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Universality of intervals of line graph order

by

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Abstract

We prove that for every $d \geq 3$ the homomorphism order of the class of line graphs of finite graphs with maximal degree d is universal. This means that every finite or countably infinite partially ordered set may be represented by line graphs of graphs with maximal degree d ordered by the existence of a homomorphism.

1 Introduction

An (undirected) *graph G* is a pair $G = (V_G, E_G)$ such that E_G is a set of 2-element subsets of V_G. We denote by V_G *the set of vertices of G* and by E_G *the set of edges of G*. We consider only finite graphs.

For given graphs G and H a *homomorphism* $f : G \to H$ is a mapping $f : V_G \to V_H$ such that $\{u, v\} \in V_G$ implies $\{f(u), f(v)\} \in V_H$. We denote the existence of a homomorphism $f : G \to H$ by $G \to H$. This allows us to consider the existence of a homomorphism, $\to$, to be a (binary) relation on the class of finite graphs.

The relation $\to$ is reflexive (identity is a homomorphism) and transitive (composition of two homomorphisms is still a homomorphism). Thus the existence of a homomorphism induces a quasi-order of the class of all finite graphs. This quasi-order can be transformed, in a standard way, to a partial order by considering only the isomorphism types of vertex-inclusion minimal elements of each equivalency class of $\to$ (the graph cores). The resulting partial order is known as the *homomorphism order* of graphs.

This partial order generalizes graph coloring and its rich structure is a fruitful area of research, see the Partial Order of Graphs and Homomorphism chapter in the monograph of Hell and Nešetřil [5]. The richness of homomorphism order is seen from a perhaps surprising fact that every countable (finite or infinite) partial order can be found as its suborder. This property of partial order is known as *universality*. The existence of such countable partial orders may seem counter-intuitive: there are uncountably many different partial orders and they are all "packed" into a

single countable structure. The existence of such partial order is given by the classical Fraïssé Theorem. There are however just few known explicit representations of such partial orders, see [4, 6, 7]. The universality of homomorphism order of graphs was first proved by Hedrlín and Pultr in a categorical setting as a culmination of several papers (see [18] for a complete proof).

It is interesting to observe that almost all naturally defined partial orders of graphs fails to be universal for simple reasons — by the absence of infinite increasing chains, decreasing chains or anti-chains. Several variants to graph homomorphisms are considered in our sequel paper [1] and only locally constrained homomorphisms are shown to produce universal orders. It is also a deep result of Robertson and Seymour that the graph minor order is a well quasi-order and thus not universal [20].

Although the universality is a quite rare property of orders on graphs, it is a very robust property of the homomorphism order. Hubička and Nešetřil showed that even homomorphism order of oriented paths is universal [9]. This result can be easily used to show the universality of other classes of graphs, such as planar graphs or series-parallel graphs [8]. This shows that complex graphs or complex computational problems (homomorphism testing is polynomial on oriented paths) are not needed to build universal partial orders. Something that was not anticipated by Nešetřil and Zhu a decade earlier [17].

Recently, D. E. Roberson proposed a systematic study of homomorphism order of the class of line graphs [19]. Here, a *line graph* of an undirected graph G, denoted by $L(G)$, is a graph $H = (V_H, E_H)$ such that $V_H = E_G$ and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G. Because edges of G play the role of vertices of $L(G)$, we will refer vertices of line graphs as *nodes*.

The classical Vizing theorem gives an insight into the structure of homomorphism order of line graphs in terms of *chromatic index* $\chi'(G)$ that is the chromatic number of $L(G)$, i.e. the minimum number of colors needed to color edges of a graph G such that edges with a common vertex receive different colors:

Theorem 1.1 (Vizing [22]). *For any graph G of maximum degree d it holds that $\chi'(G) \leq d+1$.*

Since the line graph of a graph with a vertex of degree d contains a d-clique, the Vizing theorem splits graphs into two classes. *Vizing class 1* contains the graphs whose chromatic index is the same as the maximal degree of a vertex, while *Vizing class 2* contains the remaining graphs.

The approach taken by Roberson [19] divides the class of line graphs into intervals. By $[K_n, K_{n+1})_{\mathcal{L}}$ we denote the class of all line graphs $L(G)$ such that $K_n \leq L(G) < K_{n+1}$. The line graphs in each interval have a particularly simple characterization:

Corollary 1.2. *The intervals $[K_d, K_{d+1})_{\mathcal{L}}$ consist of line graphs of graphs whose maximum degree is d.*

Proof. The existence of a $(d+1)$-edge coloring is equivalent with $L(G) \leq K_{d+1}$. Note that for the Vizing class 1 we indeed have $L(G) \leq K_d \leq K_{d+1}$.

As G contains a vertex of degree d, we have $K_d \leq L(G)$, indeed $K_d \subseteq L(G)$. On the other hand, a clique on $d+1 \geq 4$ vertices can be formed only from $d+1$ edges sharing a common vertex, hence $K_{d+1} \not\leq L(G)$. The same argument used for $K_d \leq L(G)$ implies that G contains a vertex of degree d. □

The line graphs can be considered as almost perfect graphs (a *perfect graph* is a graph in which the chromatic number of every induced subgraph equals to the size of the largest clique of that

subgraph). The homomorphism order of the class of perfect graphs is a trivial chain, since the core of every perfect graph is a clique. The almost-perfectness of the class of line graphs suggests that the homomorphism order of this class may be more constrained in its structure than the homomorphism order of graphs in general, and indeed many of the results about properties of the homomorphism order can not be easily restricted to the line graphs.

Roberson, in [19], showed that the homomorphism order of line graphs contains many gaps. This is a first important difference from the structure of the homomorphism order of graphs which was shown (up to one exception) to be dense by Welzl [23]. Roberson also asked whether every interval $[K_d, K_{d+1})_L$, $d \geq 3$ contains infinitely many incomparable elements. The answer is trivially negative for graphs with maximal degree 1 and 2. We give an affirmative answer to this problem. Indeed, we show:

Theorem 1.3. *The homomorphism order of line graphs is universal on every interval $[K_d, K_{d+1})_{\mathcal{L}}$ for $d \geq 3$.*

This further develops the results on the universality of the homomorphism order of special classes of graphs (see e.g. [8, 9, 15, 16]), and on universal partially ordered structures in general (see e.g. [10, 4, 6, 12, 13, 7, 11]).

As a special case, the universality of interval $[K_3, K_4)_L$ follows from the construction given by Šámal [21]. This is not an obvious observation — one has to carefully check that for the graphs constructed in [21] the existence of circulation coincide with the existence of a homomorphism of line graphs. Our proof use a new approach based on a new divisibility argument which we have introduced for a similar occasion [1]. This argument leads to a simpler construction without the need of complex gadgets (Blanuša snarks) used by Šámal [21].

The paper is organized as follows. In Section 2 we briefly review our construction from [1] about the universality of the divisibility order. In Section 3 we prove basic properties of a "dragon" graph that is a particularly simple example of Vizing class 2 graph that is also a graph core. Section 4 briefly reviews the indicator construction and Section 5 contains the proof of our main result. In Section 6 we discuss an extension of our construction to d-regular graphs and some additional observations on the homomorphism order of line graphs.

2 A particular universal partial order

Let $(P, \leq_P)$ be a partial order where P consists of finite set of integers and for $A, B \in P$ we put $A \leq_P B$ if and only if for every $a \in A$ there is $b \in B$ such that b divides a. We make use of the following:

Theorem 2.1 ([1]). *The order $(P, \leq_P)$ is a universal partial order.*

To make the paper self-contained we give a short proof of this result.

We say that a countable partial order is *past-finite* if every down-set is finite. Similarly a countable partial order is *future-finite* if every up-set is finite. Again, we say that a countable partial order is *past-finite-universal*, if it contains every past-finite partial order as a suborder. The *future-finite-universal* orders are defined analogously.

Let $P_f(A)$ denote the set of all finite subsets of A. The following lemma extends a well known fact about representing finite partial orders by sets ordered by subset relation.

Lemma 2.2. *Let A be countably infinite set. Then $(P_f(A), \subseteq)$ is a past-finite-universal partial order.*

Proof. Let be given an arbitrary past-finite set $(Q, \leq_Q)$. Without loss of generality we assume that $Q \subseteq A$. Assign to every $x \in Q$ a set $E(x) = \{y \in Q; y \leq x\}$. It is easy to verify that E is an embedding from $E : (Q, \leq_Q) \to (P_{\text{fin}}(A), \subseteq)$. $\qquad\square$

By the *divisibility partial order*, denoted by $(\mathbb{Z}, \leq_d)$, we mean a partial order where vertices are natural numbers and n is smaller than m if n is divisible by m.

Lemma 2.3. *The divisibility partial order $(\mathbb{Z}, \leq_d)$ is future-finite-universal.*

Proof. Denote by $\mathbb{P}$ the set of all prime numbers. Apply Lemma 2.2 for $A = \mathbb{P}$. Observe that $A \in P_f(\mathbb{P})$ is a subset of $B \in P_f(\mathbb{P})$ if and only if $\prod_{p \in A} p$ divides $\prod_{p \in B} p$. $\qquad\square$

Proof of Theorem 2.1. Let be given any partial order $(Q, \leq_Q)$. Without loss of generality we may assume that $Q \subseteq \mathbb{P}$. This way we enforce the linear order $\leq$ on elements of Q. Think of $\leq$ as a specification of the time of creation of the elements of Q.

We define two new orders on elements of Q: $\leq_f$, the *forwarding order* and $\leq_b$, the *backwarding order*:

1. We put $x \leq_f y$ if and only if $x \leq_Q y$ and $x \leq y$.

2. We put $x \leq_b y$ if and only if $x \leq_Q y$ and $x \geq y$.

Thus we decompose the partial order $(Q, \leq_Q)$ into $(Q, \leq_f)$ and $(Q, \leq_b)$. For every vertex $x \in Q$ both sets $\{y \mid y \leq_f x\}$ and $\{y \mid x \leq_b y\}$ are finite. It follows that $(Q, \leq_f)$ is past-finite and $(Q, \leq_b)$ is future-finite.

Since $(\mathbb{Z}, \leq_d)$ is future-finite-universal (Lemma 2.3), there is an embedding $E : (Q, \leq_b) \to (\mathbb{Z}, \leq_d)$. We put for every $x \in Q$:

$$U(x) = \{E(y) \mid y \leq_f x\}.$$

We show that U is an embedding $U : (Q, \leq_Q) \to (P, \leq_P)$.

First we show that $U(x) \leq_P U(y)$ imply $x \leq_Q y$. From the definition of $\leq_P$ we know that there is $w \in Q$, $E(w) \in U(y)$, such that $E(x) \leq_d E(w)$. The definition of U, $E(w) \in U(y)$ if and only if $w \leq_f y$. By the definition of E, $E(x) \leq_d E(w)$ if and only if $x \leq_b w$. It follows that $x \leq_b w \leq_f y$ and thus also $x \leq_Q w \leq_Q y$ and consequently $x \leq_Q y$.

To show that $x \leq_Q y$ imply $U(x) \leq_P U(y)$ we consider two cases.

1. When $x \leq y$ then $U(x) \subseteq U(y)$ and thus also $U(x) \leq_P U(y)$.

2. Assume $x > y$ and take any $w \in Q$, $E(w) \in U(x)$. From the construction of $U(x)$ we know that $w \leq_f x \leq_Q y$. If $w \leq y$, then $E(w) \in U(y)$. In the other case $w \leq_b y$ and thus $E(w) \leq_d E(y)$. It follows that $U(x) \leq_P U(y)$.

$\qquad\square$

Example 2.1. *Consider partial order $(Q, \leq_Q)$ as specified by Figure 1. The following is the representation of $(Q, \leq_Q)$ in $(P, \leq_P)$ given by the proof of Theorem 2.1.*

$$U(3) = \{\{3\}\}, U(5) = \{\{5\}, \{3\}\},$$

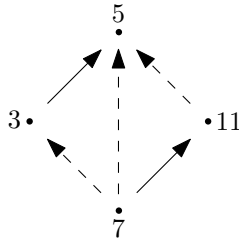$$U(7) = \{\{3, 5, 7\}\}, U(11) = \{\{3, 5, 7\}, \{5, 11\}\}.$$

Figure 1: Partial order $(Q, \leq_Q)$. Dashed lines represent backwarding edges.
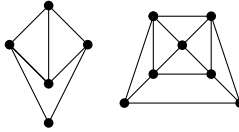

Figure 2: The 3-dragon D_3 and its line graph $L(D_3)$.

3 Dragon graphs

We use a simple gadget called d-dragon which is also used in several constructions developed by Roberson [19]. In our constructions, the parameter d specifies the maximal degree of a vertex:

Definition 3.1. *For $d \geq 3$, the d-dragon[1], denoted by D_d, is the graph created from K_{d+1} by replacing one of its edges by a path on 3 vertices.*

The 3-dragon is depicted in Figure 2.

We proceed by a simple lemma about edge-colorings of dragons.

Lemma 3.2. *For all $d \geq 3$ it holds that D_d is Vizing class 2 graph, i.e. its chromatic index is $d + 1$.*

Proof. By Vizing theorem, $L(D_d)$ is $(d + 1)$-colorable. We prove that $L(D_d)$ is not d-colorable. The number of edges of D_d is $\frac{d(d+1)}{2} + 1 = \frac{d^2+d+2}{2}$, the number of vertices is $d + 2$. We use the fact that every k-edge-coloring yields a decomposition of the graph into k disjoint matchings. We consider two cases:

1. If d is odd, then the maximum size of a matching in D_d is $\frac{d+1}{2}$, so the partition contains at least $d + 1$ matchings. Thus the chromatic number of $L(D)$ is $d + 1$.

2. If d is even, then the maximum size of a matching in D_d is $\frac{d+2}{2}$. However note that there is a vertex with degree 2, so there are at most 2 maximal matchings. The others matchings have the size at most $\frac{d}{2}$. It follows that d matching can cover at most $2(\frac{d+2}{2}) + (d - 2)\frac{d}{2} = \frac{4+d^2}{2}$ edges. For $d \geq 4$ we have $\frac{4+d^2}{2} \leq \frac{d^2+d+2}{2}$ and thus the partition contains at least $d + 1$ matchings, i.e. color classes.

$\square$

A graph G is a *core* if there is no homomorphism from G to its proper subgraphs. In our construction we will use the fact that d-dragons are cores which we show in the following lemma.

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[1]The name is derived from a visual similarity of this graph to a kite that in Czech language is called "dragon".
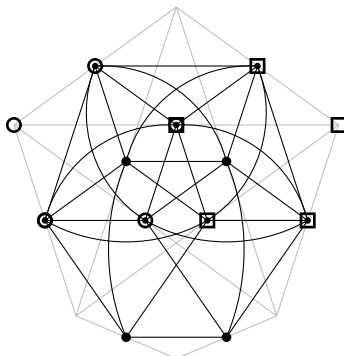
Figure 3: Line-graph of 4-dragon with cliques corresponding to the neighborhoods of two vertices distinguished.
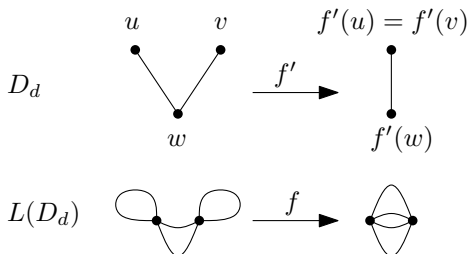

Figure 4: Mapping two d-cliques to the same target.

Lemma 3.3. *The graph $L(D_d)$, $d \geq 3$, is a core.*

Proof. For $d = 3$ observe that $L(D_3)$, depicted in Figure 2, is not 3-colorable, while each of its induced subgraphs is. Hence the statement holds for $d = 3$.

For $d \geq 4$ denote the vertices of D_d by $1, 2, \ldots, d, d+1, d+2$, where vertices $1, 2, \ldots, d+1$ have degree $d - 1$ and the vertex $d + 2$ has degree 2. The vertices of degree d correspond to d-cliques in $L(D_d)$. Each pair of those d-cliques share at most one node that corresponds to the edge connecting the original pair of vertices. Note that the shared node is unique for each such pair. Observe also that there are no other d-cliques in $L(D_d)$. This follows from the fact that the only way to create a 4-clique in a line graph is by a vertex of degree at least 4. See Figure 3.

Consider a homomorphism $f : L(D_d) \to L(D_d)$. Every homomorphism must map a d-clique to a d-clique, and thus it defines a vertex mapping $f' : \{1, 2, \ldots, d+1\} \to \{1, 2, \ldots, d+1\}$ in D_d.

Assume that there are distinct $u, v \in \{1, \ldots, d+1\}$ such that $f'(u) = f'(v)$, see Figure 4. Take any $w \in \{2, \ldots, d\} \setminus \{u, v\} \neq \emptyset$. Because the node shared by the cliques corresponding to u and w is unique, it is different from the node shared by the cliques corresponding to v and w. Consequently, the cliques corresponding to $f'(u) = f'(v)$ and $f'(w)$ share at least two nodes. Since distinct d-cliques of $L(D_d)$ may share at most one node, it follows that $f'(w) = f'(u)$. Hence f' is either a bijection or a constant function. On the other hand, f' can not be a constant function by Lemma 3.2, as otherwise such mapping would yield an edge coloring of D_d by d colors.

Since f' is a bijection on vertices $\{1, 2, \ldots, d+1\}$ of D_d, the mapping f' must be a bijection on the edges between these vertices. The only way to get a homomorphism f of the whole D_d is to extend the mapping bijectively also on edges $\{1, d+2\}$ and $\{d+1, d+2\}$. By this argument we have proved that f is an isomorphism. $\square$
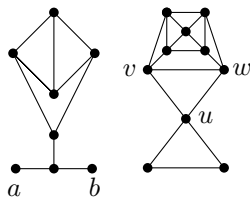
Figure 5: Indicator $I_3(a, b)$ and its line graph.

4   Indicator construction

We briefly describe the indicator technique, called often the "arrow construction" [5]. Informally, this construction means replacing every edge of a given graph G by a copy of graph I (an indicator) with two distinguished vertices identified with the endpoints of the edge. Figure 7 (left) shows result of indicator construction on graph in Figure 6 with indicator shown in Figure 5 (left). We give a precise definition of this standard notion:

An *indicator* is any graph $I = (V_I, E_I)$ with two distinguished vertices a,b.

Given a graph $G = (V_G, E_G)$, we denote by $G * I(a, b)$ the graph $H = (V_H, E_H)$, where each edge is replaced by an extra copy of $I(a, b)$, where the vertices a and b are identified with the original vertices.

Formally, to obtain V_H we first take the Cartesian product $E_G \times V_I(a, b)$ and factorize this set by the equivalence relation $\sim$ consisting of the following pairs:

$$((x, y), a) \sim ((x, y'), a),$$

$$((x, y), b) \sim ((x', y), b),$$

$$((x, y), b) \sim ((y, z), a).$$

In other words, the vertices of H are equivalence classes of the equivalence $\sim$. For a pair $(e, x) \in E \times V_I$, the symbol $[e, x]$ denotes its equivalence class.

Vertices $[e, x]$ and $[e', x']$ are adjacent in H if and only if $e = e'$ and $\{x, x'\} \in E_I$.

5   Final construction

It is a standard technique to use indicator construction to represent class of graphs which is known to be universal (such as oriented paths) within another class of graphs (such as planar graphs) by using appropriate rigid indicator, see e.g. work of Hubička and Nešetřil [8]. It is then possible to show that the structure induced by the homomorphism order is preserved by the embedding via the indicator construction.

While our construction also uses indicator, the application is not so direct. It is generally impossible to have an indicator that would turn a graph into a line graph. We use the indicator to make graphs more rigid with respect of homomorphisms of their line graphs and model the divisibility partial order directly.

Our basic building blocks are the following:

Definition 5.1. *The n-sunlet graph, denoted by S_n, is the graph on $2n$ vertices obtained by attaching n pendant edges to a cycle C_n, see Figure 6.*
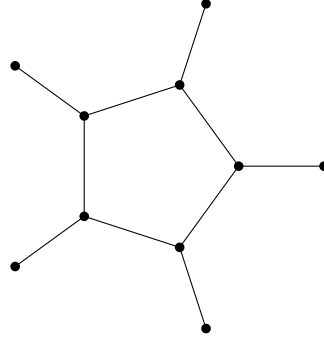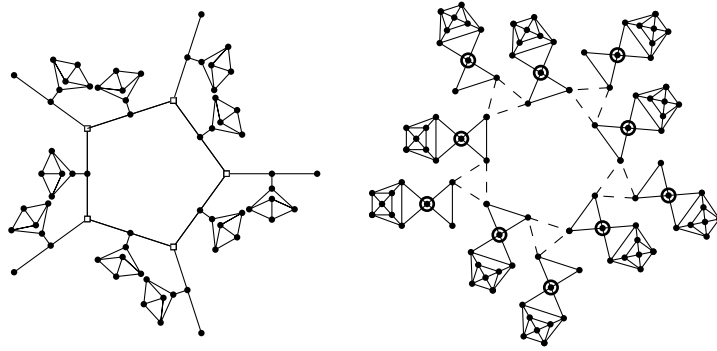
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Figure 6: The 5-sunlet S_5.

Figure 7: The graph $G_{5,3} = S_5 * I_3(a, b)$ and its line graph.

Definition 5.2. *For $d \geq 3$ the* indicator $I_d(a, b)$ *is the graph created from the disjoint union of the dragon D_d and a path on vertices a, c, b, where the vertex c is connected by an edge to the vertex of degree 2 in D_d, see Figure 5.*

The desired class of graphs to show universality of interval $[K_d, K_{d+1})_\mathcal{L}$, $d \geq 3$ consists of graphs $S_n * I_d(a, b)$ for $n \geq 3$. We abbreviate $S_n * I_d(a, b)$ by the symbol $G_{n,d}$. An example, the graph $G_{5,3}$, is shown in Figure 7. By squares are indicated vertices of degree three of the original sunlet graph. The three incident edges are in the line graph drawn as the triplets joined by the dashed triangles.

By Corollary 1.2 the graph $L(G_{n,m})$ is in the interval $[K_m, K_{m+1})_\mathcal{L}$ for every $n \geq 3$ and $m \geq 3$.

It remains to show the following property of the graphs $G_{n,m}$.

Proposition 5.3. *For every $d \geq 3$, $n \geq 3$, $n' \geq 3$ there is a homomorphism from $L(G_{n,d})$ to $L(G_{n',d})$ if and only if n is divisible n'.*

In one direction the proposition is trivial. If n is divisible by n' then the homomorphism is given by a homomorphism from S_n to $S_{n'}$ that cyclically wraps the bigger cycle around the smaller cycle. We call this homomorphism *cyclic*.

The other implication is a consequence of the following two lemmas.

The nodes of $L(G_{n,d})$ corresponding to the edges connecting the dragons with the vertices c are called *special*. In Figure 7 they are highlighted by circles.

Lemma 5.4. *For $d \geq 3$, $n \geq 3$, $n' \geq 3$ every homomorphism $f : L(G_{n,d}) \to L(G_{n',d})$ must map special vertices to special vertices.*

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Proof. In $L(G_{n,d})$ the only 2-connected components of chromatic number 4 are the line graphs of dragons. It follows that the image of every line graph of a dragon must be in a line graph of a dragon. By Lemma 3.3, the line graphs of dragons are cores, thus for any special node u holds that its two neighbors v and w in the associated dragon D_d in $L(G_{n,d})$ must be mapped into some dragon D_d from $L(G_{n',d})$, but bijectively onto the two neighbors of the attached special node. (See Figure 5.)

Since u, v and w form a triangle, the only way to complete a triangle containing $f(v)$, $f(w)$ is to map u to the adjacent special node, as such triangle cannot be completed inside the dragon. □

A triangle in $L(G_{n,d})$ is called a *connecting triangle* if it originates from a original node of degree three in S_m. In Figure 7 the connecting triangles are denoted by dashed lines.

Lemma 5.5. *For every $d \geq 3$, $n \geq 3$ and $n' \geq 3$, every homomorphism $f : L(G_{n,d}) \to L(G_{n',d})$ must map connecting triangles to connecting triangles.*

Proof. Consider an arbitrary connecting triangle of $G_{n,d}$. It has the property that each its node is adjacent to a special node. By Lemma 5.4, special nodes are preserved by the homomorphism f. The only triangles with this property in $G_{n',d}$ are precisely the connecting triangles. □

Proof of Proposition 5.3. By Lemma 5.5, the connecting triangles map to the connecting triangles. The other triangle in the cyclic configuration are those containing a special node. By Lemma 5.4 these triangles can not map to connecting triangles. Consequently, two connecting triangles joined by an edge can not map into one connecting triangle and thus the homomorphism f must be cyclic. □

Proof of Theorem 1.3. We apply Theorem 2.1 and show an embedding of $(P, \leq_P)$ to the homomorphism order of the interval $[K_d, K_{d+1})_{\mathcal{L}}$. For the chosen $d \geq 3$ we assign every $A \in P$ a line graph $L(d, A)$ consisting of the disjoint union of graphs $L(G_{a,d}), a \in A$. Since any homomorphism must map connected components to connected components, we know by Proposition 5.3 that $L(d, A)$ allows a homomorphism to $L(d, B)$ if and only if $A \leq_P B$. □

6   Concluding remarks

Our results confirm that the homomorphism order of line graphs is rich. It is interesting that our embedding considerably differs from the one used in the proof of the universality of oriented paths by Hubička and Nešetřil [9].

Our construction is based on the retrospecting of a homomorphism $f : L(G) \to L(H)$ to a vertex mapping $f' : V_G \to V_H$. We put $f'(v) = v'$ if all edges adjacent to v are mapped by f to the edges adjacent v' to H. This mapping is not always well defined. In particular:

(a) An image of the edge adjacent to a vertex of degree 1 of G is contained in the set of edges adjacent to two different vertices u, v connected by an edge in H. In this sense f' is not a function.

(b) Edges adjacent to a vertex v of degree 3 in G correspond to a triangle in $L(G)$. Because the line graph of a triangle is also a triangle, the image of these edges may thus map to a line graph of a triangle. In this case $f'(v)$ is not defined.

The basic idea behind the proof of Lemma 3.3 is the fact that f' (if it is a function) is close to a graph homomorphism $G \to H$ with two main differences:

(c) It may happen that $f'(u) = f'(v)$ for two adjacent vertices of G.

(d) For vertices of degree at least 3 the mapping f' is locally injective with the exception of (c).

A homomorphism $h : G \to H$ is *locally injective* if the restriction of the mapping h to the domain consisting of the vertex neighborhood of v and range consisting of the vertex neighborhood of v is injective. In one direction, every locally constrained homomorphism $h : G \to H$ yields a homomorphism $h' : L(G) \to L(H)$. Our observations above show that this direction can be reversed in special cases.

It is thus not a surprise that our universality proof is based on ideas developed for the proof of universality of locally injective homomorphisms. We get closer to graph homomorphisms by means of the indicator construction. In the proof of Proposition 5.3 we consider a mapping $f'' : V_G \to V_H$ that retrospects a homomorphism $L(G * I_d(a,b)) \to L(H * I_d(a,b))$ in a similar way as f'. This mapping is a locally injective homomorphism with the exception of (b) and vertices of degree 2, where the local injicitivity is not enforced. For this reason we use sunlets instead of cycles and our embedding of the divisibility partial order is based on the fact that there is a locally injective homomorphism from between sunlet graphs S_n and S_m if and only if m divides n.

With this insight it is not difficult to see that our construction can be altered to form 3-regular graphs. This can be done by adding a cycle connecting pendant vertices to every sunlet. For degrees $d > 3$, the edges connecting both inner and outer can be turned into a multi-edges of a given degree, but also the indicator needs to be modified to become d-regular except for the two vertices of degree 1. By replacing the edge connecting dragon with vertex c by a clique of the corresponding degree and by adding a separate copy of a dragon to all but one vertex (the one connected to the base).

The locally constrained homomorphism order is the main subject of several groups of authors [3, 1], see also a survey [2]. It is shown that several properties of locally injective homomorphism order are given by degree refinement matrices of the graphs considered. For a graph G, the *degree refinement matrix* is a unique matrix describing how vertices of given degree are adjacent in G, see [1, 3] for details. As a special case, for d-regular graphs G the degree refinement matrix is trivial consisting of only one value d.

Roberson [19] shows that the homomorphism order line graphs is dense above every line graph of K_n, $n \geq 2$. Our construction give many extra pairs with infinitely many different graphs strictly in between. Further such pairs can be obtained by an application of our another result [1]:

Theorem 6.1 ([1]). *Let G and H be connected graphs such that $\mathrm{drm}(G) \neq \mathrm{drm}(H)$, locally injective homomorphism $h : G \to H$ and H has no vertices of degree 1. Then:*

(a) *There exists a connected graph F that is strictly in between G and H in the locally constrained homomorphism order.*

(b) *When G has no vertices of degree 1 and H has at least one cycle with a vertex of degree greater than 2, then F can be constructed to have no vertices of degree 1 and contain a cycle with a vertex of degree greater than 2.*

Assume that G and H have no vertex of degree 2 and the degree of vertices either bounded by $d - 1$, or bounded by d and moreover that G and H are in Vizing class 1. Then graph F given by

Theorem 6.1 can be easily extended to F', where an extra edge is added to every vertex of degree 2. It is also easy to see that it cannot have vertices of degree greater than d. From the proof of Theorem 6.1 it follows that F is in Vizing class 1.

Consequently, $L(F' * I_d(a, b))$ is strictly in between $L(G * I_d(a, b))$ and $L(H * I_d(a, b))$. Afterwards, a graph strictly in between $G * I_d(a, b)$ and $F' * I_d(a, b)$ can be again constructed by applying Theorem 6.1 on G and F. This construction provides new examples of dense pairs in the homomorphism order of line graphs.

There are more results about the locally constrained homomorphism order that seem to suggest a strategy to attack problems about homomorphism order of line graphs. It appears likely that the characterization of gaps in the locally injective homomorphism order will give new gaps in the homomorphism order of line graphs. The proof of universality of the locally constrained order of connected graphs in [1] can be translated to yet another proof of the universality of homomorphism order of line graphs, this time however the graphs used are connected but not d-regular — since locally constrained homomorphism order is not universal on d-regular connected graphs. This suggests a question whether the homomorphism order of line graphs is universal on class of finite connected d-regular graphs.

Finally, the Leighton's construction of a common covering for graphs [14] may give an insight into a way of constructing a suitable product for line graphs.

References

[1] Jiří Fiala, Jan Hubička, and Yangjing Long. Constrained homomorphism orders. In preparation.

[2] Jiří Fiala and Jan Kratochvíl. Locally constrained graph homomorphisms — structure, complexity, and applications. *Computer Science Review*, 2(2):97–111, 2008.

[3] Jiří Fiala, Daniël Paulusma, and Jan Arne Telle. Matrix and graph orders derived from locally constrained graph homomorphisms. In Joanna Jedrzejowicz and Andrzej Szepietowski, editors, *MFCS*, volume 3618 of *Lecture Notes in Computer Science*, pages 340–351. Springer, 2005.

[4] Zeněk Hedrlín. On universal partly ordered sets and classes. *J. Algebra*, 11(4):503–509, 1969.

[5] Pavol Hell and Jaroslav Nešetřil. *Graphs and homomorphisms*. Oxford lecture series in mathematics and its applications. Oxford University Press, 2004.

[6] Jan Hubička and Jaroslav Nešetřil. Finite presentation of homogeneous graphs, posets and ramsey classes. *Israel Journal of Mathematics*, 149(1):21–44, 2005.

[7] Jan Hubička and Jaroslav Nešetřil. *Model Theoretic Methods in Finite Combinatorics*, chapter Some examples of universal and generic partial orders, pages 293–318. American Mathematical Society, 2011.

[8] Jan Hubička and Jaroslav Nešetřil. Finite paths are universal. *Order*, 21(3):181–200, 2004.

[9] Jan Hubička and Jaroslav Nešetřil. Universal partial order represented by means of oriented trees and other simple graphs. *European Journal of Combinatorics*, 26(5):765–778, 2005.

[10] John B Johnston. Universal infinite partially ordered sets. *Proceedings of the American Mathematical Society*, 7(3):507–514, 1956.

[11] Lonard Kwuida and Erkko Lehtonen. On the homomorphism order of labeled posets. *Order*, 28(2):251–265, 2011.

[12] Erkko Lehtonen. Labeled posets are universal. *European Journal of Combinatorics*, 29(2):493–506, 2008.

[13] Erkko Lehtonen and Jaroslav Nešetřil. Minors of boolean functions with respect to clique functions and hypergraph homomorphisms. *European Journal of Combinatorics*, 31(8):1981–1995, 2010.

[14] Frank Thomson Leighton. Finite common coverings of graphs. *Journal of Combinatorial Theory, Series B*, 33(3):231 – 238, 1982.

[15] Jaroslav Nešetřil and Yared Nigussie. Minimal universal and dense minor closed classes. *European Journal of Combinatorics*, 27(7):1159–1171, 2006.

[16] Jaroslav Nešetřil and Yared Nigussie. Density of universal classes of series-parallel graphs. *Journal of Graph Theory*, 54(1):13–23, 2007.

[17] Jaroslav Nešetřil and Xuding Zhu. Path homomorphisms. *Mathematical Proceedings of the Cambridge Philosophical Society*, 120:207–22, 1996.

[18] Aleš Pultr and Věra Trnková. *Combinatorial, algebraic, and topological representations of groups, semigroups, and categories*. North-Holland mathematical library. North-Holland Publishing Coompany, 1980.

[19] David E. Roberson. *Variations on a Theme: Graph Homomorphisms*. PhD thesis, University of Waterloo, 2013.

[20] Neil Robertson and Paul D. Seymour. Graph minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325–357, November 2004.

[21] Robert Šámal. Cycle-continuous mappings order structure. In Jaroslav Nešetřil and Marco Pellegrini, editors, *The Seventh European Conference on Combinatorics, Graph Theory and Applications*, volume 16 of *CRM Series*, pages 513–520. Scuola Normale Superiore, 2013.

[22] Vadim G Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz*, 3(7):25–30, 1964.

[23] Emo Welzl. Color-families are dense. *Theoretical Computer Science*, 17(1):29 – 41, 1982.