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Abstract

We provide a new upper bound for the critical density of Activated Random Walk in case of biased distribution of jumps. With finite sleeping rate the bound is strictly less than one in one dimension for all initial particles distribution and in higher dimension for some initial particles distributions. This answers a question and *Dickman, Rolla and Sidoravicius* (2010) in case of bias.

Introduction

Interacting particle systems are favourable models to study non-equilibrium phenomena, as they provide a simple example of phase transitions in systems maintained far from equilibrium. In the present article we consider Activated Random Walk (ARW) on the lattice. This is a continuous time interacting particle system with conserved number of particles, where each particle can be in one of two states: A (active) or S (inactive, sleeping). Each A particle performs an independent, continuous time random walk on $\mathbb{Z}^d$ with jump rate 1. The jumps have a probability density $p(\cdot)$ on $\mathbb{Z}^d$ and are identically and independently distributed. Every A particle has an exponential clock with rate $\lambda > 0$. When the clock rings, if the particle does not share the site with other particles, the transition $A \to S$ occurs, otherwise nothing happens. Particles in the A-state do not interact among themselves. S particles do not move and remain sleeping until the instant when an other particle is present at the same vertex. At such an instant the particle which is in S-state flips to the A-state, giving the transition A+S $\to$ 2A. As we consider initial configurations with only active particles, from the previous rules it follows that sleeping particles can be observed only if they occupy the site alone.

In ARW a phase transition arises from a conflict between the spread of the activity and a tendency of the activity to die out. The transition point separates an *active phase* from *local fixation*. We say that ARW exhibits *local fixation* if for any finite set $V \subset \mathbb{Z}^d$, there exists almost surely a finite time t_V such that after this time the set V contains no active particles. In case there is no local fixation, we say that ARW *stays active*. A numerical analysis of ARW in the two regimes has been provided in [1]. In [2] it has been proved that the system is monotonic under variation of certain parameters. In particular if the particles density μ increases, it is less likely for the system to fixate locally. From monotonicity in μ it follows that there exists a unique critical density μ_c that separates the two phases, namely for all $\mu < \mu_c$ the model exhibits local fixation and for all $\mu > \mu_c$ it stays active. It is conjectured that the critical density should depend only on λ and it should not depend on the specific distribution of the initial configuration, although this has not been proved so far, as far as we know.

In several works an analytical estimation for μ_c has been provided under different assumptions. At the current state of the art, as far as we know, it is known that in one dimension $\frac{\lambda}{1+\lambda} \leq \mu_c \leq 1$ [2]. In the more special case of totally asymmetric jumps on the nearest neighbour, i. e. $p(1) = 1$, it is known that $\mu_c = \frac{\lambda}{1+\lambda}$ and that at μ_c the process stays active [4]. For what concerns the model in $d \geq 2$, it is known that $\mu_c \leq 1$ for any value of λ [5, 6]. In the limit $\lambda \to \infty$ the exact value of μ_c is known in all dimensions. Namely, ARW shows local fixation a.s. for all $\mu < 1$ and stays active a.s. for all $\mu \geq 1$ [4, 5, 6]. The fact that $\mu_c \leq 1$ is intuitively obvious, since if $\mu > 1$ simply there is no space for all particles to stabilize. In the present article we provide a new upper bound for the critical density μ_c in case of biased jumps distribution with arbitrary finite support. In case of finite λ, our upper bound it is strictly less than 1. In one dimension it does not depend on the initial particles distribution. In higher dimensions it holds for a special choice of the initial particles distribution. Our result shows that, even if there is enough place for all particles to fall asleep, particles motion prevents the system to locally fixate and this makes the critical density strictly less than 1. Our main results are stated in Theorems 1.1 and Corollary 1.3.

We end this introductory section presenting the structure of the article. In Section 1 we define rigorously the model and we state our results, Theorem 1.1, Theorem 1.2 and Corollary 1.3. In Section 2 we describe the strategy of the proof in one dimension. In Section 3 we present the Diaconis-Fulton graphical representation for the dynamics of ARW following [2]. In Section 4 we prove Theorems 1.1 and 1.2.

1 Definition and result

The state of the ARW at time $t \geq 0$ is given by $\eta_t \in {\mathbb{N}_{0\rho}}^{\mathbb{Z}^d}$, where $\mathbb{N}_{0\rho} = \mathbb{N}_0 \cup \{\rho\}$. For all $x \in \mathbb{Z}^d$, $\eta_t(x)$ represents the number of particles at site x at time t. In particular $\eta_t(x) = \rho$ if the site x at time t is occupied by only one passive particle and $\eta_t(x) \in \mathbb{N}_0$ represents the number of active particles. Following [2], we define an order relation for ρ, setting $0 < \rho < 1 < 2 \ldots$. We also let $|\rho| = 1$, so that $|\eta_t(x)|$ counts the number of particles regardless of their state. The addition is defined by $\rho + 0 = \rho$, and $\rho + k = k + 1$ if $k \geq 1$, providing the $A + S \to 2A$ transition. The $A \to S$ transition is represented by $\rho \cdot k$, where $\rho \cdot 1 = \rho$ and $\rho \cdot k = k$ if $k \geq 2$. Subtractions involving ρ are not defined as sleeping particles cannot leave a site without becoming active first. Finally we define the operator $[\,\cdot\,]^*$, which counts the number of active particles, $[\eta_t(x)]^* = \eta_t(x)$ if $\eta_t(x) \geq 1$ or 0 otherwise.

The dynamics of the model can be viewed as the action of two types of operators, "move " and "sleep" at every site, with rate independent one site from the other one. For each site x, we have the transitions $\eta \to \tau_{xy}\eta$ at rate $[\eta_t(x)]^* p(y - x)$, where $\tau_{xy}\eta \in \mathbb{N}_{0\rho}^{\mathbb{Z}^d}$,

$$\tau_{xy}\eta(z) = \begin{cases} \eta(z) + 1 & \text{if } z = y, \\ \eta(z) - 1 & \text{if } z = x, \\ \eta(z) & \text{if } z \neq x \text{ and } z \neq y. \end{cases} \tag{1}$$

and the transition $\eta \to \tau_{x\rho}\eta$ at rate $\lambda[\eta_t(x)]^*$, where $\tau_{x\rho}\eta \in \mathbb{N}_{0\rho}^{\mathbb{Z}^d}$,

$$\tau_{x\rho}\eta(z) = \begin{cases} \eta(z) \cdot \rho & \text{if } z = x, \\ \eta(z) & \text{if } z \neq x. \end{cases} \tag{2}$$

The initial configuration η_0 is distributed according to ν and it is the product of identical measures. We denote by μ the density of particles at time 0, namely $\nu(|\eta_0(\mathbf{0})|)$. We further write ν_M for the distribution of the truncated configuration η^M given by $\eta^M(x) = \eta_0(x)$ for $|x| < M$ and $\eta^M(x) = 0$ otherwise, and $\mathbb{P}^\nu_M = \mathbb{P}^{\nu_M}$. $\mathbb{P}^\nu_M$ is well defined and corresponds to the evolution of a countable-state Markov chain whose configurations contain only finitely many particles. It follows from a construction due to Andjel that, if ν is a product measure with density $\nu(|\eta(0)|) < \infty$ then $\mathbb{P}^\nu$ is well defined, and, moreover,

$$\mathbb{P}^\nu(E) = \lim_{M\to\infty} \mathbb{P}^\nu_M(E) \tag{3}$$

for any event E that depends on a finite space-time window [8].

Our results are Theorem 1.1, Theorem 1.2 and Corollary1.3. Define the *expected jump*,

$$\mathbf{m} = \sum_{z\in\mathbb{Z}^d} p(z)z, \tag{4}$$

and call $x(i)$ a discrete-time random walk in $\mathbb{Z}^d$ starting from the origin and with jumps distributed according to $p(\,\cdot\,)$. For every $\epsilon > 0$ define the probability,

$$K(\epsilon) = P(\ \forall i > 0 \quad |\frac{x(i)}{i} - \mathbf{m}| < \epsilon\ ), \tag{5}$$

As a consequence of the law of large numbers this probability is positive for all ϵ. If $\epsilon = |\mathbf{m}|$ we write simply $K := K(|\mathbf{m}|)$. In the specific case of one dimension, call

$$\delta = P(\ \forall i \geq 1 \quad |x(i) - x(0)| \geq 1). \tag{6}$$

This equals the probability that the walker starting from the origin will be in position $x = -\frac{m}{|m|}$ at time 1 and that it will never reach the origin again. As a consequence of the law of large numbers δ is positive if $\mathbf{m} \neq \mathbf{0}$. Call $W = \{z \in \mathbb{Z}^d\ :\ p(z) > 0\}$. Recall that we consider initial configurations with all active particles and that $\mu = \nu(|\eta_0(\mathbf{0})|)$. The following theorem presents our estimation for μ_c in case of one dimension.

Theorem 1.1. *Consider ARW in $\mathbb{Z}$ with halting rate λ, jumps distributed according to $p(\,\cdot\,)$ such that W is finite, initial distribution given by i.i.d. random variables in $\mathbb{N}_0$ with expectation μ and variance $V < \infty$. Then $\mu_c \leq \frac{1}{\frac{\delta}{1+\lambda}+1}$.*

The following theorem and corollary present our estimation for μ_c in case of dimension greater than 1.

Theorem 1.2. *Consider ARW in $\mathbb{Z}^d$ with halting rate λ, biased jumps distribution $p(\,\cdot\,)$ with finite support W, initial distribution given by a product of i.i.d. random variables in $\mathbb{N}_0$ with expectation μ, variance $V < \infty$ and $\nu_0 = \nu(\eta(\mathbf{0}) = 0)$. If $\mu > \frac{\nu_0(1+\lambda)}{K}$, then ARW stays active almost surely.*

As a direct consequence of the previous theorem, the best upper bound for μ_c is given if one considers the initial particles distribution as in the following corollary.

Corollary 1.3. *Consider ARW in $\mathbb{Z}^d$ under the same hypothesis of the previous theorem, with initial configuration distributed as the product of i.i.d. random variables with distribution $\nu(\eta(\mathbf{0}) = 1) = \mu$, $\nu(\eta(\mathbf{0}) = 0) = 1 - \mu$ and $\mu \leq 1$. Then $\mu_c \leq \frac{1}{\frac{K}{1+\lambda}+1}$.*

In case of ARW with different initial particles distribution, it follows from Theorem 1.2 that if $\nu_0(1 + \lambda) < 1$ and the bias is strong enough, then μ_c is strictly less than one.

2 Description of the proof

Our proofs rely on the discrete Diaconis-Fulton representation for the dynamics of ARW. As it was proved in [2], local fixation for ARW is related to the stability properties of this representation, which leaves aside the chronological order of events.

At every site an infinite sequence of i.i.d. random variables is defined. Their outcomes are some operators acting on the current configuration of particles, moving one particle from one site to the other one or trying to let the particle fall asleep. Depending on the particles configuration, only some of the instructions are *legal*, i.e. using an instruction on a site which is empty or which hosts a sleeping particle is not allowed.

Local fixation for the dynamics of ARW is related to the number of instructions that must be used in order to stabilize the initial particles configuration. For stabilizing the initial configuration in a set we mean using the instructions in that set until the configuration contains only sleeping particles in that set. Denote by B_L a compact subset of $\mathbb{Z}^d$ such that $B_L \uparrow \mathbb{Z}^d$ as $L \to \infty$. For every $x \in \mathbb{Z}^d$, denote $m_{B_L,\eta}(x)$ the number of instructions that must be used at x in order to stabilize the configuration η in B_L and $\xi_{B_L,\eta}$ the stabilized configuration. A first important property of the representation is *commutativity*, i.e. $\xi_{B_L,\eta}$ and $m_{B_L,\eta}$ do not depend on the order followed in using the instructions, under the restriction that only legal instructions can be used. The probability distribution of m_{η,B_L} is denoted by $\mathcal{P}^\nu$, which is the joint probability distribution of the set of instructions and of ν, the probability distribution of the initial particles configuration. A second crucial property of the representation is the following. If there exist one site $x \in B_L$ and a positive constant K such that for every $L \in \mathbb{N}$,

$$\mathcal{P}^\nu(m_{B_L,\eta}(x) > K\,L) \geq K,$$

then ARW stays active a.s. The strategy of the proof of our theorems consists in defining a proper procedure of stabilization of the set B_L that allows to see that this fact holds even in case $\mu < 1$.

In order to prepare the reader to the proof we explain the main idea in one dimension.

Call $\eta(x)$ the number of particles initially present at $x \in \mathbb{Z}$, assuming they can be only active. Assume for them the following probability distribution, $\nu(\eta(x) = 1) = \mu$ and $\nu(\eta(x) = 0) = 1 - \mu$ independently for every $x \in \mathbb{Z}$, where $\mu < 1$ is also the expected number of particles per site. Assume for simplicity jumps on nearest neighbours and a bias to the right and consider the interval $B_L = [-L + 1, 0] \cap \mathbb{Z}$. Call N_L the total number of particles in B_L and $H_L = L - N_L$ the total number of empty sites in B_L.

We stabilize the configuration in B_L doing the following: starting from the leftmost particle and moving to the right, we move always the same particle until it falls asleep in B_L or until it leaves B_L. This means using always the first unused instruction on the site where the particle is located, following its trajectory. When the particle falls asleep in B_L or it leaves the set, we consider the following particle in the order and we do the same. We call a particle of the initial configuration "good" if it falls asleep on the right of its initial location or if it leaves B_L from the right. As the jumps distribution is biased to the right, the probability of every particle of being good is bounded from below by a positive constant, say $\varphi(p(\,\cdot\,), \lambda)$, which does not depend on the current particles configuration. This means that G_L, the total number of good particles, can be stochastically dominated from below by a binomial distribution with parameters φ and N_L, where N_L is the total number of particles in B_L. As consequence of the CLT there exists $K > 0$ such that for all $L \in \mathbb{N}$,

$$\mathcal{P}^{\nu}(N_L > L\mu,\, G_L > N_L\,\varphi) \geq K.$$

Observe also that if $G_L > H_L$, then at least $G_L - H_L$ particles must leave B_L from the right, because good particles can fall asleep only on sites which were empty in the initial configuration. Thus if $\mu > \frac{1}{1+\varphi}$, then for all $L \in \mathbb{N}$, $\mathcal{P}^{\nu}(m_{B_L,\eta}(0) > L\,[\mu(\varphi+1)-1]\,) \geq K$. This implies that the process stays active a.s.

The stabilization procedure is composed of two parts. In the first part we consider a general initial distribution of particles with density μ and we reduce the initial configuration to a new configuration which has only one particle per site. This is necessary as the argument we have just presented gives the best estimation in case the number of empty sites H_L is small. A control on the number of particles leaving B_L is needed in order the argument to work. In the second part we apply the strategy described above. The proof with all details is presented in Section 4.

The proof of Theorem 1.2, that involves ARW in higher dimension, is based on the same idea. Namely if the density of good particles is greater than the density of holes then a number of particles proportional to L^d must leave the set. Then, using a technique from [6] and choosing the shape of the B_L in a proper way, we show that under the hypothesis of the Theorem 1.2 at least a number of particles growing with L crosses the origin with positive probability.

3 Diaconis-Fulton representation

In this section we describe the Diaconis-Fulton graphical representation for the dynamics of ARW. We follow [2].

Let $\eta \in {\mathbb{N}_{0\rho}}^{\mathbb{Z}^d}$ denote the configuration of particles. A site $x \in \mathbb{Z}^d$ is *stable* in the configuration η if $\eta(x) \in \{0, \rho\}$ and it is *unstable* if $\eta(x) \geq 1$.

Define the set of independent *instructions* $\mathcal{I} = (\tau^{x,j} : x \in \mathbb{Z}^d,\, j \in \mathbb{N})$, where $\tau^{x,j} = \tau_{xy}$ with probability $\frac{p(y-x)}{1+\lambda}$ or $\tau^{x,j} = \tau_{x\rho}$ with probability $\frac{\lambda}{1+\lambda}$.

Let $h = (h(x) : x \in \mathbb{Z}^d)$ count the number of instructions read at each site. We say that we *use* an instruction at x when we act on the current configuration of particles η through the operator Φ_x, which is defined as,

$$\Phi_x(\eta, h) = (\tau^{x,h(x)+1}\,\eta,\, h + \delta_x). \tag{7}$$

The operation Φ_x is *legal* for η if x is unstable in η, i.e. $\eta(x) \geq 1$, otherwise it is *illegal*.

Finally we denote by $\mathcal{P}^\nu$ the joint law of η and $\mathcal{I}$, where η has distribution ν and it is independent from $\mathcal{I}$.

Properties. We now describe the properties of this representation. Later we discuss how they are related to the stochastic dynamics of ARW. We follow [2]. For $\alpha = (x_1, x_2, \dots x_k)$, we write $\Phi_\alpha = \Phi_{x_k}\Phi_{x_{k-1}} \dots \Phi_{x_1}$ and we say that Φ_α is *legal* for η if Φ_{x_l} is legal for $\Phi_{(x_{l-1},\dots,x_1)}(\eta, h)$ for all $l \in \{1, 2, \dots k\}$. Let $m_\alpha = (m_\alpha(x) : x \in \mathbb{Z}^d)$ be given by,

$$m_\alpha(x) = \sum_l \mathbb{1}_{x_l = x},$$

the number of times the site x appears in α. We write $m_\alpha \geq m_\beta$ if $m_\alpha(x) \geq m_\beta(x)$ $\forall x \in \mathbb{Z}^d$. Analogously we write $\eta' \geq \eta$ if $\eta'(x) \geq \eta(x)$ for all $x \in \mathbb{Z}^d$. We also write $(\eta', h') \geq (\eta, h)$ if $\eta' \geq \eta$ and $h' = h$. Let η, η' be two configurations, x be a site in $\mathbb{Z}^d$ and $\Upsilon \in \mathcal{I}$ be a realization of the set of instructions. For the proof of the following properties we refer to [2].

Property 1 If α and α' are two legal sequences for η such that $m_\alpha = m_{\alpha'}$, then $\Phi_\alpha\eta = \Phi_{\alpha'}\eta$.

Property 2 $\Phi_\alpha\eta\,(x)$ is non-increasing in $m_\alpha(x)$ and non-decreasing in $m_\alpha(z)$, $z \neq x$.

Property 3 If x is unstable in η and $\eta'(x) \geq \eta(x)$, then x is unstable in η'.

Property 4 If $\eta' \geq \eta$ then $\Phi_x\eta' \geq \Phi_x\eta$.

Consequences. Let V be a finite subset of $\mathbb{Z}^d$. A configuration η is said to be *stable* in V if all the sites $x \in V$ are stable. We say that α is contained in V if all its elements are in V and we say that α *stabilizes* η in V if every $x \in V$ is stable in $\Phi_\alpha\eta$.

Lemma 1 (Least Action Principle) If α and β are legal sequences for η such that β is contained in V and α stabilizes η in V, then $m_\beta \leq m_\alpha$.

Lemma 2 (Abelian Property) If α and β are both legal sequences for η that are contained in V and stabilize η in V, then $m_\alpha = m_\beta$. In particular, $\Phi_\alpha \eta = \Phi_\beta \eta$.

By Lemma 2, $m_{V,\eta} = m_\alpha$ and $\xi_{V,\eta} = \Phi_\alpha \eta$ are well defined.

Lemma 3 (Monotonicity) If $V \subset V'$ and $\eta \leq \eta'$, then $m_{V,\eta} \leq m_{V',\eta'}$.

By monotonicity, the limit $m_\eta = \lim_{V_n \uparrow \mathbb{Z}^d} m_{V,\eta}$ exists and does not depend on the particular sequence $V_n \uparrow \mathbb{Z}^d$. A configuration η is said to be stabilizable if $m_\eta(x) < \infty$ for every $x \in \mathbb{Z}^d$. The following lemma connects the dynamics of ARW to the stability property of the representation.

Lemma 4 Let ν be a translation-invariant, ergodic distribution with finite density $\nu(\eta(\mathbf{0}))$. Then $\mathbb{P}^\nu($ the system locally fixates $) = \mathcal{P}^\nu(m_\eta(\mathbf{0}) < \infty) \in \{0, 1\}$.

4 Upper bound for critical density

Proof of Theorem 1.1

Without loss of generality we assume $\mathbf{m} > 0$ and we consider then the set $B_L = [-L+1,\, 0] \cap \mathbb{Z}$. The case $\mathbf{m} < 0$ can be recovered repeating the same procedure for the set $B_L = [0,\, L-1] \cap \mathbb{Z}$.

First we transform the initial configuration of particles in such a way that the set contains all active particles isolated, i.e. there are no sites in B_L hosting more than one particle or sleeping particles. After that we use the argument presented in Section 2.

Step 1 - Preparation of the initial configuration Consider an initial configuration η, a realization of the set of instructions $\mathcal{I}$ and assign an arbitrary order to particles in η. Following this order, only if the particle shares the site where it is located with other particles, then we move this particle until it reaches an empty site or until it leaves B_L. This means that, starting from the initial position of the particle in η, we use the instruction on the site were the particle is located until the particle moves to a new site (instructions "sleep" have no effect as the site hosts more than one particle), and then we continue this procedure until the particle reaches an empty site or it leaves the set B_L. If the particle does not share the site with other particles, we don't do anything. Then we consider the following particle in the order and we do the same. Call $\eta' \in \Sigma$ the final configuration obtained after this procedure has been applied for all particles. Call η'_{B_L} the set of coordinates of η' in B_L. Clearly $\eta'(x) \in \{0, 1\}$ for all $x \in B_L$.

Claim The configuration η'_{B_L} does not depend on the order followed in the previous procedure.

The claim follows from the Abelian property of the Diaconis-Fulton representation and its proof follows the same steps of the proof of Lemma 2. The following proposition states that with probability bounded from below by a strictly positive constant independent on L, the number of particles leaving B_L during Step 1 is bounded from above by a positive constant c.

Proposition 1. *Call $N_L = \sum_{x \in B_L} \eta(x)$ and, referring to the procedure described above, call $N'_L = \sum_{x \in B_L} \eta'(x)$. Under the hypothesis of Theorem 1.1 there exist two positive constants c and K such that for all $L \in \mathbb{N}$,*

$$\mathcal{P}^{\nu} \left(N_L - N'_L \le c \right) \ge K. \tag{8}$$

We postpone the proof of the proposition and we proceed with the proof of the theorem.

Step 2: Stabilization procedure We consider the configuration η' obtained applying the procedure described in the first step and we assign a new order to particles in B_L. The first particle in the order is the leftmost particle in B_L, the second particle in the order is its closest particle to the right, and so on. The last particle in the order is the rightmost particle in B_L. Following now this order, we move every particle similarly to Step 1, using always the first non-used instruction on the site where the particle is located until it falls asleep in B_L or until it leaves B_L. Once a particle has fallen asleep or it has left B_L, we consider the next particle in the order and we do the same.

We call a particle in the configuration η' *good* if it falls asleep on a site located on the right of its initial location or if it leaves B_L from the right. By independence of the instructions, the probability of every particle of being good does not depend on the trajectory of the particles moved before, but it depends only on their positions in B_L. We can bound from below the probability of every particle of being good by a strictly positive constant, $\frac{\delta}{1+\lambda}$, which does not depend on the location of the other particles. This constant, defined in Section 1, equals the probability that the first instruction is "go right" and that the particle never returns to its initial site. Thus call G_L the total number of good particles. This random variable can be stochastically dominated from below by a binomial distribution with parameters $\frac{\delta}{1+\lambda}$ and N'_L. As a consequence of the Central Limit Theorem, there exists a positive constant K_1 such that for every N'_L the following is true,

$$\mathcal{P}^\nu(\, G_L \;\geq\; N'_L \frac{\delta}{1+\lambda} \mid N'_L \,) \geq K_1. \tag{9}$$

Again as a consequence of the Central Limit Theorem and of Proposition 1, there exist two positive constants c and K_2 such that for every L the following is true,

$$\mathcal{P}^\nu(\, N'_L \; \geq \mu L - c\,) \geq K_2. \tag{10}$$

Call $H_L := L - N'_L$ the total number of holes in B_L after the first step and observe that if $G_L > H_L$, then at least $G_L - H_L$ particles leave B_L from the right. Observe also from (9) and (10) that for every positive integer $L \in \mathbb{N}$,

$$\mathcal{P}^\nu(G_L - H_L \geq [\,\mu\,(\frac{\delta}{1+\lambda} + 1) - 1]\,L \,-\, c) \geq K_1 K_2. \tag{11}$$

Call then N^z_L the number of particles that leave B_L jumping away from $z \in \partial B_L$, where $\partial B_L := \{v \in B_L \text{ s.t. } \exists\, x \in \mathbb{Z}^d/B_L \text{ s.t. } x \in v + W\}$ is the boundary of B_L and $v + W$ is the set W translated by a vector v. Call also simply $C := \mu\,(\frac{\delta}{1+\lambda} + 1) - 1$, which is strictly positive under the hypothesis of the theorem. From (11) it follows that for every $L \in \mathbb{N}$,

$$\begin{aligned} &\sum_{z \in \partial B_L} \mathcal{P}^\nu(m_{B_L,\eta}(z) > \frac{CL - c}{|\partial B_L|}) \geq \\ &\mathcal{P}^\nu(\,\exists z \in \partial B_L \text{ s.t. } \; N^z_L \; > \; \frac{CL - c}{|\partial B_L|}) \geq K_1 K_2. \end{aligned} \tag{12}$$

Then for every $L \in \mathbb{N}$ there exists $v_L \in \partial B_L$ such that,

$$\mathcal{P}^\nu(m_{B_L,\eta}(v_L) > \frac{CL - c}{k_2}) \geq \frac{K_1 K_2}{k_2}, \tag{13}$$

where k_2 is a positive constant independent on L which bounds from above $|\partial B_L|$ (recall that W is finite). Calling $B'_L = B_L - v_L$, by translation invariance and by monotonicity we conclude that for every $L \in \mathbb{N}$,

$$\begin{aligned} &\mathcal{P}^\nu(m_\eta(0) > \frac{CL - c}{k_2}) \geq \mathcal{P}^\nu(\, m_{B'_L,\eta}(0) > \frac{CL - c}{k_2}) = \\ &\mathcal{P}^\nu(\, m_{B_L,\eta}(v_L) > \frac{CL - c}{k_2}) \geq \frac{K_1 K_2}{k_2}. \end{aligned} \tag{14}$$

By Lemma 4 almost sure non local-fixation follows.

Proof of Proposition 1 We prove the proposition by contradiction. Assume the statement is wrong, i.e. $\forall c > 0$,

$$\inf_{L \in \mathbb{N}} \{\, \mathcal{P}^\nu(\, N_L - N'_L \; \leq c\,) \,\} = 0. \tag{15}$$

This means that $\forall c > 0$ there exists L^* such that

$$\mathcal{P}^\nu(\, N_{L^*} - N'_{L^*} \; > c\,) \geq \frac{1}{2}. \tag{16}$$

This means that for every c there exists L^* such that with probability at least $\frac{1}{2}$ at least c particles leave B_{L^*} after Step 1. Among these particles, call M_L^z the ones that leave B_L jumping away from $z \in \partial B_L$, where ∂B_L has been defined above. From (16) it follows that,

$$\sum_{z\in\partial B_{L^*}} \mathcal{P}^\nu(m_{B_{L^*},\eta}(z) > \frac{c}{|\partial B_{L^*}|}) \geq \mathcal{P}^\nu(\, \exists z \in \partial B_{L^*} \text{ s.t. } \; M^z_{L^*} \; > \; \frac{c}{|\partial B_{L^*}|}\,) \geq \frac{1}{2}. \tag{17}$$

Then for all $c > 0$ there exists L^* and $v_{L^*} \in \partial B_{L^*}$ such that,

$$\mathcal{P}^\nu(m_{B_{L^*},\eta}(v_{L^*}) > \frac{c}{k_2}) \geq \frac{1}{2k_2}, \tag{18}$$

where, as before, k_2 is a positive constant independent on L which bounds from above $|\partial B_L|$. Calling $B'_{L^*} = B_{L^*} - v_{L^*}$, using again monotonicity and translation invariance we conclude that for all $c > 0$,

$$\mathcal{P}^\nu(m_\eta(0) > \frac{c}{k_2}) \geq \mathcal{P}^\nu(m_{B_{L^*},\eta}(v_{L^*}) > \frac{c}{k_2}) \geq \frac{1}{2k_2}. \tag{19}$$

As c is arbitrarily larger, from Lemma 4 almost sure non local-fixation for ARW follows. Now observe that for every L the probability distribution of the random variables N_L, N'_L, and M^z_L for all $z \in \partial B_L$ does not depend on the value of the parameter λ, as sleeping instructions encountered while applying the procedure described above have no effect. As (19) holds also for ARW in the limit $\lambda \to \infty$ and as we know from [4] that ARW with $\lambda = \infty$ locally fixates if $\mu < 1$, then we find a contradiction.

Proof of Theorem 1.2

We present the proof in case of dimension 2. The proof in case of higher dimension follows the same steps. The set B_L is defined in Figure 1, it depends on a positive integer L and on a positive real number g that will be defined later. In the figure we assumed for simplicity that $\mathbf{m}$ is parallel to the axis x_1. The definition of B_L in case of $\mathbf{m}$ with different orientation can be recovered just by a rotation centred in the origin. Consider a ralization η of the initial configuration. We assign an order to particles in B_L according to the following condition. Imagine that every particle of η in B_L is intersected by an hyperplane orthogonal to $\mathbf{m}$. Different sets of particles will be intersected by the same hyperplane. Then for every pair of particles belonging to distinct hyperplanes, the particle which belongs to the hyperplane closer to the origin must appear later in the order. The order relation among particles belonging to the same hyperplane is irrelevant.

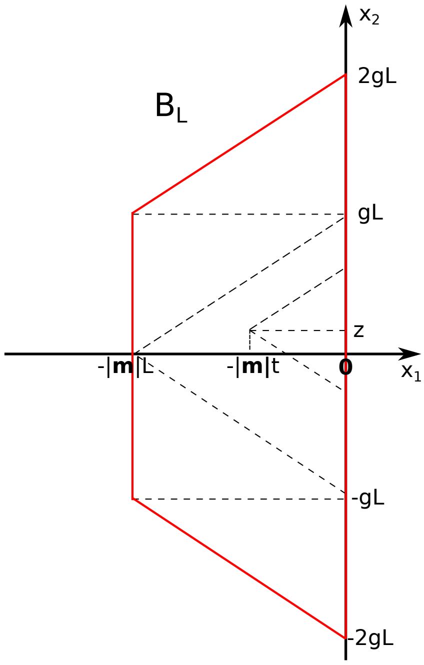

Figure 1: Representation of B_L in two dimensions, in case $\mathbf{m}$ is parallel to x_1. B_L corresponds to the set inside the red lines and it depends on g, $\mathbf{m}$ and L. Referring to equation (21) in the text, a walk starting from $(-|\mathbf{m}|t, z)$ has a probability $P(\mathcal{G}^{\epsilon}_{-t,-z})$ of being *good* and of leaving B_L from the origin.

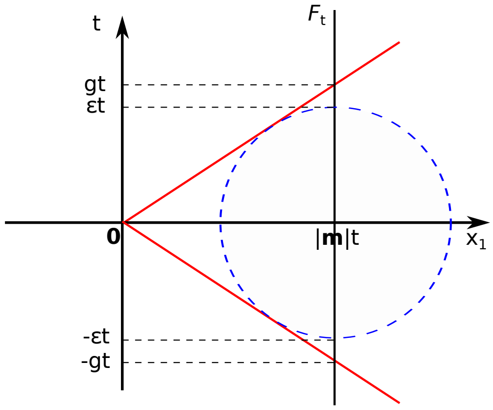

Figure 2: As defined in the text, a *good* particle is located inside the blue circle of radius ϵt and centred in $\mathbf{0} + \mathbf{m}t$, where $\mathbf{0}$ is its initial location, for every $t \in \mathbb{N}$. This means that the particle cannot leave the region inside the red lines. If $\epsilon < |\mathbf{m}|$, then the region delimited by the red lines is a cone and the good particle cannot return to the origin.

Following this order we repeat the same procedure presented in the second step of the proof of Theorem 1.1. Namely, starting from the initial position of the particle, we use the first unused instruction on the site where the particle is located until the particle falls asleep in B_L or until it leaves B_L and then we do the same for the following particle in the order. Consider then the i-th particle in the order and call $(\, z_i(t)\,)_{t=0}^{\tau_i}$ the trajectory of such particle. Thus $z_i(0)$ corresponds to the initial position of the particle and $z_i(\tau_i)$ corresponds to the site in B_L where the particle falls asleep or to the last site in B_L visited by the particle before jumping away from the set. Now fix an arbitrary $0 < \epsilon < |\mathbf{m}|$. We call the i-th particle *good* if its trajectory satisfies the following conditions; **(1)** the particle leaves its initial position at the first step and **(2)** for all $0 < t \leq \tau_i, \;\; |z_i(t) - z_i(0) - \mathbf{m}\, t| < \epsilon\, t$. Observe that if $\epsilon < |\mathbf{m}|$, every good particle does not return to its initial position by definition. Referring to Figure 2, observe also that if the particle is good, then it remains always inside the infinite cone of radius $g(\epsilon, |\mathbf{m}|)\, t$, with t going to infinity, where $g(\epsilon, |\mathbf{m}|)$ is a positive real number depending on ϵ and on $|\mathbf{m}|$. We choose g in the definition of B_L equal to this number. This ensures that every good particle in B_L can leave B_L only from the hyperplane intersecting the origin. By independence of the instructions the probability of a particle of being good does not depend on the trajectory of the particles moved previously, but it depends only on their position.

We want to estimate the number of good particles that leave B_L jumping away from the origin. Call this number G_L. We use the idea of the ghost explorers from [6, 7]. Thus to each particle that leaves its initial position at the first step we associate a walk that begins together with the particle but continues indefinitely. Let W_L be the number of such walks with a *good* trajectory and visiting the origin as last site in B_L. Namely, denoting by $(\, w_i(t)\,)_{t=0}^{q_i}$ the walk associated with the i-th particle in the order and by q_i the first time the walk leaves B_L, we say that the walk is good and leaves B_L from the origin if the following two conditions are satisfied; **(1)** for all integers $0 < t \leq q_i$, $|w_i(t) - w_i(0) - \mathbf{m}\, t| < \epsilon\, t$, and **(2)** the last site visited in B_L before leaving the set is the origin. Let R_L be the number of walks having a *good* trajectory and which leave B_L jumping away from the origin, but that leave B_L as *ghost* (i.e. after stopping in the original model). Thus $G_L = W_L - R_L$ counts the number of good particles that leave B_L jumping outside the set from the origin.

We start with the estimation of the expectation $E[W_L]$. Call then F_t the hyperplane orthogonal to $\mathbf{m}$, having a distance $|\mathbf{m}|t$ from the origin and pointed by $\mathbf{m}$, as in Figure 2. Assume for simplicity of the exposition that $\mathbf{m}$ is parallel to x_1. For a discrete time random walk with the same jumps distribution of our ARW model, call $\mathcal{G}^{\epsilon}_{t,z}$ the event {the walk starting from the origin is good} $\cap$ {the first site of F_t visited by the walk is $(|\mathbf{m}|t, z)$}. Clearly, for all $t > 0$ such that F_t intersects some site,

$$\sum_{z=-\lceil gt \rceil}^{\lfloor gt \rfloor} P(\mathcal{G}^{\epsilon}_{t,z}) = K(\epsilon), \tag{20}$$

as the sum is over the probability of disjoint events. Thus for a fixed initial configuration η the expectation $E[\, W_L \,|\, \eta\,]$ can be bounded by,

$$\begin{aligned} \sum_{t \in (-L,0]} \sum_{z=-\lceil gt \rceil}^{\lfloor gt \rfloor} \eta(-t|\mathbf{m}|, z)\, \frac{P(\mathcal{G}^{\epsilon}_{-t,-z})}{1+\lambda} \; &\leq E[\, W_L \,|\, \eta\,] \\ \leq \sum_{t \in (-L,0]} \sum_{z=-\lceil gt \rceil}^{\lfloor gt \rfloor} \eta(-t|\mathbf{m}|, z)\, P(\mathcal{G}^{\epsilon}_{-t,-z}), \end{aligned} \tag{21}$$

where, given η, W_L can be intended as a sum over indicator functions and the previous corresponds to the expectation of such variable. Observe also that in the previous expression the first sum is over all t in $(-L, 0]$ such that $|\mathbf{m}|t$ is an integer. The factor $\frac{1}{1+\lambda}$ is a lower bound for the probability that the particle to which the walk is associated can move at least one step (only under this circumstance the associated walk starts). For non isolated particles, this probability is 1, as sleeping instructions have no effect, and this gives the upper bound in the previous expression. See also Figure 1. Recalling that the initial particles distribution is a product measure and using (20) we get,

$$L\,\mu|\mathbf{m}|\frac{K(\epsilon)}{1+\lambda}\ \le\ E[W_L]\ \le\ \ L\,\mu|\mathbf{m}|K(\epsilon). \tag{22}$$

See also Figure 1. The expectation $E[R_L]$ is harder to calculate, but note that each ghost that contributes to R_L can be tied up to the unique site where the particle stops in the original model and the ghost starts. Observe also that as we are counting only good particles, ghosts can start only from sites of B_L that are empty in η. Thus, by the strong Markov property, if we start an independent walk from each empty site of B_L and we call $\tilde{R}_L$ the number of such walks that leave B_L jumping away from the origin, we get that R_L is stochastically dominated from above by $\tilde{R}_L$. Consider again Figure 2 and, considering again a discrete time random walk with the same jumps distribution of our ARW model, call $\mathcal{R}^{\epsilon}_{t,z}$ the event {the first site in F_t visited by a walk starting from the origin is $(|\mathbf{m}|t,z)$}. Clearly for all $t>0$ such that F_t intersects some site,

$$\sum_{z=-\infty}^{\infty} P(\,\mathcal{R}^{\epsilon}_{t,z}\,) = 1, \tag{23}$$

as the walk crosses F_t almost surely and we sum over disjoint events. Thus, for a fixed configuration $\eta \in \Sigma$,

$$E[\,\tilde{R}_L\,|\,\eta\,] \le \sum_{t\in(-L,0]}\ \sum_{z=-\lceil w_t\rceil}^{\lfloor w_t\rfloor} \mathbb{1}^{e}_{(-t\,|\mathbf{m}|,z)}(\eta)\,P(\,\mathcal{R}^{\epsilon}_{-t,-z}\,), \tag{24}$$

where $w_t := \max\{w\in\mathbb{R}\,:\,(-|\mathbf{m}t|,w)\ \in\ B_L\}$, $\mathbb{1}^{e}_{(x,y)}(\eta)$ equals one if and only if the site $(x,y)\in\mathbb{Z}^2$ is empty for the configuration η and, as before, the first sum is over all $t\in(-L,0]$ such that $t|\mathbf{m}|$ is an integer. Using (23) and the fact that the initial particles distribution is a product measure, we conclude that,

$$E[\tilde{R}_L] \le\ L\,|\mathbf{m}|\nu_0. \tag{25}$$

Call then $C(\epsilon) := |\mathbf{m}|(\frac{\mu K(\epsilon)}{1+\lambda} - \nu_0)$, which is positive for every $\mu > \frac{\nu_0(1+\lambda)}{K(\epsilon)}$. Now we show that the probability of the event $\{W_L - R_L < \frac{C(\epsilon)}{3}L\}$ tends to 0 as $L\to\infty$. As $m_{B_L,\eta}(0) \ge W_L - R_L$, by Lemma 3 and Lemma 4 this implies that ARW stays active for all $\mu > \frac{\nu_0(1+\lambda)}{K(\epsilon)}$. Thus,

$$\begin{aligned}&\mathcal{P}^{\nu}(W_L - R_L < \frac{C(\epsilon)}{3}L)\ \le\ \mathcal{P}^{\nu}(W_L - R_L < \frac{E[W_L-R_L]}{3})\ \le\\ &\mathcal{P}^{\nu}(E[W_L]-W_L > \frac{E[W_L-R_L]}{3}) + \mathcal{P}^{\nu}(R_L - \mathbb{E}[R_L] > \frac{E[W_L-R_L]}{3}),\end{aligned} \tag{26}$$

where for the second inequality we used the union bound. Fix now $k \in \mathbb{N}$ and observe that there exists L_0 such that for all $L > L_0$, $\frac{E[W_L - R_L]}{3} \geq k\sqrt{E[W_L]}$ and $\frac{E[W_L - R_L]}{3} \geq k\sqrt{E[\tilde{R}_L]}$. Observe also that $Var[W_L] \leq VE[W_L]$ and that $Var[\tilde{R}_L] \leq VE[\tilde{R}_L]$. Thus by the Chebyshev inequality we conclude that for every $L > L_0$,

$$\mathcal{P}^\nu(E[W_L] - W_L > \frac{E[W_L - R_L]}{3}) \leq \mathcal{P}^\nu(E[W_L] - W_L > k\sqrt{\frac{Var[W_L]}{V}}) \leq \frac{V}{k^2}, \tag{27}$$

and that,

$$\mathcal{P}^\nu(R_L - \mathbb{E}[R_L] > \frac{E[W_L - R_L]}{3}) \leq \mathcal{P}^\nu(\tilde{R}_L - \mathbb{E}[\tilde{R}_L] > k\sqrt{\frac{Var[\tilde{R}_L]}{V}}) \leq \frac{V}{k^2}. \tag{28}$$

Recall now that our construction works for any ϵ positive, but strictly less than $|\mathbf{m}|$. We can then choose $\epsilon < |\mathbf{m}|$, but arbitrarily close to $|\mathbf{m}|$. Since k was arbitrary we conclude that if $\mu > \frac{\nu_0(1+\lambda)}{K}$ then ARW stays active almost surely.

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