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Abstract

In this paper we introduce a tensor subspace based format for the representation of a tensor in a tensor space. To do this we use a property of minimal subspaces which allows us to describe the tensor representation by means of a rooted tree. By using the tree structure and the dimensions of the associated minimal subspaces, we introduce the set of tensors in a tree based format with either bounded or fixed tree based rank. This class contains the Tucker format and the Hierarchical Tucker format (including the Tensor Train format). In particular, any tensor of the topological tensor space under consideration admits best approximations in the set of tensors in the tree based format with bounded tree based rank. Moreover, we show that the set of tensors in the tree based format with fixed tree based rank is an analytic Banach manifold. The local chart representation of the manifold is often crucial for an algorithmic treatment of high-dimensional time-dependent PDEs and minimisation problems. We also show, under some natural assumptions, that the tangent (Banach) space at a given tensor is a complemented subspace in the natural ambient tensor Banach space and hence the set of tensors in the tree based format with fixed tree based rank is an immersed submanifold. This fact allows us to extend the Dirac-Frenkel variational principle in the framework of topological tensor spaces.

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1 Introduction

Tensor formats based on subspaces are widely used in scientific computation. Their constructions are usually based on a hierarchy of tensor product subspaces spanned by orthonormal bases, because in most cases a hierarchical representation fits to the structure of the mathematical model and improves its computational implementation.

Two of the most popular formats are the Tucker format and the Hierarchical Tucker format [14] (HT for short). It is possible to show that the Tensor Train format [25] (TT for short), introduced originally by Vidal [30], is a particular class of the HT format (see, e.g. Chapter 12 in [15]). An important feature of these formats, in the framework of topological tensor spaces, is the existence of a best approximation in each fixed set of tensors with bounded rank [7]. In particular, it allows to construct, on a theoretical level, iterative minimisation methods for nonlinear convex problems over reflexive tensor Banach spaces [8].

It is well known that the Tucker format is also well applicable to the discretisation of differential equations in the framework of quantum chemical problems or of multireference Hartree and Hartree-Fock methods (MR-HF) in quantum dynamics [21]. In particular, it can be shown that the set of Tucker tensors of fixed rank forms an immersed finite-dimensional manifold [18]. Then the numerical treatment of this class of problems follows the general concepts of differential equations on manifolds [12]. Recently, similar results have been obtained for the TT format [16] and the HT format [28] (see also [3]). The term "matrix-product state" (MPS) was introduced in quantum physics (see, e.g., [29]). The related tensor representation can be found already in [30] without a special naming of the representation. The method has been reinvented by Oseledets and Tyrtyshnikov (see [24], [25], and [26]) and called "TT decomposition". For matrix product systems (MPS), the differential geometry in a finite-dimensional complex Hilbert space setting is covered in [13].

Some natural questions arise in the framework of topological tensor spaces. The first one is: is it possible to introduce a class of tensors containing Tucker, HT (and hence the TT) tensors with fixed and bounded rank ? A second question is: if such a class exists, is it possible to construct a parametrisation for the set of tensors of fixed rank in order to show that it is a true manifold even in infinite dimensional case? Finally, if the answers to both questions are yes, we would like to ask the following question: is the set of tensors of fixed rank an immersed submanifold of the topological tensor space, as ambient manifold, under consideration ?

The main goal of this paper is the study of the geometric structure of tensor representations based on subspaces. The paper is organised as follows. Sect. 2 is devoted to preliminary definitions and results about Banach spaces and Banach manifolds. Next, from Sect. 3 to Sect. 6, we give the contributions of this paper. More precisely,

- In Sect. 3, we introduce a generalisation, at algebraic and topological levels, of the hierarchical tensor format in order to include the Tucker tensors (among others) in that class. Moreover, we characterise the minimal subspaces in that class extending the previous results obtained in [7].
- In Sect. 4, we show that the set of tensors with fixed rank is an analytic Banach manifold and its geometric structure is independent on the ambient tensor Banach space under consideration.
- In Sect. 5, we discuss the choice of a norm in the ambient tensor Banach space in order to show that the set of tensors with fixed rank is a immersed submanifold of that space (considered as Banach manifold). To this end we assume the existence of a norm at each node of the tree not weaker than the injective norm constructed from the Banach spaces associated with the sons of that node. This assumption generalises the condition used in [7] to prove the existence of a best approximation in the Tucker case. More precisely, under this assumption,
  - we construct a linear isomorphism, at each point in the manifold of tensors with fixed rank, from the tangent space at that point to a closed linear subspace of the ambient tensor Banach space, this subspace being given explicitly,
  - we show that the set of tensors with fixed rank is an immersed submanifold and
  - we also provide a proof of the existence of best approximation in the set of tensors with bounded rank.
- In Sect. 6, we give a formalisation in this framework of the multi–configuration time–dependent Hartree MCTDH method (see [21]) in tensor Banach spaces.

2 Banach manifolds

In the following, X is a Banach space with norm $\|\cdot\|$. The dual norm $\|\cdot\|_{X^*}$ of X^* is

$$\|\varphi\|_{X^*} = \sup\{|\varphi(x)| : x \in X \text{ with } \|x\|_X \leq 1\} = \sup\{|\varphi(x)| \,/\, \|x\|_X : 0 \neq x \in X\}\,. \tag{2.1}$$

By $\mathcal{L}(X, Y)$ we denote the space of continuous linear mappings from X into Y. The corresponding operator norm is written as $\|\cdot\|_{Y \leftarrow X}$.

Definition 2.1 *Let X be a Banach space. We say that $P \in \mathcal{L}(X, X)$ is a projection if $P \circ P = P$. In this situation we also say that P is a projection from X onto $P(X)$ parallel to* $\operatorname{Ker} P$.

From now on, we will denote $P \circ P = P^2$. Observe that if P is a projection then $I_X - P$ is also a projection. Moreover, $I_X - P$ is parallel to $P(X) := \operatorname{Im} P$.

Observe that each projection gives rise to a pair of closed subspaces, namely $U = \operatorname{Im} P$ and $V = \operatorname{Ker} P$ such that $X = U \oplus V$. It allows us to introduce the following two definitions.

Definition 2.2 *We will say that a subspace U of a Banach space X is a complemented subspace if U is closed and there exists V in X such that $X = U \oplus V$ and V is also a closed subspace of X. This subspace V is called a (topological) complement of U and (U, V) is a pair of complementary subspaces.*

Corresponding to each pair (U, V) of complementary subspaces, there is a projection P mapping X onto U along V, defined as follows. Since for each x there exists a unique decomposition $x = u + v$, where $u \in U$ and $v \in V$, we can define a linear map $P(u + v) := u$, where $\operatorname{Im} P = U$ and $\operatorname{Ker} P = V$. Moreover, $P^2 = P$.

Definition 2.3 *The* Grassmann manifold *of a Banach space X, denoted by $\mathbb{G}(X)$, is the set of all complemented subspaces of X.*

$U \in \mathbb{G}(X)$ holds if and only if U is a closed subspace and there exists a closed subspace V in X such that $X = U \oplus V$. Observe that X and $\{0\}$ are in $\mathbb{G}(X)$. Moreover, by the proof of Proposition 4.2 of [6], the following result can be shown.

Proposition 2.4 *Let X be a Banach space. The following conditions are equivalent:*

(a) $U \in \mathbb{G}(X)$.

(b) There exists $P \in \mathcal{L}(X, X)$ such that $P^2 = P$ and $\operatorname{Im} P = U$.

(c) There exists $Q \in \mathcal{L}(X, X)$ such that $Q^2 = Q$ and $\operatorname{Ker} Q = U$.

Moreover, from Theorem 4.5 in [6], the following result can be shown.

Proposition 2.5 *Let X be a Banach space. Then every finite dimensional subspace $U \in \mathbb{G}(X)$.*

Let V and U be closed subspaces of a Banach space X such that $X = U \oplus V$. From now on, we will denote by $P_{U \oplus V}$ the projection onto U along V. Then we have $P_{V \oplus U} = I_X - P_{U \oplus V}$. Let $U, U' \in \mathbb{G}(X)$. We say that U and U' have a common complementary subspace in X, if $X = U \oplus W = U' \oplus W$ for some $W \in \mathbb{G}(X)$. The following result will be useful (see Lemma 2.1 in [4]).

Lemma 2.6 *Let X be a Banach space and assume that W, U, and U' are in $\mathbb{G}(X)$. Then the following statements are equivalent:*

(a) $X = U \oplus W = U' \oplus W$, i.e., U and U' have a common complement in X.

(b) $P_{U \oplus W}|_{U'} : U' \to U$ has an inverse.

Furthermore, if $Q = \left(P_{U \oplus W}|_{U'}\right)^{-1}$, then Q is bounded and $Q = P_{U' \oplus W}|_{U}$.

Definition 2.7 *Let $\mathbb{M}$ be a set. An atlas of class C^p $(p \geq 0)$ on $\mathbb{M}$ is a family of charts with some indexing set A, namely $\{(M_\alpha, u_\alpha) : \alpha \in A\}$, having the following properties:*

AT1 $\{M_\alpha\}_{\alpha \in A}$ *is a covering*[1] *of $\mathbb{M}$, that is, $M_\alpha \subset \mathbb{M}$ for all $\alpha \in A$ and $\cup_{\alpha \in A} M_\alpha = \mathbb{M}$.*

AT2 For each $\alpha \in A$, (M_α, u_α) stands for a bijection $u_\alpha : M_\alpha \to U_\alpha$ of M_α onto an open set U_α of a Banach space X_α, and for any α and β the set $u_\alpha(M_\alpha \cap M_\beta)$ is open in X_α.

[1] The condition of an *open* covering is not needed, see [19].

AT3 Finally, if we let $M_\alpha \cap M_\beta = M_{\alpha\beta}$ and $u_\alpha(M_{\alpha\beta}) = U_{\alpha\beta}$, the transition mapping $u_\beta \circ u_\alpha^{-1} : U_{\alpha\beta} \to U_{\beta\alpha}$ is a C^p-diffeomorphism.

Since different atlases can give the same manifold, we say that two atlases are *compatible* if each chart of one atlas is compatible with the charts of the other atlas in the sense of AT3. One verifies that the relation of compatibility between atlases is an equivalence relation.

Definition 2.8 *An equivalence class of atlases of class C^p on $\mathbb{M}$ is said to define a structure of a C^p-Banach manifold on $\mathbb{M}$, and hence we say that $\mathbb{M}$ is a Banach manifold. In a similar way, if an equivalence class of atlases is given by analytic maps, then we say that $\mathbb{M}$ is an analytic Banach manifold. If X_α is a Hilbert space for all $\alpha \in A$, then we say that $\mathbb{M}$ is a Hilbert manifold.*

In condition AT2 we do not require that the Banach spaces are the same for all indices α, or even that they are isomorphic. If X_α is linearly isomorphic to some Banach space X for all α, we have the following definition.

Definition 2.9 *Let $\mathbb{M}$ be a set and X be a Banach space. We say that $\mathbb{M}$ is a C^p Banach manifold modelled on X if there exists an atlas of class C^p over $\mathbb{M}$ with X_α linearly isomorphic to X for all $\alpha \in A$.*

Example 2.10 *Every Banach space is a Banach manifold (for a Banach space Y, simply take (I_Y, Y) as atlas, where I_Y is the identity map on Y). In particular, the set of all bounded linear maps $\mathcal{L}(X, X)$ of a Banach space X is a Banach manifold.*

If X is a Banach space, then the set of all bounded linear automorphisms of X will be denoted by

$$\mathrm{GL}(X) := \{A \in \mathcal{L}(X, X) : A \text{ invertible } \}.$$

Example 2.11 *If X is a Banach space, then $\mathrm{GL}(X)$ is a Banach manifold, because it is an open set in $\mathcal{L}(X, X)$. Moreover, the map $A \mapsto A^{-1}$ is analytic (see 2.7 in [27]).*

Example 2.12 (Grassmann Banach manifold) *Let X be a Banach space. Then, following [5] (see also [27] and [22]), it is possible to construct an atlas for $\mathbb{G}(X)$. To show that the atlas is an analytic Banach manifold, denote one of the complements of $U \in \mathbb{G}(X)$ by W, i.e., $X = U \oplus W$. Then we define the* Banach Grassmannian of U relative to W *by*

$$\mathbb{G}(W, X) := \{V \in \mathbb{G}(X) : X = V \oplus W\}.$$

It is possible to introduce a bijection

$$\Psi_{U\oplus W} : \mathbb{G}(W, X) \longrightarrow \mathcal{L}(U, W)$$

as the inverse of

$$\Psi^{-1}_{U\oplus W} : \mathcal{L}(U, W) \longrightarrow \mathbb{G}(W, X),$$

defined by

$$\Psi^{-1}_{U\oplus W}(L) = G(L) := \{u + L(u) : u \in U\}.$$

Observe that $G(0) = U$ and $G(L) \oplus W = X$ for all $L \in \mathcal{L}(U, W)$. It can be shown that the collection $\{\Psi_{U\oplus W}, \mathbb{G}(W, X)\}_{U\in\mathbb{G}(X)}$ is an analytic atlas, and therefore, $\mathbb{G}(X)$ is an analytic Banach manifold. In particular, for each $U \in \mathbb{G}(X)$ the set $\mathbb{G}(W, X) \overset{\Psi_{U\oplus W}}{\cong} \mathcal{L}(U, W)$ is also a Banach manifold.

Example 2.13 *Let X be a Banach space, from Proposition 2.5, every finite dimensional subspace belongs to $\mathbb{G}(X)$. It allows to introduce $\mathbb{G}_n(X)$, the space of all n-dimensional subspaces of X. It can be shown (see [22]) that $\mathbb{G}_n(X)$ is a connected component of $\mathbb{G}(X)$, and hence it is also a Banach manifold modelled on $\mathcal{L}(U, W)$, here $U \in \mathbb{G}_n(X)$ and $X = U \oplus W$. Moreover, $\bigcup_{n\le r} \mathbb{G}_n(X)$ is also a Banach manifold for each fixed $r < \infty$.*

Example 2.14 *Let $\overline{X}$ be the Banach space obtained as the completion of the normed space $(X, \|\cdot\|)$. We say that $U \in \mathbb{G}_n(X)$ if and only if $U \in \mathbb{G}_n(\overline{X})$ and $U \subset X$. We claim that $\mathbb{G}_n(X)$ is also a Banach manifold. To prove this claim, we need to show that for each $U \in \mathbb{G}_n(X)$ such that $U \oplus W = \overline{X}$, it holds that if $U' \in \mathbb{G}(W, \overline{X})$ then $U' \subset X$. Observe that $X = U \oplus (W \cap X)$ where $W \cap X$ is a linear subspace dense in $W = W \cap \overline{X}$. Assume that the claim is not true, then there exists $U' \in \mathbb{G}(W, \overline{X})$ such that $U' \oplus W = \overline{X}$ and $U' \cap X \neq U'$. Clearly $U' \cap X \neq \{0\}$, otherwise $W \cap X = X$ a contradiction. We have $X = (U' \cap X) \oplus (W \cap X)$, which implies $\overline{X} = (U' \cap X) \oplus W$, a contradiction and the claim follows. Then the collection $\{\Psi_{U \oplus W}, \mathbb{G}(W, \overline{X})\}_{U \in \mathbb{G}_n(X)}$ is an analytic atlas, and therefore, $\mathbb{G}_n(X)$ is an analytic Banach manifold modelled on $\mathcal{L}(U, W)$, here $U \in \mathbb{G}_n(X)$ and $\overline{X} = U \oplus W$.*

Let $\mathbb{M}$ be a Banach manifold of class $\mathcal{C}^p$, $p \geq 1$. Let m be a point of $\mathbb{M}$. We consider triples (U, φ, v) where (U, φ) is a chart at m and v is an element of the vector space in which $\varphi(U)$ lies. We say that two of such triples (U, φ, v) and (V, ψ, w) are *equivalent* if the derivative of $\psi\varphi^{-1}$ at $\varphi(m)$ maps v on w. Thanks to the chain rule it is an equivalence relation. An equivalence class of such triples is called a *tangent vector of* $\mathbb{M}$ *at* m.

Definition 2.15 *The set of such tangent vectors is called* tangent space of $\mathbb{M}$ at m *and it is denoted by* $\mathbb{T}_m(\mathbb{M})$.

Each chart (U, φ) determines a bijection of $\mathbb{T}_m(\mathbb{M})$ on a Banach space, namely the equivalence class of (U, φ, v) corresponds to the vector v. By means of such a bijection it is possible to equip $\mathbb{T}_m(\mathbb{M})$ with the structure of a topological vector space given by the chart, and it is immediate that this structure is independent of the chart selected.

Example 2.16 *If X is a Banach space, then $\mathbb{T}_x(X) = X$ for all $x \in X$.*

Example 2.17 *Let X be a Banach space and take $A \in \mathrm{GL}(X)$. Then $\mathbb{T}_A(\mathrm{GL}(X)) = \mathcal{L}(X, X)$.*

Example 2.18 *For $U \in \mathbb{G}(X)$ such that $X = U \oplus W$ for some $W \in \mathbb{G}(X)$, we have $\mathbb{T}_U(\mathbb{G}(X)) = \mathcal{L}(U, W)$.*

Example 2.19 *We point out that for a Hilbert space X with associated inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, its unit sphere denoted by*

$$\mathbb{S}_X := \{x \in X : \|x\| = 1\},$$

is a Hilbert manifold of codimension one. Moreover, for each $x \in \mathbb{S}_X$, its tangent space is

$$\mathbb{T}_x(\mathbb{S}_X) = \operatorname{span}\{x\}^\perp = \{x' \in X : \langle x, x' \rangle = 0\}.$$

3 Minimal subspaces and the representation of tensors in the Tree Based Format

3.1 Tensor spaces in tree based format

Concerning the definition of the algebraic tensor space ${}_a\bigotimes_{j=1}^d V_j$ generated from vector spaces V_j $(1 \leq j \leq d)$, we refer to Greub [10]. As underlying field we choose $\mathbb{R}$, but the results hold also for $\mathbb{C}$. The suffix 'a' in ${}_a\bigotimes_{j=1}^d V_j$ refers to the 'algebraic' nature. By definition, all elements of

$$\mathbf{V} := {}_a\bigotimes_{j=1}^d V_j$$

are *finite* linear combinations of elementary tensors $\mathbf{v} = \bigotimes_{j=1}^d v_j$ $(v_j \in V_j)$.

The following notations and definitions will be useful. We recall that $L(V, W)$ is the space of linear maps from V into W, while $V' = L(V, \mathbb{R})$ is the algebraic dual of V. For metric spaces, $\mathcal{L}(V, W)$ denotes the continuous linear maps, while $V^* = \mathcal{L}(V, \mathbb{R})$ is the topological dual of V.

Let $D := \{1, \ldots, d\}$ be the index set of the 'spatial directions'. In the sequel, the index sets $D\backslash\{j\}$ will appear. Here, we use the abbreviations

$$\mathbf{V}_{[j]} := {}_a\bigotimes_{k\neq j} V_k\,, \qquad \text{where } \bigotimes_{k\neq j} \text{ means } \bigotimes_{k\in D\backslash\{j\}}\,.$$

Similarly, elementary tensors $\bigotimes_{k\neq j} v_k$ are denoted by $\mathbf{v}_{[j]}$.

For vector spaces V_j and W_j over $\mathbb{R}$, let linear mappings $A_j : V_j \to W_j$ $(1 \le j \le d)$ be given. Then the definition of the elementary tensor

$$\mathbf{A} = \bigotimes_{j=1}^{d} A_j : \ \mathbf{V} = {}_a\bigotimes_{j=1}^{d} V_j \longrightarrow \mathbf{W} = {}_a\bigotimes_{j=1}^{d} W_j$$

is given by

$$\mathbf{A}\left(\bigotimes_{j=1}^{d} v_j\right) := \bigotimes_{j=1}^{d} (A_j v_j)\,. \tag{3.1}$$

Note that (3.1) extends uniquely to a linear mapping $\mathbf{A} : \mathbf{V} \to \mathbf{W}$.

Remark 3.1 *(a) Let* $\mathbf{V} := {}_a\bigotimes_{j=1}^{d} V_j$ *and* $\mathbf{W} := {}_a\bigotimes_{j=1}^{d} W_j$*. Then the linear combinations of tensor products of linear mappings* $\mathbf{A} = \bigotimes_{j=1}^{d} A_j$ *defined by means of (3.1) form a subspace of* $L(\mathbf{V}, \mathbf{W})$*:*

$${}_a\bigotimes_{j=1}^{d} L(V_j, W_j) \subset L(\mathbf{V}, \mathbf{W}).$$

(b) The special case of $W_j = \mathbb{R}$ *for all* j *(implying* $\mathbf{W} = \mathbb{R}$*) reads as* ${}_a\bigotimes_{j=1}^{d} V_j' \subset \mathbf{V}'$ *.*
(c) If $\dim(V_j) < \infty$ *and* $\dim(W_j) < \infty$ *for all* j*, the inclusion '$\subset$' in (a) and (b) can be replaced by '$=$'. This can be easily verified by just checking the dimensions of spaces involved.*

Often, mappings $\mathbf{A} = \bigotimes_{j=1}^{d} A_j$ will appear, where most of the A_j are the identity (and therefore $V_j = W_j$). If $A_k \in L(V_k, W_k)$ for one k and $A_j = id$ for $j \neq k$, we use the following notation:

$$\mathbf{id}_{[k]} \otimes A_k := \underbrace{id \otimes \ldots \otimes id}_{k-1 \text{ factors}} \otimes A_k \otimes \underbrace{id \otimes \ldots \otimes id}_{d-k \text{ factors}} \in L(\mathbf{V}, \mathbf{V}_{[k]} \otimes_a W_k),$$

provided that it is obvious which component k is meant. By the multiplication rule $\left(\bigotimes_{j=1}^{d} A_j\right) \circ \left(\bigotimes_{j=1}^{d} B_j\right) = \bigotimes_{j=1}^{d} (A_j \circ B_j)$ and since $id \circ A_j = A_j \circ id$, the following identity[2] holds for $j \neq k$:

$$\begin{aligned} & id \otimes \ldots \otimes id \otimes A_j \otimes id \otimes \ldots \otimes id \otimes A_k \otimes id \otimes \ldots \otimes id \\ &= (\mathbf{id}_{[j]} \otimes A_j) \circ (\mathbf{id}_{[k]} \otimes A_k) \\ &= (\mathbf{id}_{[k]} \otimes A_k) \circ (\mathbf{id}_{[j]} \otimes A_j) \end{aligned}$$

(in the first line we assume $j < k$). Proceeding inductively with this argument over all indices, we obtain

$$\mathbf{A} = \bigotimes_{j=1}^{d} A_j = (\mathbf{id}_{[1]} \otimes A_1) \circ \cdots \circ (\mathbf{id}_{[d]} \otimes A_d).$$

If $W_j = \mathbb{R}$, i.e., if $A_j = \varphi_j \in V_j'$ is a linear form, then $\mathbf{id}_{[j]} \otimes \varphi_j \in L(\mathbf{V}, \mathbf{V}_{[j]})$ is used as symbol for $id \otimes \ldots \otimes id \otimes \varphi_j \otimes id \otimes \ldots \otimes id$ defined by

$$(\mathbf{id}_{[j]} \otimes \varphi_j)\left(\bigotimes_{k=1}^{d} v_k\right) = \varphi_j(v_j) \cdot \bigotimes_{k\neq j} v_k.$$

[2]Note that the meaning of $\mathbf{id}_{[j]}$ and $\mathbf{id}_{[k]}$ may differ: in the second line of (3.2), $(\mathbf{id}_{[k]} \otimes A_k) \in L(\mathbf{V}, \mathbf{V}_{[k]} \otimes_a W_k)$ and $(\mathbf{id}_{[j]} \otimes A_j) \in L\left(\mathbf{V}_{[k]} \otimes_a W_k, \mathbf{V}_{[j,k]} \otimes_a W_j \otimes_a W_k\right)$, whereas in the third one $(\mathbf{id}_{[j]} \otimes A_j) \in L(\mathbf{V}, \mathbf{V}_{[j]} \otimes_a W_j)$ and $(\mathbf{id}_{[k]} \otimes A_k) \in L\left(\mathbf{V}_{[j]} \otimes_a W_j, \mathbf{V}_{[j,k]} \otimes_a W_j \otimes_a W_k\right)$. Here $\mathbf{V}_{[j,k]} = {}_a\bigotimes_{l\in D\backslash\{j,k\}} V_l$.

Thus, if $\boldsymbol{\varphi} = \otimes_{j=1}^d \varphi_j \in \bigotimes_{j=1}^d V_j'$, we can also write

$$\boldsymbol{\varphi} = \otimes_{j=1}^d \varphi_j = (\mathbf{id}_{[1]} \otimes \varphi_1) \circ \cdots \circ (\mathbf{id}_{[d]} \otimes \varphi_d).$$

Consider again the splitting of $\mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$ into $\mathbf{V} = V_j \otimes_a \mathbf{V}_{[j]}$ with $\mathbf{V}_{[j]} := {}_a\bigotimes_{k\neq j} V_k$. For a linear form $\boldsymbol{\varphi}_{[j]} \in \mathbf{V}_{[j]}'$, the notation $id_j \otimes \boldsymbol{\varphi}_{[j]} \in L(\mathbf{V}, V_j)$ is used for the mapping

$$(id_j \otimes \boldsymbol{\varphi}_{[j]}) \left(\bigotimes_{k=1}^d v_k \right) = \boldsymbol{\varphi}_{[j]} \left(\bigotimes_{k\neq j} v_k \right) \cdot v_j.$$

If $\boldsymbol{\varphi}_{[j]} = \bigotimes_{k\neq j} \varphi_k \in {}_a\bigotimes_{k\neq j} V_k'$ is an elementary tensor[3], $\boldsymbol{\varphi}_{[j]} \left(\bigotimes_{k\neq j} v^{(k)} \right) = \prod_{k\neq j} \varphi_k \left(v^{(k)} \right)$ holds in (3.2). Finally, we can write (3.2) as

$$\boldsymbol{\varphi} = \otimes_{j=1}^d \varphi_j = \varphi_j \circ (id_j \otimes \boldsymbol{\varphi}_{[j]}) \qquad \text{for } 1 \leq j \leq d.$$

We introduce the abbreviation TBF for 'tree based format'. For instance, a TBF tensor is a tensor represented in the tree based format, etc. The tree based rank will be abbreviated by TB rank. To introduce the underlying tree we use the following example.

Example 3.2 *Let us consider $D = \{1, 2, 3, 4, 5, 6\}$, then*

$$\mathbf{V}_D = {}_a\bigotimes_{j=1}^6 V_j = \left({}_a\bigotimes_{j=1}^3 V_j \right) \otimes_a \left({}_a\bigotimes_{j=4}^5 V_j \right) \otimes_a V_6 = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6.$$

Observe that $\mathbf{V}_D = {}_a\bigotimes_{j=1}^6 V_j$ can be represented by the tree given in Figure 3.1 and $\mathbf{V}_D = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6$ by the tree given in Figure 3.2. We point out that there are other trees to describe the tensor representation $\mathbf{V}_D = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6$, because

$$\mathbf{V}_D = \left({}_a\bigotimes_{j=1}^3 V_j \right) \otimes_a \left({}_a\bigotimes_{j=4}^5 V_j \right) \otimes_a V_6 = \left(V_1 \otimes_a \left({}_a\bigotimes_{j=2}^3 V_j \right) \right) \otimes_a \left({}_a\bigotimes_{j=4}^5 V_j \right) \otimes_a V_6,$$

that is, $\mathbf{V}_{123} = {}_a\bigotimes_{j=1}^3 V_j = V_1 \otimes_a \mathbf{V}_{23}$ (see Figure 3.3).

The above example motivates the following definition.

Definition 3.3 *The tree T_D is called* a dimension partition tree of D *if*

(a) all vertices $\alpha \in T_D$ are non–empty subsets of D,

(b) D is the root of T_D,

(c) every vertex $\alpha \in T_D$ with $\#\alpha \geq 2$ has at least two sons. Moreover, if $S(\alpha) \subset 2^D$ denotes the set of sons of α then $\alpha = \cup_{\beta \in S(\alpha)} \beta$ where $\beta \cap \beta' = \emptyset$ for all $\beta, \beta' \in S(\alpha)$, $\beta \neq \beta'$.

If $S(\alpha) = \emptyset$, α is called a *leaf*. The set of leaves is denoted by $\mathcal{L}(T_D)$. An easy consequence of Definition 3.3 is that the set of leaves $\mathcal{L}(T_D)$ coincides with the singletons of D, i.e., $\mathcal{L}(T_D) = \{\{j\} : j \in D\}$.

Example 3.4 *Consider $D = \{1, 2, 3, 4, 5, 6\}$. Take*

$$T_D = \{D, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\} \text{ and } S(D) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$$

(see Figure 3.1). Then $S(D) = \mathcal{L}(T_D)$.

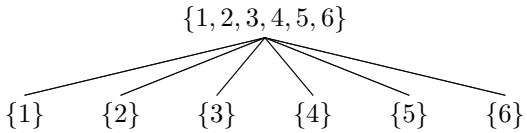

Figure 3.1: A dimension partition tree related to $\mathbf{V}_D = {}_a\bigotimes_{j=1}^{6} V_j$.

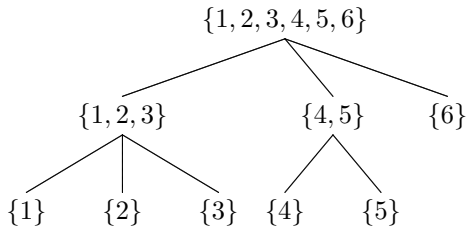

Figure 3.2: A dimension partition tree related with $\mathbf{V}_D = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6$.

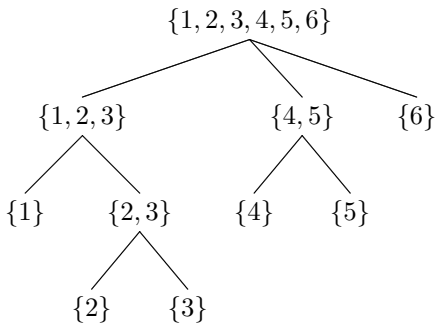

Figure 3.3: A dimension partition tree related with $\mathbf{V}_D = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6$ where $\mathbf{V}_{123} = V_1 \otimes_a \mathbf{V}_{23}$.

Example 3.5 *In Figure 3.2 we have a tree which corresponds to* $\mathbf{V}_D = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6$. *Here* $D = \{1,2,3,4,5,6\}$ *and*

$$T_D = \{D, \{1,2,3\}, \{4,5\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\},$$

$$S(D) = \{\{1,2,3\}, \{4,5\}, \{6\}\},\ S(\{4,5\}) = \{\{4\}, \{5\}\},\ S(\{1,2,3\}) = \{\{1\}, \{2\}, \{3\}\}.$$

Moreover $\mathcal{L}(T_D) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$.

Finally we give the definition of a TBF tensor.

Definition 3.6 *Let* D *be a finite index set and* T_D *be a partition tree. Let* V_j *be a vector space for* $j \in D$, *and consider for each* $\alpha \in T_D \setminus \mathcal{L}(T_D)$ *a tensor space* $\mathbf{V}_\alpha := {}_a\bigotimes_{\beta \in S(\alpha)} \mathbf{V}_\beta$. *Then the collection of vector spaces* $\{\mathbf{V}_\alpha\}_{\alpha \in T_D \setminus \{D\}}$ *is called a representation of the tensor space* $\mathbf{V}_D = {}_a\bigotimes_{\alpha \in S(D)} \mathbf{V}_\alpha$ *in* tree based format.

Observe that we can write $\mathbf{V}_D = {}_a\bigotimes_{\alpha \in S(D)} \mathbf{V}_\alpha = {}_a\bigotimes_{j \in D} V_j$. A first property of TBF tensors is the independence of the representation of the algebraic tensor space $\mathbf{V}_D$ with respect to the tree T_D.

Lemma 3.7 *Let* D *be a finite index set and* T_D *be a partition tree. Let* V_j *be a vector space for* $j \in D$. *Assume that* $\{\mathbf{V}_\alpha\}_{\alpha \in T_D \setminus \{D\}}$ *is a representation of the tensor space* $\mathbf{V}_D = {}_a\bigotimes_{\alpha \in S(D)} \mathbf{V}_\alpha$ *in the tree based format. Then for each* $\alpha_1 \in T_D \setminus \{D\}$ *there exist* $\alpha_2, \ldots, \alpha_m \in T_D \setminus \{D, \alpha_1\}$ *such that* $D = \cup_{i=1}^m \alpha_i$, $\alpha_i \cap \alpha_j = \emptyset$ *and* $\mathbf{V}_D = {}_a\bigotimes_{i=1}^m \mathbf{V}_{\alpha_i}$.

3.2 Minimal subspaces for TBF tensors

Let V_j be a vector space for $j \in D$, where D is a finite index set, and $\alpha_1, \ldots, \alpha_m \subset 2^D \setminus \{D, \emptyset\}$, be such that $\alpha_j \cap \alpha_j = \emptyset$ for all $i \neq j$ and $D = \bigcup_{j=1}^m \alpha_i$. For $\mathbf{v} \in {}_a\bigotimes_{i=1}^m \mathbf{V}_{\alpha_i}$ we define the minimal subspace of $\mathbf{v}$ on each $\mathbf{V}_{\alpha_i} := {}_a\bigotimes_{j \in \alpha_i} V_j$ for $1 \leq i \leq m$, as follows.

Definition 3.8 *For a tensor* $\mathbf{v} \in {}_a\bigotimes_{j \in D} V_j = {}_a\bigotimes_{i=1}^m \mathbf{V}_{\alpha_i}$ *the* minimal subspaces *denoted by* $U_{\alpha_i}^{\min}(\mathbf{v}) \subset \mathbf{V}_{\alpha_i}$, *for* $1 \leq i \leq m$, *are defined by the property that* $\mathbf{v} \in {}_a\bigotimes_{i=1}^m \mathbf{U}_{\alpha_i}$ *implies* $U_{\alpha_i}^{\min}(\mathbf{v}) \subset \mathbf{U}_{\alpha_i}$, *while* $\mathbf{v} \in {}_a\bigotimes_{i=1}^m U_{\alpha_i}^{\min}(\mathbf{v})$.

The minimal subspaces are useful to introduce the following sets of tensor representations based on subspaces. Fix $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{N}^d$. Then we define *the set of Tucker tensors with bounded rank* $\mathbf{r}$ *in* $\mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$ by

$$\mathcal{T}_{\mathbf{r}}(\mathbf{V}) := \left\{\mathbf{v} \in \mathbf{V} : \dim U_j^{\min}(\mathbf{v}) \leq r_j,\ 1 \leq j \leq d\right\},$$

and *the set of Tucker tensors with fixed rank* $\mathbf{r}$ *in* $\mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$ by

$$\mathcal{M}_{\mathbf{r}}(\mathbf{V}) := \left\{\mathbf{v} \in \mathbf{V} : \dim U_j^{\min}(\mathbf{v}) = r_j,\ 1 \leq j \leq d\right\}.$$

Then $\mathcal{M}_{\mathbf{r}}(\mathbf{V}) \subset \mathcal{T}_{\mathbf{r}}(\mathbf{V}) \subset \mathbf{V}$ holds.

The next characterisation of $U_{\alpha_j}^{\min}(\mathbf{v})$ for $1 \leq j \leq m$ is due to [15] (it is included in the proof of Lemma 6.12). Since we assume that $\mathbf{V}_{\alpha_j}$ are vector spaces for $1 \leq j \leq m$, then we may consider the subspaces

$$U_{\alpha_j}^{I}(\mathbf{v}) := \left\{(id_{\alpha_j} \otimes \boldsymbol{\varphi}_{[\alpha_j]})(\mathbf{v}) :\ \boldsymbol{\varphi}_{[\alpha_j]} \in {}_a\bigotimes_{k \neq j} \mathbf{V}'_{\alpha_k}\right\}$$

and

$$U_{\alpha_j}^{II}(\mathbf{v}) := \left\{(id_{\alpha_j} \otimes \boldsymbol{\varphi}_{[\alpha_j]})(\mathbf{v}) :\ \boldsymbol{\varphi}_{[\alpha_j]} \in {}_a\bigotimes_{k \neq j} U_{\alpha_k}^{\min}(\mathbf{v})'\right\},$$

for $1 \leq j \leq m$. Moreover, if $\mathbf{V}_{\alpha_j}$ are normed spaces for $1 \leq j \leq m$ we can also consider

$$U_{\alpha_j}^{III}(\mathbf{v}) := \left\{(id_{\alpha_j} \otimes \boldsymbol{\varphi}_{[\alpha_j]})(\mathbf{v}) :\ \boldsymbol{\varphi}_{[\alpha_j]} \in {}_a\bigotimes_{k \neq j} \mathbf{V}^*_{\alpha_k}\right\},$$

and

$$U_{\alpha_j}^{IV}(\mathbf{v}) := \left\{(id_{\alpha_j} \otimes \boldsymbol{\varphi}_{[\alpha_j]})(\mathbf{v}) :\ \boldsymbol{\varphi}_{[\alpha_j]} \in {}_a\bigotimes_{k \neq j} U_{\alpha_k}^{\min}(\mathbf{v})^*\right\},$$

[3]Recall that an elementary tensor is a tensor of the form $v_1 \otimes \cdots \otimes v_d$.

Theorem 3.9 *Assume that $\mathbf{V}_{\alpha_j}$ are vector spaces for $1 \le j \le m$. Then the following statements hold.*

(a) For any $\mathbf{v} \in \mathbf{V} = {}_a\bigotimes_{j=1}^m \mathbf{V}_{\alpha_j}$, it holds

$$U_{\alpha_j}^{\min}(\mathbf{v}) = U_{\alpha_j}^{I}(\mathbf{v}) = U_{\alpha_j}^{II}(\mathbf{v}),$$

for $1 \le j \le m$.

(b) Assume that $\mathbf{V}_{\alpha_j}$ are normed spaces for $1 \le j \le m$. Then for any $\mathbf{v} \in \mathbf{V} = {}_a\bigotimes_{j=1}^m \mathbf{V}_{\alpha_j}$, it holds

$$U_{\alpha_j}^{\min}(\mathbf{v}) = U_{\alpha_j}^{III}(\mathbf{v}) = U_{\alpha_j}^{IV}(\mathbf{v}),$$

for $1 \le j \le m$.

Let $D = \cup_{i=1}^m \alpha_i$ be a given partition. Assume that $\alpha_1 = \cup_{j=1}^n \beta_j$ is also a given partition, then we have minimal subspaces $U_{\beta_j}^{\min}(\mathbf{v}) \subset \mathbf{V}_{\beta_j} = {}_a\bigotimes_{k\in\beta_j} V_k$ for $1 \le j \le n$ and $U_{\alpha_i}^{\min}(\mathbf{v}) \subset \mathbf{V}_{\alpha_i} = {}_a\bigotimes_{k\in\alpha_i} V_k$ for $1 \le i \le m$. Observe that $\mathbf{V}_{\alpha_1} = {}_a\bigotimes_{j=1}^n \mathbf{V}_{\beta_j}$, where

$$\mathbf{v} \in {}_a\bigotimes_{i=1}^m U_{\alpha_i}^{\min}(\mathbf{v}) \text{ and } \mathbf{v} \in \left({}_a\bigotimes_{j=1}^n U_{\beta_j}^{\min}(\mathbf{v})\right) \otimes_a \left({}_a\bigotimes_{i=2}^m U_{\alpha_i}^{\min}(\mathbf{v})\right).$$

Example 3.10 *Let us consider $D = \{1,2,3,4,5,6\}$ and the partition tree T_D given in Figure 3.2. Take $\mathbf{v} \in {}_a\bigotimes_{j\in D} V_j = \mathbf{V}_{\alpha_1} \otimes_a \mathbf{V}_{\alpha_2} \otimes_a \mathbf{V}_{\alpha_3}$, where $\alpha_1 = \{1,2,3\}$, $\alpha_2 = \{4,5\}$, and $\alpha_3 = \{6\}$. Then we can conclude that there are minimal subspaces $\mathbf{U}_{\alpha_\nu}^{\min}(\mathbf{v})$ for $\nu = 1,2,3$, such that $\mathbf{v} \in {}_a\bigotimes_{\nu=1}^3 \mathbf{U}_{\alpha_\nu}^{\min}(\mathbf{v})$ and also minimal subspaces $U_j^{\min}(\mathbf{v})$ for $j \in D$, such that $\mathbf{v} \in {}_a\bigotimes_{j\in D} U_j^{\min}(\mathbf{v})$*

The relation between $U_j^{\min}(\mathbf{v})$ and $\mathbf{U}_{\alpha_\nu}^{\min}(\mathbf{v})$ is as follows (see Corollary 2.9 of [7]).

Proposition 3.11 *Let V_j be a vector space for $j \in D$, where D is a finite index set, and $D = \cup_{i=1}^m \alpha_i$ be a given partition. Let $\mathbf{v} \in {}_a\bigotimes_{j\in D} V_j$. For a partition $\alpha_1 = \cup_{j=1}^m \beta_j$ it holds*

$$U_{\alpha_1}^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{j=1}^m U_{\beta_j}^{\min}(\mathbf{v}).$$

The following result gives us the relationship between a basis of $U_{\alpha_1}^{\min}(\mathbf{v})$ and a basis of $U_{\beta_j}^{\min}(\mathbf{v})$ for $1 \le j \le m$.

Proposition 3.12 *Let V_j be a vector space for $j \in D$, where D is a finite index set. Let $\alpha \subset D$ such that $\alpha = \bigcup_{i=1}^m \alpha_i$, where $\emptyset \ne \alpha_i$ are pairwise disjoint for $1 \le i \le m$. Let $\mathbf{v} \in {}_a\bigotimes_{j\in D} V_j$. The following statements hold.*

(a) For each $1 \le i \le m$, it holds

$$\begin{aligned} U_{\alpha_i}^{\min}(\mathbf{v}) &= \operatorname{span}\left\{ \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)}\right)(\mathbf{v}_\alpha) : \mathbf{v}_\alpha \in U_\alpha^{\min}(\mathbf{v}) \text{ and } \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)} \in {}_a\bigotimes_{k\ne i} U_{\alpha_k}^{\min}(\mathbf{v})' \right\} \\ &= \operatorname{span}\left\{ \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)}\right)(\mathbf{v}_\alpha) : \mathbf{v}_\alpha \in U_\alpha^{\min}(\mathbf{v}) \text{ and } \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)} \in {}_a\bigotimes_{k\ne i} \mathbf{V}'_{\alpha_k} \right\}. \end{aligned}$$

(b) Assume that $\mathbf{V}_\alpha := {}_a\bigotimes_{i=1}^m \mathbf{V}_{\alpha_i}$ and $\mathbf{V}_{\alpha_i}$, for $1 \le i \le m$, are normed spaces. For each $1 \le i \le m$ it holds

$$\begin{aligned} U_{\alpha_i}^{\min}(\mathbf{v}) &= \operatorname{span}\left\{ \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)}\right)(\mathbf{v}_\alpha) : \mathbf{v}_\alpha \in U_{\alpha_i}^{\min}(\mathbf{v}) \text{ and } \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)} \in {}_a\bigotimes_{k\ne i} U_{\alpha_k}^{\min}(\mathbf{v})^* \right\} \\ &= \operatorname{span}\left\{ \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)}\right)(\mathbf{v}_\alpha) : \mathbf{v}_\alpha \in U_\alpha^{\min}(\mathbf{v}) \text{ and } \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)} \in {}_a\bigotimes_{k\ne i} \mathbf{V}^*_{\alpha_k} \right\} \end{aligned}$$

Proof. Statements (a) and (b) follows in a similar way. Let $\gamma = D \setminus \alpha$ and write $\gamma = \bigcup_{i=1}^{n} \gamma_i$, where $\emptyset \neq \gamma_i \subset D$ are pairwise disjoint for $i = 1, 2, \ldots, n$. In particular, to prove (b), we observe that

$$\mathbf{V}_D = \mathbf{V}_\alpha \otimes_a \mathbf{V}_\gamma = \left({}_a\bigotimes_{i=1}^{m} \mathbf{V}_{\alpha_i} \right) \otimes_a \left({}_a\bigotimes_{j=1}^{n} \mathbf{V}_{\gamma_j} \right).$$

Then, by Theorem 3.9(b), using $U_{\alpha_i}^{IV}(\mathbf{v})$, we have

$$U_\alpha^{\min}(\mathbf{v}) = \left\{ (id_\alpha \otimes \boldsymbol{\varphi}^{(\gamma)})(\mathbf{v}) : \boldsymbol{\varphi}^{(\gamma)} \in {}_a\bigotimes_{j=1}^{m} U_{\gamma_j}^{\min}(\mathbf{v})^* \right\} \text{ and}$$

$$U_{\alpha_i}^{\min}(\mathbf{v}) = \left\{ (id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(D\setminus\alpha_i)})(\mathbf{v}) : \boldsymbol{\varphi}^{(D\setminus\alpha_i)} \in \left({}_a\bigotimes_{k\neq i} U_{\alpha_k}^{\min}(\mathbf{v})^* \right) \otimes_a \left({}_a\bigotimes_{j=1}^{m} U_{\gamma_j}^{\min}(\mathbf{v})^* \right) \right\}$$

for $1 \leq i \leq m$. Take $\mathbf{v}_\alpha \in U_\alpha^{\min}(\mathbf{v})$. Then there exists $\boldsymbol{\varphi}^{(\gamma)} \in {}_a\bigotimes_{j=1}^{m} U_{\gamma_j}^{\min}(\mathbf{v})^*$ such that $\mathbf{v}_\alpha = \left(id_\alpha \otimes \boldsymbol{\varphi}^{(\gamma)}\right)(\mathbf{v})$. Now, for $\boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)} \in {}_a\bigotimes_{k\neq i} U_{\alpha_k}^{\min}(\mathbf{v})^*$, we have

$$\left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)}\right)(\mathbf{v}_\alpha) = \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)} \otimes \boldsymbol{\varphi}^{(D\setminus\alpha)}\right)(\mathbf{v}),$$

and hence $\left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)}\right)(\mathbf{v}_\alpha) \in U_{\alpha_i}^{\min}(\mathbf{v})$. Now, take $\mathbf{v}_{\alpha_i} \in U_{\alpha_i}^{\min}(\mathbf{v})$, then there exists

$$\boldsymbol{\varphi}^{(D\setminus\alpha_i)} \in \left({}_a\bigotimes_{k\neq i} U_{\alpha_k}^{\min}(\mathbf{v})^* \right) \otimes_a \left({}_a\bigotimes_{j=1}^{m} U_{\gamma_j}^{\min}(\mathbf{v})^* \right)$$

such that $\mathbf{v}_{\alpha_i} = \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(D\setminus\alpha_i)}\right)(\mathbf{v})$. Then $\boldsymbol{\varphi}^{(D\setminus\alpha_i)} = \sum_{i=1}^{r} \boldsymbol{\psi}_i^{(\alpha\setminus\alpha_i)} \otimes \boldsymbol{\phi}_i^{(\gamma)}$, where $\boldsymbol{\phi}_i^{(\gamma)} \in {}_a\bigotimes_{j=1}^{m} U_{\gamma_j}^{\min}(\mathbf{v})^*$ and $\boldsymbol{\psi}_i^{(\alpha\setminus\alpha_i)} \in {}_a\bigotimes_{k\neq i} U_{\alpha_k}^{\min}(\mathbf{v})^*$ for $1 \leq i \leq r$. Thus,

$$\begin{aligned}
\mathbf{v}_{\alpha_i} &= \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(D\setminus\alpha_i)}\right)(\mathbf{v}) \\
&= \sum_{i=1}^{r} \left(id_{\alpha_i} \otimes \boldsymbol{\psi}_i^{(\alpha\setminus\alpha_i)} \otimes \boldsymbol{\phi}_i^{(\gamma)}\right)(\mathbf{v}) \\
&= \sum_{i=1}^{r} \left(id_{\alpha_i} \otimes \boldsymbol{\psi}_i^{(\alpha\setminus\alpha_i)}\right)\left((id_\alpha \otimes \boldsymbol{\phi}_i^{(\gamma)})(\mathbf{v})\right).
\end{aligned}$$

Observe that $(id_\alpha \otimes \boldsymbol{\phi}_i^{(\gamma)})(\mathbf{v}) \in U_\alpha^{\min}(\mathbf{v})$. Hence the other inclusion holds and the first equality of second statement is proved. To show the second one, proceed in a similar way by using Theorem 3.9(b) and the definition of $U_{\alpha_j}^{III}(\mathbf{v})$. ∎

From now on, given $\emptyset \neq \alpha \subset D$, we will denote $\mathbf{V}_\alpha := {}_a\bigotimes_{j\in\alpha} V_j$, $r_\alpha := \dim U_\alpha^{\min}(\mathbf{v})$ and $U_D^{\min}(\mathbf{v}) := \operatorname{span}\{\mathbf{v}\}$. Observe that for each $\mathbf{v} \in \mathbf{V}_D$ we have that $(\dim U_\alpha^{\min}(\mathbf{v}))_{\alpha\in 2^D\setminus\{\emptyset\}}$ is in $\mathbb{N}^{2^{\#D}-1}$.

Definition 3.13 *Let D be a finite index set and T_D be a partition tree. Let V_j be a vector space for $j \in D$, Assume that $\{\mathbf{V}_\alpha\}_{\alpha\in T_D\setminus\{D\}}$ is a representation of the tensor space $\mathbf{V}_D = {}_a\bigotimes_{\alpha\in S(D)} \mathbf{V}_\alpha$ in the tree based format. Then for each $\mathbf{v} \in \mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$ we define its* tree based rank (TB rank) *by the tuple* $(\dim U_\alpha^{\min}(\mathbf{v}))_{\alpha\in T_D} \in \mathbb{N}^{\#T_D}$.

In order to characterise the tensors $\mathbf{v} \in \mathbf{V}_D$ satisfying $(\dim \mathbf{U}_\alpha^{\min}(\mathbf{v}))_{\alpha\in T_D} = \mathfrak{r}$, for a fixed $\mathfrak{r} := (r_\alpha)_{\alpha\in T_D} \in \mathbb{N}^{\#T_D}$, we introduce the following definition.

Definition 3.14 *We will say that $\mathfrak{r} := (r_\alpha)_{\alpha\in T_D} \in \mathbb{N}^{\#T_D}$ is an* admissible tuple for T_D, *if there exists $\mathbf{v} \in \mathbf{V}_D \setminus \{\mathbf{0}\}$ such that $\dim U_\alpha^{\min}(\mathbf{v}) = r_\alpha$ for all $\alpha \in T_D \setminus \{D\}$.*

3.3 The representations of tensors of fixed TB rank

Before introducing the representation of a tensor of fixed TB rank we need to define the set of coefficients of that tensors. To this end, we recall the definition of the 'matricisation' (or 'unfolding') of a tensor in a finite-dimensional setting.

Definition 3.15 *For $\alpha \subset 2^D$, and $\beta \subset \alpha$ the map $\mathcal{M}_\beta$ is defined as the isomorphism*

$$\begin{array}{rccc} \mathcal{M}_\beta : & \mathbb{R}^{\times_{\mu\in\alpha} r_\mu} & \to & \mathbb{R}^{(\prod_{\mu\in\beta} r_\mu)\times(\prod_{\delta\in\alpha\setminus\beta} r_\delta)}, \\ & C_{(i_\mu)_{\mu\in\alpha}} & \mapsto & C_{(i_\mu)_{\mu\in\beta},(i_\delta)_{\delta\in\alpha\setminus\beta}} \end{array}$$

It allows to introduce the following definition.

Definition 3.16 *For $\alpha \subset 2^D$, let $C^{(\alpha)} \in \mathbb{R}^{\times_{\mu\in\alpha} r_\mu}$. We say that $C^{(\alpha)} \in \mathbb{R}_*^{\times_{\mu\in\alpha} r_\mu}$ if and only if*

$$\prod_{\mu\in\alpha} \left(\det\left(\mathcal{M}_\mu(C^{(\alpha)})\mathcal{M}_\mu(C^{(\alpha)})^T\right) + \det\left(\mathcal{M}_\mu(C^{(\alpha)})^T\mathcal{M}_\mu(C^{(\alpha)})\right)\right) > 0,$$

where $\mathcal{M}_\mu(C^{(\alpha)}) \in \mathbb{R}^{r_\mu\times(\prod_{\delta\in\alpha\setminus\{\mu\}} r_\delta)}$ for each $\mu \in \alpha$. We point out that this condition is equivalent that all $\mathcal{M}_\mu(C^{(\alpha)})$ have maximal rank.

Since the determinant is a continuous function, $\mathbb{R}_*^{\times_{\mu\in\alpha} r_\mu}$ is an open set in $\mathbb{R}^{\times_{\mu\in\alpha} r_\mu}$, and hence a finite-dimensional manifold. Moreover, the tangent space $\mathbb{T}_{C^{(\alpha)}}\left(\mathbb{R}_*^{\times_{\mu\in\alpha} r_\mu}\right) = \mathbb{R}^{\times_{\mu\in\alpha} r_\mu}$ for all $C^{(\alpha)} \in \mathbb{R}_*^{\times_{\mu\in\alpha} r_\mu}$ (cf. Definition 2.15).

Definition 3.17 *Let T_D be a given dimension partition tree and fix some tuple $\mathfrak{r} \in \mathbb{N}^{T_D}$ for T_D. The* set of TBF tensors of bounded TB rank $\mathfrak{r}$ *is defined by*

$$\mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D) := \left\{\mathbf{v} \in \mathbf{V}_D : \ \dim U_\alpha^{\min}(\mathbf{v}) \le r_\alpha \text{ for all } \alpha \in T_D \ \right\}, \tag{3.3}$$

and the set of TBF tensors of fixed TB rank $\mathfrak{r}$ *is defined by*

$$\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) := \left\{\mathbf{v} \in \mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D) : \dim U_\alpha^{\min}(\mathbf{v}) = r_\alpha \text{ for all } \alpha \in T_D\right\}. \tag{3.4}$$

Note that $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) = \emptyset$ for an inadmissible tuple $\mathfrak{r}$. For $\mathfrak{r}, \mathfrak{s} \in \mathbb{N}^{T_D}$ we write $\mathfrak{s} \le \mathfrak{r}$ if and only if $s_\alpha \le r_\alpha$ for all $\alpha \in T_D$. Then we have

$$\mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D) = \{\mathbf{0}\} \cup \bigcup_{\mathfrak{s}\le\mathfrak{r}} \mathcal{FT}_{\mathfrak{s}}(\mathbf{V}_D).$$

Next we give some useful examples.

Example 3.18 (Tucker format) *Consider the partition tree over $D := \{1, \dots, d\}$, where $S(D) = \mathcal{L}(T_D) = \{\{j\} : 1 \le j \le d\}$. Let $(r_D, r_1, \dots, r_d)$ be admissible, then $r_D = 1$ and $r_j \le \dim V_j$ for $1 \le j \le d$. Thus we can write*

$$\mathcal{BT}_{(1,r_1,\dots,r_d)}(\mathbf{V}_D) = \mathcal{T}_{(r_1,\dots,r_d)}(\mathbf{V}_D)$$

and

$$\mathcal{FT}_{(1,r_1,\dots,r_d)}(\mathbf{V}_D) = \mathcal{M}_{(r_1,\dots,r_d)}(\mathbf{V}_D).$$

Example 3.19 (Tensor Train format) *Consider a binary partition tree over $D := \{1, \dots, d\}$ given by*

$$T_D = \{D, \{\{j\} : 1 \le j \le d\}, \{\{j+1, \dots, d\} : 1 \le j \le d-2\}\}.$$

In particular, $S(\{j, \dots, d\}) = \{\{j\}, \{j+1, \dots, d\}\}$ for $1 \le j \le d-1$. This tensor based format is related to the following chain of inclusions:

$$\mathbf{U}_D^{\min}(\mathbf{v}) \subset \mathbf{U}_1^{\min}(\mathbf{v}) \otimes_a \mathbf{U}_{2\cdots d}^{\min}(\mathbf{v}) \subset \mathbf{U}_1^{\min}(\mathbf{v}) \otimes_a \mathbf{U}_2^{\min}(\mathbf{v}) \otimes_a \mathbf{U}_{3\cdots d}^{\min}(\mathbf{v}) \subset \cdots \subset {}_a\bigotimes_{j\in D} \mathbf{U}_j^{\min}(\mathbf{v}) \ .$$

Let V_j be vector spaces for $j \in D$ and T_D be a tree. Let $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. Then $\dim U_\alpha^{\min}(\mathbf{v}) = r_\alpha$, for each $\alpha \in T_D \setminus \{D\}$. Since $\mathbf{v} \in {}_a\bigotimes_{\alpha\in S(D)} U_\alpha^{\min}(\mathbf{v})$, there exists $C^{(D)} \in \mathbb{R}^{\bigtimes_{\alpha\in S(D)} r_\alpha}$ such that

$$\mathbf{v} = \sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}^{(\alpha)}_{i_\alpha}, \tag{3.5}$$

where $\{\mathbf{u}^{(\alpha)}_{i_\alpha} : 1 \le i_\alpha \le r_\alpha\}$ is a basis of $U_\alpha^{\min}(\mathbf{v})$. For each $\alpha \in S(D)$ we set

$$\mathbf{U}^{(\alpha)}_{i_\alpha} := \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(D)\\ \beta\neq\alpha}} C^{(D)}_{(i_\beta)_{\beta\in S(D)}} \bigotimes_{\substack{\beta\in S(D)\\ \beta\neq\alpha}} u^{(\beta)}_{i_\beta}, \tag{3.6}$$

then (3.5) can be written as

$$\mathbf{v} = \sum_{1\le i_\alpha\le r_\alpha} \mathbf{u}^{(\alpha)}_{i_\alpha} \otimes \mathbf{U}^{(\alpha)}_{i_\alpha}. \tag{3.7}$$

Let

$$U^{\min}_{S(D)\setminus\{\alpha\}}(\mathbf{v}) := \{(\mathbf{id}_{[\alpha]} \otimes \varphi_\alpha)(\mathbf{v}) : \varphi_\alpha \in U_\alpha^{\min}(\mathbf{v})^*\}.$$

We claim that $\{\mathbf{U}^{(\alpha)}_{i_\alpha} : 1 \le i_\alpha \le r_\alpha\}$ are linearly independent. To prove the claim assume that $\mathbf{U}^{(\alpha)}_1$ is a linear combination of $\{\mathbf{U}^{(\alpha)}_{i_\alpha} : 2 \le i_\alpha \le r_\alpha\}$, then $\mathbf{U}^{(\alpha)}_1 = \sum_{2\le i_\alpha\le r_\alpha} \lambda_{i_\alpha} \mathbf{U}^{(\alpha)}_{i_\alpha}$ where $\lambda_{i_\alpha} \neq 0$ for some $2 \le i_\alpha \le r_\alpha$. Thus,

$$\mathbf{v} = \sum_{2\le i_\alpha\le r_\alpha} (\mathbf{u}^{(\alpha)}_{i_\alpha} + \lambda_{i_\alpha}\mathbf{u}^{(\alpha)}_1) \otimes \mathbf{U}^{(\alpha)}_{i_\alpha},$$

since $\{\mathbf{u}^{(\alpha)}_{i_\alpha} + \lambda_{i_\alpha}\mathbf{u}^{(\alpha)}_1 : 2 \le i_\alpha \le r_\alpha\}$ are linearly independent we have $\dim U_\alpha^{\min}(\mathbf{v}) < r_\alpha$, a contradiction. Since $\{\mathbf{U}^{(\alpha)}_{i_\alpha} : 1 \le i_\alpha \le r_\alpha\}$ are linearly independent for each $\alpha \in S(D)$, from (3.7) we have that

$$U^{\min}_{S(D)\setminus\{\alpha\}}(\mathbf{v}) = \operatorname{span}\{\mathbf{U}^{(\alpha)}_{i_\alpha} : 1 \le i_\alpha \le r_\alpha\},$$

and from (3.6), we deduce that $\mathcal{M}_\alpha(C^{(D)})$ maps a basis into another one for each $\alpha \in S(D)$ and hence $C^{(D)} \in \mathbb{R}_*^{\bigtimes_{\beta\in S(D)} r_\beta}$. We remark, that if $S(D) = \mathcal{L}(T_D)$, then (3.5) gives us the classical Tucker representation.

Now, assume $S(D) \neq \mathcal{L}(T_D)$. Then, for each $\mu \in T_D\setminus\{D\}$ such that $S(\mu) \neq \emptyset$, thanks to Proposition 3.11, we have

$$U_\mu^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{\beta\in S(\mu)} U_\beta^{\min}(\mathbf{v}) .$$

Consider $\{\mathbf{u}^{(\mu)}_{i_\mu} : 1 \le i_\mu \le r_\mu\}$ a basis of $U_\mu^{\min}(\mathbf{v})$ and $\{\mathbf{u}^{(\beta)}_{i_\beta} : 1 \le i_\beta \le r_\beta\}$ a basis of $U_\beta^{\min}(\mathbf{v})$ for $\beta \in S(\mu)$ and $1 \le i_\mu \le r_\mu$. Then, there exists $C^{(\mu)} \in \mathbb{R}^{r_\mu\times(\bigtimes_{\beta\in S(\alpha)} r_\beta)}$ such that

$$\mathbf{u}^{(\mu)}_{i_\mu} = \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(\mu)}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta\in S(\mu)}} \bigotimes_{\beta\in S(\mu)} \mathbf{u}^{(\beta)}_{i_\beta}. \tag{3.8}$$

for $1 \le i_\mu \le r_\mu$. Since $\{\mathbf{u}^{(\mu)}_{i_\mu} : 1 \le i_\mu \le r_\mu\}$ is a basis, we can identify $C^{(\mu)}$ with the matrix $\mathcal{M}_\mu(C^{(\mu)})$, in the non-compact Stiefel manifold $\mathbb{R}_*^{r_\mu\times(\prod_{\beta\in S(\mu)} r_\beta)}$, which is the set of matrices in $\mathbb{R}^{r_\mu\times(\prod_{\beta\in S(\alpha)} r_\beta)}$ whose rows are linearly independent (see 3.1.5 in [1]). In a similar way as in the root case, for each fixed $1 \le i_\mu \le r_\mu$ and $\beta \in S(\mu)$, we introduce

$$\mathbf{U}^{(\beta)}_{i_\mu,i_\beta} := \sum_{\substack{1\le i_\delta\le r_\delta\\ \delta\in S(\mu)\\ \delta\neq\beta}} C^{(\mu)}_{i_\mu,(i_\delta)_{\delta\in S(\mu)}} \bigotimes_{\substack{\delta\in S(\mu)\\ \delta\neq\beta}} \mathbf{u}^{(\delta)}_{i_\delta}, \tag{3.9}$$

where $1 \leq i_\beta \leq r_\beta$. Hence, we can write (3.8) as

$$\mathbf{u}_{i_\mu}^{(\mu)} = \sum_{1 \leq i_\beta \leq r_\beta} \mathbf{u}_{i_\beta}^{(\beta)} \otimes \mathbf{U}_{i_\mu, i_\beta}^{(\beta)},$$

where $1 \leq i_\mu \leq r_\mu$ and $\beta \in S(\mu)$. From Proposition 3.12(a), we have

$$\begin{aligned} U_\beta^{\min}(\mathbf{v}) &= \operatorname{span}\left\{ (id_\beta \otimes \varphi^{(\mu \setminus \beta)})(\mathbf{u}_{i_\mu}^{(\mu)}) : 1 \leq i_\mu \leq r_\mu \text{ and } \varphi^{(\mu \setminus \beta)} \in {}_a \bigotimes_{\delta \in S(\mu) \setminus \{\beta\}} U_\delta^{\min}(\mathbf{v})' \right\} \\ &= \operatorname{span}\left\{ (id_\beta \otimes \varphi^{(\mu \setminus \beta)})(\mathbf{u}_{i_\mu}^{(\mu)}) : 1 \leq i_\mu \leq r_\mu \text{ and } \varphi^{(\mu \setminus \beta)} \in {}_a \bigotimes_{\delta \in S(\mu) \setminus \{\beta\}} \mathbf{V}_\delta' \right\}, \end{aligned}$$

and hence $U_\beta^{\min}(\mathbf{u}_{i_\mu}^{(\mu)}) \subset U_\beta^{\min}(\mathbf{v})$ for $1 \leq i_\mu \leq r_\mu$. Let us consider $\{\varphi_{i_\beta}^{(\beta)} : 1 \leq i_\beta \leq r_\beta\} \subset U_\beta^{\min}(\mathbf{v})'$ a dual basis of the finite dimensional space $\{\mathbf{u}_{i_\beta}^{(\beta)} : 1 \leq i_\beta \leq r_\beta\}$, that is, $\varphi_{i_\beta}^{(\beta)}(\mathbf{u}_{j_\beta}^{(\beta)}) = \delta_{i_\beta, j_\beta}$ for all $1 \leq i_\beta, j_\beta \leq r_\beta$, and $\beta \in S(\mu)$. Thus, we have

$$\left(id_\beta \otimes \bigotimes_{\substack{\delta \in S(\mu) \\ \delta \neq \beta}} \varphi_{j_\delta}^{(\delta)} \right) (\mathbf{u}_{i_\mu}^{(\mu)}) = \sum_{1 \leq j_\beta \leq r_\beta} C_{i_\mu, (j_\delta)_{\delta \in S(\mu)}}^{(\mu)} \mathbf{u}_{j_\beta}^{(\beta)} \in U_\beta^{\min}(\mathbf{v})$$

for each multi-index $(j_\delta)_{\delta \in S(\mu) \setminus \beta} \in \bigtimes_{\substack{\delta \in S(\mu) \\ \delta \neq \beta}} \{1, \ldots, r_\delta\}$. Then, for $\beta \in S(\mu)$,

$$U_\beta^{\min}(\mathbf{v}) = \operatorname{span}\left\{ \left(id_\beta \otimes \bigotimes_{\substack{\delta \in S(\mu) \\ \delta \neq \beta}} \varphi_{j_\delta}^{(\delta)} \right) (\mathbf{u}_{i_\mu}^{(\mu)}) : (j_\delta)_{\delta \in S(\mu) \setminus \beta} \in \bigtimes_{\substack{\delta \in S(\mu) \\ \delta \neq \beta}} \{1, \ldots, r_\delta\},\ 1 \leq i_\mu \leq r_\mu \right\}$$

with $\dim U_\beta^{\min}(\mathbf{v}) = r_\beta$ if and only if $\operatorname{rank} \mathcal{M}_\beta(C^{(\mu)}) = r_\beta$ for $\beta \in S(\mu)$. Finally, we have $C^{(\mu)} \in \mathbb{R}_*^{r_\mu \times (\bigtimes_{\delta \in S(\mu)} r_\delta)}$ for all $\mu \in T_D \setminus \mathcal{L}(T_D)$. In a similar way, by using $id_{S(\mu) \setminus \beta} \otimes \varphi_{j_\beta}^{(\beta)}$ for $1 \leq j_\beta \leq r_\beta$, over $\mathbf{u}_{i_\mu}^{(\mu)}$ it can be proved that

$$U_{S(\mu) \setminus \{\beta\}}^{\min}(\mathbf{u}_{i_\mu}^{(\mu)}) = \operatorname{span}\left\{ \mathbf{U}_{i_\mu, i_\beta}^{(\beta)} : 1 \leq i_\beta \leq r_\beta \right\}$$

for $1 \leq i_\mu \leq r_\mu$ and also

$$U_{S(\mu) \setminus \{\beta\}}^{\min}(\mathbf{v}) = \operatorname{span}\left\{ \mathbf{U}_{i_\mu, i_\beta}^{(\beta)} : 1 \leq i_\beta \leq r_\beta,\ 1 \leq i_\mu \leq r_\mu. \right\}.$$

Now, we claim that $\left\{ \mathbf{U}_{i_\mu, i_\beta}^{(\beta)} : 1 \leq i_\beta \leq r_\beta \right\}$ are linearly independent in ${}_a \bigotimes_{\delta \neq \beta} \mathbf{V}_\delta$ for $1 \leq i_\mu \leq r_\mu$ and $\beta \in S(\mu)$. Otherwise, there exist λ_{i_β} for $1 \leq i_\beta \leq r_\beta$ not all identically zero such that $\sum_{1 \leq i_\beta \leq r_\beta} \lambda_{i_\beta} \mathbf{U}_{i_\mu, i_\beta}^{(\beta)} = \mathbf{0}$. Take $\mathbf{w}_\beta \in \mathbf{V}_\beta \setminus \{\mathbf{0}\}$ and then

$$\mathbf{w}_\beta \otimes \left(\sum_{1 \leq i_\beta \leq r_\beta} \lambda_{i_\beta} \mathbf{U}_{i_\mu, i_\beta}^{(\beta)} \right) = \sum_{1 \leq i_\beta \leq r_\beta} \lambda_{i_\beta} \mathbf{w}_\beta \otimes \mathbf{U}_{i_\mu, i_\beta}^{(\beta)} = \mathbf{0}.$$

Observe that

$$\sum_{1 \leq i_\beta \leq r_\beta} \left(\lambda_{i_\beta} \mathbf{w}_\beta \otimes \mathbf{U}_{i_\mu, i_\beta}^{(\beta)} \right) = \sum_{\substack{1 \leq i_\delta \leq r_\delta \\ \delta \in S(\mu)}} C_{i_\mu, (i_\delta)_{\delta \in S(\mu)}}^{(\mu)} \lambda_{i_\beta} \mathbf{w}_\beta \otimes \left(\bigotimes_{\substack{\delta \neq \beta \\ \delta \in S(\mu)}} \mathbf{u}_{i_\delta}^{(\delta)} \right) = \mathbf{0},$$

for $1 \le i_\mu \le r_\mu$ and $\beta \in S(\mu)$, take a dual basis of $\{\varphi^{(\delta)}_{i_\delta} : 1 \le i_\delta \le r_\delta\} \subset \mathbf{V}^*_\delta$ of $\{\mathbf{u}^{(\delta)}_{i_\delta} : 1 \le i_\delta \le r_\delta\} \subset \mathbf{V}_\delta$ where $\varphi^{(\delta)}_{i_\delta}(\mathbf{u}^{(\delta)}_{j_\delta}) = \delta_{i_\delta, j_\delta}$ for all $1 \le i_\delta, j_\delta \le r_\delta$. Then we obtain

$$id_\beta \otimes \left(\bigotimes_{\delta \in S(\mu)\setminus\{\beta\}} \varphi^{(\delta)}_{i_\delta} \right) \left(\sum_{1 \le i_\beta \le r_\beta} \left(\lambda_{i_\beta} \mathbf{w}_\beta \otimes \mathbf{U}^{(\beta)}_{i_\mu, i_\beta} \right) \right) = \sum_{1 \le i_\beta \le r_\beta} C^{(\mu)}_{i_\mu, (i_\delta)_{\delta \in S(\mu)}} \lambda_{i_\beta} \mathbf{w}_\beta = \mathbf{0},$$

that is, $\mathcal{M}_\beta(C^{(\mu)})^T \mathbf{z}_\beta = \mathbf{0}$, where $\mathbf{z}_\beta := (\lambda_{i_\beta} \mathbf{w}_\beta)^{r_\beta}_{i_\beta=1}$. Since $\operatorname{rank} \mathcal{M}_\beta(C^{(\mu)}) = r_\beta$, then $\dim \operatorname{Ker} \mathcal{M}_\beta(C^{(\mu)})^T = 0$, and hence $\mathbf{z}_\beta = (\lambda_{i_\beta} \mathbf{w}_\beta)^{r_\beta}_{i_\beta=1} = (\mathbf{0})^{r_\beta}_{i_\beta=1}$ for $\beta \in S(\gamma)$, a contradiction. In consequence,

$$\dim U^{\min}_{S(\mu)\setminus\{\beta\}}(\mathbf{u}^{(\mu)}_{i_\mu}) = \dim U^{\min}_\beta(\mathbf{u}^{(\mu)}_{i_\mu}) = r_\beta$$

for $1 \le i_\mu \le r_\mu$ and $\beta \in S(\mu)$. Hence $U^{\min}_\beta(\mathbf{v}) = U^{\min}_\beta(\mathbf{u}^{(\mu)}_{i_\mu})$ holds for $1 \le i_\mu \le r_\mu$ and $\beta \in S(\mu)$.

From (3.5) and (3.8) we obtain the Tucker representation of $\mathbf{v}$, when $S(D) \neq \mathcal{L}(T_D)$, as

$$\mathbf{v} = \sum_{\substack{1 \le i_k \le r_k \\ k \in \mathcal{L}(T_D)}} \left(\sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in T_D \setminus \{D\} \\ \alpha \notin \mathcal{L}(T_D)}} C^{(D)}_{(i_\alpha)_{\alpha \in S(D)}} \prod_{\substack{\mu \in T_D \setminus \{D\} \\ S(\mu) \neq \emptyset}} C^{(\mu)}_{i_\mu, (i_\beta)_{\beta \in S(\mu)}} \right) \bigotimes_{k \in \mathcal{L}(T_D)} u^{(k)}_{i_k}. \tag{3.10}$$

Moreover, necessary conditions for $\mathfrak{r} \in \mathbb{N}^{\# T_D}$ to be admissible are

$$\begin{array}{ll} r_D = 1, & \\ r_{\{j\}} \le \dim V_j & \text{for } \{j\} \in \mathcal{L}(T_D), \\ r_\alpha \le \prod_{\beta \in S(\alpha)} r_\beta & \text{for } \alpha \in T_D \setminus \mathcal{L}(T_D), \\ r_\delta \le r_\alpha \prod_{\beta \in S(\alpha)\setminus\{\delta\}} r_\beta & \text{for } \alpha \in T_D \setminus \mathcal{L}(T_D) \text{ and } \delta \in S(\alpha). \end{array}$$

Example 3.20 *Let us consider $D = \{1, 2, 3, 4, 5, 6\}$, then*

$$\mathbf{V}_D = {}_a\bigotimes_{j=1}^{6} V_j = \left({}_a\bigotimes_{j=1}^{3} V_j \right) \otimes_a \left({}_a\bigotimes_{j=4}^{5} V_j \right) \otimes_a V_6 = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6.$$

It is well known (see [7]) that $\mathbf{v} \in {}_a\bigotimes_{j=1}^{6} U^{\min}_j(\mathbf{v})$ and $\mathbf{v} \in U^{\min}_{123}(\mathbf{v}) \otimes_a U^{\min}_{45}(\mathbf{v}) \otimes_a U^{\min}_6(\mathbf{v})$. From Proposition 3.12 we have

$$U^{\min}_D(\mathbf{v}) \subset U^{\min}_{123}(\mathbf{v}) \otimes_a U^{\min}_{45}(\mathbf{v}) \otimes_a U^{\min}_6(\mathbf{v}) \subset {}_a\bigotimes_{j=1}^{6} U^{\min}_j(\mathbf{v}) \ .$$

Moreover, we can write

$$\mathbf{v} = \sum_{i_{123}=1}^{r_{123}} \sum_{i_{45}=1}^{r_{45}} \sum_{i_6=1}^{r_6} C^{(D)}_{i_{123}, i_{45}, i_6} \, \mathbf{u}^{(123)}_{i_{123}} \otimes \mathbf{u}^{(45)}_{i_{45}} \otimes u^{(6)}_{i_6}, \quad C^{(D)} \in \mathbb{R}^{r_{123} \times r_{45} \times r_6}_*$$

where

$$\mathbf{u}^{(123)}_{i_{123}} = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \sum_{i_3=1}^{r_3} C^{(123)}_{i_{123}, i_1, i_2, i_3} \, u^{(1)}_{i_1} \otimes u^{(2)}_{i_2} \otimes u^{(3)}_{i_3}, \quad C^{(123)} \in \mathbb{R}^{r_{123} \times r_1 \times r_2 \times r_3}_*$$

and

$$\mathbf{u}^{(45)}_{i_{45}} = \sum_{i_4=1}^{r_4} \sum_{i_5=1}^{r_5} C^{(45)}_{i_{45}, i_4, i_5} \, u^{(4)}_{i_4} \otimes u^{(5)}_{i_5}, \quad C^{(45)} \in \mathbb{R}^{r_{45} \times r_4 \times r_5}_*.$$

Finally

$$\mathbf{v} = \sum_{i_1=1}^{r_1} \cdots \sum_{i_6=1}^{r_6} \underbrace{\left(\sum_{i_{123}=1}^{r_{123}} \sum_{i_{45}=1}^{r_{45}} C^{(D)}_{i_{123},i_{45},i_6} C^{(123)}_{i_{123},i_1,i_2,i_3} C^{(45)}_{i_{45},i_4,i_5} \right)}_{v_{i_1,\ldots,i_6}} \bigotimes_{k=1}^{6} u^{(k)}_{i_k},$$

where $u^{(k)}_{i_k} \in U^{\min}_k(\mathbf{v})$ *for* $1 \le k \le 6$.

The procedure, given a basis of $U^{\min}_\alpha(\mathbf{v})$ for $\alpha \in T_D \setminus \{D\}$, used to obtain (3.10), is completely characterised by a finite tuple of tensors

$$\mathfrak{C}(\mathbf{v}) := (C^{(\alpha)})_{\alpha \in T_D \setminus \mathcal{L}(T_D)} \in \bigtimes_{\alpha \in T_D \setminus \mathcal{L}(T_D)} \mathbb{R}^{r_\alpha \times \left(\bigtimes_{\beta \in S(\alpha)} r_\beta\right)},$$

where $C^{(D)} \in \mathbb{R}_*^{\bigtimes_{\alpha \in S(D)} r_\alpha}$ and $C^{(\mu)} \in \mathbb{R}_*^{r_\mu \times \left(\bigtimes_{\beta \in S(\mu)} r_\beta\right)}$, for each $\mu \in T_D \setminus \{D\}$ such that $S(\mu) \neq \emptyset$. From now on, to simplify the notation, we introduce for an admissible $\mathfrak{r} \in \mathbb{N}^{T_D}$ the product vector space

$$\mathbb{R}^{\mathfrak{r}} := \bigtimes_{\alpha \in T_D \setminus \mathcal{L}(T_D)} \mathbb{R}^{r_\alpha \times \left(\bigtimes_{\beta \in S(\alpha)} r_\beta\right)},$$

with $r_D = 1$. It allows us to introduce its open subset $\mathbb{R}^{\mathfrak{r}}_*$, and hence a manifold, defined as

$$\mathbb{R}^{\mathfrak{r}}_* := \left\{ \mathfrak{C} \in \mathbb{R}^{\mathfrak{r}} : \begin{array}{l} C^{(D)} \in \mathbb{R}_*^{\bigtimes_{\alpha \in S(D)} r_\alpha} \text{and } C^{(\mu)} \in \mathbb{R}_*^{r_\mu \times \left(\bigtimes_{\beta \in S(\mu)} r_\beta\right)} \\ \text{for each } \mu \in T_D \setminus \{D\} \text{ such that } S(\mu) \neq \emptyset. \end{array} \right\}.$$

Before characterising the "local coordinates" of a tensor $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ we need to introduce topological TBF tensors.

3.4 Topological TBF tensors

First, we recall the definition of some topological tensor spaces and we will give some examples.

Definition 3.21 *We say that* $\mathbf{V}_{\|\cdot\|}$ *is a* Banach tensor space *if there exists an algebraic tensor space* $\mathbf{V}$ *and a norm* $\|\cdot\|$ *on* $\mathbf{V}$ *such that* $\mathbf{V}_{\|\cdot\|}$ *is the completion of* $\mathbf{V}$ *with respect to the norm* $\|\cdot\|$*, i.e.,*

$$\mathbf{V}_{\|\cdot\|} := {}_{\|\cdot\|}\bigotimes_{j=1}^{d} V_j = \overline{{}_a\bigotimes_{j=1}^{d} V_j}^{\|\cdot\|}.$$

If $\mathbf{V}_{\|\cdot\|}$ *is a Hilbert space, we say that* $\mathbf{V}_{\|\cdot\|}$ *is a* Hilbert tensor space.

Next, we give some examples of Banach and Hilbert tensor spaces.

Example 3.22 *For* $I_j \subset \mathbb{R}$ $(1 \le j \le d)$ *and* $1 \le p < \infty$*, the Sobolev space* $H^{N,p}(I_j)$ *consists of all univariate functions* f *from* $L^p(I_j)$ *with bounded norm*[4]

$$\|f\|_{N,p;I_j} := \left(\sum_{n=0}^{N} \int_{I_j} |\partial^n f|^p \, \mathrm{d}x \right)^{1/p},$$

whereas the space $H^{N,p}(\mathbf{I})$ *of* d*-variate functions on* $\mathbf{I} = I_1 \times I_2 \times \ldots \times I_d \subset \mathbb{R}^d$ *is endowed with the norm*

$$\|f\|_{N,p} := \Big(\sum_{0 \le |\mathbf{n}| \le N} \int_{\mathbf{I}} |\partial^{\mathbf{n}} f|^p \, \mathrm{d}\mathbf{x} \Big)^{1/p}$$

[4]It suffices to have in (3.22) the terms $n = 0$ and $n = N$. The derivatives are to be understood as weak derivatives.

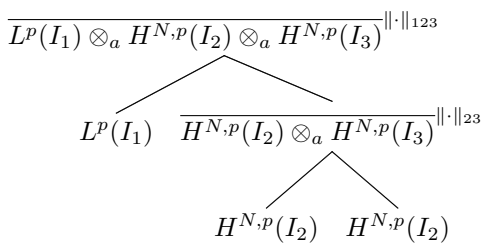

Figure 3.4: A representation in the topological tree based format for the tensor Banach space $\overline{L^p(I_1) \otimes_a H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{123}}$. Here $\|\cdot\|_{23}$ and $\|\cdot\|_{123}$ are given norms.

with $\mathbf{n} \in \mathbb{N}_0^d$ *being a multi-index of length* $|\mathbf{n}| := \sum_{j=1}^d n_j$. *For* $p > 1$ *it is well known that* $H^{N,p}(I_j)$ *and* $H^{N,p}(\mathbf{I})$ *are reflexive and separable Banach spaces. Moreover, for* $p = 2$, *the Sobolev spaces* $H^N(I_j) := H^{N,2}(I_j)$ *and* $H^N(\mathbf{I}) := H^{N,2}(\mathbf{I})$ *are Hilbert spaces. As a first example,*

$$H^{N,p}(\mathbf{I}) = {}_{\|\cdot\|_{N,p}}\bigotimes_{j=1}^d H^{N,p}(I_j)$$

is a Banach tensor space. Examples of Hilbert tensor spaces are

$$L^2(\mathbf{I}) = {}_{\|\cdot\|_{0,2}}\bigotimes_{j=1}^d L^2(I_j) \quad \text{and} \quad H^N(\mathbf{I}) = {}_{\|\cdot\|_{N,2}}\bigotimes_{j=1}^d H^N(I_j) \ \text{ for } N \in \mathbb{N}.$$

In the definition of a tensor Banach space ${}_{\|\cdot\|}\bigotimes_{j\in D} V_j$ we have not fixed, whether V_j, for $j \in D$, are complete or not. This leads us to introduce the following definition.

Definition 3.23 *Let* D *be a finite index set and* T_D *be a dimension partition tree. Let* $(V_j, \|\cdot\|_j)$ *be a normed space such that* $V_{j_{\|\cdot\|_j}}$ *is a Banach space obtained as the completion of* V_j, *for* $j \in D$, *and consider a representation* $\{\mathbf{V}_\alpha\}_{\alpha\in T_D\setminus\{D\}}$ *of the tensor space* $\mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$ *where for each* $\alpha \in T_D \setminus \mathcal{L}(T_D)$ *we have a tensor space* $\mathbf{V}_\alpha = {}_a\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_\beta$. *If for each* $\alpha \in T_D \setminus \mathcal{L}(T_D)$ *there exists a norm* $\|\cdot\|_\alpha$ *defined on* $\mathbf{V}_\alpha$ *such that* $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} V_\beta$ *is a tensor Banach space, we say that* $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ *is a representation of the tensor Banach space* $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$ *in the* topological tree based format.

Since $\mathbf{V}_\alpha = {}_a\bigotimes_{j\in\alpha} V_j$,

$$\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\alpha\in S(D)} \mathbf{V}_\alpha = {}_{\|\cdot\|_\alpha}\bigotimes_{j\in\alpha} V_j$$

holds for all $\alpha \in T_D \setminus \mathcal{L}(T_D)$.

Example 3.24 *Figure 3.4 gives an example of a representation in the topological tree-based format for an anisotropic Sobolev space.*

Remark 3.25 *Observe that a tree as given in Figure 3.5 is not included in the definition of the topological tree based format. Moreover, for a tensor* $\mathbf{v} \in L^p(I_1) \otimes_a (H^{N,p}(I_2) \otimes_{\|\cdot\|_{23}} H^{N,p}(I_3))$, *we have* $U_{23}^{\min}(\mathbf{v}) \subset H^{N,p}(I_2) \otimes_{\|\cdot\|_{23}} H^{N,p}(I_3)$. *However, in the topological tree based representation of Figure 3.4, for a given* $\mathbf{v} \in L^p(I_1) \otimes_a H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)$ *we have* $U_{23}^{\min}(\mathbf{v}) \subset H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)$, *and hence* $U_{23}^{\min}(\mathbf{v}) \subset U_2^{\min}(\mathbf{v}) \otimes_a U_3^{\min}(\mathbf{v})$.

The difference between the tensor spaces involved in Figure 3.4 and Figure 3.5 is the following. For all $\beta \in T_D \setminus \mathcal{L}(T_D)$, if $\|\cdot\|_\beta$ is also a norm on the tensor space ${}_a\bigotimes_{\eta\in S(\beta)} \mathbf{V}_{\eta_{\|\cdot\|_\eta}}$, we have

$${}_{\|\cdot\|_\beta}\bigotimes_{\eta\in S(\beta)} \mathbf{V}_{\eta_{\|\cdot\|_\eta}} \supset \mathbf{V}_{\beta_{\|\cdot\|_\beta}} = {}_{\|\cdot\|_\beta}\bigotimes_{\eta\in S(\beta)} \mathbf{V}_\eta = {}_{\|\cdot\|_\beta}\bigotimes_{j\in\beta} V_j \ .$$

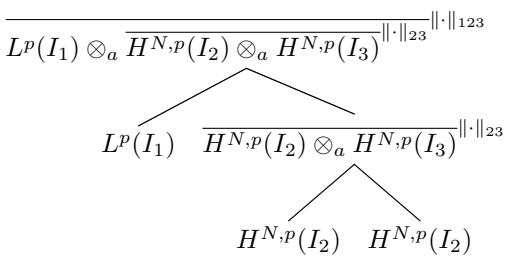

Figure 3.5: A representation for the tensor Banach space $\overline{L^p(I_1) \otimes_a \overline{H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{23}}}^{\|\cdot\|_{123}}$, using a tree. Here $\|\cdot\|_{23}$ and $\|\cdot\|_{123}$ are given norms.

A desirable property for the tensor product is that if $\|\cdot\|_\alpha$ is also a norm on the tensor space ${}_a\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_{\beta_{\|\cdot\|_\beta}}$, then

$$ {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_{\beta_{\|\cdot\|_\beta}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_\beta = {}_{\|\cdot\|_\alpha}\bigotimes_{j\in\alpha} V_j \tag{3.11} $$

must be true for all $\alpha \in T_D \setminus \mathcal{L}(T_D)$. To precise these ideas, we introduce the following definitions and results.

Let $\|\cdot\|_j$, $1 \le j \le d$, be the norms of the vector spaces V_j appearing in $\mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$. By $\|\cdot\|$ we denote the norm on the tensor space $\mathbf{V}$. Note that $\|\cdot\|$ is not determined by $\|\cdot\|_j$, for $j \in D$, but there are relations which are 'reasonable'. Any norm $\|\cdot\|$ on ${}_a\bigotimes_{j=1}^d V_j$ satisfying

$$ \left\| \bigotimes_{j=1}^d v_j \right\| = \prod_{j=1}^d \|v_j\|_j \qquad \text{for all } v_j \in V_j \ \ (1 \le j \le d) \tag{3.12} $$

is called a *crossnorm*. As usual, the dual norm of $\|\cdot\|$ is denoted by $\|\cdot\|^*$. If $\|\cdot\|$ is a crossnorm and also $\|\cdot\|^*$ is a crossnorm on ${}_a\bigotimes_{j=1}^d V_j^*$, i.e.,

$$ \left\| \bigotimes_{j=1}^d \varphi^{(j)} \right\|^* = \prod_{j=1}^d \|\varphi^{(j)}\|_j^* \qquad \text{for all } \varphi^{(j)} \in V_j^* \ \ (1 \le j \le d)\,, \tag{3.13} $$

then $\|\cdot\|$ is called a *reasonable crossnorm*.

Remark 3.26 *Eq. (3.12) implies the inequality* $\|\bigotimes_{j=1}^d v_j\| \lesssim \prod_{j=1}^d \|v_j\|_j$ *which is equivalent to the continuity of the multilinear tensor product mapping*[5] *between normed spaces:*

$$ \bigotimes : \bigtimes_{j=1}^d \left(V_j, \|\cdot\|_j\right) \longrightarrow \left({}_a\bigotimes_{j=1}^d V_j \ , \|\cdot\| \right), \tag{3.14} $$

defined by $\bigotimes((v_1,\dots,v_d)) = \bigotimes_{j=1}^d v_j$*, the product space being equipped with the product topology induced by the maximum norm* $\|(v_1,\dots,v_d)\| = \max_{1\le j\le d} \|v_j\|_j$.

The following result is a consequence of Lemma 4.34 of [15].

[5]Recall that a multilinear map T from $\bigtimes_{j=1}^d (V_j, \|\cdot\|_j)$ equipped with the product topology to a normed space $(W, \|\cdot\|)$ is continuous if and only if $\|T\| < \infty$, with

$$ \|T\| := \sup_{\substack{(v_1,\dots,v_d)\\ \|(v_1,\dots,v_d)\|\le 1}} \|T(v_1,\dots,v_d)\| = \sup_{\substack{(v_1,\dots,v_d)\\ \|v_1\|_1\le 1,\dots,\|v_d\|_d\le 1}} \|T(v_1,\dots,v_d)\| = \sup_{(v_1,\dots,v_d)} \frac{\|T(v_1,\dots,v_d)\|}{\|v_1\|_1 \dots \|v_d\|_d}. $$

Lemma 3.27 *Let $(V_j, \|\cdot\|_j)$ be normed spaces for $1 \le j \le d$. Assume that $\|\cdot\|$ is a norm on the tensor space ${}_a\bigotimes_{j=1}^d V_{j_{\|\cdot\|_j}}$ such that the tensor product map*

$$\bigotimes : \bigtimes_{j=1}^{d} \left(V_{j_{\|\cdot\|_j}}, \|\cdot\|_j\right) \longrightarrow \left({}_a\bigotimes_{j=1}^{d} V_{j_{\|\cdot\|_j}} , \|\cdot\| \right) \tag{3.15}$$

is continuous. Then (3.14) is also continuous and

$${}_{\|\cdot\|}\bigotimes_{j=1}^{d} V_{j_{\|\cdot\|_j}} = {}_{\|\cdot\|}\bigotimes_{j=1}^{d} V_j$$

holds.

Definition 3.28 *Assume that for each $\alpha \in T_D \backslash \mathcal{L}(T_D)$ there exists a norm $\|\cdot\|_\alpha$ defined on ${}_a\bigotimes_{\beta\in S(\alpha)} V_{\beta_{\|\cdot\|_\beta}}$. We will say that the tensor product map $\bigotimes$ is T_D-continuous if the map*

$$\bigotimes : \bigtimes_{\beta\in S(\alpha)} (V_{\beta_{\|\cdot\|_\beta}}, \|\cdot\|_\beta) \to \left({}_a\bigotimes_{\beta\in S(\alpha)} V_{\beta_{\|\cdot\|_\beta}} , \|\cdot\|_\alpha \right)$$

is continuous for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$.

The next result gives the conditions to have (3.11).

Theorem 3.29 *Assume that we have a representation $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ in the topological tree based format of the tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{\alpha\in S(D)} \mathbf{V}_\alpha$, such that for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$, the norm $\|\cdot\|_\alpha$ is also defined on ${}_a\bigotimes_{\beta\in S(\alpha)} V_{\beta_{\|\cdot\|_\beta}}$ and the tensor product map $\bigotimes$ is T_D-continuous. Then*

$${}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_{\beta_{\|\cdot\|_\beta}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_\beta = {}_{\|\cdot\|_\alpha}\bigotimes_{j\in\alpha} V_j ,$$

holds for all $\alpha \in T_D \setminus \mathcal{L}(T_D)$.

Proof. From Lemma 3.27, if the tensor product map

$$\bigotimes : \bigtimes_{\beta\in S(\alpha)} (\mathbf{V}_{\beta_{\|\cdot\|_\beta}}, \|\cdot\|_\beta) \longrightarrow ({}_a\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_{\beta_{\|\cdot\|_\beta}} , \|\cdot\|_\alpha)$$

is continuous, then

$${}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_{\beta_{\|\cdot\|_\beta}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} V_\beta ,$$

holds. Since $\mathbf{V}_\alpha = {}_a\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_\beta = {}_a\bigotimes_{j\in\alpha} V_j$, the theorem follows. ∎

Example 3.30 *Assume that the tensor product maps*

$$\bigotimes : (L^p(I_1), \|\cdot\|_{0,p;I_1}) \times (H^{N,p}(I_2) \otimes_{\|\cdot\|_{23}} H^{N,p}(I_3), \|\cdot\|_{23}) \to (L^p(I_1) \otimes_a (H^{N,p}(I_2) \otimes_{\|\cdot\|_{23}} H^{N,p}(I_3)), \|\cdot\|_{123})$$

and

$$\bigotimes : (H^{N,p}(I_2), \|\cdot\|_{N,p;I_1}) \times (H^{N,p}(I_3), \|\cdot\|_{N,p;I_2}) \to (H^{N,p}(I_2) \otimes_a H^{N,p}(I_3), \|\cdot\|_{23})$$

are continuous. Then the trees of Figure 3.4 and Figure 3.5 are the same.

Now, assume that we have a representation $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ of the Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{\alpha\in S(D)}\mathbf{V}_\alpha$, in the topological tree based format. Take $\mathbf{V}_D = {}_a\bigotimes_{\alpha\in S(D)}\mathbf{V}_\alpha$ and $\mathbf{v}\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. From Example 2.14, the finite dimensional subspace $U_\alpha^{\min}(\mathbf{v})\subset\mathbf{V}_\alpha\subset\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$ belongs to the Banach manifold $\mathbb{G}_{r_\alpha}(\mathbf{V}_\alpha)$ for $\alpha\in T_D\setminus\{D\}$ (see also Example 2.13). Hence, $(U_\alpha^{\min}(\mathbf{v}))_{\alpha\in T_D\setminus\{D\}}$ belongs to the product manifold

$$\mathbb{G}_{\mathfrak{r}}(T_D) := \bigtimes_{\alpha\in T_D\setminus\{D\}}\mathbb{G}_{r_\alpha}(\mathbf{V}_\alpha).$$

In consequence, under the above assumption, every $\mathbf{v}\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is completely characterised by

$$\left(\mathfrak{C}(\mathbf{v}),(U_\alpha^{\min}(\mathbf{v}))_{\alpha\in T_D\setminus\{D\}}\right)\in\mathbb{R}_*^{\mathfrak{r}}\times\mathbb{G}_{\mathfrak{r}}(T_D).$$

We remark that it allows to define a surjective map

$$\varrho_{\mathfrak{r}}:\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)\longrightarrow\mathbb{R}_*^{\mathfrak{r}}\times\mathbb{G}_{\mathfrak{r}}(T_D),\quad \mathbf{v}\mapsto\varrho_{\mathfrak{r}}(\mathbf{v}):=\left(\mathfrak{C}(\mathbf{v}),(U_\alpha^{\min}(\mathbf{v}))_{\alpha\in T_D\setminus\{D\}}\right),$$

that will be useful in the next section to define a manifold structure on $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$.

4 The manifold of TBF tensors of fixed TB rank

Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of a Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{\alpha\in S(D)}\mathbf{V}_\alpha$, in the topological tree based format. Set $\mathbf{V}_D := {}_a\bigotimes_{\alpha\in S(D)}\mathbf{V}_\alpha$.

Now, fix $\mathbf{v}\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ and consider a basis $\{\mathbf{u}_{i_\alpha}^{(\alpha)}:1\le i_\alpha\le r_\alpha\}$ of $U_\alpha^{\min}(\mathbf{v})$ for each $\alpha\in T_D\setminus\{D\}$ such that $\mathbf{v}$ can be represented by means (3.5) and (3.8). Thus $\mathbf{v}$ is completely characterised by $\mathfrak{C}\in\mathbb{R}_*^{\mathfrak{r}}$ and $(\{\mathbf{u}_{i_\alpha}^{(\alpha)}:1\le i_\alpha\le r_\alpha\})_{\alpha\in T_D\setminus\{D\}}$. Assume a decomposition into a direct sum

$$\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = U_\alpha^{\min}(\mathbf{v})\oplus W_\alpha^{\min}(\mathbf{v})$$

for $\alpha\in T_D\setminus\{D\}$. From Example 2.14 we have for each $\alpha\in T_D\setminus\{D\}$ a set

$$\mathbb{G}(W_\alpha^{\min}(\mathbf{v}),\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) = \{U_\alpha\in\mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}):U_\alpha\oplus W_\alpha^{\min}(\mathbf{v})=\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}$$

and a bijective map $\Psi_{U_\alpha^{\min}(\mathbf{v})\oplus W_\alpha^{\min}(\mathbf{v})}:\mathbb{G}(W_\alpha^{\min}(\mathbf{v}),\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})\longrightarrow\mathcal{L}(U_\alpha^{\min}(\mathbf{v}),W_\alpha^{\min}(\mathbf{v}))$. Clearly, the map

$$\mathbf{\Psi}_{\mathbf{v}}:\bigtimes_{\alpha\in T_D\setminus\{D\}}\mathbb{G}(W_\alpha^{\min}(\mathbf{v}),\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})\to\bigtimes_{\alpha\in T_D\setminus\{D\}}\mathcal{L}(U_\alpha^{\min}(\mathbf{v}),W_\alpha^{\min}(\mathbf{v})),$$

defined as $\mathbf{\Psi}_{\mathbf{v}} := \bigtimes_{\alpha\in T_D\setminus\{D\}}\Psi_{U_\alpha^{\min}(\mathbf{v})\oplus W_\alpha^{\min}(\mathbf{v})}$ is also bijective. Furthermore, it is a local chart for $\mathfrak{U}(\mathbf{v}) := \{U_\alpha^{\min}(\mathbf{v})\}_{\alpha\in T_D\setminus\{D\}}$ in $\mathbb{G}_{\mathfrak{r}}(T_D)$, such that $\mathbf{\Psi}_{\mathbf{v}}(\mathfrak{U}(\mathbf{v}))=\mathfrak{o}:=(0)_{\alpha\in T_D\setminus\{D\}}$. To simplify the notation, for each $\mathbf{v}\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ we will use

$$\begin{aligned}\mathcal{L}_{T_D}(\mathbf{v}) &:= \bigtimes_{\alpha\in T_D\setminus\{D\}}\mathcal{L}\left(U_\alpha^{\min}(\mathbf{v}),W_\alpha^{\min}(\mathbf{v})\right)\\ &= \{\mathfrak{L}:=\{L_\alpha\}_{\alpha\in T_D\setminus\{D\}}:L_\alpha\in\mathcal{L}\left(U_\alpha^{\min}(\mathbf{v}),W_\alpha^{\min}(\mathbf{v})\right)\},\end{aligned}$$

which is a closed subspace of the Banach space

$$\mathcal{L}_{T_D} := \bigtimes_{\alpha\in T_D\setminus\{D\}}\mathcal{L}\left(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}},\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\right),$$

and

$$\mathbb{G}_{\mathfrak{r}}(\mathfrak{U}(\mathbf{v})) := \bigtimes_{\alpha\in T_D\setminus\{D\}}\mathbb{G}(W_\alpha^{\min}(\mathbf{v}),\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}),$$

which is a local neighbourhood of $\mathfrak{U}(\mathbf{v})$ in the manifold $\mathbb{G}_{\mathfrak{r}}(T_D)$. Moreover, $\mathfrak{U}=\mathbf{\Psi}_{\mathbf{v}}^{-1}(\mathfrak{L})$ with $U_\alpha=G(L_\alpha)=\left\{\mathbf{u}_\alpha+L_\alpha(\mathbf{u}_\alpha):\mathbf{u}_\alpha\in U_\alpha^{\min}(\mathbf{v})\right\}$ for each $\alpha\in T_D\setminus\{D\}$. A useful result is the following.

Lemma 4.1 *For each $\alpha \in T_D \backslash \{D\}$, the set $\mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$ is a complemented subspace of $\mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$, and hence for each $\mathbf{v} \in \mathbf{V}_D$, the set $\mathcal{L}_{T_D}(\mathbf{v})$ is a complemented subspace of $\mathcal{L}_{T_D}$.*

Proof. Observe that the map

$$\Pi_\alpha : \mathcal{L}\left(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\right) \to \mathcal{L}\left(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\right)$$

defined by

$$\Pi_\alpha(L_\alpha) = P_{W_\alpha^{\min}(\mathbf{v}) \oplus U_\alpha^{\min}(\mathbf{v})} L_\alpha P_{U_\alpha^{\min}(\mathbf{v}) \oplus W_\alpha^{\min}(\mathbf{v})}$$

is a projection onto $\mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$. ∎

Now, let $\pi_2 : \mathbb{R}_*^{\mathfrak{r}} \times \mathbb{G}_{\mathfrak{r}}(T_D) \longrightarrow \mathbb{G}_{\mathfrak{r}}(T_D)$ be the morphism $\pi_2(\mathfrak{C}, \mathfrak{U}) := \mathfrak{U}$. Then we introduce the map

$$\Lambda_{T_D} := \pi_2 \circ \varrho_{\mathfrak{r}} : \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \longrightarrow \mathbb{G}_{\mathfrak{r}}(T_D) \subset \mathbb{G}(T_D), \quad \mathbf{w} \mapsto \mathfrak{U}(\mathbf{w}) := (U_\alpha^{\min}(\mathbf{w}))_{\alpha \in T_D \backslash \{D\}},$$

and observe that for each $\mathbf{w} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ we have

$$\Lambda_{T_D}^{-1}(\Lambda_{T_D}(\mathbf{w})) = \left\{\mathbf{u} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) : U_\alpha^{\min}(\mathbf{u}) = U_\alpha^{\min}(\mathbf{w}) \text{ for all } \alpha \in T_D \setminus \{D\}\right\}.$$

We define the local neighbourhood of $\mathbf{v}$, denoted by $\mathcal{U}(\mathbf{v})$, in $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ as

$$\mathcal{U}(\mathbf{v}) := \Lambda_{T_D}^{-1}\left(\mathbb{G}_{\mathfrak{r}}(\mathfrak{U}(\mathbf{v}))\right) = \varrho_{\mathfrak{r}}^{-1}\left(\mathbb{R}_*^{\mathfrak{r}} \times \mathbb{G}_{\mathfrak{r}}(\mathfrak{U}(\mathbf{v}))\right) \subset \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D).$$

Observe that for each $\mathbf{w} \in \mathcal{U}(\mathbf{v})$ we have

$$\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = U_\alpha^{\min}(\mathbf{w}) \oplus W_\alpha^{\min}(\mathbf{v}),$$

where $U_\alpha^{\min}(\mathbf{w}) \in \mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\|\cdot\|_\alpha})$, for each $\alpha \in T_D \setminus \{D\}$. Since

$$\mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \overset{\Psi_{U_\alpha^{\min}(\mathbf{v}) \oplus W_\alpha^{\min}(\mathbf{v})}}{\cong} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})),$$

there exists a unique $L_\alpha \in \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$ such that

$$\Psi_{U_\alpha^{\min}(\mathbf{v}) \oplus W_\alpha^{\min}(\mathbf{v})}(U_\alpha^{\min}(\mathbf{w})) = L_\alpha$$

for each $\alpha \in T_D \setminus \{D\}$. Moreover, we claim that

$$U_\alpha^{\min}(\mathbf{w}) = \operatorname{span}\{L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) + \mathbf{u}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$$

holds for all $\alpha \in T_D \setminus \{D\}$. To prove the claim, we only need to show that

$$\{L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) + \mathbf{u}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$$

are linearly independent in $U_\alpha^{\min}(\mathbf{w})$. If the last statement is not true, we may assume without loss of generality that

$$L_\alpha(\mathbf{u}_1^{(\alpha)}) + \mathbf{u}_1^{(\alpha)} = \sum_{k=2}^{r_\alpha} \lambda_k (L_\alpha(\mathbf{u}_k^{(\alpha)}) + \mathbf{u}_k^{(\alpha)}),$$

i.e.,

$$L_\alpha(\mathbf{u}_1^{(\alpha)}) - \sum_{k=2}^{r_\alpha} \lambda_k L_\alpha(\mathbf{u}_k^{(\alpha)}) = \sum_{k=2}^{r_\alpha} \lambda_k \mathbf{u}_k^{(\alpha)} - \mathbf{u}_1^{(\alpha)}.$$

The left-hand side is in $W_\alpha^{\min}(\mathbf{v})$ and the right-hand side is in $U_\alpha^{\min}(\mathbf{w})$. Since $W_\alpha^{\min}(\mathbf{v}) \cap U_\alpha^{\min}(\mathbf{w}) = \{\mathbf{0}\}$ we then have a contradiction and the claim follows.

Take $\mathbf{w} \in \mathcal{U}(\mathbf{v})$, for each $\mathbf{u} \in \Lambda_{T_D}^{-1}(\Lambda_{T_D}(\mathbf{w})) \subset \mathcal{U}(\mathbf{v})$, we fix the basis

$$\{\mathbf{w}_{i_\alpha}^{(\alpha)} := \mathbf{u}_{i_\alpha}^{(\alpha)} + L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) : 1 \le i_\alpha \le r_\alpha\}$$

of $U_\alpha^{\min}(\mathbf{w})$ for each $\alpha \in T_D \setminus \{D\}$. Then we define $\xi_{\mathbf{w}} : \Lambda_{T_D}^{-1}(\Lambda_{T_D}(\mathbf{w})) \longrightarrow \mathbb{R}_*^{\mathfrak{r}}$ by

$$\xi_{\mathbf{w}}(\mathbf{u}) := \mathfrak{C}(\mathbf{u}) = (C^{(\alpha)}(\mathbf{u}))_{\alpha \in T_D \setminus \mathcal{L}(T_D)},$$

where

$$\mathbf{u} = \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha \in S(D)}}(\mathbf{u}) \bigotimes_{\alpha \in S(D)} (L_\alpha(\mathbf{u}^{(\alpha)}_{i_\alpha}) + \mathbf{u}^{(\alpha)}_{i_\alpha})$$

and, if $S(D) \neq \mathcal{L}(T)$, for each $\mu \in T_D \setminus \{D\}$ such that $S(\mu) \neq \emptyset$ we have

$$L_\mu(\mathbf{u}^{(\mu)}_{i_\mu}) + \mathbf{u}^{(\mu)}_{i_\mu} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\mu)}} C^{(\mu)}_{i_\mu, (i_\beta)_{\beta \in S(\mu)}}(\mathbf{u}) \bigotimes_{\beta \in S(\mu)} (L_\beta(\mathbf{u}^{(\beta)}_{i_\beta}) + \mathbf{u}^{(\beta)}_{i_\beta}),$$

for $1 \le i_\mu \le r_\mu$. Clearly, $\xi_{\mathbf{w}}$ is one-to-one. On the other hand, given $\mathfrak{B} \in \mathbb{R}_*^{\mathfrak{r}}$, we can construct $\mathbf{u} \in \Lambda_{T_D}^{-1}(\Lambda_{T_D}(\mathbf{w}))$ satisfying $\mathfrak{B} = \mathfrak{C}(\mathbf{u})$. Thus we can conclude that $\xi_{\mathbf{w}}$ is a bijection which is independent of $\mathbf{w}$.

It allows us to define a local chart $\Theta_{\mathbf{v}} : \mathcal{U}(\mathbf{v}) \longrightarrow \mathbb{R}_*^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v})$ by

$$\Theta_{\mathbf{v}}(\mathbf{w}) := (\xi_{\mathbf{w}}(\mathbf{w}), \mathbf{\Psi}_{\mathbf{v}} \circ \Lambda_{T_D}(\mathbf{w})) = (\mathfrak{C}(\mathbf{w}), \mathbf{\Psi}_{\mathbf{v}}(\mathfrak{U}(\mathbf{w}))).$$

More precisely, $\Theta_{\mathbf{v}}(\mathbf{w}) = (\mathfrak{C}(\mathbf{w}), \mathfrak{L})$ if and only if

$$\mathbf{w} = \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha \in S(D)}}(\mathbf{w}) \bigotimes_{\alpha \in S(D)} (L_\alpha(\mathbf{u}^{(\alpha)}_{i_\alpha}) + \mathbf{u}^{(\alpha)}_{i_\alpha}), \tag{4.1}$$

where, if $S(D) \neq \mathcal{L}(T_D)$, for each $\mu \in T_D \setminus \{D\}$ such that $S(\mu) \neq \emptyset$ we have

$$L_\mu(\mathbf{u}^{(\mu)}_{i_\mu}) + \mathbf{u}^{(\mu)}_{i_\mu} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\mu)}} C^{(\mu)}_{i_\mu, (i_\beta)_{\beta \in S(\mu)}}(\mathbf{w}) \bigotimes_{\beta \in S(\mu)} (L_\beta(\mathbf{u}^{(\beta)}_{i_\beta}) + \mathbf{u}^{(\beta)}_{i_\beta}) \tag{4.2}$$

for $1 \le i_\mu \le r_\mu$. Proceeding iteratively along the tree, we obtain, for $S(D) \neq \mathcal{L}(T_D)$, a Tucker format representation of $\mathbf{w}$ given by

$$\mathbf{w} = \sum_{\substack{1 \le i_k \le r_k \\ k \in \mathcal{L}(T_D)}} \left(\sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D) \\ \alpha \notin \mathcal{L}(T_D)}} C^{(D)}_{(i_\alpha)_{\alpha \in S(D)}}(\mathbf{w}) \prod_{\substack{\mu \in T_D \setminus \{D\} \\ S(\mu) \neq \emptyset}} C^{(\mu)}_{i_\mu, (i_\beta)_{\beta \in S(\mu)}}(\mathbf{w}) \right) \bigotimes_{k \in \mathcal{L}(T_D)} (L_k(u^{(k)}_{i_k}) + u^{(k)}_{i_k}).$$

The next result shows that the collection $\{\Theta_{\mathbf{v}}, \mathcal{U}(\mathbf{v})\}_{\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)}$ is an atlas for $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$.

Theorem 4.2 *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha \in T_D \setminus \{D\}}$ be a representation of a Banach tensor space*

$$\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D} \bigotimes_{\alpha \in S(D)} \mathbf{V}_\alpha \ ,$$

in the topological tree based format. Then the collection $\{\Theta_{\mathbf{v}}, \mathcal{U}(\mathbf{v})\}_{\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)}$ is an analytic atlas for $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. Furthermore, the set $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ of TBF tensors with fixed TB rank is an analytic Banach manifold modelled on $\mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{w})$, here $\mathbf{w} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$.

Proof. Clearly, $\{\mathcal{U}(\mathbf{v})\}_{\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)}$ is a covering of $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V})$ and AT1 is true. Take $(\mathfrak{C}, \mathfrak{L}) \in \mathbb{R}_*^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v})$. By using (4.1)-(4.2), we can construct $\mathbf{w} \in \mathcal{U}(\mathbf{v})$ such that $\Theta_{\mathbf{v}}(\mathbf{w}) = (\mathfrak{C}, \mathfrak{L})$, and in consequence $\Theta_{\mathbf{v}}$ is surjective. Now, consider that $\Theta_{\mathbf{v}}(\mathbf{u}) = \Theta_{\mathbf{v}}(\mathbf{w})$. Since $U_\alpha^{\min}(\mathbf{u}) = U_\alpha^{\min}(\mathbf{w})$ for all $\alpha \in T_D \setminus \{D\}$ and $\mathfrak{C}(\mathbf{v}) = \mathfrak{C}(\mathbf{w})$,

also from (4.1)-(4.2) we can conclude that $\mathbf{w} = \mathbf{u}$. In consequence, AT2 in Definition 2.7 holds. Finally for $\mathbf{v}, \mathbf{u} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ consider $\mathcal{U}(\mathbf{v}, \mathbf{u}) := \mathcal{U}(\mathbf{v}) \cap \mathcal{U}(\mathbf{u})$. Observe that $\mathbf{w} \in \mathcal{U}(\mathbf{v}, \mathbf{u})$ if and only if

$$U_\alpha^{\min}(\mathbf{w}) \in \mathbb{G}(W_\alpha^{\min}(\mathbf{u}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \cap \mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \text{ for } \alpha \in T_D.$$

Then we need to show that

$$\Theta_{\mathbf{v}} \circ \Theta_{\mathbf{u}}^{-1} : \Theta_{\mathbf{u}}\left(\mathcal{U}(\mathbf{v}, \mathbf{u})\right) \longrightarrow \Theta_{\mathbf{v}}\left(\mathcal{U}(\mathbf{v}, \mathbf{u})\right)$$

is a diffeomorphism. To this end, assume that $\{\mathbf{u}_{i_\alpha}^{(\alpha)} : 1 \leq i_\alpha \leq r_\alpha\}$ is a basis of $U_\alpha^{\min}(\mathbf{v})$ and $\{\mathbf{z}_{i_\alpha}^{(\alpha)} : 1 \leq i_\alpha \leq r_\alpha\}$ is a basis of $U_\alpha^{\min}(\mathbf{u})$ for $\alpha \in T_D \setminus \{D\}$. For each $(\mathfrak{C}, \mathfrak{L}) \in \Theta_{\mathbf{u}}\left(\mathcal{U}(\mathbf{v}, \mathbf{u})\right)$, let $\mathbf{w} \in \mathcal{U}(\mathbf{v}, \mathbf{u})$ be such that $\Theta_{\mathbf{u}}(\mathbf{w}) = (\mathfrak{C}, \mathfrak{L})$, and

$$\Theta_{\mathbf{v}} \circ \Theta_{\mathbf{u}}^{-1}\left(\mathfrak{C}, \mathfrak{L}\right) = \Theta_{\mathbf{v}}(\mathbf{w}) = (\mathfrak{B}, \mathfrak{N}).$$

Now, we describe the transformation $\mathfrak{C} \mapsto \mathfrak{B}$. We have

$$\begin{aligned}
\mathbf{w} &= \sum_{\substack{1 \leq i_\alpha \leq r_\alpha \\ \alpha \in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha \in S(D)}}(\mathbf{w}) \bigotimes_{\alpha \in S(D)} (L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) + \mathbf{u}_{i_\alpha}^{(\alpha)}) \\
&= \sum_{\substack{1 \leq i_\alpha \leq r_\alpha \\ \alpha \in S(D)}} D^{(D)}_{(i_\alpha)_{\alpha \in S(D)}}(\mathbf{w}) \bigotimes_{\alpha \in S(D)} (N_\alpha(\mathbf{z}_{i_\alpha}^{(\alpha)}) + \mathbf{z}_{i_\alpha}^{(\alpha)}),
\end{aligned}$$

where

$$U_\alpha^{\min}(\mathbf{w}) = \operatorname{span}\{\mathbf{u}_{i_\alpha}^{(\alpha)} + L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) : 1 \leq i_\alpha \leq r_\alpha\},$$

$$L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) + \mathbf{u}_{i_\alpha}^{(\alpha)} = \sum_{\substack{1 \leq i_\beta \leq r_\beta \\ \beta \in S(\alpha)}} C^{(\alpha)}_{i_\alpha, (i_\beta)_{\beta \in S(\mu)}}(\mathbf{w}) \bigotimes_{\beta \in S(\alpha)} (L_\beta(\mathbf{u}_{i_\beta}^{(\beta)}) + \mathbf{u}_{i_\beta}^{(\beta)}),$$

$$U_\alpha^{\min}(\mathbf{w}) = \operatorname{span}\{\mathbf{z}_{i_\alpha}^{(\alpha)} + N_\alpha(\mathbf{z}_{i_\alpha}^{(\alpha)}) : 1 \leq i_\alpha \leq r_\alpha\}$$

and

$$N_\alpha(\mathbf{z}_{i_\alpha}^{(\alpha)}) + \mathbf{z}_{i_\alpha}^{(\alpha)} = \sum_{\substack{1 \leq i_\beta \leq r_\beta \\ \beta \in S(\alpha)}} B^{(\alpha)}_{i_\alpha, (i_\beta)_{\beta \in S(\mu)}}(\mathbf{w}) \bigotimes_{\beta \in S(\alpha)} (N_\beta(\mathbf{z}_{i_\beta}^{(\beta)}) + \mathbf{z}_{i_\beta}^{(\beta)})$$

holds for $1 \leq i_\alpha \leq r_\alpha$ and $\alpha \in T_D \setminus \{D\}$. We show the existence of a linear isomorphism

$$\mathcal{A}^{(D)} : \mathbb{R}_*^{\bigtimes_{\alpha \in S(D)} r_\alpha} \to \mathbb{R}_*^{\bigtimes_{\alpha \in S(D)} r_\alpha}$$

such that $\mathcal{A}^{(D)}(C^{(D)}) = B^{(D)}$, as follows. Let $S_D := \bigotimes_{\alpha \in S(D)} S_\alpha$, where $S_\alpha \in \mathcal{L}(U_\alpha^{\min}(\mathbf{w}), U_\alpha^{\min}(\mathbf{w}))$, is defined by

$$S_\alpha(L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) + \mathbf{u}_{i_\alpha}^{(\alpha)}) := \sum_{1 \leq j_\alpha \leq i_\alpha} A^{(\alpha)}_{i_\alpha, j_\alpha}(L_\alpha(\mathbf{u}_{j_\alpha}^{(\alpha)}) + \mathbf{u}_{j_\alpha}^{(\alpha)}) = N_\alpha(\mathbf{z}_{i_\alpha}^{(\alpha)}) + \mathbf{z}_{i_\alpha}^{(\alpha)}$$

for $1 \leq i_\alpha \leq r_\alpha$ and $\alpha \in S(D)$. Clearly, $A^{(\alpha)} \in \mathrm{GL}(\mathbb{R}^{r_\alpha})$, and

$$S_D \in \mathcal{L}\left({}_a \bigotimes_{\alpha \in S(D)} U_\alpha^{\min}(\mathbf{w}) \, , \, {}_a \bigotimes_{\alpha \in S(D)} U_\alpha^{\min}(\mathbf{w}) \right)$$

is a linear isomorphism. Moreover, $\mathcal{A}^{(D)} := \bigotimes_{\alpha \in S(D)} A^{(\alpha)} : \mathbb{R}_*^{\bigtimes_{\alpha \in S(D)} r_\alpha} \to \mathbb{R}_*^{\bigtimes_{\alpha \in S(D)} r_\alpha}$ and $B^{(D)} = \mathcal{A}^{(D)}(C^{(D)})$. Proceeding in a similar way for each $\alpha \in T_D \setminus \{\{D\} \cup \mathcal{L}(T_D)\}$, we can construct a linear isomorphism $\mathcal{A}^{(\alpha)} : \mathbb{R}_*^{r_\alpha \times (\bigtimes_{\beta \in S(\alpha)} r_\beta)} \to \mathbb{R}_*^{r_\alpha \times (\bigtimes_{\beta \in S(\alpha)} r_\beta)}$ such that $\mathcal{A}^{(\alpha)}(C^{(\alpha)}) = B^{(\alpha)}$. The above construction allows us to define a map $\mathfrak{A} : \mathbb{R}_*^{\mathfrak{r}} \to \mathbb{R}_*^{\mathfrak{r}}$ given by $\mathfrak{A}(\mathfrak{C}) = \mathfrak{B}$, which is also a linear isomorphism and we can write

$$\Theta_{\mathbf{v}} \circ \Theta_{\mathbf{u}}^{-1}\left(\mathfrak{C}, \mathfrak{L}\right) = (\mathfrak{A}(\mathfrak{C}), \mathfrak{N}) = (\mathfrak{A}(\mathfrak{C}), \mathbf{\Psi}_{\mathbf{v}} \circ \Lambda_{T_D}(\mathbf{w}))).$$

Since $\Lambda_{T_D}(\mathbf{w}) = \mathfrak{U}(\mathbf{w})$ and $U_\alpha^{\min}(\mathbf{w}) = \Psi^{-1}_{U_\alpha^{\min}(\mathbf{u})\oplus W_\alpha^{\min}(\mathbf{u})}(L_\alpha)$ for each $\alpha \in T_D \setminus \{D\}$, we obtain

$$\Theta_{\mathbf{v}} \circ \Theta_{\mathbf{u}}^{-1}(\mathfrak{C}, \mathfrak{L}) = \left(\mathfrak{A}(\mathfrak{C}), \left(\mathbf{\Psi}_{\mathbf{v}} \circ \mathbf{\Psi}_{\mathbf{u}}^{-1}\right)(\mathfrak{L})\right).$$

From [5] we know that $\Psi_{U_\alpha^{\min}(\mathbf{v})\oplus W_\alpha^{\min}(\mathbf{v})} \circ \Psi^{-1}_{U_\alpha^{\min}(\mathbf{u})\oplus W_\alpha^{\min}(\mathbf{u})}$ mapping from

$$\Psi_{U_\alpha^{\min}(\mathbf{u})\oplus W_\alpha^{\min}(\mathbf{u})}\left(\mathbb{G}(W_\alpha^{\min}(\mathbf{u}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \cap \mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})\right) \subset \mathcal{L}(U_\alpha^{\min}(\mathbf{u}), W_\alpha^{\min}(\mathbf{u}))$$

to

$$\Psi_{U_\alpha^{\min}(\mathbf{v})\oplus W_\alpha^{\min}(\mathbf{v})}\left(\mathbb{G}(W_\alpha^{\min}(\mathbf{u}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \cap \mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})\right) \subset \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$$

is an analytic diffeomorphism for each $\alpha \in T_D \setminus \{D\}$. Then $\mathbf{\Psi}_{\mathbf{v}} \circ \mathbf{\Psi}_{\mathbf{u}}^{-1}$ is an analytic diffeomorphism from

$$\mathbf{\Psi}_{\mathbf{u}}\left(\bigtimes_{\alpha\in T_D\setminus\{D\}} \left(\mathbb{G}(W_\alpha^{\min}(\mathbf{u}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \cap \mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})\right)\right) \subset \mathcal{L}_{T_D}(\mathbf{u})$$

to

$$\mathbf{\Psi}_{\mathbf{v}}\left(\bigtimes_{\alpha\in T_D\setminus\{D\}} \left(\mathbb{G}(W_\alpha^{\min}(\mathbf{u}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \cap \mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})\right)\right) \subset \mathcal{L}_{T_D}(\mathbf{v}).$$

Clearly, AT3 holds and the theorem follows. ∎

Remark 4.3 *We observe that the geometric structure of manifold is independent of the choice of the norm $\|\cdot\|_D$ over the tensor space $\mathbf{V}_D$.*

Corollary 4.4 *Assume that $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$ is a Hilbert space with norm $\|\cdot\|_\alpha$ for $\alpha \in T_D \setminus \{D\}$. Then $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is an analytic Hilbert manifold modelled on $\mathbb{R}^{\mathfrak{r}} \times \bigtimes_{\alpha\in T_D\setminus\{D\}} W_\alpha^{\min}(\mathbf{w})^{r_\alpha}$, here $\mathbf{w} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$.*

Proof. We can identify each $L_\alpha \in \mathcal{L}\left(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})\right)$ with a $(\mathbf{w}_{s_\alpha}^{(\alpha)})_{s_\alpha=1}^{s_\alpha=r_\alpha} \in W_\alpha^{\min}(\mathbf{v})^{r_\alpha}$, where $\mathbf{w}_{s_\alpha}^{(\alpha)} = L_\alpha(\mathbf{u}_{(s_\alpha)}^\alpha)$ and $U_\alpha^{\min}(\mathbf{v}) = \operatorname{span}\{\mathbf{u}_{(1)}^\alpha, \ldots, \mathbf{u}_{(r_\alpha)}^\alpha\}$ for $\alpha \in T_D \setminus \{D\}$. Thus we can identify each $(\mathfrak{C}, \mathfrak{L}) \in \mathcal{U}(\mathbf{v})$ with a pair

$$(\mathfrak{C}, \mathfrak{W}) \in \mathbb{R}_*^{\mathfrak{r}} \times \bigtimes_{\alpha\in T_D\setminus\{D\}} W_\alpha^{\min}(\mathbf{v})^{r_\alpha},$$

where $\mathfrak{W} := ((\mathbf{w}_{s_\alpha}^{(\alpha)})_{s_\alpha=1}^{s_\alpha=r_\alpha})_{\alpha\in T_D\setminus\{D\}}$. We assume that $\mathbb{R}_*^{\mathfrak{r}} \times \bigtimes_{\alpha\in T_D\setminus\{D\}} W_\alpha^{\min}(\mathbf{v})^{r_\alpha}$ is an open subset of the Hilbert space $\mathbb{R}^{\mathfrak{r}} \times \bigtimes_{\alpha\in T_D\setminus\{D\}} W_\alpha^{\min}(\mathbf{v})^{r_\alpha}$ endowed with the product norm

$$\|(\mathfrak{C}, \mathfrak{W})\|_\times := \sum_{\alpha\in T_D\setminus\mathcal{L}(T_D)} \|C^\alpha\|_F + \sum_{\alpha\in T_D\setminus\{D\}} \sum_{s_\alpha=1}^{r_\alpha} \|\mathbf{w}_{s_\alpha}^{(\alpha)}\|_\alpha.$$

It allows us to define local charts, also denoted by $\Theta_{\mathbf{v}}$, by

$$\Theta_{\mathbf{v}}^{-1} : \mathbb{R}_*^{\mathfrak{r}} \times \bigtimes_{\alpha\in T_D\setminus\{D\}} W_\alpha^{\min}(\mathbf{v})^{r_\alpha} \longrightarrow \mathcal{U}(\mathbf{v}),$$

where $\Theta_{\mathbf{v}}^{-1}(\mathfrak{C}, \mathfrak{W}) = \mathbf{w}$. Here $\mathbf{w}$ is given by (4.1)–(4.2) putting $L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) = \mathbf{w}_{i_\alpha}^{(\alpha)}$, $1 \le i_\alpha \le r_\alpha$ and $\alpha \in T_D \setminus \{D\}$. Since each local chart is defined over an open subset of the Hilbert space $\mathbb{R}^{\mathfrak{r}} \times \bigtimes_{\alpha\in T_D\setminus\{D\}} W_\alpha^{\min}(\mathbf{v})^{r_\alpha}$, the corollary follows. ∎

By using the geometric structure of local charts for the manifold $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$, we can identify its tangent space at $\mathbf{v}$ with $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)) := \mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v})$. We will consider $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$ endowed with the product norm

$$|||(\mathfrak{C}, \mathfrak{L})||| := \sum_{\alpha\in T_D\setminus\mathcal{L}(T_D)} \|C^{(\alpha)}\|_F + \sum_{\alpha\in T_D\setminus\{D\}} \|L_\alpha\|_{W_\alpha^{\min}(\mathbf{v})\leftarrow U_\alpha^{\min}(\mathbf{v})}.$$

with $\|\cdot\|_F$ the Frobenius norm.

Note that $\mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$ endowed with the norm $\|\cdot\|_{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\leftarrow\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}}$ is a Banach space. Thus, even if $\mathbf{V}_{\|\cdot\|_\alpha}$ is a Hilbert space for all $\alpha \in T_D \setminus \{D\}$, the set $\mathcal{L}_{T_D}$ is a Banach space.

5 Is $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ an immersed submanifold?

We start with a brief discussion about the choice of the ambient manifold for $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ which is the milestone to give a positive answer to the question that gives the title of this section. Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of the Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in the topological tree based format. Take $\mathbf{V}_D := {}_a\bigotimes_{j\in D} V_j$ and assume the existence of two norms $\|\cdot\|_{D,1}$ and $\|\cdot\|_{D,2}$ on $\mathbf{V}_D$. Then we have $\mathbf{V}_D \subset \overline{\mathbf{V}_D}^{\|\cdot\|_{D,1}}$ and $\mathbf{V}_D \subset \overline{\mathbf{V}_D}^{\|\cdot\|_{D,2}}$. The next example illustrates this situation.

Example 5.1 *Let $V_{1_{\|\cdot\|_1}} := H^{1,p}(I_1)$ and $V_{2_{\|\cdot\|_2}} = H^{1,p}(I_2)$. Take $\mathbf{V}_D := H^{1,p}(I_1)\otimes_a H^{1,p}(I_2)$, from Theorem 4.2 we obtain that $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is a Banach manifold. However, we can consider as ambient manifold either $\overline{\mathbf{V}_D}^{\|\cdot\|_{D,1}} := H^{1,p}(I_1\times I_2)$ or $\overline{\mathbf{V}_D}^{\|\cdot\|_{D,2}} = H^{1,p}(I_1)\otimes_{\|\cdot\|_{(0,1),p}} H^{1,p}(I_2)$, where $\|\cdot\|_{(0,1),p}$ is the norm given by*

$$\|f\|_{(0,1),p} := \left(\|f\|_p^p + \left\|\frac{\partial f}{\partial x_2}\right\|_p^p\right)^{1/p}$$

for $1\le p<\infty$.

Now, the question is: what is the good choice to show that $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is an immersed submanifold? The main result of this section is to show that if for each $\alpha\in T_D\setminus\mathcal{L}(T_D)$ the norm $\|\cdot\|_\alpha$ is not weaker than the injective norm generated by the Banach spaces $\{\mathbf{V}_{\beta_{\|\cdot\|_\beta}} : \beta\in S(\alpha)\}$ then $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is an immersed submanifold of $\overline{\mathbf{V}_D}^{\|\cdot\|_D}$. To see this we need to introduce the following definitions and results.

5.1 On the differentiability of the standard inclusion map

Assume that the tensor product map $\bigotimes$ is T_D-continuous. The natural ambient space for $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is the Banach tensor space $\overline{\mathbf{V}_D}^{\|\cdot\|_D} = \mathbf{V}_{D_{\|\cdot\|_D}}$. Since the natural inclusion $\mathfrak{i} : \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \longrightarrow \mathbf{V}_{D_{\|\cdot\|_D}}$, given by $\mathfrak{i}(\mathbf{v}) = \mathbf{v}$, is an injective map we will study $\mathfrak{i}$ as a function between Banach manifolds. To this end we introduce the following definitions.

Definition 5.2 *Let X and Y be two Banach manifolds. Let $F : X\to Y$ be a map. We shall say that F is a $\mathcal{C}^r$* morphism *if given $x\in X$ there exists a chart (U,φ) at x and a chart (W,ψ) at $F(x)$ such that $F(U)\subset W$, and the map*

$$\psi\circ F\circ\varphi^{-1} : \varphi(U)\to\psi(W)$$

is a $\mathcal{C}^r$-Fréchet differentiable map.

To describe $\mathfrak{i}$ as a morphism, we proceed as follows. Given $\mathbf{v}\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$, we consider $\mathcal{U}(\mathbf{v})$, a local neighbourhood of $\mathbf{v}$, and then

$$\mathfrak{i}\circ\Theta_{\mathbf{v}}^{-1} : \mathbb{R}_*^{\mathfrak{r}}\times\mathcal{L}_{T_D}(\mathbf{v})\to\mathbf{V}_{\|\cdot\|_D},\quad (\mathfrak{C},\mathfrak{L})\mapsto \sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} (L_\alpha(\mathbf{u}^{(\alpha)}_{i_\alpha}) + \mathbf{u}^{(\alpha)}_{i_\alpha}),$$

where for each $\mu\in T_D\setminus\{D\}$ such that $S(\mu)\neq\emptyset$ we have

$$L_\mu(\mathbf{u}^{(\mu)}_{i_\mu}) + \mathbf{u}^{(\mu)}_{i_\mu} = \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(\mu)}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta\in S(\mu)}} \bigotimes_{\beta\in S(\mu)} (L_\beta(\mathbf{u}^{(\beta)}_{i_\beta}) + \mathbf{u}^{(\beta)}_{i_\beta})$$

for $1\le i_\mu\le r_\mu$.

Our next step is to recall the definition of the differential as a morphism which gives a linear map between the tangent spaces of the manifolds involved with the morphism.

Definition 5.3 *Let X and Y two Banach manifolds. Let $F : X \to Y$ be a $\mathcal{C}^r$ morphism, i.e.,*

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(W)$$

is a $\mathcal{C}^r$-Fréchet differentiable map, where (U, φ) is a chart in X at x and (W, ψ) is a chart in Y at $F(x)$. For $x \in X$, we define

$$\mathrm{T}_x F : \mathbb{T}_x(X) \longrightarrow \mathbb{T}_{F(x)}(Y), \quad v \mapsto [(\psi \circ F \circ \varphi^{-1})'(\varphi(x))]v.$$

Assume that $\mathfrak{i} \circ \Theta_{\mathbf{v}}^{-1}$ is Fréchet differentiable, then $\mathrm{T}_{\mathbf{v}}\mathfrak{i} : \mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v}) \to \mathbf{V}_{\|\cdot\|_D}$, is given by

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\dot{\mathfrak{C}}, \dot{\mathfrak{L}}) = [(\mathfrak{i} \circ \Theta_{\mathbf{v}}^{-1})'(\Theta_{\mathbf{v}}(\mathbf{v}))](\dot{\mathfrak{C}}, \dot{\mathfrak{L}}).$$

The next result says us that if the tensor product map is continuous, then it is also Fréchet differentiable.

Proposition 5.4 *Let $(V_j, \|\cdot\|_j)$ be normed spaces for $1 \le j \le d$. Assume that $\|\cdot\|$ is a norm onto the tensor space ${}_a\bigotimes_{j=1}^d V_{j_{\|\cdot\|_j}}$ such that the tensor product map (3.15) is continuous. Then it is also Fréchet differentiable and its differential is given by*

$$D\left(\bigotimes(v_1, \ldots, v_d)\right)(w_1, \ldots, w_d) = \sum_{j=1}^{d} v_1 \otimes \ldots \otimes v_{j-1} \otimes w_j \otimes v_{j+1} \otimes \cdots v_d.$$

Proof. Clearly, $D\bigotimes(v_1, \ldots, v_d)$ is a multilinear map. If we assume that the tensor product map (3.14) is continuous, that is $\|\bigotimes_{j=1}^d u_j\| \le C \prod_{j=1}^d \|u_j\|_j$ for some $C > 0$, then

$$\begin{aligned}
\|D\bigotimes(v_1, \ldots, v_d)(w_1, \ldots, w_d)\| &\le C \sum_{j=1}^{d} \|v_1\|_1 \cdots \|v_{j-1}\|_{j-1} \|w_j\|_j \|v_{j+1}\|_{j+1} \cdots \|v_d\|_d \\
&\le C \left(\sum_{j=1}^{d} \frac{\prod_{k=1}^d \|v_k\|_k}{\|v_j\|_j} \right) \max_{1 \le k \le d} \|w_k\|_k
\end{aligned}$$

shows that $D\bigotimes(v_1, \ldots, v_d)$ is also continuous. Finally,

$$\begin{aligned}
&\| \otimes (v_1 + h_1, \cdots, v_d + h_d) - \otimes(v_1, \cdots, v_d) - D \otimes (v_1, \cdots, v_d)(h_1, \cdots, h_d)\| \\
&= \sum_{\substack{i_1, i_2 = 1 \\ i_1 < i_2}}^{d} \|T_{i_1, i_2}(h_{i_1}, h_{i_2}) + \sum_{\substack{i_1, i_2, i_3 = 1 \\ i_1 < i_2 < i_3}}^{d} T_{i_1, i_2, i_3}(h_{i_1}, h_{i_2}, h_{i_3}) + \ldots + T_{1, \ldots, d}(h_1, \ldots, h_d)\| \\
&\le \| \sum_{\substack{i_1, i_2 = 1 \\ i_1 < i_2}}^{d} T_{i_1, i_2}(h_{i_1}, h_{i_2})\| + \sum_{\substack{i_1, i_2, i_3 = 1 \\ i_1 < i_2 < i_3}}^{d} \|T_{i_1, i_2, i_3}(h_{i_1}, h_{i_2}, h_{i_3})\| + \ldots + \|T_{1, \ldots, d}(h_1, \ldots, h_d)\|
\end{aligned}$$

where the $T_{i_1, \ldots, i_k}$ are multilinear maps defined by $T_{i_1, \ldots, i_k}(h_{i_1}, \ldots, h_{i_k}) = \otimes_{j=1}^d z_j$ with $z_j = h_j$ if $j \in \{i_1, \ldots, i_k\}$, and $z_j = v_j$ otherwise. Since these multilinear maps have at least two arguments, we have

$$\begin{aligned}
\|T_{i_1, \ldots, i_k}(h_{i_1}, \ldots, h_{i_k})\| &\le C \prod_{j \in \{i_1, \ldots, i_k\}} \|h_j\|_j \prod_{j \in \{1, \ldots, d\} \setminus \{i_1, \ldots, i_k\}} \|v_j\|_j \\
&\le C \max_{1 \le j \le d} \|h_j\|_j \prod_{j \in \{i_2, \ldots, i_k\}} \|h_j\|_j \prod_{j \in \{1, \ldots, d\} \setminus \{i_1, \ldots, i_k\}} \|v_j\|_j \\
&= C \|(h_1, \ldots, h_d)\| \prod_{j \in \{i_2, \ldots, i_k\}} \|h_j\|_j \prod_{j \in \{1, \ldots, d\} \setminus \{i_1, \ldots, i_k\}} \|v_j\|_j
\end{aligned}$$

which proves that $\frac{\|T_{i_1, \ldots, i_k}(h_{i_1}, \ldots, h_{i_k})\|}{\|(h_1, \ldots, h_d)\|}$ tends to zero as $(h_1, \ldots, h_d) \to 0$, and therefore $\bigotimes$ is Fréchet differentiable and the proposition follows. ∎

Recall that $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$ is a tensor Banach space. Let $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \subset \mathbf{V}_{D_{\|\cdot\|_D}}$ be such that

$$\mathbf{v} = \sum_{\substack{1\le i_\alpha \le r_\alpha \\ \alpha\in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}^{(\alpha)}_{i_\alpha},$$

where for each $\mu \in T_D \setminus (\{D\} \cup \mathcal{L}(T_D))$ we have

$$\mathbf{u}^{(\mu)}_{i_\mu} = \sum_{\substack{1\le i_\beta \le r_\beta \\ \beta\in S(\alpha)}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta\in S(\mu)}} \bigotimes_{\beta\in S(\mu)} \mathbf{u}^{(\beta)}_{i_\beta}$$

for $1 \le i_\mu \le r_\mu$. Recall that for $\alpha \in S(D)$ we have $U^{\min}_{S(D)\setminus\{\alpha\}}(\mathbf{u}^{(\mu)}_{i_\mu}) = \operatorname{span}\{\mathbf{U}^{(\alpha)}_{i_\alpha} : 1 \le i_\alpha \le r_\alpha\}$, and for $\mu \in T_D \setminus (\{D\} \cup \mathcal{L}(T_D))$ we know that $U^{\min}_\beta(\mathbf{u}^{(\mu)}_{i_\mu}) = U^{\min}_\beta(\mathbf{v})$ and $U^{\min}_{S(\mu)\setminus\{\beta\}}(\mathbf{u}^{(\mu)}_{i_\mu}) = \operatorname{span}\{\mathbf{U}^{(\beta)}_{i_\mu,i_\beta} : 1 \le i_\beta \le r_\beta\}$ for $1 \le i_\mu \le r_\mu$ and $\beta \in S(\mu)$. Hence

$$W^{\min}_\beta(\mathbf{v}) = W^{\min}_\beta(\mathbf{u}^{(\mu)}_{i_\mu}) \text{ for } 1 \le i_\mu \le r_\mu \text{ and } \beta \in S(\mu).$$

Let us define the linear subspace

$$\mathbf{Z}^{(D)}_{big}(\mathbf{v}) := {}_a\bigotimes_{\alpha\in S(D)} U^{\min}_\alpha(\mathbf{v}) \oplus \left(\bigoplus_{\alpha\in S(D)} \mathbf{Z}^{(\alpha)}_{big}(\mathbf{v}) \otimes_a U^{\min}_{S(D)\setminus\{\alpha\}}(\mathbf{v}) \right),$$

where $\mathbf{Z}^{(\gamma)}_{big}(\mathbf{v}) := W^{\min}_\gamma(\mathbf{v})$ if $\gamma \in \mathcal{L}(T_D)$ or

$$\mathbf{Z}^{(\gamma)}_{big}(\mathbf{v}) := W^{\min}_\gamma(\mathbf{v}) \cap \left({}_a\bigotimes_{\beta\in S(\gamma)} U^{\min}_\beta(\mathbf{v}) \oplus \left(\bigoplus_{\beta\in S(\gamma)} \mathbf{Z}^{(\beta)}_{big}(\mathbf{v}) \otimes_a U^{\min}_{S(\gamma)\setminus\{\beta\}}(\mathbf{v}) \right) \right),$$

if $\gamma \notin \mathcal{L}(T_D)$. The next lemma describes the tangent map $\mathrm{T}_{\mathbf{v}}\mathfrak{i}$.

Proposition 5.5 *Assume that the tensor product map $\bigotimes$ is T_D-continuous. Let $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ be such that $\Theta_{\mathbf{v}}(\mathbf{v}) = (\mathfrak{C}(\mathbf{v}), \mathfrak{o})$, where $\mathfrak{C}(\mathbf{v}) = (C^{(\alpha)})_{\alpha\in T_D\setminus\mathcal{L}(T_D)} \in \mathbb{R}^{\mathfrak{r}}$, $\mathfrak{o} = (0)_{\alpha\in T_D\setminus\{D\}} \in \mathcal{L}_{T_D}(\mathbf{v})$ and*

$$U^{\min}_\alpha(\mathbf{v}) = \operatorname{span}\{\mathbf{u}^{(\alpha)}_{i_\alpha} : 1 \le i_\alpha \le r_\alpha\}$$

for $\alpha \in T_D \setminus \{D\}$. Then the following statements hold.

(a) The map $\mathfrak{i} \circ \Theta_{\mathbf{v}}^{-1}$ from $\mathbb{R}^{\mathfrak{r}}_* \times \mathcal{L}_{T_D}(\mathbf{v})$ to $\mathbf{V}_{D_{\|\cdot\|_D}}$ is Fréchet differentiable, and hence

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i} \in \mathcal{L}\left(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)), \mathbf{V}_{D_{\|\cdot\|_D}}\right).$$

(b) Assume $(\dot{\mathfrak{C}}, \dot{\mathfrak{L}}) \in \mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$, where $\dot{\mathfrak{C}} = (\dot{C}^{(\alpha)})_{\alpha\in T_D\setminus\mathcal{L}(T_D)} \in \mathbb{R}^{\mathfrak{r}}$ and $\dot{\mathfrak{L}} = (\dot{L}_\alpha)_{\alpha\in T_D\setminus\{D\}} \in \mathcal{L}_{T_D}(\mathbf{v})$. Then $\dot{\mathbf{w}} = \mathrm{T}_{\mathbf{v}}\mathfrak{i}(\dot{\mathfrak{C}}, \dot{\mathfrak{L}})$ if and only

$$\dot{\mathbf{w}} = \sum_{\substack{1\le i_\alpha \le r_\alpha \\ \alpha\in S(D)}} \dot{C}^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}^{(\alpha)}_{i_\alpha} + \sum_{\alpha\in S(D)} \sum_{1\le i_\alpha\le r_\alpha} \left(\dot{L}_\alpha(\mathbf{u}^{(\alpha)}_{i_\alpha}) \otimes \mathbf{U}^{(\alpha)}_{i_\alpha} \right), \tag{5.1}$$

where

$$\mathbf{U}^{(\alpha)}_{i_\alpha} = \sum_{\substack{1\le i_\beta \le r_\beta \\ \beta\in S(D) \\ \beta\neq\alpha}} C^{(D)}_{(i_\beta)_{\beta\in S(D)}} \bigotimes_{\beta\in S(D)} \mathbf{u}^{(\beta)}_{i_\beta}, \tag{5.2}$$

and for each $\gamma \in T_D \setminus \{D\}$ *with* $S(\gamma) \neq \emptyset$,

$$\dot{L}_\gamma(\mathbf{u}_{i_\gamma}^{(\gamma)}) = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta \in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)} + \sum_{\beta \in S(\gamma)} \sum_{1 \le i_\beta \le r_\beta} \left(\dot{L}_\beta(\mathbf{u}_{i_\beta}^{(\beta)}) \otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta} \right), \tag{5.3}$$

where

$$\mathbf{U}^{(\beta)}_{i_\gamma,i_\beta} = \sum_{\substack{1 \le i_\delta \le r_\delta \\ \delta \in S(\mu) \\ \delta \neq \beta}} C^{(\gamma)}_{i_\mu,(i_\delta)_{\delta\in S(\gamma)}} \bigotimes_{\substack{\delta \neq \beta \\ \delta \in S(\gamma)}} \mathbf{u}_{i_\delta}^{(\delta)}, \tag{5.4}$$

for $1 \le i_\gamma \le r_\gamma$ *and* $1 \le i_\beta \le r_\beta$.

(c) The inclusion $\mathrm{T}_{\mathbf{v}}\mathbf{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))) \subset \mathbf{Z}^{(D)}_{big}(\mathbf{v})$ *holds and* $\mathrm{T}_{\mathbf{v}}\mathbf{i}$ *is an injective linear operator.*

Proof. To prove statement (a), observe that for each $\mathbf{u}_\alpha \in U_\alpha^{\min}(\mathbf{v})$, $\alpha \in T_D \setminus \{D\}$, the map

$$\Phi_{\mathbf{u}_\alpha} : \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})) \to W_\alpha^{\min}(\mathbf{v}), \quad L_\alpha \mapsto L_\alpha(\mathbf{u}_\alpha),$$

is linear and continuous, and hence Fréchet differentiable. Clearly, its differential is given by $[\Phi'_{\mathbf{u}_\alpha}(L_\alpha)](H_\alpha) = H_\alpha(\mathbf{u}_\alpha)$. Also, if the tensor product map $\bigotimes$ is T_D-continuous, by Proposition 5.4, the tensor product map $\bigotimes : \times_{\beta\in S(\gamma)}(V_{\beta_{\|\cdot\|_\beta}}, \|\cdot\|_\beta) \to \left({}_a\bigotimes_{\beta\in S(\gamma)} V_{\beta_{\|\cdot\|_\beta}}, \|\cdot\|_\gamma \right)$, for $\gamma \in T_D \setminus \mathcal{L}(T_D)$, is also Fréchet differentiable. Then, by the chain rule, the map $\Theta_{\mathbf{v}}^{-1}$ is Fréchet differentiable. Since $\mathrm{T}_{\mathbf{v}}\mathbf{i}(\dot{\mathfrak{C}}, \dot{\mathfrak{L}}) = [(\mathbf{i} \circ \Theta_{\mathbf{v}}^{-1})'(\mathfrak{C}, \mathfrak{o})](\dot{\mathfrak{C}}, \dot{\mathfrak{L}})$, (a) follows. By using the chain rule we obtain

$$\dot{\mathbf{w}} = \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} \dot{C}^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha \in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)} + \sum_{\mu \in S(D)} \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \left(\dot{L}_\mu(\mathbf{u}_{i_\mu}^{(\mu)}) \otimes \bigotimes_{\substack{\alpha \in S(D) \\ \alpha \neq \mu}} \mathbf{u}_{i_\alpha}^{(\alpha)} \right),$$

where for each $\gamma \in T_D \setminus \{D\}$ with $S(\gamma) \neq \emptyset$,

$$\dot{L}_\gamma(\mathbf{u}_{i_\gamma}^{(\gamma)}) = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta \in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)} + \sum_{\delta \in S(\gamma)} \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\gamma)}} C^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \left(\dot{L}_\delta(\mathbf{u}_{i_\delta}^{(\delta)}) \otimes \bigotimes_{\substack{\beta \neq \delta \\ \beta \in S(\gamma)}} \mathbf{u}_{i_\beta}^{(\beta)} \right)$$

holds for $1 \le i_\gamma \le r_\gamma$. From (3.5), (3.6), (3.8) and (3.9), we obtain (5.1) and (5.3) and statement (b) is proved.

To prove the first statement of (c), observe that $\dot{L}_\gamma(\mathbf{u}_{i_\gamma}^{(\gamma)}) \in \mathbf{Z}^{(\gamma)}_{big}(\mathbf{v})$ for $1 \le i_\gamma \le r_\gamma$ and $\gamma \in T_D \setminus \{D\}$, that is, $\dot{\mathbf{w}} \in \mathbf{Z}^{(D)}_{big}(\mathbf{v})$. Then the inclusion

$$\mathrm{T}_{\mathbf{v}}\mathbf{i}\,(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)) \subset \mathbf{Z}^{(D)}_{big}(\mathbf{v}),$$

follows. Now, consider that

$$\mathrm{T}_{\mathbf{v}}\mathbf{i}\left((\dot{C}^{(\alpha)})_{\alpha\in T_D\setminus\mathcal{L}(T_D)}, (\dot{L}_\beta)_{\beta\in T_D\setminus\{D\}} \right) = \mathbf{0},$$

that is,

$$\mathbf{0} = \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} (\dot{C}^{(D)})_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha \in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)} + \sum_{\alpha \in S(D)} \sum_{1 \le i_\alpha \le r_\alpha} \left(\dot{L}_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) \otimes \mathbf{U}_{i_\alpha}^{(\alpha)} \right).$$

Thus,

$$\sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}} (\dot{C}^{(D)})_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}^{(\alpha)}_{i_\alpha} = \mathbf{0},$$

$$\sum_{1\le i_\alpha\le r_\alpha} \left(\dot{L}_\alpha(\mathbf{u}^{(\alpha)}_{i_\alpha}) \otimes \mathbf{U}^{(\alpha)}_{i_\alpha}\right) = \mathbf{0} \text{ for } \alpha\in S(D),$$

and hence $\dot{C}^{(D)} = \mathfrak{o}$, because $\left\{\bigotimes_{\alpha\in S(D)} \mathbf{u}^{(\alpha)}_{i_\alpha}\right\}$ is a basis of ${}_a\bigotimes_{\alpha\in S(D)} U^{\min}_\alpha(\mathbf{v})$, and $\dot{L}_\alpha(\mathbf{u}^{(\alpha)}_{i_\alpha}) \otimes \mathbf{U}^{(\alpha)}_{i_\alpha} = \mathbf{0}$ for $1\le i_\alpha\le r_\alpha$, because the $\{\mathbf{U}^{(\alpha)}_{i_\alpha} : 1\le i_\alpha\le r_\alpha\}$ are linearly independent for $\alpha\in S(D)$. Then $\dot{L}_\alpha = 0$ for all $\alpha\in S(D)$. Proceeding inductively from the root to the leaves, assume that for $\gamma\in T_D\setminus\{D\}$ such that $\gamma\notin\mathcal{L}(T_D)$, we have

$$\mathbf{0} = \dot{L}_\gamma(\mathbf{u}^{(\gamma)}_{i_\gamma}) = \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(\gamma)}} (\dot{C}^{(\gamma)})_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta\in S(\gamma)} \mathbf{u}^{(\beta)}_{i_\beta} + \sum_{\beta\in S(\gamma)} \sum_{1\le i_\beta\le r_\beta} \left(\dot{L}_\beta(\mathbf{u}^{(\beta)}_{i_\beta}) \otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta}\right),$$

for $1\le i_\gamma\le r_\gamma$. Thus,

$$\sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(\gamma)}} (\dot{C}^{(\gamma)})_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta\in S(\gamma)} \mathbf{u}^{(\beta)}_{i_\beta} = \mathbf{0},$$

$$\sum_{1\le i_\beta\le r_\beta} \left(\dot{L}_\beta(\mathbf{u}^{(\beta)}_{i_\beta}) \otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta}\right) = \mathbf{0} \text{ for } \beta\in S(\gamma).$$

As in the root case we obtain $\dot{C}^{(\gamma)} = \mathfrak{o}$ from the first equation and

$$\sum_{1\le i_\beta\le r_\beta} \left(\dot{L}_\beta(\mathbf{u}^{(\beta)}_{i_\beta}) \otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta}\right) = \mathbf{0},$$

for $1\le i_\gamma\le r_\gamma$ and $\beta\in S(\gamma)$, from the second one. Since $\{\mathbf{U}^{(\beta)}_{i_\gamma,i_\beta} : 1\le i_\beta\le r_\beta\}$ are linearly independent for $1\le i_\gamma\le r_\gamma$ and $\beta\in S(\gamma)$ we have $\dot{L}_\beta(\mathbf{u}^{(\beta)}_{i_\beta}) = \mathbf{0}$ for $1\le i_\beta\le r_\beta$ and $\beta\in S(\gamma)$. We conclude, that

$$\left((\dot{C}^{(\alpha)})_{\alpha\in T_D\setminus\mathcal{L}(T_D)}, (\dot{L}_\beta)_{\beta\in T_D\setminus\{D\}}\right) = ((\mathfrak{o})_{\alpha\in T_D\setminus\mathcal{L}(T_D)}, (0)_{\beta\in T_D\setminus\{D\}}),$$

and, in consequence, $\mathrm{T}_{\mathbf{v}}\mathbf{i}$ is injective. ∎

The next corollary says us that for Tucker tensors the linear subspace $\mathbf{Z}^{(D)}_{big}(\mathbf{v})$ characterises the tangent space at $\mathbf{v}$ in the manifold inside the tensor space $\mathbf{V}_{D_{\|\cdot\|_D}}$.

Corollary 5.6 *Assume that $S(D) = \mathcal{L}(T_D)$ and the tensor product map $\bigotimes$ is T_D-continuous. Let $\mathbf{v}\in\mathcal{M}_{\mathbf{r}}(\mathbf{V}_D)$, then $\mathrm{T}_{\mathbf{v}}\mathbf{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{M}_{\mathbf{r}}(\mathbf{V}_D))) = \mathbf{Z}^{(D)}_{big}(\mathbf{v})$ and hence*

$$\mathbf{Z}^{(D)}_{big}(\mathbf{v}) = {}_a\bigotimes_{\alpha\in S(D)} U^{\min}_\alpha(\mathbf{v}) \oplus \left(\bigoplus_{\alpha\in S(D)} W^{\min}_\alpha(\mathbf{v}) \otimes_a U^{\min}_{S(D)\setminus\{\alpha\}}(\mathbf{v})\right)$$

is linearly isomorphic to $\mathbb{T}_{\mathbf{v}}(\mathcal{M}_{\mathbf{r}}(\mathbf{V}_D))$.

Proof. First, we claim that $\mathbf{Z}^{(D)}_{big}(\mathbf{v}) \subset \mathrm{T}_{\mathbf{v}}\mathbf{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{M}_{\mathbf{r}}(\mathbf{V}_D)))$. Then the corollary follows from Proposition 5.5(c) and the above claim. To prove the claim take $\mathbf{w}\in\mathbf{Z}^{(D)}_{big}(\mathbf{v})$. Then we can write

$$\mathbf{w} = \sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}} (\dot{C}^{(D)})_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}^{(\alpha)}_{i_\alpha} + \sum_{\alpha\in S(D)} \sum_{1\le i_\alpha\le r_\alpha} \left(\mathbf{w}^{(\alpha)}_{i_\alpha} \otimes \mathbf{U}^{(\alpha)}_{i_\alpha}\right),$$

where $\mathbf{w}_{i_\alpha}^{(\alpha)} \in \mathbf{Z}_{big}^{(\alpha)}(\mathbf{v}) = W_\alpha^{\min}(\mathbf{v})$ for $1 \le i_\alpha \le r_\alpha$ and $\alpha \in S(D)$. Recall that $U_{S(D)\setminus\{\alpha\}}^{\min}(\mathbf{v}) = \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$. Now, define $\dot{L}_\alpha \in \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$ by $\dot{L}_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) := \mathbf{w}_{i_\alpha}^{(\alpha)}$ for $1 \le i_\alpha \le r_\alpha$ and $\alpha \in S(D)$. Then the claim follows from $\mathbf{w} = \mathrm{T}_{\mathbf{v}}\mathfrak{i}(\dot{C}^{(D)}, (\dot{L}_\alpha)_{\alpha\in S(D)})$. ∎

Now, our next step is to construct for each $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ a linear space, namely $\mathbf{Z}^{(D)}(\mathbf{v}) \subset \mathbf{Z}_{big}^{(D)}(\mathbf{v})$, such that $\mathbf{Z}^{(D)}(\mathbf{v}) = \mathrm{T}_{\mathbf{v}}\mathfrak{i}\left(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)$.

To this end, assume that the tensor product map is T_D-continuous and for $\gamma \in T_D \setminus (\{D\} \cup \mathcal{L}(T_D))$ consider

$$\mathbf{u}_{i_\gamma}^{(\gamma)} = \sum_{\substack{1\le i_\beta \le r_\beta \\ \beta\in S(\gamma)}} C_{i_\gamma,(i_\beta)_{\beta\in S(\alpha)}}^{(\alpha)} \bigotimes_{\beta\in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)} \in \mathcal{M}_{(r_\beta)_{\beta\in S(\gamma)}}\left({}_a\bigotimes_{\beta\in S(\gamma)} \mathbf{V}_\beta\right)$$

for $1 \le i_\gamma \le r_\gamma$ and $\beta \in S(\gamma)$. Let

$$\mathfrak{i} : \mathcal{M}_{(r_\beta)_{\beta\in S(\gamma)}}\left({}_a\bigotimes_{\beta\in S(\gamma)} \mathbf{V}_\beta\right) \longrightarrow \mathbf{V}_{\gamma_{\|\cdot\|_\gamma}}$$

be the standard inclusion map. Then we have a linear injective map

$$\mathrm{T}_{\mathbf{u}_{i_\gamma}^{(\gamma)}}\mathfrak{i} : \mathbb{R}^{\bigtimes_{\beta\in S(\gamma)} r_\beta} \times \bigtimes_{\beta\in S(\gamma)} \mathcal{L}(U_\beta^{\min}(\mathbf{v}), W_\beta^{\min}(\mathbf{v})) \to \mathbf{V}_{\gamma_{\|\cdot\|_\gamma}}$$

given by

$$\mathrm{T}_{\mathbf{u}_{i_\gamma}^{(\gamma)}}\mathfrak{i}(\dot{C}_{i_\gamma}^{(\gamma)}, (\dot{L}_\beta)_{\beta\in S(\gamma)}) = \sum_{\substack{1\le i_\beta \le r_\beta \\ \beta\in S(\gamma)}} \dot{C}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}}^{(\gamma)} \bigotimes_{\beta\in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)} + \sum_{\beta\in S(\gamma)} \sum_{1\le i_\beta\le r_\beta} \dot{L}_\beta(\mathbf{u}_{i_\beta}^{(\beta)}) \otimes \mathbf{U}_{i_\gamma,i_\beta}^{(\beta)},$$

where $\mathbf{U}_{i_\gamma,i_\beta}^{(\beta)} = \sum_{\substack{1\le i_\delta\le r_\delta \\ \delta\in S(\gamma) \\ \delta\ne\beta}} C_{i_\gamma,(i_\delta)_{\delta\in S(\gamma)}}^{(\alpha)} \bigotimes_{\delta\in S(\gamma)} \mathbf{u}_{i_\delta}^{(\delta)}$ for $1 \le i_\beta \le r_\beta$ and $\beta \in S(\gamma)$. We have a linear subspace

$$\begin{aligned}\mathbf{Z}_{big}^{(\gamma)}(\mathbf{u}_{j_\gamma}^{(\gamma)}) &:= \mathrm{T}_{\mathbf{u}_{j_\gamma}^{(\gamma)}}\mathfrak{i}\left(\mathbb{R}^{\bigtimes_{\beta\in S(\gamma)} r_\beta} \times \bigtimes_{\beta\in S(\gamma)} \mathcal{L}(U_\beta^{\min}(\mathbf{v}), W_\beta^{\min}(\mathbf{v}))\right) \\ &\cong \mathbb{R}^{\bigtimes_{\beta\in S(\gamma)} r_\beta} \times \bigtimes_{\beta\in S(\gamma)} \mathcal{L}(U_\beta^{\min}(\mathbf{v}), W_\beta^{\min}(\mathbf{v}))\end{aligned}$$

for $1 \le j_\gamma \le r_\gamma$ and following the proof of Corollary 5.6, it can be shown that

$$\mathbf{Z}_{big}^{(\gamma)}(\mathbf{u}_{j_\gamma}^{(\gamma)}) = {}_a\bigotimes_{\beta\in S(\gamma)} U_\beta^{\min}(\mathbf{v}) \oplus \left(\bigoplus_{\beta\in S(\gamma)} W_\beta^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\left\{\mathbf{U}_{j_\gamma,i_\beta} : 1 \le i_\beta \le r_\beta\right\}\right)$$

for $1 \le j_\gamma \le r_\gamma$. Let $\pi_{i_\gamma} : \mathbb{R}^{r_\gamma \times \bigtimes_{\beta\in S(\gamma)} r_\beta} \to \mathbb{R}^{\bigtimes_{\beta\in S(\gamma)} r_\beta}$ be given by $\pi_{i_\gamma}(\dot{C}^{(\gamma)}) = \dot{C}_{i_\gamma}^{(\gamma)}$, for $1 \le i_\gamma \le r_\gamma$. Then we can write

$$\dot{L}_\gamma(\mathbf{u}_{i_\gamma}^{(\gamma)}) = \mathrm{T}_{\mathbf{u}_{i_\gamma}^{(\gamma)}}\mathfrak{i}(\pi_{i_\gamma}(\dot{C}^{(\gamma)}), (\dot{L}_\beta)_{\beta\in S(\gamma)}),$$

for $1 \le i_\gamma \le r_\gamma$.

Now, for each $\gamma \in T_D \setminus \{D\}$ we define a linear subspace $\mathcal{H}_\gamma(\mathbf{v}) \subset W_\gamma^{\min}(\mathbf{v})^{r_\gamma}$ as follows. Let $\mathcal{H}_\gamma(\mathbf{v}) := W_\gamma^{\min}(\mathbf{v})^{r_\gamma}$ if $\gamma \in \mathcal{L}(T_D)$. For $\gamma \notin \mathcal{L}(T_D)$ we construct $\mathcal{H}_\gamma(\mathbf{v})$ in the following way. Let

$$\Upsilon_{\gamma,\mathbf{v}} : \mathbb{R}^{r_\gamma \times \bigtimes_{\beta\in S(\gamma)} r_\beta} \times \bigtimes_{\beta\in S(\gamma)} \mathcal{H}_\beta(\mathbf{v}) \longrightarrow W_\gamma^{\min}(\mathbf{v})^{r_\gamma}$$

be a linear map defined by

$$\Upsilon_{\gamma,\mathbf{v}}(\dot{C}^{(\gamma)}, ((\mathbf{w}_{i_\beta}^{(\beta)})_{i_\beta=1}^{r_\beta})_{\beta\in S(\gamma)}) := (\mathbf{w}_{i_\gamma}^{(\gamma)})_{i_\gamma=1}^{r_\gamma},$$

where

$$\mathbf{w}_{i_\gamma}^{(\gamma)} := \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta\in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)} + \sum_{\beta\in S(\gamma)} \sum_{1\le i_\beta\le r_\beta} \mathbf{w}_{i_\beta}^{(\beta)} \otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta},$$

for $1 \le i_\gamma \le r_\gamma$. Observe that if we define $\dot{L}_\gamma(\mathbf{u}_{i_\gamma}^{(\gamma)}) := \mathbf{w}_{i_\gamma}^{(\gamma)}$ for $1 \le i_\gamma \le r_\gamma$ and $\dot{L}_\beta(\mathbf{u}_{i_\beta}) := \mathbf{w}_{i_\beta}^{(\beta)}$ for $1 \le i_\beta \le r_\beta$ and $\beta \in S(\gamma)$, then

$$\mathbf{w}_{i_\gamma}^{(\gamma)} = \mathrm{T}_{\mathbf{u}_{i_\gamma}^{(\gamma)}} \mathfrak{i}(\pi_{i_\gamma}(\dot{C}^{(\gamma)}), (\dot{L}_\beta)_{\beta\in S(\gamma)}) \in \mathbf{Z}_{big}^{(\gamma)}(\mathbf{u}_{i_\gamma}^{(\gamma)}),$$

for $1 \le i_\gamma \le r_\gamma$, and hence by Proposition 5.5 the map $\Upsilon_{\gamma,\mathbf{v}}$ is injective. Finally, we define the linear subspace

$$\mathcal{H}_\gamma(\mathbf{v}) := \Upsilon_{\gamma,\mathbf{v}}\left(\mathbb{R}^{r_\gamma\times\bigtimes_{\beta\in S(\gamma)} r_\beta} \times \bigtimes_{\beta\in S(\gamma)} \mathcal{H}_\beta(\mathbf{v})\right).$$

For $\delta \in T_D \setminus \{D\}$ let $\Pi_{i_\delta} : W_\delta^{\min}(\mathbf{v})^{r_\delta} \to W_\delta^{\min}(\mathbf{v})$ be given by $\Pi_{i_\delta}((\mathbf{w}_{k_\delta}^{(\delta)})_{k_\delta=1}^{r_\delta}) := \mathbf{w}_{i_\delta}^{(\delta)}$ for $1 \le i_\delta \le r_\delta$. Observe, that for each $\beta \in S(\gamma)$, we can identify $(\mathbf{w}_{i_\beta}^{(\beta)})_{i_\beta=1}^{r_\beta} \in \mathcal{H}_\beta(\mathbf{v})$ with

$$\sum_{1\le i_\beta\le r_\beta} \mathbf{w}_{i_\beta}^{(\beta)} \otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta} = \sum_{1\le i_\beta\le r_\beta} \Pi_{i_\beta}((\mathbf{w}_{k_\beta}^{(\beta)})_{k_\beta=1}^{r_\beta}) \otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta}$$

for $1 \le i_\gamma \le r_\gamma$. It allows us to construct an injective linear map

$$f_{\beta,i_\gamma} : \mathcal{H}_\beta(\mathbf{v}) \longrightarrow V_{\gamma_{\|\cdot\|_\gamma}}, \quad (\mathbf{w}_{i_\beta}^{(\beta)})_{i_\beta=1}^{r_\beta} \mapsto \sum_{1\le i_\beta\le r_\beta} \mathbf{w}_{i_\beta}^{(\beta)} \otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta},$$

for $1 \le i_\gamma \le r_\gamma$. Hence $f_{\beta,i_\gamma}(\mathcal{H}_\beta(\mathbf{v}))$ is a linear subspace of $V_{\gamma_{\|\cdot\|_\gamma}}$ linearly isomorphic to $\mathcal{H}_\beta(\mathbf{v})$ for $1 \le i_\gamma \le i_\gamma$. Thus,

$$\Pi_{i_\gamma}(\mathcal{H}_\gamma(\mathbf{v})) = \begin{cases} {}_a\bigotimes_{\beta\in S(\gamma)} U_\beta^{\min}(\mathbf{v}) \oplus \left(\bigoplus_{\beta\in S(\gamma)} f_{\beta,i_\gamma}(\mathcal{H}_\beta(\mathbf{v}))\right) & \text{if } \gamma \notin \mathcal{L}(T_D),\\ W_\gamma^{\min}(\mathbf{v}) & \text{if } \gamma \in \mathcal{L}(T_D), \end{cases}$$

where

$$f_{\beta,i_\gamma}(\mathcal{H}_\beta(\mathbf{v})) = \begin{cases} \bigoplus_{i_\beta=1}^{r_\beta} \Pi_{i_\beta}(\mathcal{H}_\beta(\mathbf{v})) \otimes_a \operatorname{span}\{\mathbf{U}^{(\beta)}_{i_\gamma,i_\beta}\} & \text{if } \beta \notin \mathcal{L}(T_D)\\ \bigoplus_{i_\beta=1}^{r_\beta} W_\beta^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}^{(\beta)}_{i_\gamma,i_\beta}\} & \text{if } \beta \in \mathcal{L}(T_D) \end{cases}$$

for $1 \le i_\gamma \le r_\gamma$.

Finally, we construct a linear subspace $\mathbf{Z}^{(D)}(\mathbf{v}) \subset \mathbf{V}_{D_{\|\cdot\|_D}}$ by using a linear injective map

$$\Upsilon_{D,\mathbf{v}} : \mathbb{R}^{\bigtimes_{\alpha\in S(D)} r_\alpha} \times \bigtimes_{\alpha\in S(D)} \mathcal{H}_\alpha(\mathbf{v}) \longrightarrow \mathbf{V}_{D_{\|\cdot\|_D}}$$

defined by

$$\Upsilon_{\gamma,\mathbf{v}}(\dot{C}^{(D)}, ((\mathbf{w}_{i_\alpha}^{(\alpha)})_{i_\alpha=1}^{r_\alpha})_{\alpha\in S(D)}) := \mathbf{w}$$

where

$$\mathbf{w} := \sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}} \dot{C}^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)} + \sum_{\alpha\in S(D)} \sum_{1\le i_\alpha\le r_\alpha} \mathbf{w}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)}.$$

Then $\mathbf{Z}^{(D)}(\mathbf{v}) := \Upsilon_{D,\mathbf{v}}\left(\mathbb{R}^{\bigtimes_{\alpha\in S(D)} r_\alpha} \times \bigtimes_{\alpha\in S(D)} \mathcal{H}_\alpha(\mathbf{v})\right)$. Moreover, we can introduce for each $\alpha \in S(D)$ a linear injective map

$$f_{D,\alpha} : \mathcal{H}_\alpha(\mathbf{v}) \to \mathbf{V}_{D_{\|\cdot\|_D}}, \quad (\mathbf{w}_{i_\alpha})_{i_\alpha=1}^{r_\alpha} \mapsto \sum_{1\le i_\alpha\le r_\alpha} \mathbf{w}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)}.$$

Then $f_\alpha(\mathcal{H}_\alpha(\mathbf{v}))$ is a linear subspace in $\mathbf{V}_{D_{\|\cdot\|_D}}$ linearly isomorphic to $\mathcal{H}_\alpha(\mathbf{v})$. It is not difficult to show that

$$f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v})) = \begin{cases} \bigoplus_{i_\alpha=1}^{r_\alpha} \Pi_{i_\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\} & \text{if } \alpha \notin \mathcal{L}(T_D) \\ \bigoplus_{i_\alpha=1}^{r_\alpha} W_\alpha^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\} & \text{if } \alpha \in \mathcal{L}(T_D) \end{cases}$$

for $\alpha \in S(D)$. By construction, we have

$$\mathbf{Z}^{(D)}(\mathbf{v}) = {}_a \bigotimes_{\alpha \in S(D)} U_\alpha^{\min}(\mathbf{v}) \oplus \left(\bigoplus_{\alpha \in S(D)} f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \right).$$

Corollary 5.7 *Assume that $S(D) \neq \mathcal{L}(T_D)$ and the tensor product map $\bigotimes$ is T_D-continuous. Let $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$, then $\mathrm{T}_{\mathbf{v}}\mathbf{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))) = \mathbf{Z}^{(D)}(\mathbf{v})$ and hence it is linearly isomorphic to $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$.*

Proof. From Proposition 5.5 and the construction of $\mathbf{Z}^{(D)}(\mathbf{v})$, the inclusion $\mathrm{T}_{\mathbf{v}}\mathbf{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))) \subset \mathbf{Z}^{(D)}(\mathbf{v})$ holds. Now, take $\mathbf{w} \in \mathbf{Z}^{(D)}(\mathbf{v})$. Then we can write

$$\mathbf{w} = \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} (\dot{C}^{(D)})_{(i_\alpha)_{\alpha \in S(D)}} \bigotimes_{\alpha \in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)} + \sum_{\alpha \in S(D)} \sum_{1 \le i_\alpha \le r_\alpha} \left(\mathbf{w}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)} \right),$$

where $\dot{C}^{(D)} \in \mathbb{R}^{\bigtimes_{\alpha \in S(D)} r_\alpha}$ and $\mathbf{w}_{i_\alpha}^{(\alpha)} \in W_\alpha^{\min}(\mathbf{v})$ for $1 \le i_\alpha \le r_\alpha$. Then we can define $\dot{L}_\alpha \in \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$ by $\dot{L}_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) := \mathbf{w}_{i_\alpha}^{(\alpha)}$ for $1 \le i_\alpha \le r_\alpha$, and we have

$$(\dot{C}^{(D)}, (\dot{L}_\alpha)_{\alpha \in S(D)}) \in \mathbb{R}^{\bigtimes_{\alpha \in S(D)} r_\alpha} \times \bigtimes_{\alpha \in S(D)} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})).$$

Moreover, $\sum_{1 \le i_\alpha \le r_\alpha} \mathbf{w}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)} \in f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v}))$ for $\alpha \in S(D)$. If $\alpha \notin \mathcal{L}(T_D)$, then $(\mathbf{w}_{i_\alpha}^{(\alpha)})_{i_\alpha=1}^{r_\alpha} \in \mathcal{H}_\alpha(\mathbf{v}) = \Upsilon_{\alpha,\mathbf{v}} \left(\mathbb{R}^{r_\alpha \times \bigtimes_{\beta \in S(\gamma)} r_\beta} \times \bigtimes_{\beta \in S(\alpha)} \mathcal{H}_\beta(\mathbf{v}) \right)$. Hence there exists

$$(\dot{C}^{(\alpha)}, ((\mathbf{w}_{i_\beta}^{(\beta)})_{i_\beta=1}^{r_\beta})_{\beta \in S(\alpha)}) \in \mathbb{R}^{r_\alpha \times \bigtimes_{\beta \in S(\alpha)} r_\beta} \times \bigtimes_{\beta \in S(\alpha)} \mathcal{H}_\beta(\mathbf{v})$$

such that

$$\mathbf{w}_{i_\alpha}^{(\alpha)} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\alpha)}} \dot{C}_{i_\alpha,(i_\beta)_{\beta \in S(\alpha)}}^{(\alpha)} \bigotimes_{\beta \in S(\alpha)} \mathbf{u}_{i_\beta}^{(\beta)} + \sum_{\beta \in S(\alpha)} \sum_{1 \le i_\beta \le r_\beta} \mathbf{w}_{i_\beta}^{(\beta)} \otimes \mathbf{U}_{i_\alpha,i_\beta}^{(\beta)}$$

for $1 \le i_\alpha \le r_\alpha$. Define $\dot{L}_\beta(\mathbf{u}_{i_\beta}^{(\beta)}) := \mathbf{w}_{i_\beta}^{(\beta)}$ for $1 \le i_\beta \le r_\beta$ and $\beta \in S(\alpha)$. Then

$$(\dot{C}^{(\alpha)}, (\dot{L}_\beta)_{\beta \in S(\alpha)}) \in \mathbb{R}^{r_\gamma \times \bigtimes_{\beta \in S(\gamma)} r_\beta} \times \bigtimes_{\beta \in S(\alpha)} \mathcal{L}(U_\beta^{\min}(\mathbf{v}), W_\beta^{\min}(\mathbf{v})).$$

Moreover, $\sum_{1 \le i_\beta \le r_\beta} \mathbf{w}_{i_\beta}^{(\beta)} \otimes \mathbf{U}_{i_\alpha,i_\beta}^{(\beta)} \in f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v}))$ for $1 \le i_\alpha \le r_\alpha$. If $\beta \notin \mathcal{L}(T_D)$, then $(\mathbf{w}_{i_\beta}^{(\beta)})_{i_\beta=1}^{r_\beta} \in \mathcal{H}_\beta(\mathbf{v}) = \Upsilon_{\beta,\mathbf{v}} \left(\mathbb{R}^{r_\beta \times \bigtimes_{\gamma \in S(\beta)} r_\gamma} \times \bigtimes_{\gamma \in S(\beta)} \mathcal{H}_\gamma(\mathbf{v}) \right)$. Proceeding in a similar way from the root to the leaves, we construct $(\dot{\mathfrak{C}}, \dot{\mathfrak{L}}) \in \mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$, where $\dot{\mathfrak{C}} = (\dot{C}^{(\alpha)})_{\alpha \in T_D \setminus \mathcal{L}(T_D)} \in \mathbb{R}^{\mathfrak{r}}$ and $\dot{\mathfrak{L}} = (\dot{L}_\alpha)_{\alpha \in T_D \setminus \{D\}} \in \mathcal{L}_{T_D}(\mathbf{v})$ such that $\mathbf{w} = \mathrm{T}_{\mathbf{v}}\mathbf{i}(\dot{\mathfrak{C}}, \dot{\mathfrak{L}})$. Thus, we can conclude that $\mathbf{Z}^{(D)}(\mathbf{v}) \subset \mathrm{T}_{\mathbf{v}}\mathbf{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)))$ and the equality follows. ∎

Example 5.8 *Consider the binary tree T_D given in Figure 5.1 and consider TB ranks $\mathfrak{r} = (1, r_1, r_{23}, r_2, r_3)$. Let $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(V_1 \otimes_a V_2 \otimes_a V_3)$ and assume that the tensor product map $\bigotimes$ is T_D-continuous. Then*

$$\mathbf{Z}^{(123)}(\mathbf{v}) = \left(U_1^{\min}(\mathbf{v}) \otimes_a U_{23}^{\min}(\mathbf{v}) \right) \oplus f_{123,1}(\mathcal{H}_1(\mathbf{v})) \oplus f_{123,23}(\mathcal{H}_{23}(\mathbf{v})),$$

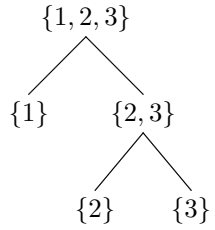

Figure 5.1: A binary tree T_D.

where

$$f_{123,1}(\mathcal{H}_1(\mathbf{v})) = \bigoplus_{i_1=1}^{r_1} W_1^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_1}^{(1)}\} \subset V_{1_{\|\cdot\|_1}} \otimes_a \left(V_{2_{\|\cdot\|_2}} \otimes_a V_{3_{\|\cdot\|_3}}\right),$$

$$f_{123,23}(\mathcal{H}_{23}(\mathbf{v})) = \bigoplus_{i_{23}=1}^{r_{23}} \operatorname{span}\{\mathbf{U}_{i_{23}}^{(23)}\} \otimes_a \Pi_{i_{23}}(\mathcal{H}_{23}(\mathbf{v})) \subset V_{1_{\|\cdot\|_1}} \otimes_a \left(V_{2_{\|\cdot\|_2}} \otimes_a V_{3_{\|\cdot\|_3}}\right),$$

and

$$\begin{aligned}\Pi_{i_{23}}(\mathcal{H}_{23}(\mathbf{v})) &= \left(U_2^{\min}(\mathbf{v}) \otimes_a U_3^{\min}(\mathbf{v})\right) \\ &\oplus \left(\bigoplus_{i_2=1}^{r_2} W_2^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_{23},i_2}^{(2)}\}\right) \oplus \left(\bigoplus_{i_3=1}^{r_3} \operatorname{span}\{\mathbf{U}_{i_{23},i_3}^{(3)}\} \otimes_a W_3^{\min}(\mathbf{v})\right),\end{aligned}$$

which is a linear subspace in $V_{2_{\|\cdot\|_2}} \otimes_a V_{3_{\|\cdot\|_3}}$.

5.2 Is the standard inclusion map an immersion?

Next we recall the definition of an immersion between manifolds.

Definition 5.9 *Let $F : X \to Y$ be a morphism between Banach manifolds and let $x \in X$. We shall say that F is an* immersion *at x, if there exists an open neighbourhood X_x of x in X such that the restriction of F to X_x induces an isomorphism from X_x onto a submanifold of Y. We say that F is an* immersion *if it is an immersion at each point of X.*

For Banach manifolds we have the following criterion for immersions (see Theorem 3.5.7 in [22]).

Proposition 5.10 *Let X, Y be Banach manifolds of class $\mathcal{C}^p$ $(p \geq 1)$. Let $F : X \to Y$ be a $\mathcal{C}^p$ morphism and $x \in X$. Then F is an immersion at x if and only if $\mathrm{T}_x F$ is injective and $\mathrm{T}_x F(\mathbb{T}_x(X)) \in \mathbb{G}(Y)$.*

A concept related with an immersion between Banach manifolds is the following.

Definition 5.11 *Assume that X and Y are Banach manifolds and let $f : X \longrightarrow Y$ be a $\mathcal{C}^r$ morphism. If f is an injective immersion, then $f(X)$ is called an* immersed submanifold *of Y.*

Recall that there exists injective immersions which are not isomorphisms onto manifolds. It allows us to introduce the following definition.

Definition 5.12 *An injective immersion $f : X \longrightarrow Y$ which is a homeomorphism onto $f(X)$ with the relative topology induced from Y is called an* embedding*. Moreover, if $f : X \longrightarrow Y$ is an embedding, then $f(X)$ is called an* embedded submanifold *of Y.*

A classical example of an immersed submanifold which is not an embedded submanifold is given by the map $f : (3\pi/4, 7\pi/4) \longrightarrow \mathbb{R}^2$, written in polar coordinates by $r = \cos 2\theta$. It can be see that f is an injective immersion however $f(3\pi/4, 7\pi/4)$ is not an open set in $\mathbb{R}^2$, because any neighborhood of 0 in $\mathbb{R}^2$ intersects $f(3\pi/4, 7\pi/4)$ in a set with "corners" which is not homeomorphic to an open interval (see Figure 5.2).

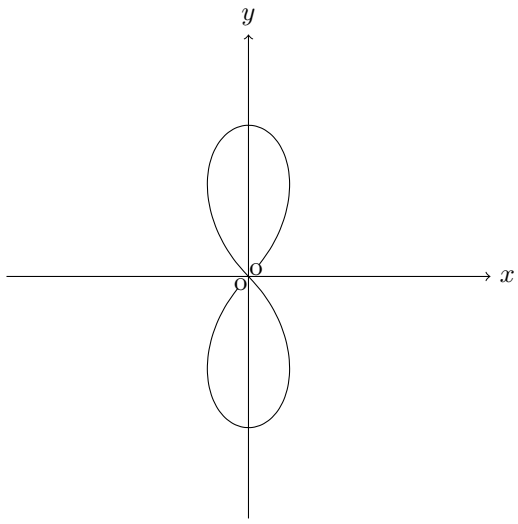

Figure 5.2: The set $f(3\pi/4, 7\pi/4)$ in $\mathbb{R}^2$. The "o" means that the lines approach without touch.

Example 5.13 *Consider the morphism $f : \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \longrightarrow \mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}$ defined by $f(\mathbf{v}) = \Theta_{\mathbf{v}}(\mathbf{v}) = (\mathfrak{C}, \mathfrak{L})$. Then in local coordinates we have $\Theta_{\mathbf{v}}^{-1} \circ f \circ id_{\mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v})} = id_{\mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v})}$. Clearly, f is injective and $\mathrm{T}_{\mathbf{v}} f(\mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v})) = id_{\mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v})}(\mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v})) = \mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v})$. From Lemma 4.1 we have that $\mathcal{L}_{T_D}(\mathbf{v}) \in \mathbb{G}(\mathcal{L}_{T_D})$ and hence $\mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v}) \in \mathbb{G}(\mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D})$. Then by Proposition 5.10 f is an immersion. Moreover, $f(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$ with the topology induced by $\mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}$ is homeomorphic to $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ when we consider in $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ the initial topology induced by f. We point out that with this topology in each local neighborhood $\mathcal{U}(\mathbf{v})$ is an open set in $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. Moreover, f is an embedding and $f(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$ is an embedded submanifold of $\mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}$.*

From the above example we have that even the manifold $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is a subset of $\mathbf{V}_{D_{\|\cdot\|_D}}$ its geometric structure is fully compatible with topology of the Banach space $\mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}$.

Finally, to show that $\mathfrak{i}$ is an inmersion, and hence $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$ is an immersed submanifold of $\mathbf{V}_{D_{\|\cdot\|_D}}$, we need to prove that $\mathrm{T}_{\mathbf{v}}\mathfrak{i}$ is injective and $\mathrm{T}_{\mathbf{v}}\mathfrak{i}\,(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))) \in \mathbb{G}(\mathbf{V}_{\|\cdot\|_D})$. To do this we need a stronger condition than the T_D-continuity of the tensor product. Grothendieck [11] named the following norm $\|\cdot\|_\vee$ the *injective norm*.

Definition 5.14 *Let V_i be a Banach space with norm $\|\cdot\|_i$ for $1 \leq i \leq d$. Then for $\mathbf{v} \in \mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$ define $\|\cdot\|_{\vee(V_1,\ldots,V_d)}$ by*

$$\|\mathbf{v}\|_{\vee(V_1,\ldots,V_d)} := \sup\left\{\frac{|(\varphi_1 \otimes \varphi_2 \otimes \ldots \otimes \varphi_d)(\mathbf{v})|}{\prod_{j=1}^d \|\varphi_j\|_j^*} : 0 \neq \varphi_j \in V_j^*, 1 \leq j \leq d\right\}. \tag{5.5}$$

It is well known that the injective norm is a reasonable crossnorm (see Lemma 1.6 in [20] and (3.12)-(3.13)). Further properties are given by the next proposition (see Lemma 4.96 and 4.2.4 in [15]).

Proposition 5.15 *Let V_i be a Banach space with norm $\|\cdot\|_i$ for $1 \leq i \leq d$, and $\|\cdot\|$ be a norm on $\mathbf{V} := {}_a\bigotimes_{j=1}^d V_j$. The following statements hold.*

(a) For each $1 \leq j \leq d$ introduce the tensor Banach space $\mathbf{X}_j := {}_{\|\cdot\|_{\vee(V_1,\ldots,V_{j-1},V_{j+1},\ldots,V_d)}}\bigotimes_{k\neq j} V_k$. Then

$$\|\cdot\|_{\vee(V_1,\ldots,V_d)} = \|\cdot\|_{\vee(V_j,\mathbf{X}_j)} \tag{5.6}$$

holds for $1 \leq j \leq d$.

(b) *The injective norm is the weakest reasonable crossnorm on* $\mathbf{V}$, *i.e., if* $\|\cdot\|$ *is a reasonable crossnorm on* $\mathbf{V}$, *then*

$$\|\cdot\| \gtrsim \|\cdot\|_{\vee(V_1,\ldots,V_d)} \tag{5.7}$$

(c) *For any norm* $\|\cdot\|$ *on* $\mathbf{V}$ *satisfying* $\|\cdot\|_{\vee(V_1,\ldots,V_d)} \lesssim \|\cdot\|$, *the map (3.14) is continuous, and hence Fréchet differentiable.*

Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of the Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in the topological tree based format. Take $\mathbf{V}_D := {}_a\bigotimes_{j\in D} V_j$ and assume that the tensor product map $\bigotimes$ is T_D-continuous. From Theorem 3.29, we may assume that we have a tensor Banach space

$$\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} V_{\beta_{\|\cdot\|_\beta}}$$

for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$, and a Banach space $V_{j_{\|\cdot\|_j}}$ for $j \in \mathcal{L}(T_D)$. Let $\alpha \in T_D \setminus \mathcal{L}(T_D)$. To simplify the notation, we write for $A, B \subset S(\alpha)$

$$\|\cdot\|_{\vee(A)} := \|\cdot\|_{\vee(\{\mathbf{V}_{\delta_{\|\cdot\|_\delta}}:\delta\in A\})},$$

and

$$\|\cdot\|_{\vee(A,\vee(B))} := \|\cdot\|_{\vee(\{\mathbf{V}_{\delta_{\|\cdot\|_\delta}}:\delta\in A\},\mathbf{X}_B)}$$

where

$$\mathbf{X}_B := {}_{\|\cdot\|_{\vee(B)}}\bigotimes_{\beta\in B} \mathbf{V}_{\beta_{\|\cdot\|_\beta}} .$$

From Proposition 5.15(a), we can write

$$\|\cdot\|_{\vee(S(\alpha))} = \|\cdot\|_{\vee(\beta,\vee(S(\alpha)\setminus\beta))}$$

for each $\beta \in S(\alpha)$. From now on, we assume that

$$\|\cdot\|_\alpha \gtrsim \|\cdot\|_{\vee(S(\alpha))} \text{ for each } \alpha \in T_D \setminus \mathcal{L}(T_D), \tag{5.8}$$

holds. Recall that Proposition 5.15(c) implies that the tensor product map $\bigotimes$ is T_D-continuous. Since $\|\cdot\|_\alpha \gtrsim \|\cdot\|_{\vee(\beta,\vee(S(\alpha)\setminus\beta))}$ holds for each $\beta \in S(\alpha)$, the tensor product map

$$\bigotimes : (\mathbf{V}_{\beta_{\|\cdot\|_\beta}}, \|\cdot\|_\beta) \times \left({}_{\|\cdot\|_{\vee(S(\alpha)\setminus\beta)}}\bigotimes_{\delta\in S(D)\setminus\{\beta\}} \mathbf{V}_{\delta_{\|\cdot\|_\delta}} , \|\cdot\|_{\vee(S(\alpha)\setminus\beta)} \right) \to (\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \|\cdot\|_\alpha)$$

is also continuous for each $\beta \in S(\alpha)$. A first useful result is the following lemma.

Lemma 5.16 *Assume that (5.8) holds. Let* $\alpha \in T_D \setminus \mathcal{L}(T_D)$ *and take* $\beta \in S(\alpha)$. *If* $W_\beta \in \mathbb{G}(\mathbf{V}_{\beta_{\|\cdot\|_\beta}})$ *satisfies* $\mathbf{V}_{\beta_{\|\cdot\|_\beta}} = U_\beta \oplus W_\beta$ *for some finite dimensional subspace* U_β *in* $\mathbf{V}_{\beta_{\|\cdot\|_\beta}}$, *then* $W_\beta \otimes_a U_{[\beta]} \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$ *for every finite dimensional subspace* $U_{[\beta]} \subset {}_a\bigotimes_{\delta\in S(\alpha)\setminus\beta} \mathbf{V}_{\delta_{\|\cdot\|_\delta}}$.

Proof. First, observe that if W_β is a finite dimensional subspace, then $W_\beta \otimes_a U_{[\beta]}$ is also finite dimensional, and hence the lemma follows. Thus, assume that W_β is an infinite dimensional closed subspace of $\mathbf{V}_{\beta_{\|\cdot\|_\beta}}$, and to simplify the notation write

$$\mathbf{X}_\beta := {}_{\|\cdot\|_{\vee(S(\alpha)\setminus\beta)}}\bigotimes_{\delta\in S(D)\setminus\{\beta\}} \mathbf{V}_{\delta_{\|\cdot\|_\delta}} .$$

If $U_{[\beta]} \subset \mathbf{X}_\beta$ is a finite dimensional subspace, then there exists $W_{[\beta]} \in \mathbb{G}(\mathbf{X}_\beta)$ such that $\mathbf{X}_\beta = U_{[\beta]} \oplus W_{[\beta]}$. Since the tensor product map

$$\bigotimes : (\mathbf{V}_{\beta_{\|\cdot\|_\beta}}, \|\cdot\|_\beta) \times \left(\mathbf{X}_\beta, \|\cdot\|_{\vee(S(\alpha)\setminus\beta)}\right) \to (\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \|\cdot\|_\alpha)$$

is continuous and by Lemma 3.18 in [7], for each elementary tensor $\mathbf{v}_\beta \otimes \mathbf{v}_{[\beta]} \in \mathbf{V}_{\beta_{\|\cdot\|_\beta}} \otimes_a \mathbf{X}_\beta$ we have

$$\begin{aligned}\|(id_\beta \otimes P_{U_{[\beta]}\oplus W_{[\beta]}})(\mathbf{v}_\beta \otimes \mathbf{v}_{[\beta]})\|_\alpha &\leq C\sqrt{\dim U_{[\beta]}}\,\|\mathbf{v}_\beta\|_\beta \|\mathbf{v}_{[\beta]}\|_{\vee(S(\alpha)\setminus\beta)} \\ &= C\,\sqrt{\dim U_{[\beta]}}\,\|\mathbf{v}_\beta \otimes \mathbf{v}_{[\beta]}\|_{\vee(S(\alpha))} \\ &\leq C'\,\sqrt{\dim U_{[\beta]}}\,\|\mathbf{v}_\beta \otimes \mathbf{v}_{[\beta]}\|_\alpha.\end{aligned}$$

Thus, $(id_\beta \otimes P_{U_{[\beta]}\oplus W_{[\beta]}})$ is continuous over $\mathbf{V}_{\beta_{\|\cdot\|_\beta}} \otimes_a \mathbf{X}_\beta$, and hence in $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$. Now, take into account that

$$id_\beta = P_{U_\beta \oplus W_\beta} + P_{W_\beta \oplus U_\beta},$$

and hence

$$id_\beta \otimes P_{U_{[\beta]}\oplus W_{[\beta]}} = P_{U_\beta \oplus W_\beta} \otimes P_{U_{[\beta]}\oplus W_{[\beta]}} + P_{W_\beta \oplus U_\beta} \otimes P_{U_{[\beta]}\oplus W_{[\beta]}}.$$

Observe that $id_\beta \otimes P_{U_{[\beta]}\oplus W_{[\beta]}}$ and $P_{U_\beta \oplus W_\beta} \otimes P_{U_{[\beta]}\oplus W_{[\beta]}}$ are continuous linear maps over $\mathbf{V}_{\beta_{\|\cdot\|_\beta}} \otimes_a \mathbf{X}_\beta$, and then $P_{W_\beta \oplus U_\beta} \otimes P_{U_{[\beta]}\oplus W_{[\beta]}}$ is a continuous linear map over $\mathbf{V}_{\beta_{\|\cdot\|_\beta}} \otimes_a \mathbf{X}_\beta$. Thus,

$$\mathcal{P}_\alpha := \overline{P_{W_\beta \oplus U_\beta} \otimes P_{U_{[\beta]}\oplus W_{[\beta]}}} \in \mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$$

and $\mathcal{P}_\alpha \circ \mathcal{P}_\alpha = \mathcal{P}_\alpha$. Since $\mathcal{P}_\alpha(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) = W_\beta \otimes_a U_{[\beta]}$, by Proposition 2.4 the lemma follows. ■

Lemma 5.17 *Let X be a Banach space and assume that $U, V \in \mathbb{G}(X)$. If $U \cap V = \{0\}$, then $U \oplus V \in \mathbb{G}(X)$. Moreover, $U \cap V \in \mathbb{G}(X)$ holds.*

Proof. To prove the first statement assume that $U \cap V = \{0\}$. Since $U, V \in \mathbb{G}(X)$ there exist $U', V' \in \mathbb{G}(X)$, such that $X = U \oplus U' = V \oplus V'$. Then $U = X \cap U = (V \oplus V') \cap U = U \cap V'$ and $V = X \cap V = (U \oplus U') \cap V = V \cap U'$. In consequence, we can write

$$U \oplus V \oplus (U' \cap V') = (U \cap V') \oplus (V \cap U') \oplus (U' \cap V') = (U \oplus U') \cap (V \oplus V') = X,$$

and the first statement follows. To prove the second one, observe that $X = (U \cap V) \oplus (U \cap V') \oplus (V \cap U') \oplus (U' \cap V')$. ■

A very useful consequence of the above two lemmas is the following.

Theorem 5.18 *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha \in T_D \setminus \{D\}}$ be a representation of a tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in the topological tree based format and assume that (5.8) holds. Then $\mathbf{Z}^{(D)}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$, and hence $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is an immersed submanifold of $\mathbf{V}_{D_{\|\cdot\|_D}}$.*

Proof. Since the tensor product map is T_D-continuous, Proposition 5.5 gives us the differentiability of $\mathrm{T}_{\mathbf{v}} i$. Assume first that $S(D) = \mathcal{L}(T_D)$. From Corollary 5.6 we have

$$\mathbf{Z}^{(D)}(\mathbf{v}) = {}_a\bigotimes_{\alpha\in S(D)} U_\alpha^{\min}(\mathbf{v}) \oplus \left(\bigoplus_{\alpha\in S(D)} W_\alpha^{\min}(\mathbf{v}) \otimes_a U_{S(D)\setminus\{\alpha\}}^{\min}(\mathbf{v})\right).$$

For each $\alpha \in S(D)$ we have $W_\alpha^{\min}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$ and $U_{S(D)\setminus\{\alpha\}}^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{\delta\in S(D)\setminus\{\alpha\}} \mathbf{V}_{\delta_{\|\cdot\|_\delta}}$ is a finite dimensional subspace. From Lemma 5.16 we have $W_\alpha^{\min}(\mathbf{v}) \otimes_a U_{S(D)\setminus\{\alpha\}}^{\min}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$ for all $\alpha \in S(D)$. Since ${}_a\bigotimes_{\alpha\in S(D)} U_\alpha^{\min}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$, by Lemma 5.17, we obtain that $\mathbf{Z}^{(D)}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$.

Now, assume that $S(D) \neq \mathcal{L}(T_D)$. Then

$$\mathbf{Z}^{(D)}(\mathbf{v}) = {}_a\bigotimes_{\alpha\in S(D)} U_\alpha^{\min}(\mathbf{v}) \oplus \left(\bigoplus_{\alpha\in S(D)} f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v}))\right).$$

and

$$f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v})) = \begin{cases} \bigoplus_{i_\alpha=1}^{r_\alpha} \Pi_{i_\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\} & \text{if } \alpha \notin \mathcal{L}(T_D) \\ \bigoplus_{i_\alpha=1}^{r_\alpha} W_\alpha^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\} & \text{if } \alpha \in \mathcal{L}(T_D) \end{cases}$$

for $\alpha \in S(D)$. For $\alpha \in \mathcal{L}(T_D)$ we have $W_\alpha^{\min}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$ and $\operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\}$ is a finite dimensional subspace for $1 \leq i_\alpha \leq r_\alpha$, and from Lemma 5.16, $W_\alpha^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\} \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$ for $1 \leq i_\alpha \leq r_\alpha$. By Lemma 5.17, $f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$. Otherwise, if $\alpha \notin \mathcal{L}(T_D)$ then

$$f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v})) = \bigoplus_{i_\alpha=1}^{r_\alpha} \Pi_{i_\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\},$$

where

$$\Pi_{i_\alpha}(\mathcal{H}_\alpha(\mathbf{v})) = {}_a \bigotimes_{\beta \in S(\alpha)} U_\beta^{\min}(\mathbf{v}) \oplus \left(\bigoplus_{\beta \in S(\alpha)} f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v})) \right)$$

for $1 \leq i_\alpha \leq r_\alpha$. Now,

$$f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v})) = \begin{cases} \bigoplus_{i_\beta=1}^{r_\beta} \Pi_{i_\beta}(\mathcal{H}_\beta(\mathbf{v})) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha,i_\beta}^{(\beta)}\} & \text{if } \beta \notin \mathcal{L}(T_D) \\ \bigoplus_{i_\beta=1}^{r_\beta} W_\beta^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha,i_\beta}^{(\beta)}\} & \text{if } \beta \in \mathcal{L}(T_D) \end{cases}$$

for $1 \leq i_\alpha \leq r_\alpha$. Clearly, if $\beta \in \mathcal{L}(T_D)$ then $f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v})) \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$ for $1 \leq i_\alpha \leq r_\alpha$. Then we can write,

$$\Pi_{i_\alpha}(\mathcal{H}_\alpha(\mathbf{v})) = {}_a \bigotimes_{\beta \in S(\alpha)} U_\beta^{\min}(\mathbf{v}) \oplus \left(\bigoplus_{\substack{\beta \in S(\alpha) \\ \beta \in \mathcal{L}(T_D)}} f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v})) \right) \oplus \left(\bigoplus_{\substack{\beta \in S(\alpha) \\ \beta \notin \mathcal{L}(T_D)}} f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v})) \right)$$

for $1 \leq i_\alpha \leq r_\alpha$. Starting by the leaves, that is $\gamma \in \mathcal{L}(T_D)$, we have that always $\Pi_{i_\gamma}(\mathcal{H}_\gamma(\mathbf{v})) = W_\gamma^{\min}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{\gamma_{\|\cdot\|_\gamma}})$ for $1 \leq i_\gamma \leq r_\gamma$, and hence for $\delta \in T_D$ such that $\gamma \in S(\delta)$ we obtain $f_{\gamma,i_\delta}(\mathcal{H}_\gamma(\mathbf{v})) \in \mathbb{G}(\mathbf{V}_{\delta_{\|\cdot\|_\delta}})$ for $1 \leq i_\delta \leq r_\delta$. Proceeding inductively from the leaves to the root, we obtain that $f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v})) \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$, for $\beta \in S(\alpha)$ with $\beta \notin \mathcal{L}(T_D)$ and $1 \leq i_\alpha \leq r_\alpha$. Lemma 5.17 says us that $\Pi_{i_\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$ for $1 \leq i_\alpha \leq r_\alpha$. From Lemma 5.16 and Lemma 5.17 we obtain that $f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$. Also by Lemma 5.17, we have $\mathbf{Z}^{(D)}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$ and hence this proves the theorem. ■

Example 5.19 *Recall the topological tensor spaces introduced in the Example 3.22. Let $I_j \subset \mathbb{R}$ $(1 \leq j \leq d)$ and $1 \leq p < \infty$. Given tree T_D, for $\alpha \in T_D$ let $\mathbf{I}_\alpha := \times_{j \in \alpha} I_j$, and hence $L^p(\mathbf{I}_\alpha)$ is a tensor Banach space for all $\alpha \in T_D$. In this example we denote the usual norm of $L^p(\mathbf{I}_\alpha)$ by $\|\cdot\|_{\alpha,p}$. Since $\|\cdot\|_{\alpha,p}$ is a reasonable crossnorm (see Example 4.72 in [15]), then (5.8) holds for all $\alpha \in T_D$. From Theorem 5.18 we obtain that $\mathcal{FT}_{\mathfrak{r}}\left({}_a\bigotimes_{j=1}^d L^p(I_j)\right)$ is an immersed submanifold of $L^p(\mathbf{I}_D)$.*

Example 5.20 *Now, we return to Example 5.1. From Example 4.42 in [15] we know that the norm $\|\cdot\|_{(0,1),p}$ is a crossnorm on $H^{1,p}(I_1) \otimes_a H^{1,p}(I_2)$, and hence it is not weaker than the injective norm. In consequence, from Theorem 5.18, we obtain that $\mathcal{FT}_{\mathfrak{r}}(H^{1,p}(I_1) \otimes_a H^{1,p}(I_2))$ is an immersed submanifold in $H^{1,p}(I_1) \otimes_{\|\cdot\|_{(0,1),p}} H^{1,p}(I_2)$.*

Since in a reflexive Banach space every closed linear subspace is proximinal (see p. 61 in [9]), we have the corollary.

Corollary 5.21 *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha \in T_D \setminus \{D\}}$ be a representation of a reflexive tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j \in D} V_j$, in the topological tree based format and assume that (5.8) holds. Let $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$, then for each $\dot{\mathbf{u}} \in \mathbf{V}_{D_{\|\cdot\|_D}}$ there exists $\dot{\mathbf{v}}_{best} \in \mathbf{Z}^{(D)}(\mathbf{v})$ such that*

$$\|\dot{\mathbf{u}} - \dot{\mathbf{v}}_{best}\| = \min_{\dot{\mathbf{v}} \in \mathbf{Z}^{(D)}(\mathbf{v})} \|\dot{\mathbf{u}} - \dot{\mathbf{v}}\|. \tag{5.9}$$

5.3 On the best $\mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D)$-approximation

To end this section we would discuss about the $\mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D)$-best approximation. In Corollary 4.4 in [7] the following result, which is re-stated here using the notations of the present paper, is proved as a consequence of a similar result showed for tensors in Tucker format with bounded rank.

Theorem 5.22 *Let $\mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$ and let $\{\mathbf{V}_{\alpha_j\,\|\cdot\|_{\alpha_j}} : 2 \le j \le d\} \cup \{V_{j_{\|\cdot\|_j}} : 1 \le j \le d\}$ for $d \ge 3$, be a representation of a reflexive Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in topological tree based format such that*

(a) $\|\cdot\|_D \gtrsim \|\cdot\|_{\vee(V_{1_{\|\cdot\|_j}},\ldots,V_{d_{\|\cdot\|_d}})}$,

(b) $\mathbf{V}_{\alpha_d} = V_{d-1} \otimes_a V_d$, and $\mathbf{V}_{\alpha_j} = V_{j-1} \otimes_a \mathbf{V}_{\alpha_{j+1}}$, for $2 \le j \le d-1$, and

(c) $\|\cdot\|_{\alpha_j} := \|\cdot\|_{\vee(V_{j-1\,\|\cdot\|_{j-1}},\ldots,V_{d_{\|\cdot\|_d}})}$ for $2 \le j \le d$.

Then for each $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}}$ there exists $\mathbf{u}_{best} \in \mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D)$ such that

$$\|\mathbf{v} - \mathbf{u}_{best}\|_D = \min_{\mathbf{u}\in\mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D)} \|\mathbf{v} - \mathbf{u}\|_D.$$

It seems clear that tensor Banach spaces as we show in Example 3.5 are not included in this framework. Observe that this choice of norms satisfies (5.8). So a natural question is if for a representation in the topological tree based format of a reflexive Banach space satisfying (5.8) the statement of Theorem 5.22 is also true. To prove this, we will reformulate some of the results given in [7]. In the aforementioned paper, the milestone to prove the existence of a best approximation is the extension of the definition of minimal subspace for tensors $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}} \setminus \mathbf{V}_D$. To do this the authors use a similar result to the following lemma (see Lemma 3.8 in [7]).

Lemma 5.23 *Let $V_{j_{\|\cdot\|_j}}$ be a Banach space for $j \in D$, where D is a finite index set, and $\alpha_1, \ldots, \alpha_m \subset 2^D \setminus \{D, \emptyset\}$, be such that $\alpha_i \cap \alpha_j = \emptyset$ for all $i \ne j$ and $D = \bigcup_{i=1}^m \alpha_i$. Assume that if $\#\alpha_i \ge 2$ for some $1 \le i \le m$, then $\mathbf{V}_{\alpha_i\,\|\cdot\|_{\alpha_i}}$ is a tensor Banach space. Consider the tensor space*

$$\mathbf{V}_D := {}_a\bigotimes_{i=1}^{m} \mathbf{V}_{\alpha_i\,\|\cdot\|_{\alpha_i}}$$

endowed with the injective norm $\|\cdot\|_{\vee(\mathbf{V}_{\alpha_1\,\|\cdot\|_{\alpha_1}},\ldots,\mathbf{V}_{\alpha_m\,\|\cdot\|_{\alpha_m}})}$. Fix $1 \le k \le m$, then given $\boldsymbol{\varphi}_{[\alpha_k]} \in {}_a\bigotimes_{i\ne k} \mathbf{V}^*_{\alpha_i\,\|\cdot\|_{\alpha_i}}$ the map $id_{\alpha_k} \otimes \boldsymbol{\varphi}_{[\alpha_k]}$ belongs to $\mathcal{L}\left(\mathbf{V}_D, \mathbf{V}_{\alpha_k\,\|\cdot\|_{\alpha_k}}\right)$. Moreover, $\overline{id_{\alpha_k} \otimes \boldsymbol{\varphi}_{[\alpha_k]}} \in \mathcal{L}(\overline{\mathbf{V}_D}^{\|\cdot\|}, \mathbf{V}_{\alpha_k\,\|\cdot\|_{\alpha_k}})$ for any norm satisfying

$$\|\cdot\| \gtrsim \|\cdot\|_{\vee(\mathbf{V}_{\alpha_1\,\|\cdot\|_{\alpha_1}},\ldots,\mathbf{V}_{\alpha_m\,\|\cdot\|_{\alpha_m}})}.$$

Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of the Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in the topological tree based format and assume that (5.8) holds. Then the tensor product map is T_D-continuous and, by Theorem 3.29,

$$\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} V_{\beta_{\|\cdot\|_\beta}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} V_\beta = {}_{\|\cdot\|_\alpha}\bigotimes_{j\in\alpha} V_j \ ,$$

holds for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$. Observe, that $\mathbf{V}^*_{\alpha_{\|\cdot\|_\alpha}} \subset \mathbf{V}^*_\alpha$ for all $\alpha \in S(D)$. Take $\mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$. Since $\|\cdot\|_D \gtrsim \|\cdot\|_{\vee(S(D))}$, from Lemma 5.23 and Proposition 3.12(b), we can extend for $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}} \setminus \mathbf{V}_D$, the definition of minimal subspace for each $\alpha \in S(D)$ as

$$U^{\min}_\alpha(\mathbf{v}) := \left\{ \overline{(id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})}(\mathbf{v}) : \boldsymbol{\varphi}_{[\alpha]} \in {}_a\bigotimes_{\beta\in S(D)\setminus\{\alpha\}} \mathbf{V}^*_\beta \right\}.$$

Observe that $\overline{(id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})} \in \mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$. Recall that if $\mathbf{v} \in \mathbf{V}_D$ and $\alpha \notin \mathcal{L}(T_D)$, from Proposition 3.11, we have $U_\alpha^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{\beta\in S(\alpha)} U_\beta^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_\beta$. Moreover, by Proposition 3.12(b), for $\beta \in S(\alpha)$ we have

$$U_\beta^{\min}(\mathbf{v}) = \operatorname{span}\left\{(id_\beta \otimes \boldsymbol{\varphi}_{[\beta]})(\mathbf{v}_\alpha) : \mathbf{v}_\alpha \in U_\alpha^{\min}(\mathbf{v}) \text{ and } \boldsymbol{\varphi}_{[\beta]} \in {}_a\bigotimes_{\delta\in S(\alpha)\setminus\{\beta\}} \mathbf{V}_\delta^*\right\}$$
$$= \operatorname{span}\left\{(id_\beta \otimes \boldsymbol{\varphi}_{[\beta]}) \circ (id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})(\mathbf{v}) : \boldsymbol{\varphi}_{[\alpha]} \in {}_a\bigotimes_{\mu\in S(D)\setminus\{\alpha\}} \mathbf{V}_\mu^* \text{ and } \boldsymbol{\varphi}_{[\beta]} \in {}_a\bigotimes_{\delta\in S(\alpha)\setminus\{\beta\}} \mathbf{V}_\delta^*\right\}.$$

Thus, $(id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})(\mathbf{v}) \in U_\alpha^{\min}(\mathbf{v}) \subset \mathbf{V}_\alpha \subset \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$, and hence

$$(id_\beta \otimes \boldsymbol{\varphi}_{[\beta]}) \circ (id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})(\mathbf{v}) \in U_\beta^{\min}(\mathbf{v}) \subset \mathbf{V}_\beta \subset \mathbf{V}_{\beta_{\|\cdot\|_\beta}},$$

when $\#\beta \geq 2$. However, if $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}} \setminus \mathbf{V}_D$ then $\overline{(id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})}(\mathbf{v}) \in U_\alpha^{\min}(\mathbf{v}) \subset \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$. Since $\|\cdot\|_\alpha \gtrsim \|\cdot\|_{\vee(S(\alpha))}$ also by Lemma 5.23 we have $\overline{id_\beta \otimes \boldsymbol{\varphi}_{[\beta]}} \in \mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\beta_{\|\cdot\|_\beta}})$. In consequence, a natural extension of the definition of minimal subspace $U_\beta^{\min}(\mathbf{v})$, for $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}} \setminus \mathbf{V}_D$, is given by

$$U_\beta^{\min}(\mathbf{v}) := \operatorname{span}\left\{\overline{(id_\beta \otimes \boldsymbol{\varphi}_{[\beta]})} \circ \overline{(id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})}(\mathbf{v}) : \boldsymbol{\varphi}_{[\alpha]} \in {}_a\bigotimes_{\mu\in S(D)\setminus\{\alpha\}} \mathbf{V}_\mu^* \text{ and } \boldsymbol{\varphi}_{[\beta]} \in {}_a\bigotimes_{\delta\in S(\alpha)\setminus\{\beta\}} \mathbf{V}_\delta^*\right\}.$$

To simplify the notation, we can write

$$\overline{(id_\beta \otimes \boldsymbol{\varphi}_{[\beta,\alpha]})}(\mathbf{v}) := \overline{(id_\beta \otimes \boldsymbol{\varphi}_{[\beta]})} \circ \overline{(id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})}(\mathbf{v})$$

where $\boldsymbol{\varphi}_{[\beta,\alpha]} := \boldsymbol{\varphi}_{[\alpha]} \otimes \boldsymbol{\varphi}_{[\beta]} \in \left({}_a\bigotimes_{\mu\in S(D)\setminus\{\alpha\}} \mathbf{V}_\mu^*\right) \otimes_a \left({}_a\bigotimes_{\delta\in S(\alpha)\setminus\{\beta\}} \mathbf{V}_\delta^*\right)$ and $\overline{(id_\beta \otimes \boldsymbol{\varphi}_{[\beta,\alpha]})} \in \mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}}, \mathbf{V}_{\beta_{\|\cdot\|_\beta}})$. Proceeding inductively, from the root to the leaves, we define the minimal subspace $U_j^{\min}(\mathbf{v})$ for each $j \in \mathcal{L}(T_D)$ such that there exists $\eta \in T_D \setminus \{D\}$ with $j \in S(\eta)$ as

$$U_j^{\min}(\mathbf{v}) := \operatorname{span}\left\{\overline{(id_j \otimes \boldsymbol{\varphi}_{[j,\eta,\ldots,\beta,\alpha]})}(\mathbf{v}) : \boldsymbol{\varphi}_{[j,\eta,\ldots,\beta,\alpha]} \in \mathbf{W}_j\right\},$$

where

$$\mathbf{W}_j := \left({}_a\bigotimes_{\mu\in S(D)\setminus\{\alpha\}} \mathbf{V}_\mu^*\right) \otimes_a \left({}_a\bigotimes_{\delta\in S(\alpha)\setminus\{\beta\}} \mathbf{V}_\delta^*\right) \otimes_a \cdots \otimes_a \left({}_a\bigotimes_{k\in S(\eta)\setminus\{j\}} V_k^*\right).$$

With this extension the following result it can be shown (see Lemma 3.13 in [7]).

Lemma 5.24 *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of the Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in the topological tree based format and assume that (5.8) holds. Let $\{\mathbf{v}_n\}_{n\geq 0} \subset \mathbf{V}_{D_{\|\cdot\|_D}}$ with $\mathbf{v}_n \rightharpoonup \mathbf{v}$, and $\mu \in T_D \setminus (\{D\} \cup \mathcal{L}(T_D))$ then for each $\gamma \in S(\mu)$ we have*

$$\overline{(id_\gamma \otimes \boldsymbol{\varphi}_{[\gamma,\mu,\cdots,\beta,\alpha]})}(\mathbf{v}_n) \rightharpoonup \overline{(id_\gamma \otimes \boldsymbol{\varphi}_{[\gamma,\mu,\cdots,\beta,\alpha]})}(\mathbf{v}) \text{ in } \mathbf{V}_{\gamma_{\|\cdot\|_\gamma}},$$

for all $\boldsymbol{\varphi}_{[\gamma,\mu,\cdots,\beta,\alpha]} \in \left({}_a\bigotimes_{\mu\in S(D)\setminus\{\alpha\}} \mathbf{V}_\mu^*\right) \otimes_a \left({}_a\bigotimes_{\delta\in S(\alpha)\setminus\{\beta\}} \mathbf{V}_\delta^*\right) \otimes_a \cdots \otimes_a \left({}_a\bigotimes_{\eta\in S(\mu)\setminus\{\gamma\}} V_\eta^*\right)$.

Then in a similar way as Theorem 3.15 in [7] the following theorem can be shown.

Theorem 5.25 *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of the Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in the topological tree based format and assume that (5.8) holds. Let $\{\mathbf{v}_n\}_{n\geq 0} \subset \mathbf{V}_{D_{\|\cdot\|_D}}$ with $\mathbf{v}_n \rightharpoonup \mathbf{v}$, then*

$$\dim \overline{U_\alpha^{\min}(\mathbf{v})}^{\|\cdot\|_\alpha} = \dim U_\alpha^{\min}(\mathbf{v}) \leq \liminf_{n\to\infty} \dim U_\alpha^{\min}(\mathbf{v}_n),$$

for all $\alpha \in T_D \setminus \{D\}$.

Now, following the proof of Theorem 4.1 in [7] we obtain the final theorem.

Theorem 5.26 *Let* $\mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$ *and let* $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ *be a representation of a reflexive Banach tensor space* $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$ *, in the topological tree based format and assume that (5.8) holds. Then for each* $\mathbf{v}\in\mathbf{V}_{D_{\|\cdot\|_D}}$ *there exists* $\mathbf{u}_{best}\in\mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D)$ *such that*

$$\|\mathbf{v}-\mathbf{u}_{best}\|_D = \min_{\mathbf{u}\in\mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D)} \|\mathbf{v}-\mathbf{u}\|_D.$$

6 On the Dirac–Frenkel variational principle on tensor Banach spaces

6.1 Model Reduction in tensor Banach spaces

In this section we consider the abstract ordinary differential equation in a reflexive tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}}$, given by

$$\dot{\mathbf{u}}(t) = \mathbf{F}(t,\mathbf{u}(t)), \text{ for } t\geq 0 \tag{6.1}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{6.2}$$

where we assume $\mathbf{u}_0 \neq \mathbf{0}$ and $\mathbf{F} : [0,\infty)\times\mathbf{V}_{D_{\|\cdot\|_D}} \longrightarrow \mathbf{V}_{D_{\|\cdot\|_D}}$ satisfying the usual conditions. Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in the topological tree based format and assume that (5.8) holds. As usual we will consider $\mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$. We want to approximate $\mathbf{u}(t)$, for $t\in I := (0,\varepsilon)$ for some $\varepsilon>0$, by a differentiable curve $t\mapsto\mathbf{v}_r(t)$ from I to $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$, where $\mathfrak{r}\in\mathbb{N}^{T_D}$ is such that $\mathbf{v}_r(0)=\mathbf{u}(0)=\mathbf{u}_0\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$.

Our main goal is to construct a Reduced Order Model of (6.1)–(6.2) over the Banach manifold $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. Since $\mathbf{F}(t,\mathbf{v}_r(t))$ in $\mathbf{V}_{D_{\|\cdot\|_D}}$, for each $t\in I$, and $\mathbf{Z}^{(D)}(\mathbf{v}_r(t))$ is a closed linear subspace in $\mathbf{V}_{D_{\|\cdot\|_D}}$, we have the existence of a $\dot{\mathbf{v}}_r(t)\in\mathbf{Z}^{(D)}(\mathbf{v}_r(t))$ such that

$$\|\dot{\mathbf{v}}_r(t)-\mathbf{F}(t,\mathbf{v}_r(t))\|_D = \min_{\dot{\mathbf{v}}(t)\in\mathbf{Z}^{(D)}(\mathbf{v}_r(t))} \|\dot{\mathbf{v}}(t)-\mathbf{F}(t,\mathbf{v}_r(t))\|_D,$$

It is well known that, if $\mathbf{V}_{D_{\|\cdot\|_D}}$ is a Hilbert space, then $\dot{\mathbf{v}}_r(t)=\mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t,\mathbf{v}_r(t)))$, where

$$\mathcal{P}_{\mathbf{v}_r(t)} = \mathcal{P}_{\mathbf{Z}^{(D)}(\mathbf{v}_r(t))\oplus\mathbf{Z}^{(D)}(\mathbf{v}_r(t))^\perp}$$

is called the *metric projection.* It has the following important property: $\dot{\mathbf{v}}_r(t)=\mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t,\mathbf{v}_r(t)))$ if and only if

$$\langle\dot{\mathbf{v}}_r(t)-\mathbf{F}(t,\mathbf{v}_r(t)),\dot{\mathbf{v}}(t)\rangle_D = 0 \text{ for all } \dot{\mathbf{v}}(t)\in\mathbf{Z}^{(D)}(\mathbf{v}_r(t)).$$

The concept of a metric projection can be extended to the Banach space setting. To this end we recall the following definitions. A Banach space X with norm $\|\cdot\|$ is said to be *strictly convex* if $\|x+y\|/2<1$ for all $x,y\in X$ with $\|x\|=\|y\|=1$ and $x\neq y$. It is *uniformly convex* if $\lim_{n\to\infty}\|x_n-y_n\|=0$ for any two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ such that $\|x_n\|=\|y_n\|=1$ and $\lim_{n\to\infty}\|x_n+y_n\|/2=1$. It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space X is said to be *smooth* if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for all $x,y\in U:=\{z\in X:\|z\|=1\}$. Finally, a Banach space X is said to be *uniformly smooth* if its modulus of smoothness

$$\rho(\tau) = \sup_{x,y\in U}\left\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2}-1\right\},\ \tau>0,$$

satisfies the condition $\lim_{\tau\to 0}\rho(\tau)=0$. In uniformly smooth spaces, and only in such spaces, the norm is uniformly Fréchet differentiable. A uniformly smooth Banach space is smooth. The converse is true if the

Banach space is finite-dimensional. It is known that the space L^p $(1 < p < \infty)$ is a uniformly convex and uniformly smooth Banach space.

Let $\langle \cdot, \cdot \rangle : X \times X^* \longrightarrow \mathbb{R}$ denote the duality map, i.e.,

$$\langle x, f \rangle := f(x).$$

The normalised duality mapping $J : X \longrightarrow 2^{X^*}$ is defined by

$$J(x) := \{f \in X^* : \langle x, f \rangle = \|x\|^2 = (\|f\|^*)^2\}.$$

Notice that, in a Hilbert space, the duality mapping is the identity operator. The duality mapping J has the following properties (see [2]):

(a) If X is smooth, the map J is single-valued;

(b) if X is smooth, then J is norm–to–weak* continuous;

(c) if X is uniformly smooth, then J is uniformly norm–to–norm continuous on each bounded subset of X.

Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha \in T_D \setminus \{D\}}$ be a representation of reflexive and strictly convex tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j \in D} V_j$, in the topological tree based format and assume that (5.8) holds. For $\mathbf{F}(t, \mathbf{v}_r(t))$ in $\mathbf{V}_{D_{\|\cdot\|_D}}$, with a fixed $t \in I$, it is known that the set

$$\left\{ \dot{\mathbf{v}}_r(t) : \|\dot{\mathbf{v}}_r(t) - \mathbf{F}(t, \mathbf{v}_r(t))\|_D = \min_{\dot{\mathbf{v}}(t) \in \mathbf{Z}^{(D)}(\mathbf{v}_r(t))} \|\dot{\mathbf{v}}(t) - \mathbf{F}(t, \mathbf{v}_r(t))\|_D \right\}$$

is always a singleton. Let $\mathcal{P}_{\mathbf{v}_r(t)}$ be the mapping of $\mathbf{V}_{D_{\|\cdot\|_D}}$ onto $\mathbf{Z}^{(D)}(\mathbf{v}_r(t))$ defined by $\dot{\mathbf{v}}_r(t) := \mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t, \mathbf{v}_r(t)))$ if and only if

$$\|\dot{\mathbf{v}}_r(t) - \mathbf{F}(t, \mathbf{v}_r(t))\|_D = \min_{\dot{\mathbf{v}}(t) \in \mathbf{Z}^{(D)}(\mathbf{v}_r(t))} \|\dot{\mathbf{v}}(t) - \mathbf{F}(t, \mathbf{v}_r(t))\|_D.$$

It is also called *the metric projection.* The classical characterisation of the metric projection allows us to state the next result.

Theorem 6.1 *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha \in T_D \setminus \{D\}}$ be a representation of reflexive and strictly convex tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j \in D} V_j$, in the topological tree based format and assume that (5.8) holds. Then for each $t \in I$ we have*

$$\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t, \mathbf{v}_r(t)))$$

if and only if

$$\langle \dot{\mathbf{v}}_r(t) - \dot{\mathbf{v}}(t), J(\mathbf{F}(t, \mathbf{v}_r(t)) - \dot{\mathbf{v}}_r(t)) \rangle \geq 0 \text{ for all } \dot{\mathbf{v}}(t) \in \mathbf{Z}^{(D)}(\mathbf{v}_r(t)).$$

An alternative approach is the use of the so-called *generalised projection operator* (see also [2]). To formulate this, we will use the following framework. Let T_D a given tree and assume that for each $\alpha \in T_D$ we have a Banach space $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$, such that (5.8) holds and where $\mathbf{V}_{D_{\|\cdot\|_D}}$ is a reflexive, strictly convex and smooth tensor Banach space. Following [17], we can define a function $\phi : \mathbf{V}_{D_{\|\cdot\|_D}} \times \mathbf{V}_{D_{\|\cdot\|_D}} \longrightarrow \mathbb{R}$ by

$$\phi(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}\|_D^2 - 2\langle \mathbf{u}, J(\mathbf{v}) \rangle + \|\mathbf{v}\|_D^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality map and J is the normalised duality mapping. It is known that the set

$$\left\{ \dot{\mathbf{v}}_r(t) : \phi(\dot{\mathbf{v}}_r(t), \mathbf{F}(t, \mathbf{v}_r(t))) = \min_{\dot{\mathbf{v}}(t) \in \mathbf{Z}^{(D)}(\mathbf{v}_r(t))} \phi(\dot{\mathbf{v}}(t), \mathbf{F}(t, \mathbf{v}_r(t))) \right\}$$

is always a singleton. It allows us to define a map $\Pi_{\mathbf{v}_r(t)} : \mathbf{V}_{D_{\|\cdot\|_D}} \longrightarrow \mathbf{Z}^{(D)}(\mathbf{v}_r(t))$ by $\dot{\mathbf{v}}_r(t) := \Pi_{\mathbf{v}_r(t)}(\mathbf{F}(t, \mathbf{v}_r(t)))$ if and only if

$$\phi(\dot{\mathbf{v}}_r(t), \mathbf{F}(t, \mathbf{v}_r(t))) = \min_{\dot{\mathbf{v}}(t) \in \mathbf{Z}^{(D)}(\mathbf{v}_r(t))} \phi(\dot{\mathbf{v}}(t), \mathbf{F}(t, \mathbf{v}_r(t))).$$

The map $\Pi_{\mathbf{v}_r(t)}$ is called *the generalised projection.* It coincides with the metric projection when $\mathbf{V}_{D_{\|\cdot\|_D}}$ is a Hilbert space.

Remark 6.2 *We point out that, in general, the operators $\mathcal{P}_{\mathbf{v}_r(t)}$ and $\Pi_{\mathbf{v}_r(t)}$ are nonlinear in Banach (not Hilbert) spaces.*

Again, a classical characterisation of the generalised projection give us the following theorem.

Theorem 6.3 *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of reflexive, strictly convex and smooth tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in the topological tree based format and assume that (5.8) holds. Then for each $t \in I$ we have*

$$\dot{\mathbf{v}}_r(t) = \Pi_{\mathbf{v}_r(t)}(\mathbf{F}(t, \mathbf{v}_r(t)))$$

if and only if

$$\langle \dot{\mathbf{v}}_r(t) - \dot{\mathbf{v}}(t), J(\mathbf{F}(t, \mathbf{v}_r(t))) - J(\dot{\mathbf{v}}_r(t))\rangle \geq 0 \text{ for all } \dot{\mathbf{v}}(t) \in \mathbf{Z}^{(D)}(\mathbf{v}_r(t)).$$

6.2 The time–dependent Hartree method

Let $\langle\cdot,\cdot\rangle_j$ be a scalar product defined on V_j $(1 \leq j \leq d)$, i.e., V_j is a pre-Hilbert space. Then $\mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$ is again a pre-Hilbert space with a scalar product which is defined for elementary tensors $\mathbf{v} = \bigotimes_{j=1}^d v^{(j)}$ and $\mathbf{w} = \bigotimes_{j=1}^d w^{(j)}$ by

$$\langle \mathbf{v}, \mathbf{w}\rangle = \left\langle \bigotimes_{j=1}^d v^{(j)}, \bigotimes_{j=1}^d w^{(j)} \right\rangle := \prod_{j=1}^d \left\langle v^{(j)}, w^{(j)} \right\rangle_j \qquad \text{for all } v^{(j)}, w^{(j)} \in V_j. \tag{6.3}$$

This bilinear form has a unique extension $\langle\cdot,\cdot\rangle : \mathbf{V}\times\mathbf{V} \to \mathbb{R}$. One verifies that $\langle\cdot,\cdot\rangle$ is a scalar product, called the *induced scalar product.* Let $\mathbf{V}$ be equipped with the norm $\|\cdot\|$ corresponding to the induced scalar product $\langle\cdot,\cdot\rangle$. As usual, the Hilbert tensor space $\mathbf{V}_{\|\cdot\|} = {}_{\|\cdot\|}\bigotimes_{j=1}^d V_j$ is the completion of $\mathbf{V}$ with respect to $\|\cdot\|$. Since the norm $\|\cdot\|$ is derived via (6.3), it is easy to see that $\|\cdot\|$ is a reasonable and even uniform crossnorm.

Let us consider in $\mathbf{V}_{\|\cdot\|}$ a flow generated by a densely defined operator $A \in L(\mathbf{V}_{\|\cdot\|}, \mathbf{V}_{\|\cdot\|})$. More precisely, there exists a collection of bijective maps $\boldsymbol{\varphi}_t : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$, here $\mathcal{D}(A)$ denotes the domain of A, satisfying

(i) $\boldsymbol{\varphi}_0 = \mathbf{id}$,

(ii) $\boldsymbol{\varphi}_{t+s} = \boldsymbol{\varphi}_t \circ \boldsymbol{\varphi}_s$, and

(iii) for $\mathbf{u}_0 \in \mathcal{D}(A)$, the map $t \mapsto \boldsymbol{\varphi}_t$ is differentiable as a curve in $\mathbf{V}_{\|\cdot\|}$, and $\mathbf{u}(t) := \boldsymbol{\varphi}_t(\mathbf{u}_0)$ satisfies

$$\dot{\mathbf{u}} = A\mathbf{u},$$
$$\mathbf{u}(0) = \mathbf{u}_0.$$

In this framework we want to study the approximation of a solution $\mathbf{u}(t) = \boldsymbol{\varphi}_t(\mathbf{u}_0) \in \mathbf{V}_{\|\cdot\|}$ by a curve $\mathbf{v}_r(t) := \lambda(t) \otimes_{j=1}^d v_j(t)$ in the Hilbert manifold $\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$, also called in [21] the *Hartree manifold.* The time–dependent Hartree method consists in the use of the Dirac–Frenkel variational principle on the Hartree manifold. More precisely, we want to solve the following Reduced Order Model:

$$\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(A\mathbf{v}_r(t)) \text{ for } t \in I,$$
$$\mathbf{v}_r(0) = \mathbf{v}_0,$$

with $\mathbf{v}_0 = \lambda_0 \otimes_{j=1}^d v_0^{(j)} \in \mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$ being an approximation of $\mathbf{u}_0$[6]. By using the characterisation of the metric projection in a Hilbert space, for each $t > 0$ we would like to find $\dot{\mathbf{v}}_r(t) \in \mathrm{T}_{\mathbf{v}_r(t)}\mathbf{i}\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V}))\right)$ such that

$$\langle \dot{\mathbf{v}}_r(t) - A\mathbf{v}_r(t), \dot{\mathbf{v}}(t)\rangle = 0 \text{ for all } \dot{\mathbf{v}}(t) \in \mathrm{T}_{\mathbf{v}_r(t)}\mathbf{i}\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V}))\right), \tag{6.4}$$

$$\mathbf{v}_r(0) = \mathbf{v}_0 = \lambda_0 \otimes_{j=1}^d v_0^{(j)},$$

and where, without loss of generality, we may assume $\|v_0^{(j)}\|_j = 1$ for $1 \leq j \leq d$. A first result is the following.

[6]Indeed, $\mathbf{v}_0$ can be chosen as the best approximation of $\mathbf{u}_0$ in $\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$ because $\mathcal{M}_{(1,\ldots,1)}(\mathbf{V}) = \mathcal{T}_{(1,\ldots,1)}(\mathbf{V}) \setminus \{\mathbf{0}\}$.

Lemma 6.4 *Let $\mathbf{v} \in \mathcal{C}^1(I, \mathcal{U}(\mathbf{v}_0))$, where $\mathbf{v}(0) = \mathbf{v}_0 \in \mathcal{M}_{(1,\dots,1)}(\mathbf{V})$ and $(\mathcal{U}(\mathbf{v}_0), \Theta_{\mathbf{v}_0})$ is a local chart for $\mathbf{v}_0$ in $\mathcal{M}_{(1,\dots,1)}(\mathbf{V})$. Assume that $\mathbf{v}$ is also a $\mathcal{C}^1$-morphism between the manifolds $I \subset \mathbb{R}$ and $\mathcal{U}(\mathbf{v}_0) \subset \mathcal{M}_{(1,\dots,1)}(\mathbf{V})$ such that $\mathbf{v}(t) = \lambda(t) \bigotimes_{j=1}^d v_j(t)$ for some $\lambda \in \mathcal{C}^1(I, \mathbb{R})$ and $v_j \in \mathcal{C}^1(I, V_j)$ for $1 \le j \le d$. Then*

$$\dot{\mathbf{v}}(t) = \dot{\lambda}(t) \bigotimes_{j=1}^d v_j(t) + \lambda(t) \sum_{j=1}^d \dot{v}_j(t) \otimes \bigotimes_{k \neq j} v_k(t) = \mathrm{T}_{\mathbf{v}(t)}\mathbf{i}(\mathrm{T}_t\mathbf{v}(1)). \tag{6.5}$$

Moreover, if $v_j(t) \in \mathbb{S}_{V_j}$, i.e., $\|v_j(t)\|_j = 1$, for $t \in I$ and $1 \le j \le d$, then $\dot{v}_j(t) \in \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$ for $t \in I$ and $1 \le j \le d$.

Proof. First at all, we recall that by the construction of $\mathcal{U}(\mathbf{v}_0)$ it follows that $W_j^{\min}(\mathbf{v}_0) = W_j^{\min}(\mathbf{v}(t))$ and that $U_j^{\min}(\mathbf{v}_0) = \mathrm{span}\{v_0^{(j)}\}$ is linearly isomorphic to $U_j^{\min}(\mathbf{v}(t))$ for all $t \in I$ and $1 \le j \le d$. Assume $\Theta_{\mathbf{v}_0}(\mathbf{v}(t)) = (\lambda(t), L_1(t), \dots, L_d(t))$, i.e.,

$$\mathbf{v}(t) := \lambda(t) \bigotimes_{j=1}^d (id_j + L_j(t))\,(v_0^{(j)}),$$

where $\lambda \in \mathcal{C}^1(I, \mathbb{R} \setminus \{0\})$, $L_j \in \mathcal{C}^1(I, \mathcal{L}(U_j^{\min}(\mathbf{v}_0), W_j^{\min}(\mathbf{v}_0)))$ and $(id_j + L_j(t))(v_0^{(j)}) \in U_j^{\min}(\mathbf{v}(t))$ for $1 \le j \le d$. We point out that the linear map $\mathrm{T}_t\mathbf{v} : \mathbb{R} \to \mathbb{T}_{\mathbf{v}(t)}(\mathcal{M}_{(1,\dots,1)}(\mathbf{V}))$ is characterised by

$$\mathrm{T}_t\mathbf{v}(1) = (\Theta_{\mathbf{v}_0} \circ \mathbf{v})'(t) = (\dot{\lambda}(t), \dot{L}_1(t), \dots, \dot{L}_d(t)). \tag{6.6}$$

Since $L_j \in \mathcal{C}^1(I, \mathcal{L}(U_j^{\min}(\mathbf{v}_0), W_j^{\min}(\mathbf{v}_0)))$ then $\dot{L}_j \in \mathcal{C}^0(I, \mathcal{L}(U_j^{\min}(\mathbf{v}_0), W_j^{\min}(\mathbf{v}_0)))$. Observe that $U_j^{\min}(\mathbf{v}_0)$ and $U_j^{\min}(\mathbf{v}(t))$ have $W_j^{\min}(\mathbf{v}_0)$ as a common complement. From Lemma 2.6 we know that

$$P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)}|_{U_j^{\min}(\mathbf{v}(t))} : U_j^{\min}(\mathbf{v}(t)) \longrightarrow U_j^{\min}(\mathbf{v}_0)$$

is a linear isomorphism. We can write

$$L_j(t) = L_j(t) P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)} \text{ and } \dot{L}_j(t) = \dot{L}_j(t) P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)},$$

and then in (6.6) we identify $\dot{L}_j(t) \in \mathcal{L}(U_j^{\min}(\mathbf{v}_0), W_j^{\min}(\mathbf{v}_0)))$ with

$$\dot{L}_j(t) P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)}|_{U_j^{\min}(\mathbf{v}(t))} \in \mathcal{L}(U_j^{\min}(\mathbf{v}(t)), W_j^{\min}(\mathbf{v}_0))).$$

Introduce $v_j(t) := (id_j + L_j(t))(v_0^{(j)})$ for $1 \le j \le d$. Then

$$\dot{L}_j(t)(v_j(t)) = \dot{L}_j(t) P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)}|_{U_j^{\min}(\mathbf{v}(t))}(v_0^{(j)} + L_j(t)(v_0^{(j)})) = \dot{L}_j(t)(v_0^{(j)})$$

holds for all $t \in I$ and $1 \le j \le d$. Hence

$$\dot{v}_j(t) = \dot{L}_j(t)(v_0^{(j)}) = \dot{L}_j(t)(v_j(t)) \tag{6.7}$$

holds for all $t \in I$ and $1 \le j \le d$. From Lemma 5.5(b) and (6.6), we have

$$\mathrm{T}_{\mathbf{v}(t)}\mathbf{i}(\mathrm{T}_t\mathbf{v}(1)) = \dot{\lambda}(t) \bigotimes_{j=1}^d v_j(t) + \lambda(t) \sum_{j=1}^d \dot{L}_j(t)(v_j(t)) \otimes \bigotimes_{k \neq j} v_k(t),$$

and, by using (6.7) for $\mathbf{v}(t) = \lambda(t) \bigotimes_{j=1}^d v_j(t)$, we obtain (6.5).

To prove the second statement, recall that $U_j^{\min}(\mathbf{v}(t)) = \mathrm{span}\,\{v_j(t)\}$ and $V_j = U_j^{\min}(\mathbf{v}(t)) \oplus W_j^{\min}(\mathbf{v}_0)$ for $1 \le j \le d$. Then we consider

$$W_j^{\min}(\mathbf{v}_0) = \mathrm{span}\,\{v_j(t)\}^{\perp} = \{u_j \in V_j : \langle u_j, v_j(t) \rangle_j = 0\} \text{ for } 1 \le j \le d,$$

and hence $\langle \dot{v}_j(t)), v_j(t)\rangle_j = 0$ holds for $1 \leq j \leq d$. From Remark 2.19, we have $(\dot{v}_1(t), \ldots, \dot{v}_d(t)) \in \mathcal{C}(I, \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j}))$, because of $W_j^{\min}(\mathbf{v}_0) = \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$ for $1 \leq j \leq d$. ■

Before stating the next result, we introduce for $\mathbf{v}_r(t) = \lambda(t) \bigotimes_{j=1}^d v_j(t)$ the following time dependent bilinear forms

$$\mathrm{a}_k(t; \cdot, \cdot) : V_k \times V_k \longrightarrow \mathbb{R},$$

by

$$\mathrm{a}_k(t; z_k, y_k) := \left\langle A \left(z_k \otimes \bigotimes_{j \neq k} v_j(t) \right), \left(y_k \otimes \bigotimes_{j \neq k} v_j(t) \right) \right\rangle$$

for each $1 \leq k \leq d$. Now, we will show the next result (compare with Theorem 3.1 in [21]).

Theorem 6.5 (Time dependent Hartree method) *The solution* $\mathbf{v}_r(t) = \lambda(t) \bigotimes_{j=1}^d v_j(t)$ *for* $(v_1(t), \ldots, v_d(t)) \in \times_{j=1}^d \mathbb{S}_{V_j}$ *of*

$$\begin{aligned} \dot{\mathbf{v}}_r(t) &= \mathcal{P}_{\mathbf{v}_r(t)}(A\mathbf{v}_r(t)) \text{ for } t \in I, \\ \mathbf{v}_r(0) &= \mathbf{v}_0, \end{aligned}$$

satisfies

$$\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \mathrm{a}_j(t; v_j(t), \dot{w}_j(t)) = 0 \quad \text{for all } \dot{w}_j(t) \in \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j}), \quad 1 \leq j \leq d,$$

and

$$\lambda(t) = \lambda_0 \exp\left(\int_0^t \left\langle A \left(\otimes_{j=1}^d v_j(s) \right), \otimes_{j=1}^d v_j(s) \right\rangle ds \right).$$

Proof. From Lemma 6.4 we have $\mathbb{T}_{\mathbf{v}_r(t)}\left(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})\right) = \mathbb{R} \times \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$, Thus, for each $\dot{\mathbf{w}}(t) \in \mathrm{T}_{\mathbf{v}(t)} i \left(\mathbb{T}_{\mathbf{v}(t)}\left(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})\right)\right)$ there exists $(\dot{\beta}(t), \dot{w}_1(t), \ldots, \dot{w}_d(t)) \in \mathbb{R} \times \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$, such that

$$\dot{\mathbf{w}}(t) = \dot{\beta}(t) \bigotimes_{j=1}^d v_j(t) + \lambda(t) \sum_{j=1}^d \dot{w}_j(t) \otimes \bigotimes_{k \neq j} v_k(t).$$

Then (6.4) holds if and only if

$$\left\langle \dot{\mathbf{v}}_r(t) - A\mathbf{v}_r(t), \dot{\beta}(t) \bigotimes_{j=1}^d v_j(t) + \lambda(t) \sum_{j=1}^d \dot{w}_j(t) \otimes \bigotimes_{k \neq j} v_k(t) \right\rangle = 0$$

for all $(\dot{\beta}(t), \dot{w}_1(t), \ldots, \dot{w}_d(t)) \in \mathbb{R} \times \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$. Then

$$\begin{aligned} \dot{\lambda}(t)\dot{\beta}(t) + \lambda(t)^2 \sum_{j=1}^d \left(\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \langle A \bigotimes_{s=1}^d v_s(t), \dot{w}_j(t) \otimes \bigotimes_{k \neq j} v_k(t)\rangle \right) \\ -\lambda(t)\dot{\beta}(t)\langle A \bigotimes_{j=1}^d v_j(t), \bigotimes_{j=1}^d v_j(t)\rangle = 0, \end{aligned}$$

i.e.,

$$\begin{aligned} &\dot{\beta}(t) \left(\dot{\lambda}(t) - \lambda(t)\langle A \bigotimes_{j=1}^d v_j(t), \bigotimes_{j=1}^d v_j(t)\rangle \right) \\ &+\lambda(t)^2 \textstyle\sum_{j=1}^d \left(\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \langle A \bigotimes_{s=1}^d v_s(t), \dot{w}_j(t) \otimes \bigotimes_{k \neq j} v_k(t)\rangle \right) = 0 \end{aligned} \tag{6.8}$$

holds for all $\dot{\beta}(t) \in \mathbb{R}$ and $(\dot{w}_1(t), \ldots, \dot{w}_d(t)) \in \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$. If $\lambda(t)$ solves the differential equation

$$\begin{aligned} \dot{\lambda}(t) &= \left\langle A \left(\otimes_{j=1}^d v_j(t) \right), \otimes_{j=1}^d v_j(t) \right\rangle \lambda(t) \\ \lambda(0) &= \lambda_0, \end{aligned}$$

i.e.,

$$\lambda(t) = \lambda_0 \exp\left(\int_0^t \left\langle A\left(\otimes_{j=1}^d v_j(s)\right), \otimes_{j=1}^d v_j(s)\right\rangle ds\right),$$

then the first term of (6.8) is equal to 0. Therefore, from (6.8) we obtain that for all $j \in D$,

$$\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \langle A \bigotimes_{s=1}^{d} v_s(t), \dot{w}_j(t) \otimes \bigotimes_{k\neq j} v_k(t)\rangle = 0,$$

that is,

$$\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \mathrm{a}_j(t; v_j(t), \dot{w}_j(t)) = 0$$

holds for all $\dot{w}_j(t) \in \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$, and the theorem follows. ■

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