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by

Miroslav Bačák and Simeon Reich

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THE ASYMPTOTIC BEHAVIOR OF A CLASS OF NONLINEAR SEMIGROUPS IN HADAMARD SPACES

MIROSLAV BAČÁK AND SIMEON REICH

Abstract. We study a nonlinear semigroup associated to a nonexpansive mapping on a Hadamard space and establish its weak convergence to a fixed point. A discrete-time counterpart of such a semigroup, the proximal point algorithm, turns out to have the same asymptotic behavior. This complements several results in the literature – both classical and more recent ones. As an application, we obtain a new approach to heat flows in singular spaces for discrete, as well as continuous times.

1. Introduction and Main Results

Throughout the paper, the symbol $(\mathcal{H}, d)$ stands for a Hadamard space, that is, a complete geodesic metric space of nonpositive curvature. Given a nonexpansive mapping $F\colon \mathcal{H} \to \mathcal{H}$, we study the asymptotic behavior of its resolvent and of the nonlinear semigroup it generates.

As a motivation, we first recall some known results in gradient flow theory in Hadamard spaces. Let $f\colon \mathcal{H} \to (-\infty, \infty]$ be a convex lower semicontinuous (lsc) function. Given $\lambda > 0$, we define the *resolvent* of f by

$$J_\lambda x := \operatorname*{arg\,min}_{y \in \mathcal{H}} \left[f(y) + \frac{1}{2\lambda} d(x,y)^2 \right], \qquad x \in \mathcal{H}, \tag{1}$$

and put $J_0 x := x$ for each $x \in \mathcal{H}$. The *gradient flow semigroup* corresponding to f is defined by

$$S_t x := \lim_{n \to \infty} \left(J_{\frac{t}{n}} \right)^n x, \qquad x \in \overline{\operatorname{dom}} f, \tag{2}$$

for every $t \in [0, \infty)$. Gradient flow semigroups in Hadamard spaces have been studied by several authors [18, 21, 25, 4, 5, 6] and the theory can be extended to more general metric spaces [1].

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If $C \subset \mathcal{H}$ is a convex set, we denote the corresponding metric projection by P_C. The set of minimizers of a function $f\colon \mathcal{H} \to (-\infty, \infty]$ is denoted by $\operatorname{Min} f$.

Theorem 1.1. [17, Theorem 3.1.1] *Let $f\colon \mathcal{H} \to (-\infty,\infty]$ be a convex lsc function and $x_0 \in \mathcal{H}$. Assume there exists a sequence $(\lambda_n) \subset (0,\infty)$ with $\lambda_n \to \infty$ such that $(J_{\lambda_n} x_0)$ is a bounded sequence. Then f attains its minimum and*

$$\lim_{\lambda\to\infty} J_\lambda x_0 = P_{\operatorname{Min} f}(x_0).$$

Recall that the *proximal point algorithm* (PPA, for short) starting at a point $x_0 \in \mathcal{H}$ generates the sequence

$$x_n := J_{\lambda_n} x_{n-1}, \qquad n \in \mathbb{N}, \tag{3}$$

where $\lambda_n > 0$ for each $n \in \mathbb{N}$. In contrast to Theorem 1.1, it is known that the PPA converges only weakly.

Theorem 1.2. [4, Theorem 1.4] *Let $f\colon \mathcal{H} \to (-\infty,\infty]$ be a convex lsc function attaining its minimum on $\mathcal{H}$. Then for an arbitrary starting point $x_0 \in \mathcal{H}$ and any sequence of positive reals (λ_n) such that $\sum_1^\infty \lambda_n = \infty$, the sequence $(x_n) \subset \mathcal{H}$ defined by* (3) *converges weakly to a minimizer of f.*

It is not surprising that the gradient flow behaves in the same way.

Theorem 1.3. [4, Theorem 1.5] *Let $f\colon \mathcal{H} \to (-\infty,\infty]$ be a convex lsc function attaining its minimum on $\mathcal{H}$. Then, given a starting point $x_0 \in \overline{\operatorname{dom}}\, f$, the gradient flow $x_t := S_t x_0$ converges weakly to a minimizer of f as $t \to \infty$.*

In a Hilbert space H, one can define the resolvent and the semigroup for an arbitrary maximally monotone operator $A\colon H \to 2^H$. The situation described above then corresponds to the case $A := \partial f$ for a convex lsc function $f\colon H \to (-\infty,\infty]$. In particular, the semigroup in (2) provides us with a solution to the parabolic problem

$$\begin{aligned} \dot{u}(t) &\in -\partial f(u(t)), \quad t \in (0,\infty), \\ u(0) &= u_0 \in H, \end{aligned}$$

for a curve $u : [0,\infty) \to H$. Indeed, in this case $u(t) := S_t u_0$.

In the present paper, we prove analogs of the above gradient flow results which in a Hilbert space H correspond to another important instance of a maximal monotone operator, namely $A := I - F$, where $F\colon H \to H$ is nonexpansive and $I\colon H \to H$ is the identity.

Let $F\colon \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping. We now define its resolvent and the semigroup it generates as in [25]. Given a point $x \in \mathcal{H}$ and a number $\lambda > 0$, the mapping $G_{x,\lambda} : \mathcal{H} \to \mathcal{H}$ defined by

$$G_{x,\lambda}(y) := \frac{1}{1+\lambda}x + \frac{\lambda}{1+\lambda}Fy, \quad y \in \mathcal{H}, \tag{4}$$

is a strict contraction with Lipschitz constant $\frac{\lambda}{1+\lambda}$, and hence has a unique fixed point, which will be denoted $R_\lambda x$. The mapping $x \mapsto R_\lambda x$ is called the *resolvent* of F.

It is known that the limit

$$T_t x := \lim_{n\to\infty} \left(R_{\frac{t}{n}}\right)^n x, \qquad x \in \mathcal{H}, \tag{5}$$

exists uniformly with respect to t on each bounded subinterval of $[0,\infty)$. Moreover, the family (T_t) is a a strongly continuous semigroup of nonexpansive mappings [25]. This definition appeared in [23, Theorem 8.1] in a similar context, namely, for a coaccretive operator on a hyperbolic space.

The following result is a counterpart of Theorem 1.1. It was proved for the Hilbert ball in [13, Theorem 24.1] and for a bounded Hadamard space in [19, Theorem 26]. The latter proof however works also without the boundedness assumption, as we demonstrate in Section 3 for the reader's convenience.

Theorem 1.4. *Let $F\colon \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping and $x \in \mathcal{H}$. If there exists a sequence $(\lambda_n) \subset (0,\infty)$ such that $\lambda_n \to \infty$ and the sequence $(R_{\lambda_n} x)_{n\in\mathbb{N}}$ is bounded, then* $\operatorname{Fix} F$ *is nonempty and*

$$\lim_{\lambda\to\infty} R_\lambda x = P_{\operatorname{Fix} F}(x). \tag{6}$$

Conversely, if $\operatorname{Fix} F \neq \emptyset$*, then the curve $(R_\lambda x)_{\lambda\in(0,\infty)}$ is bounded.*

Our results are presented in Proposition 1.5 and Theorem 1.6 below. In Proposition 1.5, we give an algorithm which finds a fixed point of F. It is a counterpart of Theorem 1.2. For a general form of this algorithm in Hilbert space, see [10, Theorem 23.41]. See also [12, Theorem 2.6], [23, Corollary 7.10] and [24, Theorem 4.7]. The best result in Hadamard spaces works only with $\lambda_n = \lambda > 0$; see [2, Theorem 6.4].

Proposition 1.5. *Let $F\colon \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping with at least one fixed point and let $(\lambda_n) \subset (0,\infty)$ be a sequence satisfying $\sum_n \lambda_n^2 = \infty$. Given a point $x_0 \in \mathcal{H}$, put*

$$x_n := R_{\lambda_n} x_{n-1}, \qquad n \in \mathbb{N}. \tag{7}$$

Then the sequence (x_n) converges weakly to a fixed point of F.

Note that the assumption $\sum_n \lambda_n^2 = \infty$ appears also in Hilbert spaces; see [10, Theorem 23.41].

We also study the asymptotic behavior of the nonlinear semigroup defined in (5). A Hilbert ball version of this result appears in [22].

Theorem 1.6. *Let $F\colon \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping with at least one fixed point and let $x_0 \in \mathcal{H}$. Then $T_t x_0$ converges weakly to a fixed point of F as $t \to \infty$.*

As noted in [9, Remark, p. 7], there exists a counterexample in Hilbert space showing that the convergence in Theorem 1.6 is not strong in general. This counterexample is based on an earlier work of J.-B. Baillon [8].

In Section 6, we apply Proposition 1.5 and Theorem 1.6 to harmonic mapping theory in singular spaces and obtain the convergence of a heat flow to a solution to a Dirichlet problem under very mild assumptions. To this end, we construct discrete and continuous heat flows by (7) and (5), respectively, with F being the nonlinear Markov operator. To the best of our knowledge, these constructions are new and complement the existing approaches, for instance, the gradient flow of the energy functional.

2. Preliminaries

Here we will recall some basic definitions and fact on Hadamard spaces. More information can be found in the books [3, 11, 17]

Throughout the paper, the space $(\mathcal{H}, d)$ is Hadamard, that is, it is a complete geodesic metric space satisfying

$$d(x,\gamma_t)^2 \leq (1-t)d(x,\gamma_0)^2 + td(x,\gamma_1)^2 - t(1-t)d(\gamma_0,\gamma_1)^2, \tag{8}$$

for any $x \in \mathcal{H}$, any geodesic $\gamma\colon [0,1] \to \mathcal{H}$, and any $t \in [0,1]$. Given a closed and convex set $C \subset \mathcal{H}$ and a point $x \in \mathcal{H}$, there exists a unique point $c \in C$ such that

$$d(x,c) = d(x,C) := \inf_{y\in C} d(x,y).$$

We denote this point c by $P_C x$ and call the mapping $P_C\colon \mathcal{H} \to C$ the *metric projection* of $\mathcal{H}$ onto the set C.

Given a bounded sequence $(x_n) \subset \mathcal{H}$, put

$$\omega(x;(x_n)) := \limsup_{n\to\infty} d(x,x_n)^2, \qquad x \in \mathcal{H}. \tag{9}$$

Then the function ω defined in (9) has a unique minimizer, which we call the *asymptotic center* of (x_n). We shall say that $(x_n) \subset \mathcal{H}$ *weakly converges* to a point $x \in \mathcal{H}$ if x is the asymptotic center of each subsequence of (x_n). We use the notation $x_n \xrightarrow{w} x$. Clearly, if $x_n \to x$, then $x_n \xrightarrow{w} x$. If there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \xrightarrow{w} z$ for some $z \in \mathcal{H}$, we say that z is a *weak cluster point* of the sequence (x_n).

Proposition 2.1. [7, Proposition 3.3] *Let $C \subset \mathcal{H}$ be a closed convex set. Assume $(x_n) \subset \mathcal{H}$ is a Fejér monotone sequence with respect to C. Then we have:*

(i) *(x_n) is bounded,*
(ii) *$d(x_{n+1}, C) \leq d(x_n, C)$ for each $n \in \mathbb{N}$.*
(iii) *(x_n) weakly converges to some $x \in C$ if and only if all weak cluster points of (x_n) belong to C.*
(iv) *(x_n) converges to some $x \in C$ if and only if $d(x_n, C) \to 0$.*

For each $\lambda > 0$ and $x \in \mathcal{H}$, we have $R_\lambda x = x$ if and only if $Fx = x$. Furthermore, we have the following estimate.

Lemma 2.2. [25, Lemma 3.4] *Let $F\colon \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping. Then its resolvent satisfies*

$$d(x, R_\lambda x) \leq \lambda d(x, Fx), \tag{10}$$

for every $\lambda \in (0,\infty)$.

Proof. Since $R_\lambda x$ is a fixed point of the contraction $G_{x,\lambda}$, it can be approximated by the Banach contraction principle. Therefore,

$$\begin{aligned} d\left(x, R_\lambda x\right) &\leq \sum_{n=1}^{\infty} d\left(G_{x,\lambda}^{n-1}(x), G_{x,\lambda}^{n}(x)\right) \\ &\leq d\left(x, G_{x,\lambda}(x)\right) \sum_{n=1}^{\infty} \left(\frac{\lambda}{1+\lambda}\right)^{n-1} \\ &\leq (1+\lambda) d\left(x, G_{x,\lambda}(x)\right), \end{aligned}$$

and we are done, because the right-hand side is equal to $\lambda d(x, Fx)$. □

Consequently,

$$d\left(x, R_{\frac{t}{n}}^{n} x\right) \leq \sum_{j=0}^{n-1} d\left(R_{\frac{t}{n}}^{j} x, R_{\frac{t}{n}}^{j+1} x\right) \leq nd\left(x, R_{\frac{t}{n}} x\right) = td(x, Fx),$$

and taking the limit $n \to \infty$, we obtain

$$d\left(x, T_t x\right) \leq td(x, Fx). \tag{11}$$

3. Proof of Theorem 1.4

Proof of Theorem 1.4. To simplify our notation, put $x_\lambda := R_\lambda x$ for each $\lambda \in (0, \infty)$. Fix now $0 < \mu < \lambda$ and let

$$\overline{\triangle}\left(\overline{x}, \overline{Fx_\lambda}, \overline{Fx_\mu}\right) \subset \mathbb{R}^2$$

be a comparison triangle for $\triangle\left(x, Fx_\lambda, Fx_\mu\right)$. We have

$$\left\|\overline{Fx_\lambda} - \overline{Fx_\mu}\right\| = d\left(Fx_\lambda, Fx_\mu\right) \leq d\left(x_\lambda, x_\mu\right) \leq \left\|\overline{x_\lambda} - \overline{x_\mu}\right\|.$$

Without loss of generality we can assume $\overline{x} = 0 \in \mathbb{R}^2$. From this and the fact that $\overline{x_\lambda} = \frac{\lambda}{1+\lambda}\overline{F\left(x_\lambda\right)}$ and $\overline{x_\mu} = \frac{\mu}{1+\mu}\overline{Fx_\mu}$ we further obtain

$$\left\langle \frac{1+\lambda}{\lambda}\overline{x_\lambda} - \frac{1+\mu}{\mu}\overline{x_\mu}, \frac{1+\lambda}{\lambda}\overline{x_\lambda} - \frac{1+\mu}{\mu}\overline{x_\mu} \right\rangle \leq \left\|\overline{x_\lambda} - \overline{x_\mu}\right\|.$$

A simple computation yields

$$\begin{aligned} \left(\frac{1+\lambda}{\lambda} - \frac{1+\mu}{\mu}\right)^2 \left\|\overline{x_\mu}\right\| + \left(\frac{(1+\lambda)^2}{\lambda^2} - 1\right) \left\|\overline{x_\lambda} - \overline{x_\mu}\right\| \\ \leq 2\left(\frac{1+\mu}{\mu} - \frac{1+\lambda}{\lambda}\right) \frac{1+\lambda}{\lambda} \left\langle \overline{x_\mu}, \overline{x_\lambda} - \overline{x_\mu} \right\rangle. \end{aligned}$$

Consequently,

$$\left\langle \overline{x_\mu}, \overline{x_\lambda} - \overline{x_\mu} \right\rangle \geq 0.$$

Since

$$\left\|\overline{x_\lambda}\right\|^2 = \left\|\overline{x_\mu}\right\|^2 + \left\|\overline{x_\lambda} - \overline{x_\mu}\right\|^2 + 2\left\langle \overline{x_\mu}, \overline{x_\lambda} - \overline{x_\mu} \right\rangle,$$

we have

$$\|\overline{x_\mu}\| \leq \|\overline{x_\lambda}\|,$$

and

$$d(x_\lambda, x_\mu)^2 \leq \|\overline{x_\lambda} - \overline{x_\mu}\|^2 \leq \|\overline{x_\lambda} - \overline{x}\|^2 - \|\overline{x_\mu} - \overline{x}\|^2 . \tag{12}$$

The monotonicity of $\lambda \mapsto \|\overline{x_\lambda}\|$ and the boundedness of the sequence $\{\overline{x_{\lambda_n}}\}$ yield the boundedness of the curve $(x_\lambda)_{\lambda\in(0,\infty)}$. Inequality (12) therefore implies that

$$d(x_\lambda, x_\mu)^2 \to 0, \quad \text{as } \lambda, \mu \to \infty.$$

Let $z \in \mathcal{H}$ be the limit point of (x_λ). By continuity we have

$$d(z, Fz) = \lim_{\lambda\to\infty} d(x_\lambda, Fx_\lambda) = \lim_{\lambda\to\infty} \frac{1}{1+\lambda} d(x, Fx_\lambda) = 0,$$

which means that $z \in \operatorname{Fix} F$.

We will now show $z = P_{\operatorname{Fix} F}(x)$. Let $p \in \operatorname{Fix} F$ and repeat the above argument with the triangle $\triangle(x, p, F(x_\mu))$. We obtain

$$d(x,p)^2 = \|\overline{x} - \overline{p}\|^2 \geq \|\overline{x} - \overline{x_\mu}\|^2 + \|\overline{x_\mu} - \overline{p}\|^2 \geq d(x, x_\mu)^2 + d(x_\mu, p)^2,$$

and after taking the limit $\mu \to \infty$, we have

$$d(x,p)^2 \geq d(x,z)^2 + d(z,p)^2,$$

which completes the proof that $z = P_{\operatorname{Fix} F}(x)$.

Finally, it is easy to see that if $\operatorname{Fix} F \neq \emptyset$, then $(x_\lambda)_{\lambda\in(0,\infty)}$ is bounded. □

4. Proof of Proposition 1.5

Proof of Theorem 1.5. Let $x \in \mathcal{H}$ be a fixed point of F. Then, for each $n \in \mathbb{N}$, we have

$$d(x_{n-1}, x) \geq d(R_{\lambda_n} x_{n-1}, R_{\lambda_n} x) = d(x_n, x),$$

which verifies the Fejér monotonicity of (x_n) with respect to $\operatorname{Fix} F$. Put

$$\beta_n := \frac{1}{1+\lambda_n}.$$

Inequality (8) yields

$$\begin{aligned} d(x, x_n)^2 &\leq \beta_n d(x, x_{n-1})^2 + (1-\beta_n) d(x, Fx_n)^2 \\ &\quad - \beta_n(1-\beta_n) d(x_{n-1}, Fx_n)^2 \\ &\leq \beta_n d(x, x_{n-1})^2 + (1-\beta_n) d(x, x_n)^2 - \beta_n d(x_{n-1}, x_n)^2, \end{aligned}$$

which gives

$$d(x_{n-1}, x_n)^2 \leq d(x, x_{n-1})^2 - d(x, x_n)^2$$

and hence

$$\lambda_n^2 \frac{d(x_{n-1}, x_n)^2}{\lambda_n^2} \leq d(x, x_{n-1})^2 - d(x, x_n)^2 . \tag{13}$$

By the triangle inequality, we have

$$\begin{aligned} d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, Fx_{n+1}\right) &\le d\left(x_n, Fx_n\right) + d\left(Fx_n, Fx_{n+1}\right) \\ &\le d\left(x_n, Fx_n\right) + d\left(x_n, x_{n+1}\right) \end{aligned}$$

and therefore

$$\frac{d\left(x_n, x_{n+1}\right)}{\lambda_{n+1}} = d\left(x_{n+1}, Fx_{n+1}\right) \le d\left(x_n, Fx_n\right) = \frac{d\left(x_{n-1}, x_n\right)}{\lambda_n}. \tag{14}$$

Summing up (13) over $n = 1, \ldots, m$, where $m \in \mathbb{N}$, and using (14), we obtain

$$\left(\sum_{n=1}^{m} \lambda_n^2\right) \frac{d\left(x_{m-1}, x_m\right)^2}{\lambda_m^2} \le d\left(x, x_0\right)^2 - d\left(x, x_m\right)^2.$$

Taking the limit $m \to \infty$, we arrive at

$$d\left(x_m, Fx_m\right) = \frac{1}{\lambda_m} d\left(x_{m-1}, x_m\right) \to 0, \qquad \text{as } m \to \infty.$$

Assume now that $z \in \mathcal{H}$ is a weak cluster point of $\left(x_n\right)$. Then

$$\begin{aligned} \limsup_{n\to\infty} d\left(Fz, x_n\right) &\le \limsup_{n\to\infty} \left[d\left(Fz, Fx_n\right) + d\left(Fx_n, x_n\right)\right], \\ &\le \limsup_{n\to\infty} d\left(z, x_n\right) + 0. \end{aligned}$$

By the uniqueness of the weak limit, we get $z = Fz$. Finally, apply Proposition 2.1(iii) to conclude that $\left(x_n\right)$ weakly converges to a fixed point of F. □

5. Proof of Theorem 1.6

Proof of Theorem 1.6. We mimic the technique from [22] and adapt it to our situation. First observe that

$$d\left(R_\lambda x, FR_\lambda x\right) = \frac{1}{\lambda} d\left(x, R_\lambda x\right) \le d(x, Fx),$$

by (10). Hence we have

$$d(x, Fx) \ge d\left(R_{\frac{t}{n}} x, FR_{\frac{t}{n}} x\right) \ge d\left(R_{\frac{t}{n}}^n x, FR_{\frac{t}{n}}^n x\right),$$

and after taking the limit $n \to \infty$, we also have

$$d(x, Fx) \ge d\left(T_t x, FT_t x\right).$$

The semigroup property implies that

$$d\left(T_t x, FT_t x\right) \le d\left(T_s x, FT_s x\right),$$

whenever $s \le t$ and therefore the limit

$$\lim_{t\to\infty} d\left(T_t x, FT_t x\right) \tag{15}$$

exists. We will now show that this limit actually equals 0.

Let $0 \leq s \leq t$. Then inequality (11) yields

$$d\left(T_s x, T_t x\right) \leq \sum_{j=0}^{n-1} d\left(T_{s+\frac{j}{n}(t-s)} x, T_{s+\frac{j+1}{n}(t-s)} x\right)$$
$$\leq \frac{t-s}{n} \sum_{j=0}^{n-1} d\left(T_{s+\frac{j}{n}(t-s)} x, F T_{s+\frac{j}{n}(t-s)} x\right),$$

and after letting $n \to \infty$, we obtain

$$d\left(T_s x, T_t x\right) \leq \int_s^t d\left(T_r x, F T_r x\right) \mathrm{d}r. \tag{16}$$

Next we prove that

$$\lim_{t \to \infty} d\left(T_t x, F T_t x\right) \leq \frac{1}{h} \lim_{t \to \infty} d\left(T_{t+h} x, T_t x\right). \tag{17}$$

To this end, we repeatedly use the inequality

$$d\left(F R_\lambda^n x, R_\lambda^{n-k+1} x\right) \leq \frac{1}{1+\lambda} d\left(F R_\lambda^n x, R_\lambda^{n-k} x\right)$$
$$+ \frac{\lambda}{1+\lambda} d\left(R_\lambda^n x, R_\lambda^{n-k+1} x\right),$$

valid for each $1 \leq k \leq n$, to obtain

$$d\left(F R_\lambda^n x, R_\lambda^n x\right) \leq \frac{1}{(1+\lambda)^n} d\left(F R_\lambda^n x, x\right)$$
$$+ \lambda \sum_{j=0}^{n-1} \frac{1}{(1+\lambda)^{j+1}} d\left(R_\lambda^n x, R_\lambda^{n-j+1} x\right).$$

Put now $\lambda := \frac{t}{n}$ and take the limit $n \to \infty$. One arrives at

$$d\left(T_t x, F T_t x\right) \leq \int_0^t e^{-r} d\left(T_t x, T_{t-r} x\right) \mathrm{d}r + e^{-t} d\left(F T_t x, x\right).$$

Applying inequality (16) and an elementary calculation, we arrive at

$$e^t d\left(T_t x, F T_t x\right) \leq \int_0^t \left(e^r - 1\right) d\left(T_r x, F T_r x\right) \mathrm{d}r + d\left(F T_t x, x\right)$$

or

$$\left(e^t - 1\right) d\left(T_t x, F T_t x\right) \leq \int_0^t \left(e^r - 1\right) d\left(T_r x, F T_r x\right) \mathrm{d}r + d\left(T_t x, x\right).$$

Replacing t by h and then x by $T_t x$ gives

$$\left(e^h - 1\right) d\left(T_{t+h} x, F T_{t+h} x\right) \leq \int_t^{t+h} \left(e^{r-t} - 1\right) d\left(T_r x, F T_r x\right) \mathrm{d}r$$
$$+ d\left(T_{t+h} x, T_t x\right).$$

By an easy calculation we obtain

$$\begin{aligned} d\left(T_{t+h}x, T_t x\right) \geq \left(e^h - 1\right)\left[d\left(T_{t+h}x, FT_{t+h}x\right) - d\left(T_t x, FT_t x\right)\right] \\ + hd\left(T_t x, FT_t x\right), \end{aligned}$$

which proves (17). Then (17) and (16) yield

$$\begin{aligned} \lim_{t\to\infty} d\left(T_t x, FT_t x\right) &\leq \limsup_{h\to\infty} \frac{1}{h} d\left(T_h x, x\right) \\ &\leq \lim_{h\to\infty} \frac{1}{h}\int_0^h d\left(T_r x, FT_r x\right) \mathrm{d}r \\ &= \lim_{t\to\infty} d\left(T_t x, FT_t x\right). \end{aligned}$$

From this, we can see that the limit in (15) is independent of x and is therefore equal to 0, for one may choose $x \in \operatorname{Fix} F$.

To finish the proof, choose a sequence $t_n \to \infty$ and put $x_n := T_{t_n} x_0$. Since T_t is nonexpansive, we know that the sequence (x_n) is Fejér monotone with respect to $\operatorname{Fix} F$. In particular, x_n is bounded and has therefore a weak cluster point $z \in \mathcal{H}$. It suffices to show that $z \in \operatorname{Fix} F$. We easily get

$$\begin{aligned} \limsup_{n\to\infty} d\left(Fz, x_n\right) &\leq \limsup_{n\to\infty} d\left(Fz, Fx_n\right) + \limsup_{n\to\infty} d\left(Fx_n, x_n\right) \\ &\leq \limsup_{n\to\infty} d\left(z, x_n\right), \end{aligned}$$

which by the uniqueness of the weak limit gives $z = Fz$. Here we used, of course, the fact that the limit in (15) is 0.

It is easy to see that z is independent of the choice of the sequence (t_n) and therefore $T_t x_0 \stackrel{w}{\to} z$. □

6. Discrete and continuous heat flows in singular spaces

There have been considerable interest in harmonic mappings between singular spaces and several (nonequivalent) approaches have been developed. See, for example, [14, 15, 16, 18, 20, 26, 27, 28]. We will follow [26] and consider an L^2-Dirichlet problem for mappings from a measure space equipped with a symmetric Markov kernel to a Hadamard space. Under the assumption that the Markov kernel satisfies an L^2-spectral bound condition, it is shown in [26] that a Dirichlet problem has a unique solution and that an associated heat flow converges to this solution. The L^2-spectral bound condition is completely natural albeit rather strong, since the heat flow semigroup is then a contracting mapping and converges to the unique solution exponentially fast.

In the present paper, we use a somewhat different approach than [26] to construct discrete and continuous time heat flows, namely formulae (7) and (5). Then we employ Proposition 1.5 and Theorem 1.6 to obtain the convergence of these heat flows to a solution to the Dirichlet problem. Let us first formulate the Dirichlet problem for singular spaces. For the details, see the original paper [26] .

Let $(M, \mathcal{M}, \mu)$ be a measure space with σ-algebra $\mathcal{M}$, measure μ and assume that it is complete in the sense that all subsets of a μ-null set belong to $\mathcal{M}$. Given a set $D \in \mathcal{M}$, define

$$L^2(D) := \left\{ u \in L^2(M) \colon u = 0 \text{ a.e. on } M \setminus D \right\}.$$

Next, let $(\mathcal{H}, d)$ be a Hadamard space and fix a measurable mapping $h \colon M \to \mathcal{H}$, we define the nonlinear Lebesgue space $L^2(D, \mathcal{H}, h)$ of measurable mappings $f \colon M \to \mathcal{H}$ satisfying

$$d\left(f(\cdot), h(\cdot)\right)^2 \in L^2(D).$$

The space $L^2(D, \mathcal{H}, h)$, when equipped with the metric

$$d_2(f, g) := \left(\int_M d\left(f(x), g(x)\right)^2 \mu(\mathrm{d}x) \right)^{\frac{1}{2}},$$

is again a Hadamard space.

Let $p := p(x, \mathrm{d}y)$ be a Markov kernel, which is symmetric with respect to μ, that is, we have $p(x, \mathrm{d}y)\mu(\mathrm{d}x) = p(y, \mathrm{d}x)\mu(\mathrm{d}y)$ for every $x, y \in M$. Then one can define the nonlinear Markov operator $P \colon L^2(M, \mathcal{H}, h) \to L^2(M, \mathcal{H}, h)$ by

$$Pf(x) := \operatorname*{arg\,min}_{z \in \mathcal{H}} \int_M d\left(z, f(y)\right)^2 p(x, \mathrm{d}y),$$

where $f \in L^2(M, \mathcal{H}, h)$. By [26, Theorem 5.2], we know that

$$d_2\left(Pf, Pg\right) \leq d_2\left(Pf, Pg\right),$$

for every $f, g \in L^2(M, \mathcal{H}, h)$, that is, the nonlinear Markov operator is nonexpansive on $L^2(M, \mathcal{H}, h)$. A fixed point of P is called a harmonic mapping.

Remark 6.1. If $M \subset \mathbb{R}^n$ is a bounded set and $P \colon L^2(M, \mathbb{R}) \to L^2(M, \mathbb{R})$ is the usual (linear) Markov operator, then the laplacian satisfies $\Delta = I - P$, with $I \colon L^2(M, \mathbb{R}) \to L^2(M, \mathbb{R})$ being the identity, and we see that a function $f \colon M \to \mathbb{R}$ is harmonic if $\Delta f = 0$.

Since we are concerned with the Dirichlet problem, a somewhat refined notion of a nonlinear Markov operator is needed. Given $D \in \mathcal{M}$, define a new Markov kernel

$$p_D(x, \mathrm{d}y) := \chi_D(x) p(x, \mathrm{d}y) + \chi_{M \setminus D} \delta_x(\mathrm{d}y),$$

for every $x, y \in M$. Denote $P_{(D)}$ the nonlinear Markov operator associated with the kernel p_D. Then we have

$$P_{(D)} f(x) = \begin{cases} Pf(x) & \text{if } x \in D \\ f(x) & \text{if } x \in M \setminus D, \end{cases}$$

and $P_{(D)} \colon L^2(D, \mathcal{H}, h) \to L^2(D, \mathcal{H}, h)$ is also nonexpansive.

Definition 6.2 (The L^2-Dirichlet problem)**.** *Let $D \in \mathcal{M}$ and $h \colon M \to \mathcal{H}$ be a measurable mapping. Is there $f \in L^2(D, \mathcal{H}, h)$ such that $P_{(D)} f = f$?*

In [26, Theorem 6.4], the author shows that the Dirichlet problem has a unique solution provided the linear operator

$$P_{(D)}f(x) := \int_M f(y)p_D(x, \mathrm{d}y), \qquad f \in L^2(M),$$

satisfies the spectral bound condition $\lambda_k > 0$ for some $k \in \mathbb{N}$, where

$$\lambda_k := 1 - \|P_{(D)}^k\|_{L^2(D)}.$$

Under this assumption, one also gets strong (and exponentially fast) convergence of an associated heat flow to the (unique) solution to the Dirichlet problem. In contrast, we apply Proposition 1.5 and Theorem 1.6 with $F := P_{(D)}$ and do not assume any spectral bound condition. We are then able to conclude that if there exists a solution to the Dirichlet problem for a measurable mapping $h\colon M \to \mathcal{H}$, both the proximal point algorithm (discrete time heat flow) and the semigroup (continuous time heat flow) weakly converge to a mapping $f \in L^2(D, \mathcal{H}, h)$ such that $P_{(D)}f = f$.

We finish our paper by making the following conjecture.

Conjecture 6.3. *The heat flow semigroup* (T_t) *defined by* (5) *for* $F := P_{(D)}$ *coincides with the heat flow constructed in* [26, Theorem 8.1].

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Miroslav Bačák, Max Planck Institute, Inselstr. 22, 04 103 Leipzig, Germany
E-mail address: bacak@mis.mpg.de

Simeon Reich, Department of Mathematics, The Technion – Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: sreich@tx.technion.ac.il