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Abstract. We prove that in general it is not possible to extend a Lipschitz multiple valued function without increasing the Lipschitz constant, i.e. we show that there is no analog of Kirszbraun's extension theorem for Almgren's multiple valued functions

Keywords. Kirszbraun's extension theorem, multiple valued functions, geometric measure theory.

1. Introduction

Almgren's multiple valued functions play a key role in geometric measure theory since they are employed in the analysis of the branching behaviour of minimal surfaces in codimension larger than or equal to 2 (see [1] and [3]).

We recall basic definitions for multiple valued functions. Let Q be a positive integer, then

$$\mathcal{A}_Q(\mathbb{R}^n) = \left\{ \sum_{i=1}^{Q} [\![P_i]\!] : P_i \in \mathbb{R}^n,\ 1 \leq i \leq Q \right\}, \tag{1.1}$$

where $[\![P]\!]$ denotes the Dirac measure at P. This space is endowed with the L^2-Wasserstein distance: for $T_1 = \sum_{i=1}^{Q} [\![P_i]\!]$ and $T_2 = \sum_{i=1}^{Q} [\![S_i]\!]$ we define

$$\mathcal{G}(T_1, T_2) = \min_{\sigma \in \mathcal{P}_Q} \sqrt{\sum_{i=1}^{Q} |P_i - S_{\sigma(i)}|^2},$$

where $\mathcal{P}_Q$ denotes the group of permutations of $\{1, \ldots, Q\}$.

One of the main ingredients in the theory of multiple valued funtions is the following extension theorem (see Theorem 1.7 in [3]).

Theorem 1.1. *Let $B \subset \mathbb{R}^m$ be a measurable set and let $f : B \to \mathcal{A}_Q(\mathbb{R}^n)$ be Lipschitz. Then there exists a constant $C = C(m, Q) > 0$ and an extension*

$\bar{f}:\mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ *of* f *such that*

$$\mathrm{Lip}(\bar{f}) \le C\mathrm{Lip}(f).$$

In the Euclidean case, the classical Kirszbraun's extension theorem (see Theorem 2.10.43 in [2]) states that an analogous result holds with $C = 1$. More precisely, Kirszbraun's theorem states that Lipschitz functions defined on a subset of $\mathbb{R}^m$ with values in $\mathbb{R}^n$ (both endowed with the Euclidean distance) can be extended to all of $\mathbb{R}^m$ without increasing the Lipschitz constant. The conclusion may fail as soon as $\mathbb{R}^m$ or $\mathbb{R}^n$ is remetrized by a metric which is not induced by an inner product, as shown in 2.10.44 of [2].

In §2 we prove that the conclusion also fails in the setting of multiple valued functions, by exhibiting a $\sqrt{2/3}$-Lipschitz function f defined on a subset of $\mathbb{R}^2$ with values in $\mathcal{A}_2(\mathbb{R}^2)$ with the property that any Lipschitz extension $\bar{f}$ to $\mathbb{R}^2$ has Lipschitz constant at least 1.

2. Construction of the counterexample

Let $A = (0,1), B = (-\sqrt{3}/2, -1/2), C = (\sqrt{3}/2, -1/2)$ and let $P_1, \dots, P_6$ be the vertices of a regular hexagon centered at 0, with side length 1: $P_1 = (0,1)$, $P_2 = (\sqrt{3}/2, 1/2)$, $P_3 = (\sqrt{3}/2, -1/2)$, $P_4 = (0,1)$, $P_5 = (-\sqrt{3}/2, -1/2)$ and $P_6 = (-\sqrt{3}/2, 1/2)$.

Consider the map $f : \{A, B, C\} \subset \mathbb{R}^2 \to \mathcal{A}_2(\mathbb{R}^2)$ given by

$$\begin{aligned} f(A) &= [\![P_1]\!] + [\![P_4]\!], \\ f(B) &= [\![P_2]\!] + [\![P_5]\!], \\ f(C) &= [\![P_3]\!] + [\![P_6]\!]. \end{aligned}$$

The Lipschitz constant of f is $\sqrt{2/3}$. In fact, $|A-B| = |A-C| = |B-C| = \sqrt{3}$ and

$$\mathcal{G}(f(A), f(B)) = \mathcal{G}(f(A), f(C)) = \mathcal{G}(f(B), f(C)) = \sqrt{2}.$$

Now consider a map $\bar{f} : \{A, B, C\} \cup \{0\} \to \mathcal{A}_2(\mathbb{R}^2)$. We will prove that if $\bar{f}$ is an extension of f, then the Lipschitz constant of $\bar{f}$ is at least 1.

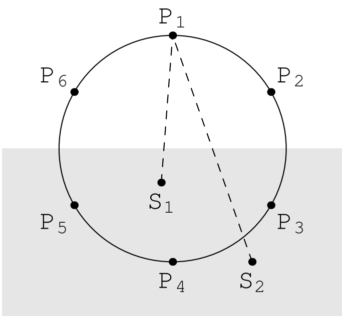

Figure 1. S_1 and S_2 must lie on different sides of $y = 0$

Indeed, let $\bar{f}(0) = [\![S_1]\!] + [\![S_2]\!]$. Assume by contradiction $\mathrm{Lip}(\bar{f}) < 1$, then S_1 and S_2 should lie on different sides of the perpendicular bisector of the line segment $\overline{P_1P_4}$ (see Figure 1). In fact, if for example S_1 and S_2 both lie in the half plane $\{y \leq 0\}$ then $|P_1 - S_i| \geq 1$ for $i = 1, 2$ which implies $\mathcal{G}(\bar{f}(0), f(A)) \geq 1$. The latter contradicts the assumption since $|A| = 1$.

Arguing analogously for $\overline{P_2P_5}$ and $\overline{P_3P_6}$ we deduce that S_1 and S_2 must lie on opposite sectors among the six determined by the three perpendicular bisectors. Without loss of generality we can assume that S_1 belongs to the intersection of the sector containing P_1 and the first orthant (see Figure 2).

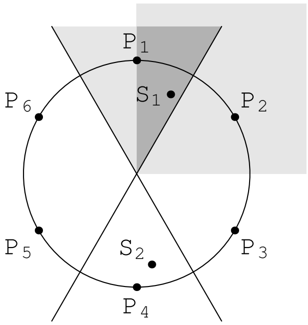

FIGURE 2.

Since $|S_1 - P_6| \leq |S_1 - P_3|$ and $|S_2 - P_6| \geq |S_2 - P_3|$, we can estimate the distance between $\bar{f}(0)$ and $f(C)$ and get

$$\mathcal{G}(\bar{f}(0), f(C))^2 = |S_1 - P_6|^2 + |S_2 - P_3|^2 \geq \frac{3}{4} + \frac{1}{4} = 1,$$

which contradicts our assumption since $|C| = 1$.

Remark 2.1. Following the proof of Theorem 1.1 in [3] one can explicitly determine the growth of the constant C depending on m and Q. It would be desirable to understand if the sharp constant has the same growth (or at least if $C(m, Q)$ goes to infinity as either m or Q goes to infinity). Clearly, just considering one-point extensions cannot lead to an answer to this question as the following general argument shows. Let (M, d_M) and (N, d_N) be two complete metric spaces, A a subset of M and $f : A \to N$ be Lipschitz continuous. Then for every $P \in M \setminus A$ there exists a Lipschitz extension $\bar{f} : A \cup \{P\} \to N$ such that

$$\mathrm{Lip}(\bar{f}) \leq 2\mathrm{Lip}(f).$$

In fact, let $S \in \bar{A}$ be a point realizing the distance between P and $\bar{A}$. Let $\bar{f}(P)$ be the value at S of the unique continuous extension of f to $\bar{A}$, denoted by $f(S)$. Then for every $y \in A \setminus \{S\}$ we get

$$\frac{d_N(\bar{f}(P), f(y))}{d_M(P, y)} = \frac{d_N(f(S), f(y))}{d_M(S, y)} \frac{d_M(S, y)}{d_M(P, y)} \leq 2\mathrm{Lip}(f),$$

because $d_M(S, y) \leq d_M(S, P) + d_M(P, y)$ and $d_M(S, P) \leq d_M(P, y)$ by the definition of S.

References

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