N(\nu_M, \nu_M)\phi^2\right)d\mu_M,$$

for any $\phi \in C_c^2(N)$, where ∇ and $\bar{B}$ are the Levi-Civita connection and the second fundamental form of M, respectively.

Let S_κ be an $n-$sphere in $\mathbb{R}^{n+1}$ with radius $0 < \kappa \leq 1$, namely,

$$S_\kappa = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \ x_1^2 + \cdots + x_{n+1}^2 = \kappa^2\}.$$

If $\{\theta_i\}_{i=1}^n$ be an orthonormal basis of S_κ, then the sectional curvature of S_κ is

$$K_S(\theta_i, \theta_j) = \frac{1}{\kappa^2} \qquad \text{for } i \neq j.$$

Let $CS_\kappa = \mathbb{R} \times_\rho S_\kappa$ be the cone over S_κ with vertex o, which has the metric

$$\sigma_C = d\rho^2 + \kappa^2\rho^2 d\theta^2,$$

where $d\theta^2$ is the standard metric on $\mathbb{S}^n(1)$.

Let $\{e_\alpha\}_{\alpha=1}^n \bigcup \{\frac{\partial}{\partial\rho}\}$ be an orthonormal basis at the considered point of CS_κ away from the vertex, then the sectional curvature and Ricci curvature of CS_κ are

(2.6)
$$K_{CS_\kappa}\left(\frac{\partial}{\partial\rho}, e_\alpha\right) = 0, \quad K_{CS_\kappa}(e_\alpha, e_\beta) = \frac{1}{\rho^2}\left(\frac{1}{\kappa^2} - 1\right),$$

$$Ric_{CS_\kappa}\left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\rho}\right) = Ric_{CS_\kappa}\left(\frac{\partial}{\partial\rho}, e_\alpha\right) = 0, \quad Ric_{CS_\kappa}(e_\alpha, e_\beta) = \frac{n-1}{\rho^2}\left(\frac{1}{\kappa^2} - 1\right)\delta_{\alpha\beta}.$$

Set $\rho = r^\kappa$, then σ_C can be rewritten as a conformally flat metric

(2.7)
$$\sigma_C = \kappa^2 r^{2\kappa-2}dr^2 + \kappa^2 r^{2\kappa}d\theta^2 = \kappa^2 r^{2\kappa-2}\sum_{i=1}^{n+1}dx_i^2 = e^{2\log\kappa-2(1-\kappa)\log r}\sum_{i=1}^{n+1}dx_i^2,$$

where $r^2 = \sum_i x_i^2$.

Let Y be an $(n-1)-$dimensional minimal hypersurface in S_κ with the second fundamental form B and CY be the cone over Y in CS_κ with vertex o. For any $0 < \epsilon < 1$ denote

$$CY_\epsilon = \{tx \in S_\kappa \times \mathbb{R}| \ x \in Y, \ t \in [\epsilon, 1]\}.$$

Clearly, Y is a minimal hypersurface in S_κ if and only if CY_ϵ is minimal in CS_κ. Moreover, let $\bar{B}$ be the second fundamental form of CY_ϵ in CS_κ, then

$$|\bar{B}|^2 = \frac{1}{\rho^2}|B|^2.$$

At any considered point, we can suppose that θ_n is the unit normal vector of $Y \subset S_\kappa$ and $\{\theta_i\}_{i=1}^{n-1}$ is the orthonormal basis of TY. Let $\nu = \frac{1}{\rho}\theta_n$ be the unit normal vector of CY_ϵ. Let $d\mu$ and $d\mu_Y$ be the volume element of CY_ϵ and Y, respectively (see Chapter 6 of [30] for a more detailed argument when $\kappa = 1$).

Now, from (2.5), the index form of CY_ϵ in CS_κ becomes

(2.8)
$$I(\phi, \phi) = \int_{CY_\epsilon}\left(-\phi\Delta_{CY}\phi - |\bar{B}|^2\phi^2 - Ric_{CS_\kappa\times\mathbb{R}}(\nu, \nu)\phi^2\right)d\mu$$

for any $\phi \in C_c^2(CY \setminus \{o\})$. Noting $Ric_{S_\kappa}(\theta_i, \theta_j) = \frac{n-1}{\kappa^2}\delta_{ij}$ and

$$Ric_{CS_k}(\nu, \nu) = \frac{1}{\rho^2}Ric_{S_k}(\theta_n, \theta_n) - \frac{n-1}{\rho^2} = \frac{n-1}{\rho^2}\left(\frac{1}{\kappa^2} - 1\right).$$

When ϕ is written as $\phi(x, \rho) \in C^2(Y \times \mathbb{R})$, a simple calculation implies

(2.9)
$$\Delta_{CY}\phi = \frac{1}{\rho^2}\Delta_Y\phi + \frac{n-1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{\partial^2\phi}{\partial\rho^2},$$

then

(2.10)
$$I(\phi, \phi) = \int_\epsilon^1\left(\int_Y\left(-\Delta_Y\phi - |B|^2\phi - \frac{n-1}{\kappa^2}\phi + (n-1)\phi\right.\right.$$
$$\left.\left. - (n-1)\rho\frac{\partial\phi}{\partial\rho} - \rho^2\frac{\partial^2\phi}{\partial\rho^2}\right)\phi \ d\mu_Y\right)\rho^{n-3}d\rho.$$

When $\kappa = 1$ and Y is the Clifford minimal hypersurface in the unit $7-$sphere

$$Y = S^3\left(\frac{\sqrt{2}}{2}\right) \times S^3\left(\frac{\sqrt{2}}{2}\right),$$

then, CY is Simons' cone, proved to be unstable in [28] (see also Chapter 6 of [30]).

3. CONSTRUCTIONS OF AREA-MINIMIZING HYPERSURFACES

Let Σ be Euclidean space $\mathbb{R}^{n+1}$ with a conformally flat metric

$$ds^2 = e^{\phi(r)} \sum_{i=1}^{n+1} dx_i^2,$$

where $r = |x| = \sqrt{x_1^2 + \cdots + dx_{n+1}^2}$ and $\phi(|x|)$ is smooth in $\mathbb{R}^{n+1}$. Let F be a function on $\mathbb{R}^{n+1}$. Let $E_i = \{\frac{\partial}{\partial x_i}\}$ be a standard basis of $\mathbb{R}^{n+1}$ and $u_i = E_i u$ be the ordinary derivative in $\mathbb{R}^{n+1}$. Moreover,

$$\Gamma_{ij}^k = \frac{\phi'}{2}\left(\delta_{ik}\frac{x_j}{r} + \delta_{jk}\frac{x_i}{r} - \delta_{ij}\frac{x_k}{r}\right).$$

Denote $|\partial F|^2 = \sum_i F_i^2$. Let Δ be the standard Laplacian of $\mathbb{R}^{n+1}$, then

$$\begin{aligned}
(3.1) \quad \Delta_\Sigma F &= \sigma^{ij} F_{i,j} = e^{-\phi}\delta_{ij}\left(F_{ij} - \frac{\phi'}{2}\left(\delta_{ik}\frac{x_j}{r} + \delta_{jk}\frac{x_i}{r} - \delta_{ij}\frac{x_k}{r}\right)F_k\right) \\
&= e^{-\phi}\left(\Delta F + \frac{n-1}{2}\phi' F_i \frac{x_i}{r}\right).
\end{aligned}$$

By (2.3) we can compute $\mathfrak{L}F$ in the conformal flat metric as follows.

$$\begin{aligned}
(3.2) \quad \mathfrak{L}F &= e^{-\phi}\left(1 + e^{-\phi}|\partial F|^2\right)\left(\Delta F + \frac{n-1}{2}\phi' F_i \frac{x_i}{r}\right) - e^{-2\phi}\left(F_{ij}F_iF_j - \frac{|\partial F|^2}{2}\phi' F_i \frac{x_i}{r}\right) \\
&= e^{-\phi}\left(\left(1 + e^{-\phi}|\partial F|^2\right)\Delta F - e^{-\phi}F_{ij}F_iF_j\right) + e^{-\phi}\left(\frac{n-1}{2} + \frac{n}{2}e^{-\phi}|\partial F|^2\right)\phi' F_i \frac{x_i}{r} \\
&= e^{-2\phi}\left(|\partial F|^2\left(\Delta F + \frac{n}{2}\phi' F_i \frac{x_i}{r}\right) - F_{ij}F_iF_j\right) + e^{-\phi}\left(\Delta F + \frac{n-1}{2}\phi' F_i \frac{x_i}{r}\right).
\end{aligned}$$

Lemma 3.1. *Let $F = F(\theta, r)$ be a function with*

$$(3.3) \qquad \theta = \frac{x_{n+1}}{\sqrt{x_1^2 + \cdots + x_{n+1}^2}}, \qquad r = \sqrt{x_1^2 + \cdots + x_{n+1}^2},$$

on $[-1, 1] \times (0, \infty)$. Then we have

$$\begin{aligned}
(3.4) \quad \mathfrak{L}F &= e^{-2\phi}\left(n\left((1-\theta^2)\frac{F_\theta^2}{r^2} + F_r^2\right)\left(\frac{F_r}{r} + \frac{\phi'}{2}F_r - \frac{\theta F_\theta}{r^2}\right)\right. \\
&\quad \left. + (1-\theta^2)\frac{F_\theta^2}{r^2}\left(\frac{\theta F_\theta}{r^2} + \frac{F_r}{r}\right) + \frac{1-\theta^2}{r^2}\left(F_\theta^2 F_{rr} + F_r^2 F_{\theta\theta} - 2F_\theta F_r F_{r\theta}\right)\right) \\
&\quad + e^{-\phi}\left(F_{rr} + \frac{1-\theta^2}{r^2}F_{\theta\theta} + \frac{n}{r}F_r - \frac{n\theta}{r^2}F_\theta + \frac{n-1}{2}\phi' F_r\right).
\end{aligned}$$

Proof. For $1 \le \alpha \le n$ we have

$$\begin{aligned}
(3.5) \quad F_\alpha = \partial_{x_\alpha}F &= F_\theta \cdot \left(-\frac{x_\alpha x_{n+1}}{r^3}\right) + F_r \frac{x_\alpha}{r}, \\
F_{n+1} = \partial_{x_{n+1}}F &= F_\theta \cdot \left(\frac{1}{r} - \frac{x_{n+1}^2}{r^3}\right) + F_r \frac{x_{n+1}}{r} = F_\theta \frac{\sum_\alpha x_\alpha^2}{r^3} + F_r \frac{x_{n+1}}{r}.
\end{aligned}$$

Hence

$$(3.6) \qquad |\partial F|^2 = \sum_\alpha F_\alpha^2 + F_{n+1}^2 = F_\theta^2 \frac{\sum_\alpha x_\alpha^2}{r^4} + F_r^2 = \left(1 - \theta^2\right) \frac{F_\theta^2}{r^2} + F_r^2,$$

and

$$(3.7) \qquad \sum_{i=1}^{n+1} x_i F_i = \sum_\alpha x_\alpha F_\alpha + x_{n+1} F_{n+1} = r F_r.$$

In polar coordinates,

$$\sum_{i=1}^{n+1} dx_i^2 = dr^2 + r^2 \left(d\beta^2 + \cos^2 \beta \; dS^{n-1}\right),$$

where $\sin \beta = \theta \in [-1, 1]$ and dS^{n-1} is the standard metric in the unit sphere $\mathbb{S}^{n-1} \in \mathbb{R}^n$. Hence

$$\sum_{i=1}^{n+1} dx_i^2 = dr^2 + \frac{r^2}{1 - \theta^2} d\theta^2 + r^2(1 - \theta^2) dS^{n-1},$$

and

$$(3.8) \qquad \begin{aligned} \Delta F =& \frac{1}{r^n (1-\theta^2)^{\frac{n}{2}-1}} \left(\partial_r \left(r^n (1-\theta^2)^{\frac{n}{2}-1} F_r \right) + \partial_\theta \left(r^n (1-\theta^2)^{\frac{n}{2}-1} \frac{1-\theta^2}{r^2} F_\theta \right) \right) \\ =& F_{rr} + \frac{n}{r} F_r + \frac{1-\theta^2}{r^2} F_{\theta\theta} - \frac{n\theta}{r^2} F_\theta. \end{aligned}$$

Moreover,

$$(3.9) \qquad \begin{aligned} \sum_{1 \le i,j \le n+1} F_{ij} F_i F_j =& \frac{1}{2} \sum_i F_i \partial_i |\partial F|^2 \\ =& \frac{1}{2} \sum_\alpha \left(-\frac{x_\alpha x_{n+1}}{r^3} F_\theta + \frac{x_\alpha}{r} F_r \right) \left(-\frac{x_\alpha x_{n+1}}{r^3} \partial_\theta |\partial F|^2 + \frac{x_\alpha}{r} \partial_r |\partial F|^2 \right) \\ & + \frac{1}{2} \left(\frac{\sum_\alpha x_\alpha^2}{r^3} F_\theta + \frac{x_{n+1}}{r} F_r \right) \left(\frac{\sum_\alpha x_\alpha^2}{r^3} \partial_\theta |\partial F|^2 + \frac{x_{n+1}}{r} \partial_r |\partial F|^2 \right) \\ =& \frac{1}{2} \frac{\sum_\alpha x_\alpha^2}{r^4} F_\theta \partial_\theta |\partial F|^2 + \frac{1}{2} F_r \partial_r |\partial F|^2 \\ =& \frac{1-\theta^2}{2r^2} F_\theta \partial_\theta \left(\left(1-\theta^2\right) \frac{F_\theta^2}{r^2} + F_r^2 \right) + \frac{1}{2} F_r \partial_r \left(\left(1-\theta^2\right) \frac{F_\theta^2}{r^2} + F_r^2 \right) \\ =& -\theta(1-\theta^2) \frac{F_\theta^3}{r^4} + (1-\theta^2)^2 \frac{F_\theta F_{\theta\theta}}{r^4} + 2(1-\theta^2) \frac{F_\theta F_r F_{r\theta}}{r^2} \\ & - (1-\theta^2) \frac{F_\theta^2 F_r}{r^3} + F_r^2 F_{rr}. \end{aligned}$$

Hence by (3.2) we have

$$
\begin{aligned}
\mathfrak{L}F =&e^{-2\phi}\left(\left(\left(1-\theta^2\right)\frac{F_\theta^2}{r^2}+F_r^2\right)\left(F_{rr}+\frac{n}{r}F_r+\frac{1-\theta^2}{r^2}F_{\theta\theta}-\frac{n\theta}{r^2}F_\theta+\frac{n}{2}\phi'F_r\right)\right.\\
&-\left(-\theta(1-\theta^2)\frac{F_\theta^3}{r^4}+(1-\theta^2)^2\frac{F_\theta F_{\theta\theta}}{r^4}+2(1-\theta^2)\frac{F_\theta F_r F_{r\theta}}{r^2}-(1-\theta^2)\frac{F_\theta^2 F_r}{r^3}\right.\\
&\left.\left.+F_r^2 F_{rr}\right)\right)+e^{-\phi}\left(F_{rr}+\frac{n}{r}F_r+\frac{1-\theta^2}{r^2}F_{\theta\theta}-\frac{n\theta}{r^2}F_\theta+\frac{n-1}{2}\phi'F_r\right)\\
=&e^{-2\phi}\left(n\left(\left(1-\theta^2\right)\frac{F_\theta^2}{r^2}+F_r^2\right)\left(\frac{F_r}{r}+\frac{\phi'}{2}F_r-\frac{\theta F_\theta}{r^2}\right)\right.\\
&\left.+(1-\theta^2)\frac{F_\theta^2}{r^2}\left(\frac{\theta F_\theta}{r^2}+\frac{F_r}{r}\right)+\frac{1-\theta^2}{r^2}\left(F_\theta^2 F_{rr}+F_r^2 F_{\theta\theta}-2F_\theta F_r F_{r\theta}\right)\right)\\
&+e^{-\phi}\left(F_{rr}+\frac{1-\theta^2}{r^2}F_{\theta\theta}+\frac{n}{r}F_r-\frac{n\theta}{r^2}F_\theta+\frac{n-1}{2}\phi'F_r\right).
\end{aligned}
$$

(3.10)

$\square$

Theorem 3.2. *Let Σ be an $(n+1)-$dimensional Euclidean space $\mathbb{R}^{n+1}$, $n\geq 2$, endowed with a smooth conformally flat metric $ds^2=e^\phi\sum dx_i^2$, where $\phi'(r)\geq -2(1-\kappa)r^{-1}$ and $\kappa\geq\frac{2}{n}\sqrt{n-1}$. If*

$$
F(\theta,r)=C\theta r^p=Cx_{n+1}r^{p-1}\triangleq\mathcal{F}(x_{n+1},r)
$$

with any constant $C>0$ and $p=\frac{n}{2}\kappa-\sqrt{\frac{n^2\kappa^2}{4}-(n-1)}$, then except at the origin we have

(3.11)
$$
\mathfrak{L}\mathcal{F}(x_{n+1},r)\begin{cases}\geq 0 & \text{if}\quad (x_1,\cdots,x_n)\in\mathbb{R}^n,\ x_{n+1}\geq 0\\[2mm]\leq 0 & \text{if}\quad (x_1,\cdots,x_n)\in\mathbb{R}^n,\ x_{n+1}\leq 0\end{cases}.
$$

Proof. Since $\phi'\geq-2(1-\kappa)r^{-1}$ for $0<\kappa\leq 1$ and $F_r=Cp\theta r^{p-1}$. By (3.4) except at the origin we have

$$
\begin{aligned}
\theta\mathfrak{L}F\geq&\theta e^{-2\phi}\left(n\left(\left(1-\theta^2\right)\frac{F_\theta^2}{r^2}+F_r^2\right)\left(\frac{\kappa F_r}{r}-\frac{\theta F_\theta}{r^2}\right)\right.\\
&\left.+(1-\theta^2)\frac{F_\theta^2}{r^2}\left(\frac{\theta F_\theta}{r^2}+\frac{F_r}{r}\right)+\frac{1-\theta^2}{r^2}\left(F_\theta^2 F_{rr}+F_r^2 F_{\theta\theta}-2F_\theta F_r F_{r\theta}\right)\right)\\
&+\theta e^{-\phi}\left(F_{rr}+\frac{1-\theta^2}{r^2}F_{\theta\theta}+\left((n-1)\kappa+1\right)\frac{F_r}{r}-\frac{n}{r^2}\theta F_\theta\right).
\end{aligned}
$$

(3.12)

Furthermore, we take the derivatives of F and get

(3.13)
$$
\begin{aligned}
\theta\mathfrak{L}F\geq&C^3\theta e^{-2\phi}\left(n\left(\left(1-\theta^2\right)r^{2p-2}+\theta^2 p^2 r^{2p-2}\right)\left(\kappa\theta p r^{p-2}-\theta r^{p-2}\right)\right.\\
&\left.+(1-\theta^2)r^{2p-2}\left(\theta r^{p-2}+\theta p r^{p-2}\right)+\frac{1-\theta^2}{r^2}\left(p(p-1)\theta r^{3p-2}-2p^2\theta r^{3p-2}\right)\right)\\
&+C\theta e^{-\phi}\left(p(p-1)\theta r^{p-2}+\left((n-1)\kappa+1\right)p\theta r^{p-2}-n\theta r^{p-2}\right)\\
=&C^3\theta e^{-2\phi}\left(\left(n(\kappa p-1)+1-p^2\right)(1-\theta^2)+np^2(\kappa p-1)\theta^2\right)\theta r^{3p-4}\\
&+C\theta e^{-\phi}\left(p^2+(n-1)\kappa p-n\right)\theta r^{p-2}.
\end{aligned}
$$

Note

$$n(\kappa p - 1) + 1 - p^2 = -\left(p - \frac{n\kappa}{2}\right)^2 + \frac{n^2\kappa^2}{4} - (n-1) = 0.$$

By the definition of p, we obtain

$$
\begin{aligned}
(3.14) \qquad p =& \frac{n\kappa}{2}\left(1 - \sqrt{1 - \frac{4(n-1)}{n^2\kappa^2}}\right) = \frac{n\kappa}{2}\left(1 - \frac{n-2}{n}\sqrt{1 - \frac{4(n-1)}{(n-2)^2}\left(\frac{1}{\kappa^2} - 1\right)}\right) \\
\geq& \frac{n\kappa}{2}\left(1 - \frac{n-2}{n}\left(1 - \frac{2(n-1)}{(n-2)^2}\left(\frac{1}{\kappa^2} - 1\right)\right)\right) = \frac{1}{\kappa}\left(1 + \frac{1-\kappa^2}{n-2}\right) \geq \frac{1}{\kappa}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(3.15) \qquad \theta\mathfrak{L}F \geq& C^3 e^{-2\phi} n p^2 (\kappa p - 1)\theta^4 r^{3p-4} + Ce^{-\phi}\left(p^2 + (n-1)\kappa p - n\right)\theta^2 r^{p-2} \\
\geq& Ce^{-\phi}(p^2 - 1)\theta^2 r^{p-2} \geq 0.
\end{aligned}
$$

We complete the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.3. *There are other $\mathfrak{L}$-sub(super)harmonic functions on Σ. For instance, for all $j > 0$, $\mathfrak{L}(jx_{n+1}w^{p-1}) \geq 0$ on $x_{n+1} \geq 0$ and $\mathfrak{L}(jx_{n+1}w^{p-1}) \leq 0$ on $x_{n+1} \leq 0$, where $w = \sqrt{x_1^2 + \cdots + x_n^2}$.*

Denote $B_R = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \ x_1^2 + \cdots + x_{n+1}^2 \leq R^2\}$.

Theorem 3.4. *If $n \geq 3$ and*

$$\frac{2}{n}\sqrt{n-1} \leq \kappa < 1,$$

then any hyperplane through the origin in Σ as described in Theorem 3.2, that is, $\mathbb{R}^{n+1}$ equipped with a particular conformally flat metric, is area-minimizing.

Proof. We shall show that the hyperplane $T = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \ x_{n+1} = 0\}$ in Σ with the induced metric is area-minimizing.

Set $\tilde{\phi}(r) = \int_0^r e^{\frac{\phi(r)}{2}}dr$. Let us define $\rho = \tilde{\phi}(r)$ and $\lambda(\rho) = r\tilde{\phi}'(r)$, then the Riemannian metric in Σ can be written in polar coordinates as $ds^2 = d\rho^2 + \lambda^2(\rho)d\theta^2$, where $d\theta^2$ is the standard metric on $\mathbb{S}^n(1)$. Moreover,

$$(3.16) \qquad \frac{d\lambda}{d\rho} = \frac{d\lambda}{dr}\frac{dr}{d\rho} = \left(\tilde{\phi}' + r\tilde{\phi}''\right)\frac{1}{\tilde{\phi}'} = 1 + r(\log\tilde{\phi}')' = 1 + \frac{1}{2}r\phi' \geq 1 - (1 - \kappa) = \kappa.$$

When $n \geq 3$ and

$$q = \frac{n}{2}\kappa - \sqrt{\frac{n^2\kappa^2}{4} - (n-1)} - 1,$$

let $\mathcal{F}_j(x_{n+1}, r) = jx_{n+1}r^q$ for $j > 0$ with $r = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$. By Theorem 3.2 we obtain

$$(3.17) \qquad \mathfrak{L}\mathcal{F}_j(x_{n+1}, r)\begin{cases} \geq 0 & \text{in} \quad \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \ x_{n+1} \geq 0\} \setminus \{0\} \\ \leq 0 & \text{in} \quad \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \ x_{n+1} \leq 0\} \setminus \{0\} \end{cases}.$$

Combining (3.16) and formula (2.9) in [10], we know that any geodesic sphere centered at the origin has positive inward mean curvature. By the existence theorem for the Dirichlet

problem for minimal hypersurface in $\Sigma \times \mathbb{R}$, see Theorem 1.5 in [29], for any constant $R > 0$ and $j = 1, 2, \cdots, \infty$, there is a solution $u_j \in C^\infty(B_{jR})$ to the Dirichlet problem

$$(3.18) \qquad \begin{cases} \mathfrak{L}u_j = 0 & \text{in } B_{jR} \\ u_j = \mathcal{F}_j & \text{on } \partial B_{jR} \end{cases}.$$

By symmetry, $u_j = 0$ on $B_{R^*} \cap T$ for any fixed $R^* > 0$. Let $U = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} |\ x_{n+1} > 0\}$, then the comparison theorem on $B_{R^*} \setminus \{0\}$ implies

$$(3.19) \qquad \lim_{j\to\infty} u_j \geq \lim_{j\to\infty} \mathcal{F}_j = +\infty \qquad \text{in } B_{R^*} \cap U$$

and

$$(3.20) \qquad \lim_{j\to\infty} u_j \leq \lim_{j\to\infty} \mathcal{F}_j = -\infty \qquad \text{in } B_{R^*} \cap (\mathbb{R}^{n+1} \setminus \overline{U}).$$

Let U_j denote the subgraph of u_j in $B_{R^*} \times \mathbb{R}$, namely,

$$U_j = \{(x, t) \in B_{R^*} \times \mathbb{R} |\ t < u_j(x)\}.$$

Clearly, its characteristic function χ_{U_j} converges in $L^1_{loc}(B_{R^*} \times \mathbb{R})$ to $\chi_{U \times \mathbb{R}}$. By an analogous argument as in Lemma 9.1 in [14] for the Euclidean case, for any compact set $E \subset B_{R^*} \times \mathbb{R}$, that $\text{Graph}(u_j) \triangleq \{(x, u_j(x))|\ x \in \mathbb{R}^{n+1}\}$ is an area-minimizing hypersurface implies that $(U \times \mathbb{R}) \cap E$ is a minimizing set in E. Hence $U \times \mathbb{R}$ is a minimizing set in $B_{R^*} \times \mathbb{R} \subset \Sigma \times \mathbb{R}$. By an analogous argument as in Proposition 9.9 in [14] for the Euclidean case, U is a minimizing set in B_{R^*}, namely, the hyperplane T minimizes perimeter in B_{R^*}. Since R^* is arbitrary, we complete the proof. □

As we showed in the previous section, on the cone CS_κ the usual metric can be rewritten as a conformally flat one. Our constructions will be those modified from the cone CS_κ.

Lemma 3.5. *Let Λ be the rotational symmetric function on $\mathbb{R}^{n+1}$ defined by*

$$(3.21) \qquad \Lambda(x) = \begin{cases} \frac{\sqrt{1-\kappa^2}}{\kappa} \sqrt{x_1^2 + \cdots + x_n^2} & \text{on } \mathbb{R}^{n+1} \setminus B_1 \\ \frac{\sqrt{1-\kappa^2}}{\kappa} \left(1 - \frac{2}{\pi} \int_{|x|}^1 (\arctan \xi(s))\, ds\right) & \text{on } B_1 \end{cases},$$

where $\xi(s) = s\left(e^{\frac{1}{1-s^2}} - e\right)$. It is a smooth convex function on $\mathbb{R}^{n+1}$.

Proof. In fact, $\xi'(0) = 0$, $\xi^{(2k)}(0) = 0$ for $k \geq 0$ and $\xi^{(j)}(1) = +\infty$ for $j \geq 0$. Then on B_1

$$(3.22) \qquad \begin{aligned} \partial_i \Lambda(x) &= \frac{2\sqrt{1-\kappa^2}}{\kappa\pi} \frac{x_i}{|x|} \arctan \xi(|x|), \\ \partial_{ij}\Lambda(x) &= \frac{2\sqrt{1-\kappa^2}}{\kappa\pi}\left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right) \frac{\arctan\xi}{|x|} + \frac{2\sqrt{1-\kappa^2}}{\kappa\pi} \frac{\xi'}{1+\xi^2} \frac{x_i x_j}{|x|^2}. \end{aligned}$$

Since

$$\frac{\arctan\xi(\sqrt{t})}{\sqrt{t}} = \sum_{k=0}^\infty \frac{(-1)^k}{2k+1} t^k \left(e^{\frac{1}{1-t}} - e\right)^{2k+1}$$

in $[0, \epsilon]$ for small $\epsilon > 0$, $t^{-\frac{1}{2}} \arctan\xi(\sqrt{t})$ is a smooth function for $t \in [0, 1)$ and

$$\Lambda(x) = \frac{\sqrt{1-\kappa^2}}{\kappa}\left(1 - \frac{1}{\pi} \int_{|x|^2}^1 \frac{\arctan\xi(\sqrt{t})}{\sqrt{t}} dt\right)$$

is a smooth convex function on B_1. Denote $\Lambda(r) = \Lambda(|x|)$, then the radial derivative of Λ at 1 is

$$\lim_{r \to 1} \partial_r \Lambda(r) = \frac{2\sqrt{1-\kappa^2}}{\kappa\pi} \arctan \xi(1) = \frac{\sqrt{1-\kappa^2}}{\kappa},$$

and the higher order radial derivative of Λ at 1 is

$$\begin{aligned}
\lim_{r \to 1} (\partial_r)^{j+1} \Lambda(r) &= \frac{2\sqrt{1-\kappa^2}}{\kappa\pi} (\partial_r)^j \arctan \xi(r) \Big|_{r=1} \\
&= \frac{2\sqrt{1-\kappa^2}}{\kappa\pi} (\partial_r)^{j-1} \left(\frac{\xi'}{1+\xi^2} \right) \Big|_{r=1} = 0 \qquad \text{for } j \geq 1.
\end{aligned}$$

Hence Λ is a smooth convex function on $\mathbb{R}^{n+1}$. $\qquad\square$

Now we suppose that MCS_κ is an $(n+1)$-dimensional smooth entire graphic hypersurface in $\mathbb{R}^{n+2}$ with the defining function Λ. We see that it has non-negative sectional curvature everywhere. In fact, MCS_κ is a $\kappa-$sphere cone CS_κ with a smooth cap, which we shall call the modified $\kappa-$ sphere cone.

We already showed that the metric of the $\kappa-$sphere cone is conformally flat, and we shall now also derive this for MCS_κ.

Lemma 3.6. *The $(n+1)-$dimensional MCS_κ has a smooth conformally flat metric*

$$ds^2 = e^{\Phi(r)} \sum_{1 \leq i \leq n+1} dx_i^2$$

on $\mathbb{R}^{n+1}$ with $-\frac{2}{r}(1-\kappa) \leq \Phi' \leq 0$.

Proof. MCS_κ is defined as an entire graph on $\mathbb{R}^{n+2}$. Its induced metric can also be written in polar coordinates as

$$(3.23) \qquad\qquad ds^2 = d\rho^2 + \lambda^2(\rho)d\theta^2,$$

where $d\theta^2$ is a standard metric on $\mathbb{S}^n(1)$, and

$$(3.24) \qquad\qquad \lambda(\rho) = \begin{cases} \kappa\left(\rho + \frac{1}{\kappa} - \rho_0\right) & \text{for } \rho \geq \rho_0 \\ \zeta(\rho) & \text{for } 0 \leq \rho \leq \rho_0 \end{cases}.$$

Here

$$1 < \rho_0 = \int_0^1 \sqrt{1 + (\partial_r \Lambda)^2} dr < \frac{1}{\kappa},$$

and the inverse function of ζ satisfies

$$\zeta^{-1}(s) = \int_0^s \sqrt{1 + (\partial_r \Lambda)^2} dr,$$

where Λ is defined in the last lemma. Moreover, $\kappa \leq \zeta' \leq 1$.

Let $\psi(r)$ be a function on $\left[0, \left(\frac{1}{\kappa}\right)^{\frac{1}{\kappa}}\right)$ with $\psi\left(\left(\frac{1}{\kappa}\right)^{\frac{1}{\kappa}}\right) = \rho_0$ and

$$(3.25) \qquad\qquad \psi'(r) = \frac{1}{r}\zeta(\psi(r)) \qquad \text{on} \quad \left[0, \left(\frac{1}{\kappa}\right)^{\frac{1}{\kappa}}\right).$$

In fact, let $\tilde{\zeta}(\rho) = \int_1^\rho \frac{1}{\zeta(t)} dt$ for $\rho \in (0, \rho_0]$, then we integrate the above ordinary differential equation and obtain

$$\tilde{\zeta}(\psi(r)) - \tilde{\zeta}(\rho_0) = \log r + \frac{1}{\kappa}\log \kappa.$$

Since $\tilde{\zeta}$ is a monotonic function, we can solve the desired ψ. Note $\kappa\rho \le \zeta(\rho) \le \rho$ on $[0, \rho_0]$, comparison theorem implies that

$$\left(\frac{1}{\kappa}\right)^{-\frac{1}{\kappa}} \rho_0 r \le \psi(r) \le \kappa\rho_0 r^\kappa \qquad \text{on} \quad \left[0, \left(\frac{1}{\kappa}\right)^{\frac{1}{\kappa}}\right].$$

In particular, $\psi(0) = 0$. Since

$$\psi''(r) = \frac{\zeta'}{r}\psi' - \frac{\zeta}{r^2} = \frac{\zeta}{r^2}(\zeta' - 1),$$

then,

$$(3.26) \qquad\qquad \frac{\kappa - 1}{r} \le \frac{\psi''(r)}{\psi'(r)} = \frac{\zeta' - 1}{r} \le 0.$$

Let

$$(3.27) \qquad\qquad \rho = \tilde{\psi}(r) = \begin{cases} r^\kappa - \frac{1}{\kappa} + \rho_0 & \text{for } r \ge \left(\frac{1}{\kappa}\right)^{\frac{1}{\kappa}} \\[2mm] \psi(r) & \text{for } 0 \le r \le \left(\frac{1}{\kappa}\right)^{\frac{1}{\kappa}} \end{cases},$$

then $\tilde{\psi}$ also satisfies (3.25) and hence $\tilde{\psi}$ is smooth on $[0, \infty)$. Set

$$(3.28) \qquad\qquad e^{\Phi(r)} = \left(\tilde{\psi}'(r)\right)^2 = \begin{cases} \kappa^2 r^{2\kappa-2} & \text{for } r \ge \left(\frac{1}{\kappa}\right)^{\frac{1}{\kappa}} \\[2mm] (\psi')^2(r) & \text{for } 0 \le r \le \left(\frac{1}{\kappa}\right)^{\frac{1}{\kappa}} \end{cases},$$

then

$$(3.29) \qquad ds^2 = e^{\Phi(r)}dr^2 + e^{\Phi(r)}r^2 d\theta^2 = e^{\Phi(r)} \sum_{1 \le i \le n+1} dx_i^2,$$

where $r^2 = \sum_i x_i^2$. By (2.7) and (3.26) we have

$$-\frac{2}{r}(1 - \kappa) \le \Phi' \le 0.$$

$\square$

Now, Lemma 3.6 and Theorem 3.4 yield the following conclusion.

Theorem 3.7. *Let $n \ge 3$. If*

$$\frac{2}{n}\sqrt{n-1} \le \kappa < 1,$$

then any hyperplane through the origin in MCS_κ is area-minimizing.

Remark 3.8. *Let $\{e_\alpha\}_{\alpha=1}^n \bigcup \{\frac{\partial}{\partial\rho}\}$ be an orthonormal basis at the considered point of MCS_κ. Compared with (2.6) we calculate the sectional curvature and Ricci curvature of MCS_κ as follows (see Appendix A in [19] for instance).*

$$K_{MCS_\kappa}\left(\frac{\partial}{\partial\rho}, e_\alpha\right) = -\frac{\lambda''}{\lambda} \ge 0, \quad K_{MCS_\kappa}(e_\alpha, e_\beta) = \frac{1 - (\lambda')^2}{\lambda^2} \ge 0,$$

$$(3.30) \qquad Ric_{MCS_\kappa}\left(\frac{\partial}{\partial\rho}, e_\alpha\right) = 0, \quad Ric_{MCS_\kappa}\left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\rho}\right) = -n\frac{\lambda''}{\lambda} \ge 0,$$

$$Ric_{MCS_\kappa}(e_\alpha, e_\beta) = \left((n-1)\frac{1 - (\lambda')^2}{\lambda^2} - \frac{\lambda''}{\lambda}\right)\delta_{\alpha\beta} \ge 0.$$

In particular, for $\rho \geq \rho_0$ with $1 < \rho_0 < \frac{1}{\kappa}$ we have

$$K_{MCS_\kappa}\left(\frac{\partial}{\partial\rho}, e_\alpha\right) = 0, \quad K_{MCS_\kappa}(e_\alpha, e_\beta) = \frac{1-\kappa^2}{\kappa^2(\rho + \frac{1}{\kappa} - \rho_0)^2},$$

(3.31)
$$Ric_{MCS_\kappa}\left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\rho}\right) = Ric_{MCS_\kappa}\left(\frac{\partial}{\partial\rho}, e_\alpha\right) = 0,$$

$$Ric_{MCS_\kappa}(e_\alpha, e_\beta) = (n-1)\frac{1-\kappa^2}{\kappa^2(\rho + \frac{1}{\kappa} - \rho_0)^2}\delta_{\alpha\beta}.$$

From the construction above we see that MCS_κ is a complete simply connected manifold with non-negative sectional curvature.

Remark 3.9. *Since MCS_κ in Theorem 3.4 cannot split off a Euclidean factor $\mathbb{R}$ isometrically, the Cheeger-Gromoll splitting theorem [6] implies that it does not contain a line. Consequently, this gives a negative answer to the question (1) in [3], which is*

If M is a complete area-minimizing hypersurface in a complete simply connected manifold N of non-negative curvature, does it follow that N contains a line, that is a complete length-minimizing geodesic?

If we define for each $x \in \mathbb{R}^n$

$$\widetilde{\Lambda}(x) = \frac{2\sqrt{1-\kappa^2}}{\pi\kappa}\int_0^{|x|} \arctan s \, ds,$$

then $\widetilde{\Lambda}$ is a smooth strictly convex function on $\mathbb{R}^n$ and the hypersurface $\widetilde{\Sigma} = \{(x, \widetilde{\Lambda}(x))| \ x \in \mathbb{R}^n\}$ is a smooth manifold with positive sectional curvature everywhere. In fact, $\widetilde{\Sigma}$ can be seen as a Riemannian manifold $(\mathbb{R}^n, \tilde{\sigma})$ with

$$\tilde{\sigma} = d\rho^2 + \tilde{\lambda}^2(\rho)d\theta^2$$

in polar coordinates, where the inverse function of $\tilde{\lambda}$ satisfies

$$\tilde{\lambda}^{-1}(s) = \int_0^s \sqrt{1 + (\partial_r\widetilde{\Lambda})^2}dr = \int_0^s \sqrt{1 + \frac{4(1-\kappa^2)}{\pi^2\kappa^2}(\arctan r)^2}dr.$$

Hence

$$1 \geq \tilde{\lambda}'(s) = \left(1 + \frac{4(1-\kappa^2)}{\pi^2\kappa^2}(\arctan\tilde{\lambda}(s))^2\right)^{-\frac{1}{2}} > \kappa,$$

and

$$\tilde{\lambda}''(s) = -\left(1 + \frac{4(1-\kappa^2)}{\pi^2\kappa^2}(\arctan\tilde{\lambda}(s))^2\right)^{-\frac{3}{2}}\frac{4(1-\kappa^2)}{\pi^2\kappa^2}\arctan\tilde{\lambda}(s)\frac{\tilde{\lambda}'(s)}{1 + \tilde{\lambda}^2(s)}.$$

Clearly,

$$\lim_{s\to\infty}\frac{\tilde{\lambda}(s)}{s} = \lim_{s\to\infty}\tilde{\lambda}'(s) = \kappa, \quad \text{and} \quad \lim_{s\to\infty}(s^2\tilde{\lambda}''(s)) = -\frac{2}{\pi}(1-\kappa^2).$$

If $\{\partial_\rho\}$ and $\{e_\alpha\}_{\alpha=1}^{n-1}$ are an orthonormal basis of $\widetilde{\Sigma}$, then the sectional curvature of $\widetilde{\Sigma}$ is

$$0 < K(\partial_\rho, e_\alpha) = -\frac{\tilde{\lambda}''}{\tilde{\lambda}} \sim \frac{2(1-\kappa^2)}{\pi\kappa s^3}, \quad K(e_\alpha, e_\beta) = \frac{1 - \tilde{\lambda}'^2}{\tilde{\lambda}^2} \sim \frac{1-\kappa^2}{\kappa^2 s^2}.$$

Clearly,

$$\lim_{s\to0}\frac{1 - \tilde{\lambda}'^2(s)}{\tilde{\lambda}^2(s)} = \frac{4(1-\kappa^2)}{\pi^2\kappa^2} > 0.$$

Hence $\widetilde{\Sigma} = \{(x, \widetilde{\Lambda}(x)) \mid x \in \mathbb{R}^n\}$ has positive sectional curvature everywhere.

Theorem 3.10. *Let $n \geq 4$ and $\widetilde{\Sigma} = (\mathbb{R}^n, \tilde{\sigma})$ be a complete manifold with positive sectional curvature as above. If*

$$\frac{2}{n-1}\sqrt{n-2} \leq \kappa < 1,$$

then any hyperplane through the origin in $\widetilde{\Sigma} = (\mathbb{R}^n, \tilde{\sigma})$ is area-minimizing.

Proof. Note $\kappa < \lambda' \leq 1$, then we can rewrite the metric $\tilde{\sigma}$ similar to (3.27)(3.28)(3.29). Apply Theorem 3.4 to complete the proof. $\square$

Remark 3.11. *Our theorem above gives an example for the question (2) in [3], which is*

If N is a complete manifold of positive sectional curvature, does N ever admit an area-minimizing hypersurface?

Now scaling the manifold MCS_κ yields $\epsilon^2 MCS_\kappa$ for $\epsilon > 0$, which is $\mathbb{R}^{n+1}$ endowed with the metric

$$(3.32) \qquad \sigma_\epsilon = d\rho^2 + \epsilon^2 \lambda^2 \left(\frac{\rho}{\epsilon}\right) d\theta^2$$

in polar coordinates, where λ and $d\theta^2$ as in (3.23) and (3.24). Obviously $\epsilon\lambda\left(\frac{\rho}{\epsilon}\right) < \kappa\rho$ and $\epsilon\lambda\left(\frac{\rho}{\epsilon}\right)$ converges to $\kappa\rho$ uniformly as $\epsilon \to 0$. Hence σ_ϵ converges to σ_C as $\epsilon \to 0$, where σ_C is the metric of CS_κ defined in (2.7).

Now we can derive the result of F. Morgan in [21], obtained there by a different method due to G. R. Lawlor [18].

Proposition 3.12. *Let $n \geq 3$ and $\kappa \geq \frac{2}{n}\sqrt{n-1}$. Then any hyperplane in $(n+1)$-dimensional CS_κ through the origin is area-minimizing.*

Proof. Let T_ϵ denote the hyperplane in $\epsilon^2 MCS_\kappa$ corresponding to $T \subset MCS_\kappa$ during the re-scaling procedure. Denote $T_0 = \lim_{\epsilon \to 0} T_\epsilon \subset \lim_{\epsilon \to 0} \epsilon^2 MCS_\kappa = CS_\kappa$. Let $\mathcal{H}^n_\epsilon$ and $\mathcal{H}^n_0$ be the n-dimensional Hausdorff measures of $\epsilon^2 MCS_\kappa$ and CS_κ.

Now we consider a bounded domain $\Omega_0 \subset T_0$ and a subset set $W_0 \subset CS_\kappa$ with $\partial\Omega_0 = \partial W_0$. View Ω_0 as a set $\Omega \subset \mathbb{R}^n$ with the induced metric from T_0 and W_0 as a set W in $\mathbb{R}^{n+1}$ with the induced metric from CS_κ. Let Ω_ϵ be the set $\Omega \subset \mathbb{R}^n$ with the induced metric from T_ϵ and W_ϵ be the set W in $\mathbb{R}^{n+1}$ with the induced metric from $\epsilon^2 MCS_\kappa$. Clearly, $\Omega_0 = \lim_{\epsilon \to 0} \Omega_\epsilon$ and $W_0 = \lim_{\epsilon \to 0} W_\epsilon$ with $\partial\Omega_\epsilon = \partial W_\epsilon$.

Since T_ϵ is area-minimizing in $\epsilon^2\Sigma$, then

$$\mathcal{H}^n_\epsilon(\Omega_\epsilon) \leq \mathcal{H}^n_\epsilon(W_\epsilon).$$

$\epsilon\lambda\left(\frac{\rho}{\epsilon}\right) < \kappa\rho$ implies

$$\mathcal{H}^n_\epsilon(W_\epsilon) \leq \mathcal{H}^n_0(W_0).$$

Since also $\epsilon\lambda\left(\frac{\rho}{\epsilon}\right) \to \kappa\rho$ uniformly as $\epsilon \to 0$, we obtain

$$\mathcal{H}^n_0(\Omega_0) = \lim_{\epsilon \to 0} \mathcal{H}^n_\epsilon(\Omega_\epsilon) \leq \limsup_{\epsilon \to 0} \mathcal{H}^n_\epsilon(W_\epsilon) \leq \mathcal{H}^n_0(W_0).$$

Hence T_0 is an area-minimizing hypersurface in CS_κ. $\square$

Actually, here the number $\frac{2}{n}\sqrt{n-1}$ is optimal. Namely, if $\kappa < \frac{2}{n}\sqrt{n-1}$ then every hyperplane in CS_κ is no more area-minimizing and even not stable. This also has been proved in [21]. Let us show this fact by using the second variation formula for the volume functional.

Theorem 3.13. *Let $\kappa \in (0,1]$ and $n \geq 3$. Any hyperplane in $(n+1)$-dimensional CS_κ through the origin is area-minimizing if and only if*

$$\text{(3.33)} \qquad \kappa \geq \frac{2}{n}\sqrt{n-1}.$$

Proof. By Proposition 3.12 we only need to prove that if (3.33) fails to hold, any hyperplane in CS_κ through the origin is not area-minimizing. Let X be a totally geodesic sphere in S_κ, then X is minimal in S_κ and $P \triangleq CX$ is a hyperplane in CS_κ through the origin. Clearly, P is a minimal hypersurface in CS_κ. The second variation formula is (see also (2.10))

$$
\begin{aligned}
\text{(3.34)} \qquad I(\phi, \phi) = \int_\epsilon^1 \Bigg(& \int_X \Bigg(-\Delta_X \phi - \frac{n-1}{\kappa^2}\phi + (n-1)\phi \\
& - (n-1)\rho\frac{\partial \phi}{\partial \rho} - \rho^2\frac{\partial^2 \phi}{\partial \rho^2} \Bigg)\phi \; d\mu_X \Bigg)\rho^{n-3}d\rho,
\end{aligned}
$$

where $\phi(x,t) \in C^2(X \times_\rho \mathbb{R})$. Define a second order differential operator L by

$$L = \rho^2\frac{\partial^2}{\partial \rho^2} + (n-1)\rho\frac{\partial}{\partial \rho}.$$

If $s = \log \rho$, then

$$L = \frac{\partial^2}{\partial s^2} + (n-2)\frac{\partial}{\partial s} = e^{-\frac{n-2}{2}s}\frac{\partial^2}{\partial s^2}\left(e^{\frac{n-2}{2}s}\cdot\right) - \frac{(n-2)^2}{4}.$$

So the $k(k \geq 1)$-th eigenvalue of L on $[\epsilon, 1]$ is

$$\text{(3.35)} \qquad \frac{(n-2)^2}{4} + \left(\frac{k\pi}{\log \epsilon}\right)^2$$

with the k-th eigenfunction (see [28] or [30] for instance)

$$\rho^{\frac{2-n}{2}}\sin\left(\frac{k\pi}{\log \epsilon}\log \rho\right).$$

By the second variation formula (3.34), P is stable if and only if

$$-\frac{n-1}{\kappa^2} + n - 1 + \frac{(n-2)^2}{4} \geq 0,$$

i.e.,

$$\kappa \geq \frac{2}{n}\sqrt{n-1}.$$

$\square$

4. A class of manifolds with non-negative Ricci curvature

Let N be an $(n+1)$-dimensional complete non-compact Riemannian manifold satisfying the following three conditions:

C1) Nonnegative Ricci curvature: Ric ≥ 0;

C2) Euclidean volume growth:

$$V_N \triangleq \lim_{r \to \infty} \frac{Vol\big(B_r(x)\big)}{r^{n+1}} > 0;$$

C3) Quadratic decay of the curvature tensor: for sufficiently large $\rho = d(x, p)$, the distance from a fixed point in N,

$$|R(x)| \leq \frac{c}{\rho^2(x)}.$$

By Gromov's compactness theorem [16], for any sequence $\bar{\epsilon}_i \to 0$ there is a subsequence $\{\epsilon_i\}$ converging to zero such that $\epsilon_i N = (N, \epsilon_i \bar{g}, p)$ converges to a metric space (N_∞, d_∞) with vertex o in the pointed Gromov-Hausdorff sense. It is called the tangent cone at infinity. $N_\infty \setminus \{o\}$ is a smooth manifold with $C^{1,\alpha}$ Riemannian metric $\bar{g}_\infty (0 < \alpha < 1)$ which is compatible with the distance d_∞. The precise statements were derived in [15] and [24] on the basis of the harmonic coordinate constructions of [17]. In fact, $N_\infty \setminus \{o\}$ is a $D^{1,1}$-Riemannian manifold (see [15, 24]). For any compact domain $K \subset N_\infty \setminus \{o\}$, there exists a diffeomorphism $\Phi_i : K \to \Phi_i(K) \subset \epsilon_i N$ such that $\Phi_i^*(\epsilon_i \bar{g})$ converges as $i \to \infty$ to $\bar{g}_\infty$ in the $C^{1,\alpha}$-topology on K.

Cheeger-Colding (see Theorem 7.6 in [5]) proved that under the conditions C1) and C2) the cone N_∞ is a metric cone. $N_\infty = CX = \mathbb{R}^+ \times_\rho X$ for some n dimensional smooth compact manifold X with Diam $X \leq \pi$ and the metric

$$\bar{g}_\infty = d\rho^2 + \rho^2 s_{ij} d\theta_i d\theta_j$$

where $s_{ij} d\theta_i d\theta_j$ is the metric of X and $s_{ij} \in C^{1,\alpha}(X)$. Let ρ_i be the distance function from p to the considered point in $\epsilon_i N$. Set $B_r^i(x)$ be the geodesic ball with radius r and centered at x in $(N, \epsilon_i \bar{g})$, and $\mathcal{B}_r(x)$ be the geodesic ball with radius r and centered at x in N_∞. In particular, $X = \partial \mathcal{B}_1(o)$.

Mok-Siu-Yau [20] showed that if C1) and C2) hold, then there exists the Green function $G(p, \cdot)$ on N^{n+1} with $\lim_{r \to \infty} \sup_{\partial B_r(p)} |Gr^{n-1} - 1| = 0$ and

$$(4.1) \qquad\qquad r^{1-n} \leq G(p, x) \leq Cr^{1-n}$$

for any $n \geq 2$, $x \in \partial B_r(p)$ and some constant C. Set $\mathcal{R} = G^{\frac{1}{1-n}}$, then

$$(4.2) \qquad\qquad \Delta_N \mathcal{R}^2 = 2(n+1)|\overline{\nabla}\mathcal{R}|^2.$$

Under the additional condition C3), Colding-Minicozzi (see Corollary 4.11 in [8]) showed that

$$(4.3) \qquad \limsup_{r \to \infty} \left(\sup_{\partial B_r} \left| \frac{\mathcal{R}}{r} - 1 \right| + \sup_{\partial B_r} \left| |\overline{\nabla}\mathcal{R}| - 1 \right| \right) = 0,$$

and

$$(4.4) \qquad \limsup_{r \to \infty} \left(\sup_{\partial B_r} |\text{Hess}_{\mathcal{R}^2} - 2\bar{g}| \right) = 0,$$

where $\mathrm{Hess}_{\mathcal{R}^2}$ is the Hessian matrix of $\mathcal{R}^2$ in N. In particular, $|\overline{\nabla}\mathcal{R}| \leq C(n, V_N)$ which is a constant depending only on n, V_N.

For any $f \in C^1(\partial\mathcal{B}_1)$, we can extend f to $N_\infty \setminus \{o\}$ by defining

$$f(\rho\theta) = f(\theta)$$

for any $\rho > 0$ and $\theta \in \partial\mathcal{B}_1$. Let $\widetilde{\nabla}$ be the Levi-Civita connection of N_∞, then

$$\tag{4.5} \left\langle \widetilde{\nabla}f, \frac{\partial}{\partial\rho} \right\rangle = 0.$$

For any $K_2 > K_1 > 0$ and $\epsilon > 0$, let $\Phi_i : \overline{\mathcal{B}_{2K_2}} \setminus \mathcal{B}_{\frac{\epsilon}{2}K_1} \to \Phi_i(\overline{\mathcal{B}_{2K_2}} \setminus \mathcal{B}_{\frac{\epsilon}{2}K_1}) \subset \epsilon_i N$ be a diffeomorphism such that $\Phi_i^*(\epsilon_i\bar{g})$ converges as $i \to \infty$ to $\bar{g}_\infty$ in the $C^{1,\alpha}$-topology on $\overline{\mathcal{B}_{2K_2}} \setminus \mathcal{B}_{\frac{\epsilon}{2}K_1}$. Moreover, Φ_i is $C^{2,\alpha}$-bounded relative to harmonic coordinates with a bound independent of i (see [17]).

Let $\overline{\nabla}^i$, Δ_N^i, Hess^i, $\mathrm{Ric}_{\epsilon_i N}$ and $|R_{\epsilon_i N}|$ be the Levi-Civita connection, Laplacian operator, Hessian matrix, Ricci curvature and curvature tensor of $\epsilon_i N$, respectively, then on $\epsilon_i N$ we have the relations

$$\rho_i = \epsilon_i^{\frac{1}{2}}\rho, \qquad \overline{\nabla}^i = \overline{\nabla}, \qquad \Delta_N^i = \epsilon_i^{-1}\Delta_N, \qquad \mathrm{Hess}^i = \mathrm{Hess},$$

$$\mathrm{Ric}_{\epsilon_i N} = \epsilon_i^{-1}\mathrm{Ric}, \qquad |R_{\epsilon_i N}| = \epsilon_i^{-1}|R|, \qquad d\mu_{\epsilon_i N} = \epsilon_i^{\frac{n+1}{2}} d\mu_N$$

where ρ_i and $d\mu_{\epsilon_i N}$ are the distance function and volume element on $\epsilon_i N$, respectively, and $d\mu_N$ is the volume element on N. We see that conditions C1), C2) and C3) are all scaling invariant. Let

$$\widetilde{\mathcal{R}}_i = \sqrt{\epsilon_i}\mathcal{R} \qquad \text{on} \ \ \epsilon_i N,$$

then

$$\Delta_N^i \widetilde{\mathcal{R}}_i^2 = \Delta_N \mathcal{R}^2 = 2(n+1)|\overline{\nabla}\mathcal{R}|^2 = 2(n+1)|\overline{\nabla}^i \widetilde{\mathcal{R}}_i|^2$$

and so $\widetilde{\mathcal{R}}_i^{1-n}$ is the Green function on $\epsilon_i N$. By (4.4) we have

$$\tag{4.6} \limsup_{i\to\infty} \left(\sup_{B_{K_2}^i \setminus B_{\epsilon K_1}^i} \left| \mathrm{Hess}_{\widetilde{\mathcal{R}}_i^2}^i - 2\epsilon_i\bar{g} \right| \right) = 0.$$

For each $x \in \epsilon_i N$ there is a minimal normal geodesic γ_x^i from p to x such that $\overline{\nabla}^i \rho_i(x) = \dot{\gamma}_x^i$. When $\epsilon_i = 1$, we define $\overline{\nabla}\rho(x)$ corresponding to the normal geodesic $\dot{\gamma}_x$. Hence $\overline{\nabla}\rho(x)$ depends on the choice of γ_x^i. Note that $\rho(x)$ is just a Lipschitz function, but the definition of $\overline{\nabla}\rho(x)$ is is equivalent to the common one if ρ is C^1 at the considered point.

Now if $x \in B_{K_2}^i \setminus B_{\epsilon K_1}^i$, let $x = \gamma_x^i(t)$, $x_\epsilon = \gamma_x^i(t_\epsilon) \in \partial B_{\epsilon K_1}^i \cap \gamma_x^i$, then for any parallel vector field ξ along γ_x^i, we have

$$\tag{4.7} \overline{\nabla}_\xi^i \widetilde{\mathcal{R}}_i^2(x) - \overline{\nabla}_\xi^i \widetilde{\mathcal{R}}_i^2(x_\epsilon) = \int_{t_\epsilon}^t \overline{\nabla}_{\dot{\gamma}_x^i}^i \overline{\nabla}_\xi^i \widetilde{\mathcal{R}}_i^2(\gamma_x^i(s))ds = \int_{t_\epsilon}^t \mathrm{Hess}_{\widetilde{\mathcal{R}}_i^2}^i \left(\overline{\nabla}^i \rho_i, \xi \right) \Big|_{\gamma_x^i(s)} ds.$$

Hence

$$
\begin{aligned}
\left| \overline{\nabla}^i_\xi \widetilde{\mathcal{R}}^2_i(x) - \overline{\nabla}^i_\xi \rho^2_i(x) \right| &\leq \left| \overline{\nabla}^i_\xi \widetilde{\mathcal{R}}^2_i(x_\epsilon) - \overline{\nabla}^i_\xi \rho^2_i(x_\epsilon) \right| \\
&\quad + \int_{t_\epsilon}^t \left| \mathrm{Hess}^i_{\widetilde{\mathcal{R}}^2_i}\left(\overline{\nabla}^i \rho_i, \xi \right) \Big|_{\gamma^i_x(s)} - \mathrm{Hess}^i_{\rho^2_i}\left(\overline{\nabla}^i \rho_i, \xi \right) \Big|_{\gamma^i_x(s)} \right| ds \\
&\leq C\epsilon + \int_{t_\epsilon}^t \left| \mathrm{Hess}^i_{\widetilde{\mathcal{R}}^2_i}\left(\overline{\nabla}^i \rho_i, \xi \right) \Big|_{\gamma^i_x(s)} - 2 \left\langle \overline{\nabla}^i \rho_i, \xi \right\rangle \Big|_{\gamma^i_x(s)} \right| ds \\
&\leq C\epsilon + K_2 \sup_{B^i_{K_2} \setminus B^i_{\epsilon K_1}} \left| \mathrm{Hess}^i_{\widetilde{\mathcal{R}}^2_i}\left(\overline{\nabla}^i \rho_i, \xi \right) - 2 \left\langle \overline{\nabla}^i \rho_i, \xi \right\rangle \right|,
\end{aligned}
$$

(4.8)

where C depends only on K_1, K_2 and the manifold N. With (4.6) we obtain

$$
\limsup_{i \to \infty} \sup_{B^i_{K_2} \setminus B^i_{\epsilon K_1}} \left| \overline{\nabla}^i \widetilde{\mathcal{R}}^2_i(x) - \overline{\nabla}^i \rho^2_i(x) \right| \leq C\epsilon. \tag{4.9}
$$

Since the geodesics γ^i_x in $\epsilon_i N$ converge to a geodesic in N_∞, with (4.5) we have

$$
\limsup_{i \to \infty} \sup_{B^i_{K_2} \setminus B^i_{\epsilon K_1}} \left| \left\langle \overline{\nabla}^i(f \circ \Phi^{-1}_i), \overline{\nabla}^i \widetilde{\mathcal{R}}^2_i \right\rangle \right| \leq C_1\epsilon, \tag{4.10}
$$

and

$$
\limsup_{i \to \infty} \sup_{B^i_{K_2} \setminus B^i_{\epsilon K_1}} \left(\widetilde{\mathcal{R}}_i \left| \overline{\nabla}^i(f \circ \Phi^{-1}_i) \right| \right) < \infty. \tag{4.11}
$$

Let Π_i be the rescaling map from $(N, \bar{g})$ to $\epsilon_i N = (N, \epsilon_i \bar{g}, p)$. Now (4.10) and (4.11) are equivalent to

$$
\limsup_{i \to \infty} \sup_{B_{\frac{K_2}{\sqrt{\epsilon_i}}} \setminus B_{\frac{\epsilon K_1}{\sqrt{\epsilon_i}}}} \left| \left\langle \overline{\nabla}(f \circ \Phi^{-1}_i \circ \Pi_i), \overline{\nabla}\mathcal{R}^2 \right\rangle \right| \leq C_1\epsilon, \tag{4.12}
$$

and

$$
\limsup_{i \to \infty} \sup_{B_{\frac{K_2}{\sqrt{\epsilon_i}}} \setminus B_{\frac{\epsilon K_1}{\sqrt{\epsilon_i}}}} \left(\mathcal{R} \left| \overline{\nabla}(f \circ \Phi^{-1}_i \circ \Pi_i) \right| \right) < \infty. \tag{4.13}
$$

The theory of integral currents in metric spaces was developed by Ambrosio and Kirchheim in [2]. It provides a suitable notion of generalized surfaces in metric spaces, which extends the classical Federer-Fleming theory [11]. We shall need the compactness Theorem (see Theorem 5.2 in [2]) and the closure Theorem (see Theorem 8.5 in [2]) for normal currents in a metric space E.

Theorem 4.1. *Let $(T_h) \subset \boldsymbol{N}_k(E)$ be a bounded sequence of normal currents, and assume that for any integer $p \geq 1$ there exists a compact set $K_p \subset E$ such that*

$$
||T_h||(E \setminus K_p) + ||\partial T_h||(E \setminus K_p) < \frac{1}{p} \qquad \text{for all } h \in \mathbb{N}.
$$

Then, there exists a subsequence $(T_{h(n)})$ converging to a current $T \in \boldsymbol{N}_k(E)$ satisfying

$$
||T|| \left(E \setminus \bigcup_{p=1}^\infty K_p \right) + ||\partial T|| \left(E \setminus \bigcup_{p=1}^\infty K_p \right) = 0.
$$

Theorem 4.2. *Let $\mathcal{I}_k(E)$ be the class of integer-rectifiable currents in E. Let $(T_h) \subset \mathbf{N}_k(E)$ be a sequence weakly converging to $T \in \mathbf{N}_k(E)$. Then, the conditions*

$$T_h \in \mathcal{I}_k(E), \qquad \sup_{h \in \mathbb{N}} \mathbf{N}(T_h) < \infty$$

imply $T \in \mathcal{I}_k(E)$.

Now let M denote a minimal hypersurface in N with the induced metric g from N. Since N has nonnegative Ricci curvature, then $Vol(\partial B_r) \leq \omega_n r^n$, where ω_n is the volume of the n-dimensional unit sphere in $\mathbb{R}^{n+1}$. Suppose that M has *Euclidean volume growth at most*, namely,

$$(4.14) \qquad \limsup_{r \to \infty} \left(r^{-n} \int_{M \cap B_r} 1 d\mu \right) < +\infty,$$

where $d\mu$ is the volume element of M. Hence there is a smallest positive constant V_M such that

$$\int_{M \cap B_r} 1 d\mu \leq V_M \, r^n \qquad \text{for any } r > 0.$$

Denote $\epsilon_i M = (M, \epsilon_i g)$. For any fixed $r > 1$ let $\Phi_i : \overline{\mathcal{B}_{2r}} \setminus \mathcal{B}_{\frac{1}{2r}} \to \Phi_i(\overline{\mathcal{B}_{2r}} \setminus \mathcal{B}_{\frac{1}{2r}}) \subset \epsilon_i N$ be a diffeomorphism such that $\Phi_i^*(\epsilon_i \bar{g})$ converges as $i \to \infty$ to $\bar{g}_\infty$ in the $C^{1,\alpha}$-topology on $\overline{\mathcal{B}_{2r}} \setminus \mathcal{B}_{\frac{1}{2r}}$. We see that the minimality is also scaling invariant and $\epsilon_i M$ are also minimal hypersurfaces of $\epsilon_i N$. Since

$$\int_{M \cap B_{2r}} 1 d\mu = \int_0^{2r} Vol\left(M \cap \partial B_s \right) ds \leq V_M \, 2^n r^n$$

which is scaling invariant, there exists a sequence $l_i \in (r, 2r)$ such that $Vol\left(\epsilon_i M \cap \partial B_{l_i}^i \right) + Vol\left(\epsilon_i M \cap \partial B_{l_i^{-1}}^i \right)$ is uniformly bounded for every i.

Let $T_i = \epsilon_i M \cap \left(B_{l_i}^i \setminus B_{l_i^{-1}}^i \right)$, then $\Phi_i^{-1}(T_i)$ is a minimal hypersurface in $\left(\Phi_i^{-1}(\epsilon_i N), \Phi_i^*(\epsilon_i \bar{g}) \right)$ with the unit normal vector $\hat{\nu}_i$. Now we change the metric $\Phi_i^*(\epsilon_i \bar{g})$ to $\bar{g}_\infty$, then the hypersurface $\Phi_i^{-1}(T_i)$ induces a metric, say $\tilde{g}_i$ from $\left(\Phi_i^{-1}(\epsilon_i N), \bar{g}_\infty \right) \subset \left(N_\infty, \bar{g}_\infty \right)$. Set $\widetilde{T}_i = \left(\Phi_i^{-1}(T_i), \tilde{g}_i \right)$, and $\tilde{\nu}_i$ be the unit normal vector of smooth hypersurface $\widetilde{T}_i$ in the metric space $\left(N_\infty, \bar{g}_\infty \right)$.

$\Phi_i^*(\epsilon_i \bar{g}) \to \bar{g}_\infty$ implies $\lim_{i \to \infty} \hat{\nu}_i = \lim_{i \to \infty} \tilde{\nu}_i \triangleq \nu_0$ and these two convergences are both uniform. Then obviously

$$H^n(\widetilde{T}_i) + H^{n-1}(\partial \widetilde{T}_i)$$

is uniformly bounded. By Theorem 4.1 and 4.2 (see also [27] for compactness of currents in the Euclidean case), there is a subsequence of ϵ_{i_j} such that

$$(4.15) \qquad \widetilde{T}_{i_j} \rightharpoonup T \qquad \text{as } j \to \infty,$$

where T is an integer-rectifiable current in N_∞. Denote $\widetilde{T}_{i_j}$ by $\widetilde{T}_i$ for simplicity. Let $\mathcal{D}^n(\Omega)$ be the set containing all smooth differential n-forms with compact support in Ω. For any $\omega \in \mathcal{D}^n\left(\mathcal{B}_{2r} \setminus \mathcal{B}_{\frac{1}{r}} \right)$ we have

$$(4.16) \qquad \lim_{i \to \infty} \int_{\widetilde{T}_i} \langle \omega, \tilde{\nu}_i \rangle d\tilde{\mu}_i = \int_T \langle \omega, \nu_\infty \rangle d\mu_\infty,$$

where $d\tilde{\mu}_i$ and $d\mu_\infty$ are the volume elements of $\widetilde{T}_i$ and T, respectively, and ν_∞ is the unit normal vector of T. Since $\hat{\nu}_i \to \nu_0$ and $\tilde{\nu}_i \to \nu_0$ uniformly, then we have

$$(4.17) \qquad \int_T \langle \omega, \nu_\infty \rangle d\mu_\infty = \lim_{i\to\infty} \int_{\Phi_i^{-1}(T_i)} \langle \omega, \hat{\nu}_i \rangle d\hat{\mu}_i = \lim_{i\to\infty} \int_{T_i} \langle \omega \circ \Phi_i^{-1}, \nu_i \rangle d\mu_i,$$

where $d\hat{\mu}_i$ and $d\mu$ are the volume elements of $\Phi_i^{-1}(T_i)$ and T_i, respectively. Then we conclude that

$$(4.18) \qquad T_i = \epsilon_i M \bigcap B_{l_i}^i \setminus B_{l_i^{-1}}^i \rightharpoonup T \qquad \text{as } i \to \infty.$$

5. Non-existence of area-minimizing hypersurfaces

Before we can prove our main results, we still need volume estimates for minimal hypersurfaces. In fact, these results are interesting in their own right.

Theorem 5.1. *let M be a complete minimal hypersurface in a complete non-compact Riemannian manifold N satisfying conditions C1), C2), C3). Then*

  i) *every end E of M has infinite volume;*
  ii) *if M is a proper immersion, then M has Euclidean volume growth at least,*

$$(5.1) \qquad \liminf_{r\to\infty} \left(\frac{1}{r^n} \int_{M\cap B_r(p)} 1 d\mu \right) > 0, \qquad \text{for any } p \in N;$$

  iii) *If M has at most Euclidean volume growth, i.e.,*

$$(5.2) \qquad \limsup_{r\to\infty} \left(r^{-n} \int_{M\cap B_r} 1 d\mu \right) < \infty,$$

  then M is a proper immersion.

Proof. For any $0 < \delta \leq 1$, set $\Omega = \left(\frac{\sqrt{c}}{\delta} + 1 \right)$. For any fixed point $p \in N$ and arbitrary $q \in \partial B_{\Omega r}(p)$, we have

$$d(p, x) \geq \frac{\sqrt{c}}{\delta} r, \qquad \text{for any } x \in B_r(q).$$

Then by condition C3) the sectional curvature satisfies

$$(5.3) \qquad |K_N(x)| \leq \frac{\delta^2}{r^2}, \qquad \text{for any } x \in B_r(q).$$

Note $Vol(B_s(q)) \geq V_N s^n$ for any $s > 0$ as conditions C1),C2). By [7], for sufficiently small δ depending only on n, c, V_N the injectivity radius at q satisfies $i(q) \geq r$. Hence $\rho_q(x)$ is smooth for $x \in B_r(q) \setminus \{q\}$.

Let $\{e_i\}$ be a local orthonormal frame field of M. Then

$$(5.4) \qquad \begin{aligned} \Delta_M \rho_q^2 &= \sum_{i=1}^n \left(\nabla_{e_i} \nabla_{e_i} \rho_q^2 - (\nabla_{e_i} e_i) \rho_q^2 \right) \\ &= \sum_{i=1}^n \left(\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} \rho_q^2 - (\overline{\nabla}_{e_i} e_i) \rho_q^2 \right) + \sum_{i=1}^n \left(\overline{\nabla}_{e_i} e_i - \nabla_{e_i} e_i \right) \bar{\rho}_q^2 \\ &= \sum_{i=1}^n \text{Hess}_{\rho_q^2}(e_i, e_i). \end{aligned}$$

For any $\xi \in \Gamma(TN)$ we denote $\xi_q^T = \xi - \left\langle \xi, \frac{\partial}{\partial \rho_q} \right\rangle \frac{\partial}{\partial \rho_q}$. Combining $\mathrm{Hess}_{\rho_q^2} \left(\xi_q^T, \frac{\partial}{\partial \rho_q} \right) = 0$ and $\mathrm{Hess}_{\rho_q^2} \left(\frac{\partial}{\partial \rho_q}, \frac{\partial}{\partial \rho_q} \right) = 2$, we obtain

$$
\begin{aligned}
\mathrm{Hess}_{\rho_q^2}(e_i, e_i) =& \mathrm{Hess}_{\rho_q^2} \left((e_i)_q^T, (e_i)_q^T \right) + 2 \left\langle e_i, \frac{\partial}{\partial \rho_q} \right\rangle^2 \\
=& 2\rho_q \mathrm{Hess}_{\rho_q} \left((e_i)_q^T, (e_i)_q^T \right) + 2 \left\langle e_i, \frac{\partial}{\partial \rho_q} \right\rangle^2.
\end{aligned}
\tag{5.5}
$$

By the Hessian comparison theorem, for any $\xi \perp \frac{\partial}{\partial \rho_q}$ we have

$$
\mathrm{Hess}_{\rho_q}(\xi, \xi) \geq \frac{\delta}{r} \cot \left(\frac{\delta \rho_q}{r} \right) |\xi|^2.
$$

Since $\frac{\delta \rho_q}{r} \cot \left(\frac{\delta \rho_q}{r} \right) \leq 1$ for $\rho_q \leq r$ with sufficiently small δ, then

$$
\begin{aligned}
\Delta_M \rho_q^2 \geq& 2\rho_q \sum_{i=1}^n \frac{\delta}{r} \cot \left(\frac{\delta \rho_q}{r} \right) \left| (e_i)_q^T \right|^2 + 2 \sum_{i=1}^n \left\langle e_i, \frac{\partial}{\partial \rho_q} \right\rangle^2 \\
\geq& 2\frac{\delta \rho_q}{r} \cot \left(\frac{\delta \rho_q}{r} \right) \sum_{i=1}^n \left| (e_i)_q^T \right|^2 + 2\frac{\delta \rho_q}{r} \cot \left(\frac{\delta \rho_q}{r} \right) \sum_{i=1}^n \left\langle e_i, \frac{\partial}{\partial \rho_q} \right\rangle^2 \\
=& 2n \frac{\delta \rho_q}{r} \cot \left(\frac{\delta \rho_q}{r} \right).
\end{aligned}
\tag{5.6}
$$

For any $t \in [0,1)$ we have $\cos t \geq 1 - t$, then

$$
\left(\tan t - \frac{t}{1-t} \right)' = \frac{1}{\cos^2 t} - \frac{1}{(1-t)^2} \leq 0.
$$

So on $[0,1)$

$$
\tan t \leq \frac{t}{1-t}.
$$

Denote the extrinsic ball $D_s(q) = B_s(q) \cap M$. Hence on $D_r(q)$ we have

$$
\Delta_M \rho_q^2(x) \geq 2n \left(1 - \frac{\delta}{r} \rho_q(x) \right) = 2n - \frac{2n \delta \rho_q(x)}{r}.
\tag{5.7}
$$

Let ρ_q^M and $B_s^M(q)$ be the distance function from q and the geodesic ball with radius s and centered at q in M. Obviously, the intrinsic ball $B_s^M(q) \subset D_s(q)$ for any $s \in (0, r)$ and (5.7) is valid on $B_r^M(q)$.

Integrating (5.7) by parts on $B_s^M(q)$ yields

$$
2n \int_{B_s^M(q)} \left(1 - \frac{\delta \rho_q}{r} \right) \leq \int_{B_s^M(q)} \Delta_M \rho_q^2 = \int_{\partial B_s^M(q)} \nabla \rho_q^2 \cdot \nu \leq 2s \int_{\partial B_s^M(q)} |\nabla \rho_q|,
\tag{5.8}
$$

where ν is the normal vector to $\partial B_s^M(q)$. Then

$$
\begin{aligned}
\frac{\partial}{\partial s}\left(s^{-n}\int_{B_s^M(q)}1\right) &= -ns^{-n-1}\int_{B_s^M(q)}1 + s^{-n}\int_{\partial B_s^M(q)}1 \\
&\geq -ns^{-n-1}\int_{B_s^M(q)}1 + s^{-n}\int_{\partial B_s^M(q)}|\nabla\rho_q| \\
&\geq -ns^{-n-1}\int_{B_s^M(q)}1 + ns^{-n-1}\int_{B_s^M(q)}\left(1-\frac{\delta\rho_q}{r}\right) \\
&= -\frac{n\delta}{r}s^{-n}\int_{B_s^M(q)}1.
\end{aligned}
$$

(5.9)

Integrating the above inequality implies for $0 < s \leq r$

$$(5.10) \qquad \mathrm{vol}(B_s^M(q)) \triangleq \int_{B_s^M(q)}1 \geq \frac{\omega_{n-1}}{n}s^n e^{-\frac{n\delta s}{r}} \geq \frac{\omega_{n-1}}{n}s^n e^{-n\delta}.$$

Here ω_{n-1} is the measure of the standard $(n-1)$-dimensional unit sphere in Euclidean space.

(i) Let E be an and of M. If E is not contained in any bounded domain in N, then we choose r large enough and some $q \in \partial B_{\Omega r}(p)$. By (5.10), E then has infinite volume.

Now we suppose that $E \subset B_{R_0}(p)$ for some constant $R_0 > 0$. Since the injectivity radius at p is positive, then analogously to the above proof for (5.10) we have constants $r_p > 0$ and $C_p > 0$ such that

$$(5.11) \qquad \mathrm{vol}(B_{r_p}^M(p)) \geq C_p r_p^n.$$

Recalling (5.10), there is a constant $r_0 > 0$ so that for any $0 < r \leq r_0$ and $z \in E$ we have a constant $C_0 > 0$ such that

$$(5.12) \qquad \mathrm{vol}(B_r^M(z)) \geq C_0 r^n.$$

Since E is noncompact, then we can choose a sequence $\{z_i\}$ such that $B_{r_0}^M(z_i)\cap B_{r_0}^M(z_j) \neq \emptyset$ for $i \neq j$. Hence

$$\mathrm{vol}(E) \geq \sum_i \mathrm{vol}(B_{r_0}^M(z_i)) \geq C_0 \sum_i r_0^n = \infty.$$

(ii) Since $B_s^M(q) \subset D_s(q)$ for any point $q \in \partial B_{\Omega r}(p)$ and any $s \in (0, r)$, then with (5.10) we obtain

$$(5.13) \qquad \int_{D_s(q)}1 \geq \frac{\omega_{n-1}}{n}s^n e^{-n\delta} \qquad \text{for every } s \in (0, r].$$

Hence we conclude that (5.1) holds.

(iii) If M is not a proper immersion into N, there exist an end $E \subset M$ and a constant r_0, such that $E \subset B_{r_0}(p)$. The assumption that M has at most Euclidean volume growth implies M has finite volume, which contradicts the results in (i). $\square$

Let M be a minimal hypersurface in N with Euclidean volume growth at most. Combining (4.1)(4.3) and the definition of $\mathcal{R}$, the quantity

$$r^{-n}\int_{M\cap\{\mathcal{R}\leq r\}}|\overline{\nabla}\mathcal{R}|^2 d\mu$$

is uniformly bounded for any $r \in (0, \infty)$, then there exists a sequence $r_i \to \infty$ such that

$$(5.14) \qquad \limsup_{r \to \infty} \left(r^{-n} \int_{M \cap \{\mathcal{R} \le r\}} |\overline{\nabla} \mathcal{R}|^2 d\mu \right) = \lim_{r_i \to \infty} \left(r_i^{-n} \int_{M \cap \{\mathcal{R} \le r_i\}} |\overline{\nabla} \mathcal{R}|^2 d\mu \right).$$

Lemma 5.2. *There is a sequence $\delta_i \to 0^+$ such that for any constants $K_2 > K_1 > 0$ and $\epsilon \in (0, 1)$ and any bounded Lipschitz function f on $N \setminus B_1$ we have*

$$(5.15) \qquad \limsup_{i \to \infty} \left| \left(\frac{\delta_i}{K_2 r_i} \right)^n \int_{M \cap \{\mathcal{R} \le \frac{K_2 r_i}{\delta_i}\}} f |\overline{\nabla} \mathcal{R}|^2 - \left(\frac{\delta_i}{K_1 r_i} \right)^n \int_{M \cap \{\mathcal{R} \le \frac{K_1 r_i}{\delta_i}\}} f |\overline{\nabla} \mathcal{R}|^2 \right|$$

$$\le C\epsilon^n \sup_{N \setminus B_1} |f| + \limsup_{i \to \infty} \int_{\frac{K_1 r_i}{\delta_i}}^{\frac{K_2 r_i}{\delta_i}} \left(s^{-n-1} \int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le s\}} \mathcal{R} \nabla f \cdot \nabla \mathcal{R} \right) ds.$$

Proof. Let $\{e_i\}$ be an orthonormal basis of TM and ν be the unit normal vector of M. Then by (4.2) we have

$$(5.16) \qquad \begin{aligned} \Delta_M \mathcal{R}^2 &= \sum_{i=1}^{n} \left(\nabla_{e_i} \nabla_{e_i} \mathcal{R}^2 - (\nabla_{e_i} e_i) \mathcal{R}^2 \right) \\ &= \sum_{i=1}^{n} \left(\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} \mathcal{R}^2 - (\overline{\nabla}_{e_i} e_i) \mathcal{R}^2 \right) + \sum_{i=1}^{n} \left(\overline{\nabla}_{e_i} e_i - \nabla_{e_i} e_i \right) \mathcal{R}^2 \\ &= \Delta_N \mathcal{R}^2 - \text{Hess}_{\mathcal{R}^2}(\nu, \nu) \\ &= 2(n+1) |\overline{\nabla} \mathcal{R}|^2 - \text{Hess}_{\mathcal{R}^2}(\nu, \nu). \end{aligned}$$

By (4.4) and (4.3) there exists a sequence $\delta_i \to 0^+$ such that on $M \setminus B_{\sqrt{r_i}}$ we have

$$(5.17) \qquad \left| \Delta_M \mathcal{R}^2 - 2n |\overline{\nabla} \mathcal{R}|^2 \right| \le 2\delta_i |\overline{\nabla} \mathcal{R}|^2.$$

For any $s \ge \alpha_i r_i^{\frac{1}{2}}$ with $\alpha_i \ge 1$ and $f \in \text{Lip}(N \setminus B_1)$, integrating by parts yields
$$(5.18)$$
$$2s \int_{M \cap \{\mathcal{R}=s\}} f |\nabla \mathcal{R}| - 2\alpha_i r_i^{\frac{1}{2}} \int_{M \cap \{\mathcal{R}=\alpha_i r_i^{\frac{1}{2}}\}} f |\nabla \mathcal{R}| = \int_{M \cap \{\alpha_i r_i^{\frac{1}{2}} < \mathcal{R} \le s\}} \text{div}_M \left(f \nabla \mathcal{R}^2 \right)$$

$$= \int_{M \cap \{\alpha_i r_i^{\frac{1}{2}} < \mathcal{R} \le s\}} \nabla f \cdot \nabla \mathcal{R}^2 + \int_{M \cap \{\alpha_i r_i^{\frac{1}{2}} < \mathcal{R} \le s\}} f \Delta_M \mathcal{R}^2.$$

Hence,
(5.19)

$$\frac{\partial}{\partial s}\left(s^{-n}\int_{M\cap\{\mathcal{R}\le s\}}f|\overline{\nabla}\mathcal{R}|^2\right)$$

$$=-ns^{-n-1}\int_{M\cap\{\mathcal{R}\le s\}}f|\overline{\nabla}\mathcal{R}|^2+s^{-n}\int_{M\cap\{\mathcal{R}=s\}}f\frac{|\overline{\nabla}\mathcal{R}|^2}{|\nabla\mathcal{R}|}$$

$$=-ns^{-n-1}\int_{M\cap\{\mathcal{R}\le s\}}f|\overline{\nabla}\mathcal{R}|^2+s^{-n}\int_{M\cap\{\mathcal{R}=s\}}f|\nabla\mathcal{R}|+s^{-n}\int_{M\cap\{\mathcal{R}=s\}}f\frac{\langle\overline{\nabla}\mathcal{R},\nu\rangle^2}{|\nabla\mathcal{R}|}$$

$$=-ns^{-n-1}\int_{M\cap\{\mathcal{R}\le s\}}f|\overline{\nabla}\mathcal{R}|^2+\frac{1}{2}s^{-n-1}\int_{M\cap\{\alpha_i r_i^{\frac{1}{2}}<\mathcal{R}\le s\}}f\Delta_M\mathcal{R}^2$$

$$+\alpha_i r_i^{\frac{1}{2}}s^{-n-1}\int_{M\cap\{\mathcal{R}=\alpha_i r_i^{\frac{1}{2}}\}}f|\nabla\mathcal{R}|+s^{-n-1}\int_{M\cap\{\alpha_i r_i^{\frac{1}{2}}<\mathcal{R}\le s\}}\mathcal{R}\nabla f\cdot\nabla\mathcal{R}$$

$$+s^{-n}\int_{M\cap\{\mathcal{R}=s\}}f\frac{\langle\overline{\nabla}\mathcal{R},\nu\rangle^2}{|\nabla\mathcal{R}|}$$

$$=-ns^{-n-1}\int_{M\cap\{\mathcal{R}\le\alpha_i r_i^{\frac{1}{2}}\}}f|\overline{\nabla}\mathcal{R}|^2+\frac{1}{2}s^{-n-1}\int_{M\cap\{\alpha_i r_i^{\frac{1}{2}}<\mathcal{R}\le s\}}f\left(\Delta_M\mathcal{R}^2-2n|\overline{\nabla}\mathcal{R}|^2\right)$$

$$+\alpha_i r_i^{\frac{1}{2}}s^{-n-1}\int_{M\cap\{\mathcal{R}=\alpha_i r_i^{\frac{1}{2}}\}}f|\nabla\mathcal{R}|+s^{-n-1}\int_{M\cap\{\alpha_i r_i^{\frac{1}{2}}<\mathcal{R}\le s\}}\mathcal{R}\nabla f\cdot\nabla\mathcal{R}$$

$$+s^{-n}\int_{M\cap\{\mathcal{R}=s\}}f\frac{\langle\overline{\nabla}\mathcal{R},\nu\rangle^2}{|\nabla\mathcal{R}|}.$$

Select $f\equiv 1$ and $\alpha_i=1$ in (5.19) and integrate. Then for any $r\ge\sqrt{r_i}$ there is a constant C depending only on N and V_M such that
(5.20)

$$\left(\delta_i^{-2}r\right)^{-n}\int_{M\cap\{\mathcal{R}\le\delta_i^{-2}r\}}|\overline{\nabla}\mathcal{R}|^2-r^{-n}\int_{M\cap\{\mathcal{R}\le r\}}|\overline{\nabla}\mathcal{R}|^2$$

$$\ge-nCr_i^{\frac{n}{2}}\int_r^{\delta_i^{-2}r}s^{-n-1}ds-C\delta_i\int_r^{\delta_i^{-2}r}\frac{1}{s}ds+\int_r^{\delta_i^{-2}r}\left(s^{-n}\int_{M\cap\{\mathcal{R}=s\}}\frac{\langle\overline{\nabla}\mathcal{R},\nu\rangle^2}{|\nabla\mathcal{R}|}\right)ds$$

$$\ge-C\frac{r_i^{\frac{n}{2}}}{r^n}+2C\delta_i\log\delta_i+\int_r^{\delta_i^{-2}r}\left(s^{-n}\int_{M\cap\{\mathcal{R}=s\}}\frac{\langle\overline{\nabla}\mathcal{R},\nu\rangle^2}{|\nabla\mathcal{R}|}\right)ds.$$

Choose $r=r_i$ in the above inequality and let i go to infinity, then we obtain

$$\limsup_{r\to\infty}\left(r^{-n}\int_{M\cap\{\mathcal{R}\le r\}}|\overline{\nabla}\mathcal{R}|^2\right)-\lim_{i\to\infty}\left(r_i^{-n}\int_{M\cap\{\mathcal{R}\le r_i\}}|\overline{\nabla}\mathcal{R}|^2\right)$$

(5.21)
$$\ge\lim_{i\to\infty}\left(\left(\delta_i^{-2}r_i\right)^{-n}\int_{M\cap\{\mathcal{R}\le\delta_i^{-2}r_i\}}|\overline{\nabla}\mathcal{R}|^2\right)-\lim_{i\to\infty}\left(r_i^{-n}\int_{M\cap\{\mathcal{R}\le r_i\}}|\overline{\nabla}\mathcal{R}|^2\right)$$

$$\ge\lim_{i\to\infty}\int_{r_i}^{\delta_i^{-2}r_i}\left(s^{-n}\int_{M\cap\{\mathcal{R}=s\}}\frac{\langle\overline{\nabla}\mathcal{R},\nu\rangle^2}{|\nabla\mathcal{R}|}\right)ds.$$

which together with (5.14) implies

(5.22)
$$\lim_{i\to\infty}\int_{r_i}^{\delta_i^{-2}r_i}\left(s^{-n}\int_{M\cap\{\mathcal{R}=s\}}\frac{\langle\overline{\nabla}\mathcal{R},\nu\rangle^2}{|\nabla\mathcal{R}|}\right)ds=0,$$

namely,

$$(5.23) \qquad \lim_{i \to \infty} \int_{M \cap \{r_i < \mathcal{R} \le \delta_i^{-2} r_i\}} \frac{\langle \overline{\nabla} \mathcal{R}, \nu \rangle^2}{\mathcal{R}^n} = 0.$$

Set $|f|_0 \triangleq \sup_N f < \infty$ and $\alpha_i = \epsilon K_1 r_i^{\frac{1}{2}} \delta_i^{-1}$ for any small $\epsilon \in (0,1)$ in (5.19), then for any $r \ge \epsilon r_i \delta_i^{-1}$

$$\left| (K_2 r)^{-n} \int_{M \cap \{\mathcal{R} \le K_2 r\}} f |\overline{\nabla} \mathcal{R}|^2 - (K_1 r)^{-n} \int_{M \cap \{\mathcal{R} \le K_1 r\}} f |\overline{\nabla} \mathcal{R}|^2 \right|$$

$$\le nC |f|_0 \left(\frac{\epsilon K_1 r_i}{\delta_i} \right)^n \int_{K_1 r}^{K_2 r} s^{-n-1} ds + C \delta_i |f|_0 \int_{K_1 r}^{K_2 r} \frac{1}{s} ds$$

$$+ |f|_0 \left(\frac{\epsilon K_1 r_i}{\delta_i} \int_{M \cap \{\mathcal{R} = \frac{\epsilon K_1 r_i}{\delta_i}\}} |\nabla \mathcal{R}| \right) \int_{K_1 r}^{K_2 r} s^{-n-1} ds$$

$$+ \int_{K_1 r}^{K_2 r} \left(s^{-n-1} \int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le s\}} \mathcal{R} \nabla f \cdot \nabla \mathcal{R} \right) ds$$

$$(5.24) \qquad + |f|_0 \int_{K_1 r}^{K_2 r} \left(s^{-n} \int_{M \cap \{\mathcal{R} = s\}} \frac{\langle \overline{\nabla} \mathcal{R}, \nu \rangle^2}{|\nabla \mathcal{R}|} \right) ds$$

$$\le C |f|_0 \frac{\epsilon^n r_i^n}{\delta_i^n r^n} + C \delta_i |f|_0 \log \frac{K_2}{K_1} + \int_{K_1 r}^{K_2 r} \left(s^{-n-1} \int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le s\}} \mathcal{R} \nabla f \cdot \nabla \mathcal{R} \right) ds$$

$$+ \frac{|f|_0}{2n K_1^n r^n} \int_{M \cap \{\mathcal{R} \le \frac{\epsilon K_1 r_i}{\delta_i}\}} \Delta_M \mathcal{R}^2 + |f|_0 \int_{K_1 r}^{K_2 r} \left(s^{-n} \int_{M \cap \{\mathcal{R} = s\}} \frac{\langle \overline{\nabla} \mathcal{R}, \nu \rangle^2}{|\nabla \mathcal{R}|} \right) ds$$

$$\le C |f|_0 \frac{\epsilon^n r_i^n}{\delta_i^n r^n} + C \delta_i |f|_0 \log \frac{K_2}{K_1} + \int_{K_1 r}^{K_2 r} \left(s^{-n-1} \int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le s\}} \mathcal{R} \nabla f \cdot \nabla \mathcal{R} \right) ds$$

$$+ C_1 \frac{|f|_0}{2n \delta_i^n r^n} \epsilon^n r_i^n + |f|_0 \int_{K_1 r}^{K_2 r} \left(s^{-n} \int_{M \cap \{\mathcal{R} = s\}} \frac{\langle \overline{\nabla} \mathcal{R}, \nu \rangle^2}{|\nabla \mathcal{R}|} \right) ds.$$

Let $r = \frac{r_i}{\delta_i}$ and $i \to \infty$, then we complete the proof. $\qquad \square$

Let $\epsilon_i = \delta_i^2 r_i^{-2}$ and suppose that $\epsilon_i N$ converges to (N_∞, d_∞) without loss of generality. Let $\epsilon_i M = (M, \epsilon_i g)$ and $D_r^i(x) = \epsilon_i M \cap B_r^i(x)$. Clearly, $\epsilon_i M$ is still a minimal hypersurface in $\epsilon_i N$ with $Vol \left(\epsilon_i M \cap B_r^i(p) \right) \le V_M r^n$.

Lemma 5.3. *There exists a subsequence* $\{\epsilon_{i_j}\} \subset \{\epsilon_i\}$ *such that* $\epsilon_{i_j} M$ *converges to a cone* $CY = \mathbb{R}^+ \times_\rho Y$ *in* N_∞, *where* $Y \subset \partial \mathcal{B}_1(o)$ *is an* $(n-1)$*-dimensional Hausdorff set with* $H^{n-1}(Y) > 0$.

Proof. Note (4.18). By choosing a diagonal sequence, we can assume

$$\Phi_{i_j}^{-1} \left(\epsilon_{i_j} M \bigcap \overline{B^{i_j}}_r \setminus B_{\frac{1}{r}}^{i_j} \right) \rightharpoonup T \qquad \text{as } j \to \infty,$$

for any $r > 1$, where T is an integer-rectifiable current in N_∞. For convenience, we still write ϵ_i instead of ϵ_{i_j}.

Let f be a homogenous function in $C^1(N_\infty \setminus \{o\})$, that is,

$$f(\rho\theta) = f(\theta)$$

for any $\rho > 0$ and $\theta \in \partial\mathcal{B}_1$. Let Π_i be the re-scaling map from $(N, \bar{g})$ to $\epsilon_i N = (N, \epsilon_i\bar{g}, p)$ as before, then both of (4.12) and (4.13) hold. Now we can extend the function $f \circ \Phi_i^{-1} \circ \Pi_i$ to a uniformly bounded function F_i in $B_{\frac{K_2 r_i}{\delta_i}} = B_{\frac{K_2}{\sqrt{\epsilon_i}}}$ with $F_i = f \circ \Phi_i^{-1} \circ \Pi_i$ on $B_{\frac{K_2 r_i}{\delta_i}} \setminus B_{\frac{\epsilon K_1 r_i}{\delta_i}} = B_{\frac{K_2}{\sqrt{\epsilon_i}}} \setminus B_{\frac{\epsilon K_1}{\sqrt{\epsilon_i}}}$. Note (4.1) and the definition of $\mathcal{R}$. Hence for sufficiently large i and $s \in \left(\frac{\epsilon K_1 r_i}{\delta_i}, \frac{K_2 r_i}{\delta_i}\right)$, combining (4.12) and (4.13) we have

$$
\begin{aligned}
\int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le s\}} \mathcal{R} \nabla F_i \cdot \nabla \mathcal{R} &\le \int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le s\}} \mathcal{R} \left(\overline{\nabla} F_i \cdot \overline{\nabla} \mathcal{R} + |\overline{\nabla} F_i| \cdot |\langle \overline{\nabla} \mathcal{R}, \nu \rangle| \right) \\
(5.25) \qquad\qquad &\le \int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le s\}} \left(C_2 \epsilon + C_2 |\langle \overline{\nabla} \mathcal{R}, \nu \rangle| \right) \\
&\le C_3 \epsilon s^n + C_2 \int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le s\}} |\langle \overline{\nabla} \mathcal{R}, \nu \rangle|
\end{aligned}
$$

for some constants $C_2, C_3 > 1$. By the Cauchy inequality we get
(5.26)
$$
\begin{aligned}
&\limsup_{i \to \infty} \int_{\frac{K_1 r_i}{\delta_i}}^{\frac{K_2 r_i}{\delta_i}} \left(\frac{1}{s^{n+1}} \int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le s\}} \mathcal{R} \nabla F_i \cdot \nabla \mathcal{R} \right) ds \\
&\le \limsup_{i \to \infty} \int_{\frac{K_1 r_i}{\delta_i}}^{\frac{K_2 r_i}{\delta_i}} \left(\frac{C_3 \epsilon}{s} + \frac{C_2}{s^{n+1}} \left(\int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le s\}} \frac{\langle \overline{\nabla} \mathcal{R}, \nu \rangle^2}{\mathcal{R}^n} \int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le s\}} \mathcal{R}^n \right)^{\frac{1}{2}} \right) ds \\
&\le C_3 \epsilon \log \frac{K_2}{K_1} + C_4 \limsup_{i \to \infty} \left(\int_{\frac{K_1 r_i}{\delta_i}}^{\frac{K_2 r_i}{\delta_i}} \frac{1}{s} ds \left(\int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le \frac{K_2 r_i}{\delta_i}\}} \frac{\langle \overline{\nabla} \mathcal{R}, \nu \rangle^2}{\mathcal{R}^n} \right)^{\frac{1}{2}} \right) \\
&\le C_3 \epsilon \log \frac{K_2}{K_1} + C_4 \log \frac{K_2}{K_1} \limsup_{i \to \infty} \left(\int_{M \cap \{\frac{\epsilon K_1 r_i}{\delta_i} < \mathcal{R} \le \frac{K_2 r_i}{\delta_i}\}} \frac{\langle \overline{\nabla} \mathcal{R}, \nu \rangle^2}{\mathcal{R}^n} \right)^{\frac{1}{2}}.
\end{aligned}
$$

where C_4 is a constant. Note F_i is uniformly bounded for all i, then by Lemma 5.2 and (5.23) we obtain

$$
\begin{aligned}
&\limsup_{i \to \infty} \left| \left(\frac{\delta_i}{K_2 r_i} \right)^n \int_{M \cap \{\mathcal{R} \le \frac{K_2 r_i}{\delta_i}\}} F_i |\overline{\nabla} \mathcal{R}|^2 - \left(\frac{\delta_i}{K_1 r_i} \right)^n \int_{M \cap \{\mathcal{R} \le \frac{K_1 r_i}{\delta_i}\}} F_i |\overline{\nabla} \mathcal{R}|^2 \right| \\
(5.27) \qquad &\le C_3 \epsilon \log \frac{K_2}{K_1} + C_4 \limsup_{i \to \infty} \left(\epsilon^n \sup_{B_{\frac{K_2 r_i}{\delta_i}}} |F_i| \right) \le C_3 \epsilon \log \frac{K_2}{K_1} + C_5 \epsilon^n
\end{aligned}
$$

for some constant C_5. For any $\delta \in (0,1)$, together with (4.3) we have

(5.28)
$$\left| \frac{1}{K_2^n} \int_{T \cap (\mathcal{B}_{K_2} \setminus \mathcal{B}_{\delta K_1})} f - \frac{1}{K_1^n} \int_{T \cap (\mathcal{B}_{K_1} \setminus \mathcal{B}_{\delta K_1})} f \right|$$

$$= \lim_{i \to \infty} \left| \left(\frac{\delta_i}{K_2 r_i} \right)^n \int_{M \cap \{ \frac{\delta K_1 r_i}{\delta_i} \leq \mathcal{R} \leq \frac{K_2 r_i}{\delta_i} \}} F_i |\overline{\nabla} \mathcal{R}|^2 - \left(\frac{\delta_i}{K_1 r_i} \right)^n \int_{M \cap \{ \frac{\delta K_1 r_i}{\delta_i} \leq \mathcal{R} \leq \frac{K_1 r_i}{\delta_i} \}} F_i |\overline{\nabla} \mathcal{R}|^2 \right|$$

$$\leq \limsup_{i \to \infty} \left| \left(\frac{\delta_i}{K_2 r_i} \right)^n \int_{M \cap \{ \mathcal{R} \leq \frac{K_2 r_i}{\delta_i} \}} F_i |\overline{\nabla} \mathcal{R}|^2 - \left(\frac{\delta_i}{K_1 r_i} \right)^n \int_{M \cap \{ \mathcal{R} \leq \frac{K_1 r_i}{\delta_i} \}} F_i |\overline{\nabla} \mathcal{R}|^2 \right|$$

$$+ \limsup_{i \to \infty} \left| \left(\frac{\delta_i}{K_2 r_i} \right)^n \int_{M \cap \{ \mathcal{R} \leq \frac{\delta K_1 r_i}{\delta_i} \}} F_i |\overline{\nabla} \mathcal{R}|^2 - \left(\frac{\delta_i}{K_1 r_i} \right)^n \int_{M \cap \{ \mathcal{R} \leq \frac{\delta K_1 r_i}{\delta_i} \}} F_i |\overline{\nabla} \mathcal{R}|^2 \right|$$

$$\leq C_3 \epsilon \log \frac{K_2}{K_1} + C_5 \epsilon^n + C_5 \left(\frac{1}{K_1^n} - \frac{1}{K_2^n} \right) \limsup_{i \to \infty} \left(\frac{\delta_i^n}{r_i^n} \int_{M \cap \{ \mathcal{R} \leq \frac{\delta K_1 r_i}{\delta_i} \}} 1 d\mu \right).$$

Letting $\delta \to 0$ and $\epsilon \to 0$ implies

(5.29)
$$\frac{1}{K_2^n} \int_{T \cap \mathcal{B}_{K_2}} f = \frac{1}{K_1^n} \int_{T \cap \mathcal{B}_{K_1}} f.$$

By the argument in the proof of Theorem 19.3 in [27], the above equality means that T is a cone in N_∞ up to a set of measure zero, as f is an arbitrary homogeneous function. In fact, by Fubini's Theorem the above equality becomes

(5.30)
$$K_1^n \int_0^{K_2} \left(\int_{T \cap \partial \mathcal{B}_s} f \right) ds = K_2^n \int_0^{K_1} \left(\int_{T \cap \partial \mathcal{B}_s} f \right) ds.$$

Differentiating w.r.t. K_2 and K_1 implies

(5.31)
$$\frac{1}{K_2^{n-1}} \int_{T \cap \partial \mathcal{B}_{K_2}} f = \frac{1}{K_1^{n-1}} \int_{T \cap \partial \mathcal{B}_{K_1}} f.$$

Since $N_\infty = CX$ is a cone and any point in it can be represented by (ρ, θ) for some $\theta \in X$, then we define $\frac{1}{r} T$ by $\{ (\frac{\rho}{r}, \theta) \in N_\infty | \ (\rho, \theta) \in T \}$. So

(5.32)
$$\int_{\frac{1}{K_2} T \cap \partial \mathcal{B}_1} f = \int_{\frac{1}{K_1} T \cap \partial \mathcal{B}_1} f.$$

Hence $\frac{1}{K_2} T = \frac{1}{K_1} T$ up to a set of measure zero, namely, T is a cone, say, CY, where $Y \in \partial \mathcal{B}_1(o)$ is an $(n-1)$-dimensional Hausdorff set. By (5.1), we know $H^n(CY) > 0$, which implies $H^{n-1}(Y) > 0$. $\qquad \square$

Remark 5.4. *By a simple modification, Lemma 5.2 and Lemma 5.3 also apply to minimal submanifolds of high codimensions with Euclidean volume growth in N.*

For any $\omega \in \mathcal{D}^n \left(B_{\frac{2}{\epsilon}}^i \setminus B_\epsilon^i \right)$ let

(5.33)
$$\epsilon_i M(\omega) = \int_{\epsilon_i M} \langle \omega, \nu_i \rangle d\mu_i, \qquad CY(\omega \circ \Phi_i) = \int_T \langle \omega \circ \Phi_i, \nu_\infty \rangle d\mu_\infty,$$

where $d\mu_i$ and $d\mu_\infty$ are the volume elements of $\epsilon_i M$ and CY, and ν_i and ν_∞ are the unit normal vectors of $\epsilon_i M$ and CY.

For any sufficiently small fixed constant $\epsilon \in (0,1)$, $\epsilon_i M \bigcap \left(B^i_{\frac{2}{\epsilon}} \setminus B^i_\epsilon \right)$ converges to $CY \bigcap \left(\mathcal{B}_{\frac{2}{\epsilon}} \setminus \mathcal{B}_\epsilon \right)$ in the varifold sense. Then

$$
(5.34) \qquad \lim_{i\to\infty} \epsilon_i M \llcorner \left(B^i_{\frac{2}{\epsilon}} \setminus B^i_\epsilon \right) \left(\omega \circ \Phi_i^{-1} \right) = CY \llcorner \left(\mathcal{B}_{\frac{2}{\epsilon}} \setminus \mathcal{B}_\epsilon \right) (\omega)
$$

for any $\omega \in \mathcal{D}^n \left(\mathcal{B}_{\frac{2}{\epsilon}} \setminus \mathcal{B}_\epsilon \right)$.

Let

$$
(5.35) \qquad E_i \triangleq \left\{ x \in \epsilon_i M \bigcap \left(B^i_{\frac{2}{\epsilon}} \setminus B^i_\epsilon \right) \,\Big|\, \left| \left\langle \overline{\nabla}^i \rho_i(x), \nu_i \right\rangle \right| \geq \epsilon \right\}.
$$

If $\rho_\infty(x) = d_\infty(o,x)$ is the distance function on N_∞, then $\lim_{i\to\infty} \rho \circ \Phi_i = \rho_\infty$ in $B^i_{\frac{2}{\epsilon}} \setminus B^i_\epsilon$. For any compact set $K \in \mathcal{B}_{\frac{2}{\epsilon}} \setminus \mathcal{B}_\epsilon$ by (5.34) we have

$$
(5.36) \qquad 0 = \lim_{i\to\infty} \left(\epsilon_i M \llcorner \Phi_i(K) \right) \left(\omega^* \circ \Phi_i^{-1} \right) = \lim_{i\to\infty} \int_{\epsilon_i M \cap \Phi_i(K)} \langle \omega^* \circ \Phi_i^{-1}, \nu_i \rangle d\mu_i,
$$

where ω^* is the dual form of $\frac{\partial}{\partial \rho_\infty}$ in TN_∞. Hence for any sufficiently small $\epsilon > 0$ we conclude that for sufficiently large i there holds

$$
(5.37) \qquad H^n(E_i) < \epsilon^{n+1}.
$$

Now we assume that M is a stable minimal hypersurface in N. Then $\epsilon_i M$ is still a stable minimal hypersurface in $\epsilon_i N$. Let B^i be the second fundamental form of $\epsilon_i M$ in $\epsilon_i N$, and $Ric_{\epsilon_i N}$ the Ricci curvature of $\epsilon_i N$. For any Lipschitz function ϕ with compact support in $\epsilon_i M$ we have from (2.5)

$$
(5.38) \qquad \int_{\epsilon_i M} \left(|B^i|^2 + Ric_{\epsilon_i N}(\nu_i, \nu_i) \right) \phi^2 \leq \int_{\epsilon_i M} |\nabla^i \phi|^2,
$$

where ∇^i is the Levi-Civita connection of $\epsilon_i M$. Note that $\overline{\nabla}\rho$ and $\overline{\nabla}^i \rho_i$ have been defined in section 4. Now we suppose that there exists some sufficiently large $r_0 > 0$ such that the non-radial Ricci curvature of N satisfies

$$
(5.39) \qquad \inf_{\partial B_r} Ric \left(\xi^T, \xi^T \right) \geq \frac{\kappa'}{r^2} > 0
$$

for all $r \geq r_0$ and $n \geq 2$, where ξ is a local vector field on N, $\xi^T = \xi - \langle \xi, \overline{\nabla}\rho \rangle \overline{\nabla}\rho$ with $|\xi^T| = 1$, and κ' is a positive constant. Then

$$
\inf_{\partial B^i_s} Ric_{\epsilon_i N} \left(\eta^T, \eta^T \right) \geq \frac{\kappa'}{r^2} > 0
$$

for all $s \geq \sqrt{\epsilon_i} r_0$ and $n \geq 2$, where η is a local vector field on $\epsilon_i N$, $\eta^T = \eta - \left\langle \eta, \overline{\nabla}^i \rho_i \right\rangle \overline{\nabla}^i \rho_i$ with $|\eta^T| = 1$. Using conditions C1) and C3) which are both scaling invariant, we obtain

$$
(5.40) \qquad
\begin{aligned}
Ric_{\epsilon_i N}(\nu_i, \nu_i) &\geq Ric_{\epsilon_i N}(\nu_i^T, \nu_i^T) + 2 \left\langle \nu_i, \overline{\nabla}^i \rho_i \right\rangle Ric_{\epsilon_i N}(\nu_i^T, \overline{\nabla}^i \rho_i) \\
&\geq Ric_{\epsilon_i N}(\nu_i^T, \nu_i^T) - c' \left\langle \nu_i, \overline{\nabla}^i \rho_i \right\rangle \rho_i^{-2}
\end{aligned}
$$

for some absolute constant $c' > 0$. Let ϕ be the Lipschitz function on $\epsilon_i N$ defined by

$$
\phi(x) = (\rho_i(x))^{\frac{2-n}{2}} \sin \left(\pi \frac{\log \rho_i(x)}{\log \epsilon} \right)
$$

in $B_1^i \setminus B_\epsilon^i$ and $\phi = 0$ in other places. Here ϵ is a small positive constant less than $\min\{\frac{1}{2}, \frac{\kappa'}{2c'}\}$, which implies $\kappa'(1 - \epsilon^2) - c'\epsilon \geq \frac{\kappa'}{4}$. So from (5.35), (5.37) and (5.40)

$$
\begin{aligned}
&\int_{\epsilon_i M} Ric_{\epsilon_i N}(\nu_i, \nu_i)\phi^2 d\mu_i \\
&\geq \int_{(\epsilon_i M \setminus E_i) \cap (B_1^i \setminus B_\epsilon^i)} \left(\frac{\kappa'}{\rho_i^2} |\nu_i^T|^2 - \frac{c'}{\rho_i^2} \left\langle \nu_i, \overline{\nabla}^i \rho_i \right\rangle \right) \sin^2\left(\pi \frac{\log \rho_i}{\log \epsilon} \right) \rho_i^{2-n} d\mu_i \\
&\geq \left(\kappa'(1 - \epsilon^2) - c'\epsilon \right) \int_{(\epsilon_i M \setminus E_i) \cap (B_1^i \setminus B_\epsilon^i)} \sin^2\left(\pi \frac{\log \rho_i}{\log \epsilon} \right) \rho_i^{-n} d\mu_i \\
&\geq \left(\kappa'(1 - \epsilon^2) - c'\epsilon \right) \left(\int_{\epsilon_i M \cap (B_1^i \setminus B_\epsilon^i)} \sin^2\left(\pi \frac{\log \rho_i}{\log \epsilon} \right) \rho_i^{-n} d\mu_i - \epsilon^{-n} H^n(E_i) \right) \\
&\geq \left(\kappa'(1 - \epsilon^2) - c'\epsilon \right) \int_{\epsilon_i M \cap (B_1^i \setminus B_\epsilon^i)} \sin^2\left(\pi \frac{\log \rho_i}{\log \epsilon} \right) \rho_i^{-n} d\mu_i - \kappa'\epsilon(1 - \epsilon^2).
\end{aligned}
$$
(5.41)

Substituting this into (5.38) yields

$$
\begin{aligned}
&\left(\kappa'(1 - \epsilon^2) - c'\epsilon \right) \int_{\epsilon_i M \cap (B_1^i \setminus B_\epsilon^i)} \sin^2\left(\pi \frac{\log \rho_i}{\log \epsilon} \right) \rho_i^{-n} d\mu_i - \kappa'\epsilon(1 - \epsilon^2) \\
&\leq \int_{\epsilon_i M} Ric_{\epsilon_i N}(\nu_i, \nu_i)\phi^2 \leq \int_{\epsilon_i M} |\overline{\nabla}^i \phi|^2 \\
&\leq \int_{\epsilon_i M \cap (B_1^i \setminus B_\epsilon^i)} \left(\frac{2-n}{2} \sin\left(\pi \frac{\log \rho_i}{\log \epsilon} \right) + \frac{\pi}{\log \epsilon} \cos\left(\pi \frac{\log \rho_i}{\log \epsilon} \right) \right)^2 \rho_i^{-n} d\mu_i.
\end{aligned}
$$
(5.42)

Due to Lemma 5.3, we let $i \to \infty$, to get

$$
\begin{aligned}
&\left(\kappa'(1 - \epsilon^2) - c'\epsilon \right) \int_{CY \cap (\mathcal{B}_1 \setminus \mathcal{B}_\epsilon)} \sin^2\left(\pi \frac{\log \rho_\infty}{\log \epsilon} \right) \rho_\infty^{-n} d\mu_\infty - \kappa'\epsilon(1 - \epsilon^2) \\
&\leq \int_{CY \cap (\mathcal{B}_1 \setminus \mathcal{B}_\epsilon)} \left(\frac{2-n}{2} \sin\left(\pi \frac{\log \rho_\infty}{\log \epsilon} \right) + \frac{\pi}{\log \epsilon} \cos\left(\pi \frac{\log \rho_\infty}{\log \epsilon} \right) \right)^2 \rho_\infty^{-n} d\mu_\infty.
\end{aligned}
$$
(5.43)

Since

$$
\begin{aligned}
\int_{CY \cap (\mathcal{B}_1 \setminus \mathcal{B}_\epsilon)} \sin^2\left(\pi \frac{\log \rho_\infty}{\log \epsilon} \right) \rho_\infty^{-n} d\mu_\infty &= H^{n-1}(Y) \int_\epsilon^1 \sin^2\left(\pi \frac{\log s}{\log \epsilon} \right) \frac{1}{s} ds \\
&= \left(\log \frac{1}{\epsilon} \right) H^{n-1}(Y) \int_0^1 \sin^2(\pi t) dt,
\end{aligned}
$$
(5.44)

and $H^{n-1}(Y) > 0$, then

$$
\begin{aligned}
&\left(\kappa'(1 - \epsilon^2) - c'\epsilon \right) \left(\log \frac{1}{\epsilon} \right) H^{n-1}(Y) \int_0^1 \sin^2(\pi t) dt - \kappa'\epsilon(1 - \epsilon^2) \\
&\leq H^{n-1}(Y) \int_\epsilon^1 \left(\frac{2-n}{2} \sin\left(\pi \frac{\log s}{\log \epsilon} \right) + \frac{\pi}{\log \epsilon} \cos\left(\pi \frac{\log s}{\log \epsilon} \right) \right)^2 \frac{1}{s} ds \\
&= \left(\log \frac{1}{\epsilon} \right) H^{n-1}(Y) \int_0^1 \left(\frac{2-n}{2} \sin(\pi t) + \frac{\pi}{\log \epsilon} \cos(\pi t) \right)^2 dt \\
&= \left(\log \frac{1}{\epsilon} \right) H^{n-1}(Y) \left(\frac{(n-2)^2}{4} + \frac{\pi^2}{(\log \epsilon)^2} \right) \int_0^1 \sin^2(\pi t) dt,
\end{aligned}
$$
(5.45)

which implies

$$\kappa' \leq \frac{(n-2)^2}{4}.$$

Finally, we obtain the following results.

Theorem 5.5. *Let N be an $(n+1)$-dimensional complete Riemannian manifold satisfying conditions C1), C2) and C3), and with non-radial Ricci curvature $\inf_{\partial B_r} Ric\left(\xi^T, \xi^T\right) \geq \kappa' r^{-2}$ for a constant κ' and sufficiently large $r > 0$, where ξ is a local vector field on N with $|\xi^T| = 1$ defined in (5.39). If $\kappa' > \frac{(n-2)^2}{4}$, then N admits no complete stable minimal hypersurface with at most Euclidean volume growth.*

It is well known that area-minimizing hypersurfaces have Euclidean volume growth automatically. Let M be an n-dimensional area-minimizing hypersurface in N. Then the s-dimensional Hausdorff measure of the singular set of S is $H^s(\mathrm{Sing}\, M) = 0$ for all $s > n - 7$ (see [27] for example). We readily check that Lemmas 5.2 and 5.3 also hold for M. Namely, there is a sequence $\{\epsilon_i\}$ converging to zero such that $\epsilon_i N = (N, \epsilon_i \bar{g}, p)$ converges to a metric cone (N_∞, d_∞), and $\epsilon_i M$ converges to the cone $C\mathcal{Y} = \mathbb{R}^+ \times_\rho \mathcal{Y}$ in N_∞, where $\mathcal{Y} \in \partial \mathcal{B}_1(o)$ is an $(n-1)$-dimensional Hausdorff set.

Corollary 5.6. *Let N be an $(n+1)$-dimensional complete Riemannian manifold satisfying conditions C1), C2) and C3), and with non-radial Ricci curvature $\inf_{\partial B_r} Ric\left(\xi^T, \xi^T\right) \geq \kappa' r^{-2}$ for a constant κ' and sufficiently large $r > 0$, where ξ is a local vector field on N with $|\xi^T| = 1$ defined in (5.39). If $\kappa' > \frac{(n-2)^2}{4}$, then N admits no complete area-minimizing hypersurface.*

Remark 5.7. $\kappa = \frac{2}{n}\sqrt{n-1}$ *in Remark 3.8 is equivalent to*

$$Ric_{MCS_\kappa}\left(\xi^T, \xi^T\right) = \frac{(n-2)^2}{4(\rho + \frac{1}{\kappa} - \rho_0)^2} \qquad \text{for all } \rho \geq \rho_0,$$

where $\xi^T = \xi - \left\langle \xi, \frac{\partial}{\partial \rho} \right\rangle \frac{\partial}{\partial \rho}$, $\left|\xi^T\right| = 1$ and $\rho_0 \in (1, \frac{1}{\kappa})$ is a constant. Hence the constant κ' in Theorem 5.5 and Corollary 5.6 is optimal.

REFERENCES

1. F. J. Almgren, Jr., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. **85** (1966), 277-292.
2. L. Ambrosio, B. Kirchheim, *Currents in metric spaces*, Acta Math. **185(1)** (2000), 1-80.
3. M. Anderson, *On area-minimizing hypersurfaces in manifolds of nonnegative curvature*, Indiana Univ. Math. J. **32** (1983), 745-760.
4. E. Bombieri, E. De Giorgi and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** 1969, 243-268.
5. J. Cheeger and Tobias H. Colding, *Lower bound on Ricci curvature and almost rigidity of warped product*, Ann Math. **144(1)** (1996)189-237.
6. J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geom. **6** (1971), 119-128.
7. J. Cheeger, M. Gromov, and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geom. **17** (1982), 15-53.
8. Tobias H. Colding and William P. Minicozzi II, *Large scale Behavior of Kernels of Schrödinger Operators*, Amer. J. Math. **119 (6)** (1997), 1355-1398.
9. E. De Giorgi, *Una estensione del teorema di Bernstein*, Ann. Sc. Norm. Sup. Pisa **19** (1965), 79-85.
10. Qi Ding, *The inverse mean curvature flow in rotationally symmetric spaces*, Chin. Ann. Math. B **32B(1)** (2011), 27-44.
11. H. Federer, W.H. Fleming, *Normal and integral currents*, Ann. Math. **72(2)** (1960), 458-520.

12. D. Fischer-Colbrie, R. Schoen, The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature, Commun. Pure Appl. Math. **33** (1980), 199-211.
13. W. Fleming, On the oriented Plateau problem, Rend Circ. Mat. Palermo **11** (1962), 1-22.
14. Enrico Giusti, Minimal surfaces and functions of bounded variation, Birkhäuser Boston, Inc., 1984.
15. R. E. Greene, H. Wu, Lipschitz convergence of Riemannian manifolds, Pac. J. Math. **131** (1988), 119-141.
16. M. Gromov, J. Lafontaine and P. Pansu, Structures metriques pour les variete riemannienes, Cedic-Fernand Nathan, Paris (1981).
17. J. Jost and H. Karcher, Geometrische Methoden zur Gewinnung von a-priori-Schranken für harmonische Abbildungen, manuscr.math. **40** (1982), 27-77.
18. Gary R. Lawlor, A sufficient criterion for a cone to be area-minimizing, Mem. AMS **446** (1991).
19. Peter Li, Harmonic Functions and Applications to Complete Manifolds, University of California, Irvine, 2004, preprint.
20. Ngaiming Mok, Yum-Tong Siu and Shing-Tung Yau, The Poincaré-Lelong equation on complete Kähler manifolds, Compositio Mathematica **44** (1981), 183-218.
21. Frank Morgan, Area-minimizing surfaces in cones, Communications in analysis and geometry **10(5)** (2002), 971-983.
22. P. Nabonnand, Sur les variétés Riemanniennes complètes à courbure de Ricci positive, R. Acad. Sci. Paris **291** (1980), 591-593.
23. G. Perelman, A complete Riemannian manifold of positive Ricci curvature with Euclidean volume growth and nonunique asymptotic cone, Comparison geometry 165-166, Cambridge Univ. Press Cambridge, 1977.
24. S. Peters, Convergence of Riemannian manifolds, Compositio Math. **62** (1987), 3-16.
25. R. Schoen, L. Simon, and S. T. Yau, Curvature estimates for minimal hypersurfaces, Acta Math. **134** (1975), 275-288.
26. Ying Shen and Shunhui Zhu, Rigidity of stable minimal hypersurfaces, Math. Ann. **309** (1997), no. 1, 107-116.
27. Leon Simon, Lectures on Geometric Measure Theory, Proceedings of the center for mathematical analysis Australian national university, Vol. 3, 1983.
28. J. Simons, Minimal varieties in Riemannian manifolds, Ann. Math. **88** (1968), 62-105.
29. Joel Spruck, Interior gradient estimates and existence theorems for constant mean curvature graphs in $M^n \times \mathbb{R}$, Pure Appl. Math. Q. **3** (2007), no. 3, Special Issue: In honor of Leon Simon. Part 2, 785-800.
30. Y. L. Xin, Minimal Submanifolds and Related Topics, World Scientific Publ., (2003).

MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTR. 22, 04103 LEIPZIG, GERMANY

E-mail address: qiding@mis.mpg.de

E-mail address: 09110180013@fudan.edu.cn

MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTR. 22, 04103 LEIPZIG, GERMANY

E-mail address: jost@mis.mpg.de

INSTITUTE OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI 200433, CHINA

E-mail address: ylxin@fudan.edu.cn