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OMORI-YAU MAXIMUM PRINCIPLES, V-HARMONIC MAPS AND THEIR GEOMETRIC APPLICATIONS

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Abstract. We establish a V-Laplacian comparison theorem under the Bakry–Emery Ricci condition and then give various Omori–Yau type maximum principles on complete noncompact manifolds. We also obtain Liouville theorems for V-harmonic maps. We apply these findings to Ricci solitons and self-shrinkers.

Keywords and phrases: Omori-Yau maximum principle, V-Laplacian, V-harmonic map, Ricci soliton, self-shrinker.
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1. Introduction

In differential geometry, the relation between the geometry and the analysis of a given Riemannian manifold is one of the central topics. It is well-known that the Hopf maximum principle is a fundamental tool for studying compact manifolds. When M is noncompact, geometry and analysis on it becomes more complicated. Concretely, for the maximum principle, the difficulty arises that a, say harmonic, function need no longer attain its maximum on M. In 1967, Omori [31] proved the following generalized maximum principle to handle this problem:

Let M be a noncompact complete manifold with the sectional curvature bounded below by some constant:

$$K_M \geq -K.$$

If $u \in C^2(M)$ is bounded from above, then for any $\varepsilon > 0$, there is a point $x_\varepsilon \in M$ such that

$$|\nabla u|(x_\varepsilon) < \varepsilon, \qquad \mathrm{Hess}(u)(x_\varepsilon) < \varepsilon. \tag{1.1}$$

Restricting the sectional curvature, however, is too strong an assumption. In 1975, Yau [44] and Cheng–Yau [13] obtained a generalized maximum principle when only the Ricci curvature of M is bounded below by some constant:

Let M be a noncompact complete manifold with

$$Ric_M \geq -K$$

for some constant $K > 0$. If $u \in C^2(M)$ is bounded from above, then for any $\varepsilon > 0$, there is a point $x_\varepsilon \in M$ such that

$$u(x_\varepsilon) > \sup u - \varepsilon, \quad |\nabla u|(x_\varepsilon) < \varepsilon, \quad \Delta u(x_\varepsilon) < \varepsilon. \tag{1.2}$$

We call the above conclusion (1.2) the *Omori–Yau maximum principle.*

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This result provides a powerful and fundamental tool for the geometry and analysis on noncompact Riemannian manifolds whose Ricci curvature is bounded below by a constant, and has led to many applications in the study of such manifolds.

Therefore, it is not surprising that a lot of subsequent research activity was devoted to generalize the Omori–Yau maximum principles to include cases where the Ricci curvature is not bounded from below by a constant, but may decay at a certain rate. This started with [12] (1992), where it was proved that: *If the Ricci curvature satisfies:*

$$Ric_M \geq -K(1 + r^2 \log^2(r+2)), \quad r >> 1,$$

then for any function $u \in C^2(M)$ *bounded from above, the Omori–Yau holds.*

Since then, many extensions and/or appliciations have been obtained, see the references (not exhaustive): [1, 2, 3], [?], [23], [25], [27], [32, 33], [34, 35], [38], [45], etc.

Huang [23] and Ratto–Rigoli–Setti [35] extended this result to integral lower bounds for Ric_M: *Omori-Yau holds for* $u \in C^2(M)$ *with* $\sup u < +\infty$ *if* $Ric_M \geq -KG(r)$ *for a function* $G(r)$ *satisfying*

$$(i) G > 0, \quad G' \geq 0, \quad (ii) \int_0^{+\infty} \frac{1}{\sqrt{G}} = +\infty.$$

In another direction, the Omori–Yau maximum principle is related to the stochastic completeness of manifolds (c.f. [32], [21] etc.). Lima–Pessoa obtained the following result (Theorem 4 in [27]):

Assume that there exists a nonnegative function γ *satisfying*
h1) $\gamma(x) \to +\infty$, *as* $x \to \infty$;
h2) $\exists A > 0$ *such that* $|\nabla\gamma| < A\sqrt{G(\gamma)}(\int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1)$ *off a compact set;*
h3) $\exists B > 0$ *such that* $\Delta\gamma \leq B\sqrt{G(\gamma)}(\int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1)$ *off a compact set;*
where $G : [0, +\infty) \to [0, +\infty)$ *is a smooth function satisfying*

$$(i) G(0) = 1, \quad G' \geq 0; \quad (ii) \int_0^{+\infty} \frac{1}{\sqrt{G}} = +\infty.$$

Then for $u \in C^2(M)$ *with* $\lim_{x\to\infty} \frac{u(x)}{\phi(\gamma(x))} = 0$, *where* $\phi(t) = \log(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1)$, *Omori-Yau holds.*

The significance of the results of Omori, Yau et al. is that they indicate useful links between the geometric structures (such as curvature and volume growth) and the analytic properties of complete manifolds. From the viewpoint of differential geometry, it is therefore natural and interesting to find other geometric structures for which the Omori–Yau maximum principle holds.

In particular, a natural question is whether the Bakry–Emery Ricci tensor which has been extensively studied in recent years (see e.g. [29], [26], [41]), also admits such a maximum principle. The Bakry–Emery Ricci tensor is defined as the following tensor:

$$\text{Ric}_V := \text{Ric} - \frac{1}{2} L_V g, \tag{1.3}$$

where L_V stands for the Lie derivative with respect to some differentiable vector field V on (M, g). We shall now briefly explain its geometric significance.

The first instance concerns Ricci solitons. We recall that a Riemannian manifold (M, g) satisfying $\text{Ric}_V = \rho g$ for some constant ρ is called a *Ricci soliton* (see e.g. [5]). When $V = \nabla f$, the Ricci soliton is called a *gradient steady (shrinking, expanding resp.) soliton* if the constant ρ is zero (positive, negative resp.).

We also recall that M^m is called a *self-shrinker in the Euclidean space* $\mathbb{R}^{m+n}$ if it satisfies

$$H = -\frac{1}{2}X^N.$$

Here H is the mean curvature vector of M in $\mathbb{R}^{m+n}$, X^N is the normal part of X. Colding–Minicozzi [16] introduced the drift Laplacian operator $\mathcal{L}$

$$\mathcal{L} := \Delta - \frac{1}{2}\langle x, \nabla(\cdot)\rangle = e^{\frac{|X|^2}{4}}\mathrm{div}(e^{-\frac{|X|^2}{4}}\nabla(\cdot)) \tag{1.4}$$

for self-shrinkers M^m in $\mathbb{R}^{m+n}$.

By computations given below (see (2.11) and (3.5)), we will see that the self-shrinkers satisfy the Bakry–Emery Ricci conditions, that is, the corresponding tensors Ric_V enjoy certain lower bounds.

On the other hand, the notion of *V-harmonic maps* was introduced in [10]: Let (M, g) and (X, h) be Riemannian manifolds. Given a smooth vector field V on M, we call a map $u : M \to X$ a *V-harmonic map* if u satisfies

$$\tau_V(u) := \tau(u) + du(V) = 0, \tag{1.5}$$

where $\tau(u) = \mathrm{tr}Ddu$ is the tension field of the map u. This is a generalization of the usual harmonic map that includes the Hermitian harmonic maps, the Weyl harmonic maps from a Weyl manifold into a Riemannian manifold, the affine harmonic maps mapping from an affine manifold into a Riemannian manifold, and harmonic maps from a Finsler manifold into a Riemannian manifold, see [10] for details and references. From the Bochner formula (c.f. [9]), V-harmonic maps are closely related to the Bakry–Emery Ricci tensors Ric_V. Furthermore, by Theorem 8 in Sect. 3, we will see that the Gauss maps of self-shrinkers can be seen as V-harmonic maps.

In the light of these structures, it is therefore natural and possibly rewarding to derive Omori–Yau maximum principles under Bakry–Emery Ricci conditions and use them to obtain Liouville theorems for V-harmonic maps and then explore their geometric applications. This is the topic of our paper.

The rest of this paper is organized as follows: In Sect. 2, we first establish a V-Laplacian comparison theorem (Lemma 1), which we then use it to obtain an Omori–Yau maximum principle under a Bakry–Emery Ricci condition (Theorem 1). we also derive various Omori–Yau maximum principles for Ricci solitons (Theorems 2, 3) and self-shrinkers (Theorems 4, 5) and their applications (Theorems 6, 7). In Sect. 3, we first describe the link between V-harmonic map and the Gauss map of self-shrinkers (Theorem 8), and we then apply Liouville theorems for V-harmonic maps to derive Bernstein properties of self-shrinkers (Theorem 9–12, Corollary 4).

2. Omori–Yau maximum principles and applications

2.1. **Omori–Yau maximum principles.** We establish an Omori–Yau maximum principle under the Bakry–Emery Ricci condition. First we present the following:

Lemma 1. *Let (M^m, g) be a complete Riemannian manifold, V a smooth vector field on M with $\mathrm{Ric}_V \geq -F(r)$, where r is the distance function on M from a fixed point $x_0 \in M$, and*

$F:\mathbb{R}\to\mathbb{R}$ is a positive nondecreasing function. Suppose V satisfies

$$\langle V,\nabla r\rangle \le v(r) \tag{2.1}$$

for some nondecreasing function $v(\cdot)$. If $x\in M$ is not on the cut locus of the point x_0, then for $r(x)\ge r_0$ (r_0 is a constant),

$$\Delta_V r(x)\le \frac{m-1}{r}+\sqrt{(m-1)F(r)}+v(r). \tag{2.2}$$

Here $\Delta_V f := \Delta u + \langle V,\nabla f\rangle$ for $f\in C^2(M)$.

Proof. Let $\gamma:[0,r]\to M$ be a minimal unit speed geodesic with $\gamma(0)=x_0$, $\gamma(r)=x$. Choose a local orthonormal frame $\{e_\alpha\}$ near x such that $e_1=\dot\gamma(x)$, by parallel translation along γ we have a frame $\{e_\alpha(t)\}$. Let $J_\alpha(t)$ be the Jacobi field with $J_\alpha(0)=0$, $J_\alpha(r)=e_\alpha$, $\alpha=2,...,m$. Then

$$\begin{aligned}\Delta r(x) &= \sum_{\alpha=1}^m (e_\alpha e_\alpha r-\nabla_{e_\alpha}e_\alpha r)\\ &= -\sum_{\alpha=2}^m \nabla_{e_\alpha}e_\alpha r\\ &= \sum_{\alpha=2}^m\int_0^r \frac{d}{dt}\langle\nabla_{J_\alpha}\dot\gamma, J_\alpha\rangle dt\\ &= \sum_{\alpha=2}^m\int_0^r (|\dot J_\alpha|^2-\langle R(\dot\gamma,J_\alpha)\dot\gamma,J_\alpha\rangle)dt.\end{aligned}$$

For any piecewise smooth function $f(\cdot)$ on $[0,r]$ with $f(0)=0$, $f(r)=1$, let $X_\alpha(t):=f(t)e_\alpha(t)$, $\alpha=2,\cdots,m$. Then by the basic index lemma, we have

$$\begin{aligned}\Delta r(x) &\le \sum_{\alpha=2}^m\int_0^r (|\dot X_\alpha|^2-\langle R(\dot\gamma,X_\alpha)\dot\gamma,X_\alpha\rangle)dt\\ &= \int_0^r[(m-1)f'^2-f^2\mathrm{Ric}(\dot\gamma,\dot\gamma)]dt.\end{aligned}$$

Noting that

$$\mathrm{Ric}_V(\dot\gamma,\dot\gamma)=\mathrm{Ric}(\dot\gamma,\dot\gamma)-\frac12 L_V g(\dot\gamma,\dot\gamma)=\mathrm{Ric}(\dot\gamma,\dot\gamma)-\langle\dot V,\dot\gamma\rangle,$$

we have

$$\begin{aligned}\Delta_V r(x) &\le \int_0^r[(m-1)f'^2-f^2\mathrm{Ric}_V(\dot\gamma,\dot\gamma)]dt+\int_0^r (f^2)'\langle V,\dot\gamma\rangle dt\\ &\le \int_0^r (m-1)f'^2+F(t)f^2)dt+v(r). \qquad (2.3)\end{aligned}$$

Let f be the solution of the following ODE

$$\begin{cases} f''(t)-\dfrac{1}{m-1}F(t)f(t) = 0 \quad (0\le t\le r),\\ f(0)=0,\quad f(r)=1.\end{cases} \tag{2.4}$$

From [12] we know that (2.4) has a unique solution.

Using integration by parts and (2.4), we get

$$\int_0^r (f'(t))^2 dt = f'(r) - \frac{1}{m-1}\int_0^r F(t) f^2(t) dt. \tag{2.5}$$

Substituting (2.5) into (2.3), we obtain

$$\Delta_V r \leq (m-1) f'(r) + v(r).$$

As in [12, 23], we have

$$\Delta_V r(x) \leq \frac{m-1}{r} + \sqrt{(m-1)F(r)} + v(r).$$

∎

Based on the above estimate, we can obtain the following:

Theorem 1. *Let (M^m, g) be a complete Riemannian manifold, V a smooth vector field on M. Suppose that $\mathrm{Ric}_V \geq -F(r)$, where r is the distance function on M from some fixed point, $F : \mathbb{R} \to \mathbb{R}$ is a positive nondecreasing C^1-function and $\int_0^{+\infty} \frac{dr}{\sqrt{F(r)}} = +\infty$. Let V satisfy*

$$\langle V, \nabla r\rangle \leq v(r)$$

for some nondecreasing function $v(r)$ with $\lim_{r\to+\infty} \frac{v(r)}{\sqrt{F(r)}\left(\int_0^r \frac{dt}{\sqrt{F(t)}}+1\right)} = 0$.

Let $f \in C^2(M)$ with $\lim_{x\to\infty} \frac{f(x)}{\varphi(r(x))} = 0$, where $\varphi(t) = \log\left(\int_0^t \frac{ds}{\sqrt{F(s)}} + 1\right)$. Then there exists points $\{x_j\} \subset M$, such that

$$\begin{aligned} \lim_{j\to\infty} f(x_j) &= \sup f, \\ \lim_{j\to\infty} |\nabla f|(x_j) &= 0, \\ \lim_{j\to\infty} \Delta_V f(x_j) &\leq 0. \end{aligned}$$

Proof. Let $\{\varepsilon_j\}$ be a sequence of positive real numbers, such that $\varepsilon_j \to 0$ as $j \to \infty$. As in [27], define for any j

$$f_j(x) = f(x) - \varepsilon_j \varphi(r(x)).$$

Then by the condition on f, $f_j \to -\infty$ as $r \to +\infty$. Since the set $\{x \in M : r(x) \leq C\}$ is compact for any constant $C > 0$, so f_j has a lower bound, say A on it. Then there exists a constant $\tilde{C} > C$, such that $f_j(x) < A$ for $x \in \{x \in M : r(x) \geq \tilde{C}\}$, so f_j attains its maximum at some point $x_j \in \{x \in M : r(x) \leq \tilde{C}\}$. Hence we have

$$\nabla f_j(x_j) = 0 \quad \text{and} \quad \Delta_V f_j(x_j) \leq 0.$$

Since

$$\varphi'(r) = \left\{\sqrt{F(r)}\left(\int_0^r \frac{dt}{\sqrt{F(t)}} + 1\right)\right\}^{-1} > 0,$$

$$\varphi''(r) = -\left\{\sqrt{F(r)}\left(\int_0^r \frac{dt}{\sqrt{F(t)}} + 1\right)\right\}^{-2} \cdot \left\{\frac{F'(r)}{2\sqrt{F(r)}}\left(\int_0^r \frac{dt}{\sqrt{F(t)}} + 1\right) + 1\right\} \leq 0.$$

Thus by direct computation, we obtain

$$\begin{aligned}|\nabla f|(x_j) =&\varepsilon_j \varphi'(r(x_j))|\text{grad} r(x_j)| \le \varepsilon_j \left\{\sqrt{F(r(x_j))}\left(\int_0^{r(x_j)} \frac{dt}{\sqrt{F(t)}}+1\right)\right\}^{-1}\\ &\to\ 0 \ \text{ as } \ j\to\infty.\end{aligned}$$

By using the above V-Laplacian comparison theorem (Lemma 1), we have

$$\begin{aligned}\Delta_V f(x_j) \le&\varepsilon_j \left\{\varphi'(r(x_j))\Delta_V r(x_j) + \varphi''(r(x_j))|\text{grad} r(x_j)|^2\right\}\\ \le&\varepsilon_j \varphi'(r(x_j))\Delta_V r(x_j)\\ \le&\varepsilon_j \left\{\sqrt{F(r(x_j))}\left(\int_0^{r(x_j)} \frac{dt}{\sqrt{F(t)}}+1\right)\right\}^{-1}\cdot\left(\frac{m-1}{r(x_j)}+\sqrt{(m-1)F(r(x_j))+v(r(x_j))}\right)\\ \to&\ 0 \ \text{ as } \ j\to\infty.\end{aligned}$$

It remains to prove $\lim\limits_{j\to+\infty} f(x_j)=\sup f$. If there exists a subsequence $\{x_{j_k}\}\neq\{x_j\}$, such that $\lim\limits_{k\to+\infty} f(x_{j_k})=\sup f$, then by still denoting $\{x_{j_k}\}$ as x_j, our proof is completed. Otherwise, we claim that $\lim\limits_{j\to+\infty} f(x_j)=\sup f$ (If $\sup f=\infty$, then we claim that $\lim\limits_{j\to+\infty}\sup f(x_j)=\infty$). Indeed, if this were not true, there would exist $\hat{x}\in M$ and $\delta>0$, such that

$$f(\hat{x})>f(x_j)+\delta \tag{2.6}$$

for each $j\ge j_0$ sufficiently large.

Since

$$f(x_j)-\varepsilon_j\varphi(r(x_j))=f_j(x_j)\ge f_j(\hat{x})=f(\hat{x})-\varepsilon_j\varphi(r(\hat{x})), \tag{2.7}$$

we then have

$$f(x_j)\ge f(\hat{x})+\varepsilon_j(\varphi(r(x_j))-\varphi(r(\hat{x}))).$$

If $r(x_j)\to+\infty$ as $j\to+\infty$, then for j large enough, we have $\varphi(r(x_j))>\varphi(r(\hat{x}))$, that is $f(x_j)>f(\hat{x})$, which contradicts (2.6).

If $\{x_j\}$ lies in a compact set, then for some subsequence of j, x_j converges to a point $\bar{x}$, so that $f(\hat{x})\ge f(\bar{x})+\delta$. On the other hand, we can deduce from (2.7) that

$$f(\bar{x})\ge f(\hat{x})$$

This is also a contradiction. Thus we complete the proof. ■

Remark 1. *Choose $F(r)=c(r+2)^2(\log(r+2)+1)^2$, where c is a positive constant. Obviously, $F(r)$ is a positive nondecreasing C^1-function and satisfying $\int_0^{+\infty}\frac{dr}{\sqrt{F(r)}}=+\infty$.*

2.2. **Ricci solitons.** Note that for a complete gradient shrinking Ricci soliton (M^m, g_{ij}, f), by a suitable scaling of any shrinking soliton metric g_{ij}, it can be achieved that

$$R_{ij}+\nabla_i\nabla_j f=\frac{1}{2}g_{ij}. \tag{2.8}$$

Hamilton [22] showed that (2.8) implies

$$R+|\nabla f|^2-f=C_0. \tag{2.9}$$

If we normalize f by adding the constant C_0 to it, then (2.9) becomes

$$R + |\nabla f|^2 - f = 0. \tag{2.10}$$

Therefore we can derive the following theorem by applying Theorem 1:

Theorem 2. *Let (M^m, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton. Let $u \in C^2(M)$ with $\lim_{x\to\infty} \frac{u(x)}{\varphi(r(x))} = 0$, where r is the distance function on M from some fixed point and $\varphi(t) = \log(\log(\log(t+2)+1)+1)$. Then there exist points $\{x_j\} \subset M$, such that*

$$\begin{aligned}\lim_{j\to\infty} u(x_j) &= \sup u,\\ \lim_{j\to\infty} |\nabla u|(x_j) &= 0,\\ \lim_{j\to\infty} \Delta_f u(x_j) &\le 0.\end{aligned}$$

Proof. By (2.8) we have $Ric_{\nabla f} \ge \frac{1}{2} g$. From Corollary 2.5 in [8] (see also [5]), we know that the scalar curvature of M is nonnegative. Thus by (2.10) and Theorem 1.1 in [7], we get

$$|\nabla f| \le \frac{1}{2}(r + c).$$

Then the conclusion follows from Theorem 1.

■

Remark 2. *Fernandez-Lopez and Garcia Rio [20] showed that complete noncompact gradient shrinking Ricci solitons satisfy the Omori–Yau maximum principle for the f-Laplacian for $u \in C^2(M)$ with $\sup_M u < +\infty$. The condition on u in Theorem 2 is weaker than theirs.*

For weighted Riemannian manifolds, the Omori–Yau maximum principle for f-Laplacians was discussed in Theorem 6 [27], see also the references therein.

Recall that for a gradient steady Ricci soliton (M^m, g_{ij}, f), i.e., $\mathrm{Ric}_f \equiv 0$, there is a positive constant a such that [30]

$$|\nabla f|^2 + R = a^2, \qquad \Delta f + R = 0, \qquad R \ge 0.$$

Combining this with Theorem 1, we have

Theorem 3. *Let (M^m, g_{ij}, f) be a complete noncompact gradient steady Ricci soliton. Let $u \in C^2(M)$ with $\lim_{x\to\infty} \frac{u(x)}{\varphi(r(x))} = 0$, where r is the distance function on M from some fixed point and $\varphi(t) = \log(\log(\log(t+2)+1)+1)$. Then there exist points $\{x_j\} \subset M$, such that*

$$\begin{aligned}\lim_{j\to\infty} u(x_j) &= \sup u,\\ \lim_{j\to\infty} |\nabla u|(x_j) &= 0,\\ \lim_{j\to\infty} \Delta_f u(x_j) &\le 0.\end{aligned}$$

■

2.3. **Self-shrinkers.** For a self-shrinker M^m in $\mathbb{R}^{m+n}$, let $\{e_1, ..., e_m\}$ be a local orthonormal normal frame field on M at the considered point, then by the Gauss equation, we have

$$\mathrm{Ric}(e_i, e_i) = \sum_\alpha H^\alpha h^\alpha_{ii} - \sum_{\alpha,j}(h^\alpha_{ij})^2$$

and

$$\begin{aligned}(L_{-\frac{X^T}{2}} g)(e_i, e_i) =& -\frac{1}{2}\left(X^T\langle e_i, e_i\rangle - 2\langle L_{X^T} e_i, e_i\rangle\right) = \langle L_{X^T} e_i, e_i\rangle = \langle [X^T, e_i], e_i\rangle \\ =&\langle \overline{\nabla}_{X^T} e_i - \overline{\nabla}_{e_i} X^T, e_i\rangle = -\langle \overline{\nabla}_{e_i}(\langle X, e_j\rangle e_j), e_i\rangle \\ =& -\langle (e_i\langle X, e_j\rangle)e_j + \langle X, e_j\rangle \overline{\nabla}_{e_i} e_j, e_i\rangle = -(1 + \langle X, B(e_i, e_i)\rangle) \\ =& -(1 + \langle X^N, B_{ii}\rangle) = -(1 - 2\langle H, B_{ii}\rangle = -1 + 2\sum_\alpha H^\alpha h^\alpha_{ii}\end{aligned}$$

From the above two equalities it follows that

$$\begin{aligned}\mathrm{Ric}_{-\frac{X^T}{2}}(e_i, e_i) =& \mathrm{Ric}(e_i, e_i) - \frac{1}{2}(L_{-\frac{X^T}{2}} g)(e_i, e_i) \\ =& \sum_\alpha H^\alpha h^\alpha_{ii} - \sum_{\alpha,j}(h^\alpha_{ij})^2 + \frac{1}{2} - \sum_\alpha H^\alpha h^\alpha_{ii} \\ =& \frac{1}{2} - \sum_{\alpha,j}(h^\alpha_{ij})^2 \geq \frac{1}{2} - |B|^2.\end{aligned} \tag{2.11}$$

Theorem 4. *Let $X : M^m \to \mathbb{R}^{m+n}$ be a complete proper self-shrinker. Let $u \in C^2(M)$ with $\lim_{x\to\infty} \frac{u(x)}{\log(\sqrt{|X|^2+4}-1)} = 0$. Then there exist points $\{x_j\} \subset M$, such that*

$$\begin{aligned}\lim_{j\to\infty} u(x_j) =& \sup u, \\ \lim_{j\to\infty} |\nabla u|(x_j) =& 0, \\ \lim_{j\to\infty} \Delta_M u(x_j) \leq& 0.\end{aligned}$$

Proof. Let us choose $G(t) = t + 1$ and $\gamma = f = \frac{|X|^2}{4}$. Obviously $G(t)$ satisfies the condition of Theorem 4 in [27]. By direct computation, we have

$$f - |\nabla f|^2 = |H|^2, \tag{2.12}$$

$$\Delta_M f = \frac{n}{2} - (f - |\nabla f|^2). \tag{2.13}$$

Thus $|\nabla f| \leq \sqrt{f}$ and $\Delta_M f \leq \frac{n}{2} \leq \sqrt{f+1}$ off a compact set, together with the proper condition on X, imply that $(h1) - (h3)$ of Theorem 4 in [27] are satisfied. Hence according to Theorem 4 in [27], there exist points $\{x_j\} \subset M$, such that

$$|\nabla u|(x_j) < \frac{1}{j} \quad \text{and} \quad \Delta_M u(x_j) < \frac{1}{j}.$$

By a similar proof as of Theorem 1, we can obtain $\lim_{j\to\infty} u(x_j) = \sup u$. ■

Similarly, we can obtain the following maximum principle for the drift Laplacian operator $\mathcal{L}$ by using Theorem 1.9 in [32].

Theorem 5. *Let $X : M^m \to \mathbb{R}^{m+n}$ be a complete proper self-shrinker. Let $u \in C^2(M)$ be bounded from above on M. Then there exist points $\{x_j\} \subset M$, such that*

$$\lim_{j\to\infty} u(x_j) = \sup u,$$
$$\lim_{j\to\infty} |\nabla u|(x_j) = 0,$$
$$\lim_{j\to\infty} \mathcal{L}u(x_j) \le 0.$$

Proof. The function $G(t) = t^2 + 1$ satisfies the condition (1.17) of Theorem 1.9 in [32]. Let $\gamma = f = \frac{|X|^2}{4}$. Since X is proper, we then have

$$\gamma(x) \to \infty \quad \text{as} \quad x \to \infty$$

From (2.12) and (2.13), we can easily see that γ satisfies (1.15) and (1.16) of Theorem 1.9 in [32]. Therefore by the proof of Theorem 1.9 in [32], there exist points $\{x_j\} \subset M$, such that

$$\lim_{j\to\infty} u(x_j) = \sup u,$$

$$|\nabla u|(x_j) \le \frac{C}{j\sqrt{G(\gamma(x_j)^{\frac{1}{2}})}} = \frac{C}{j\sqrt{\gamma(x_j)+1}}, \tag{2.14}$$

$$\Delta_M u(x_j) \le \frac{C}{j}. \tag{2.15}$$

Thus from (2.14) and (2.15), we have

$$\mathcal{L}u(x_j) = \Delta_M u(x_j) - \langle \nabla f, \nabla u\rangle(x_j) \le \frac{C}{j} + \sqrt{\gamma(x_j)} \cdot \frac{C}{j\sqrt{\gamma(x_j)+1}}. \tag{2.16}$$

By (2.14) and (2.16), we can conclude that

$$\lim_{j\to\infty} |\nabla u|(x_j) = 0 \quad \text{and} \quad \lim_{j\to\infty} \mathcal{L}u(x_j) \le 0.$$

∎

In [20], the authors gave a characterization of shrinking Ricci solitons with constant scalar curvature. We can give the following characterization of self-shrinkers with constant mean curvature by using Theorem 4 or 5.

Theorem 6. *Let $X : M^n \to \mathbb{R}^{n+1}$ be a complete proper self-shrinker with nonzero mean curvature H. Then M has constant mean curvature if and only if*

$$|B|^2 \le \frac{1}{2} + C\frac{|\nabla H|^2}{H^2+1} \quad \text{for some constant } C > 0.$$

Proof. We may assume that $H > 0$. By direct computation, it is easy to get that (see [11])

$$\mathcal{L}H = (\frac{1}{2} - |B|^2)H.$$

Hence if H is a constant, then $|B|^2 = \frac{1}{2}$.

Conversely, assume that $|B|^2 \le \frac{1}{2} + C\frac{|\nabla H|^2}{H^2+1}$ for some constant $C > 0$. Let

$$f(x) = \left(\frac{1}{H(x)+1}\right)^{\frac{2}{3}}.$$

Obviously, f is a bounded function on M. By Theorem 4 or Theorem 5, there exists a sequence $\{x_j\} \subset M$, such that

$$\lim_{j\to\infty} f(x_j) = \inf_M f \quad \text{and} \quad \lim_{j\to\infty} |\nabla f|(x_j) = \lim_{j\to\infty} \frac{2|\nabla H|(x_j)}{3(H(x_j)+1)^{\frac{5}{3}}} = 0. \tag{2.17}$$

It follows that $\lim\limits_{j\to\infty} H(x_j) = \sup_M H$. Since $|B|^2 \geq \frac{H^2}{n}$, we have

$$H(x_j)^{\frac{2}{3}} \leq \frac{n}{H(x_j)^{\frac{4}{3}}}|B(x_j)|^2 \leq \frac{n}{2H(x_j)^{\frac{4}{3}}} + nC\frac{|\nabla H(x_j)|^2}{H(x_j)^{\frac{10}{3}}}.$$

Letting $j \to \infty$ in the above inequality, together with (2.17), we have $\sup_M H \leq \frac{n}{2}$.

On the other hand, we have

$$\begin{aligned} \mathcal{L}H^{C+1} =& \Delta H^{C+1} - \langle X, \nabla H^{C+1}\rangle \\ =& C(C+1)H^{C-1}|\nabla H|^2 + (C+1)H^C\mathcal{L}H \\ =& (C+1)H^{C-1}\left(C|\nabla H|^2 + H^2(\frac{1}{2} - |B|^2)\right). \end{aligned} \tag{2.18}$$

Since

$$|B|^2 \leq \frac{1}{2} + C\frac{|\nabla H|^2}{H^2+1} \leq \frac{1}{2} + C\frac{|\nabla H|^2}{H^2}.$$

Then (2.18) implies that $\mathcal{L}H^{C+1} \geq 0$.

Since M is proper, then by Theorem 4.1 in Cheng–Zhou [15], we have

$$\widehat{Vol}(B_r) = \int_{B_r} e^{-\frac{|X|^2}{4}} \leq c < +\infty.$$

It follows that

$$\int_{r_0}^{+\infty} \frac{r}{\widehat{Vol}(B_r)}dr \geq \int_{r_0}^{+\infty} \frac{r}{c}dr = +\infty.$$

This implies that M is $\frac{|X|^2}{4}$-parabolic (see [32] or [33]). So $H = \text{const}$. ■

Now, we focus on spacelike self-shrinkers in pseudo-Euclidean space $\mathbb{R}^{m+n}_n$.

Let $\mathbb{R}^{m+n}_n$ be an $(m+n)$-dimensional pseudo-Euclidean space with index n. The indefinite flat metric on $\mathbb{R}^{m+n}_n$ is defined by

$$ds^2 = \sum_{i=1}^m (dx^i)^2 - \sum_{\alpha=m+1}^{m+n}(dx^\alpha)^2.$$

Let $X : M \to \mathbb{R}^{m+n}_n$ be a spacelike m-submanifold in $\mathbb{R}^{m+n}_n$ with the second fundamental form B defined by

$$B_{UW} := (\overline{\nabla}_U W)^N$$

for $U, W \in \Gamma(TM)$. We use the notation $(\cdot)^T$ and $(\cdot)^N$ for the orthogonal projections into the tangent bundle TM and the normal bundle NM, respectively. For $\nu \in \Gamma(NM)$ we define the shape operator $A_\nu : TM \to TM$ by

$$A_\nu(U) := -(\overline{\nabla}_U \nu)^T$$

Taking the trace of B gives the mean curvature vector H of M in $\mathbb{R}^{m+n}_n$ and

$$H := \text{trace}(B) = \sum_{i=1}^{m} B_{e_ie_i},$$

where $\{e_i\}$ is a local orthonormal frame field of M. Actually, such a complete spacelike submanifold M is an entire graph (see [42]).

M^m is said to be a self-shrinker in $\mathbb{R}^{m+n}_n$ if it satisfies

$$H = -\frac{1}{2}X^N. \tag{2.19}$$

Here H is the mean curvature vector of M in $\mathbb{R}^{m+n}_n$, X^N is the normal part of X. And we define the drift operator $\mathcal{L}$ as in (1.4).

Recently, Liu–Xin [28] obtained a Bernstein theorem for spacelike self-shrinkers in pseudo-Euclidean space $\mathbb{R}^{m+n}_n$; they proved: *Let M be a space-like self-shrinker in pseudo-Euclidean space $\mathbb{R}^{m+n}_n$, which is closed with respect to the Euclidean topology. If the mean curvature of M satisfies $|H|^2 \le e^{\alpha z}$, where $\alpha < 1/8$ and $z := \|X\|^2$, then it must be an affine n-plane.*

Their method was by using integral estimates. We can also deal with this kind of problems by using an Omori–Yau maximum principle.

Theorem 7. *Let $X : M^m \to \mathbb{R}^{m+n}_n$ be a complete noncompact spacelike self-shrinker. Let r be the distance function on M from some fixed point, and $F : \mathbb{R} \to \mathbb{R}$ be a positive nondecreasing C^1-function and $\int_0^{+\infty} \frac{dr}{\sqrt{F(r)}} = +\infty$. Suppose X satisfies*

$$\langle X, \nabla r\rangle \ge -v(r) \tag{2.20}$$

for some nondecreasing function $v(r)$ satisfying $\lim_{r\to+\infty} \frac{v(r)}{\sqrt{F(r)}\left(\int_0^r \frac{dt}{\sqrt{F(t)}}+1\right)} = 0$. Then M has to be an affine m-plane.

Proof. From the self-shrinker equation, we derive

$$\nabla_{e_j} H = \frac{1}{2}\langle X, e_k\rangle B_{jk} \tag{2.21}$$

and

$$\nabla_{e_i}\nabla_{e_j} H = \frac{1}{2}B_{ij} - \langle H, B_{ik}\rangle B_{jk} + \frac{1}{2}\langle X, e_k\rangle \nabla_{e_i} B_{jk}.$$

Hence using the Codazzi equation, we obtain that

$$\begin{aligned}
\mathcal{L}|H|^2 =&\Delta|H|^2 - \frac{1}{2}\langle X, \nabla|H|^2\rangle\\
=&2\langle \nabla_{e_i}\nabla_{e_i} H, H\rangle + 2|\nabla H|^2 - \frac{1}{2}\langle X, \nabla|H|^2\rangle\\
=&|H|^2 - 2\langle H, B_{ik}\rangle^2 + \frac{1}{2}\nabla_{X^T}|H|^2 + 2|\nabla H|^2 - \frac{1}{2}\langle X, \nabla|H|^2\rangle\\
=&|H|^2 - 2\langle H, B_{ik}\rangle^2 + 2|\nabla H|^2.
\end{aligned}$$

It follows that

$$\mathcal{L}\|H\|^2 = \|H\|^2 + 2\langle H, B_{ik}\rangle^2 + 2\|\nabla H\|^2, \tag{2.22}$$

where $\|H\|^2$ is the absolute value of the norm square of the mean curvature of M. Thus direct computation gives us

$$\begin{aligned}\mathcal{L}(-\frac{1}{1+\|H\|^2}) =& \frac{\mathcal{L}\|H\|^2}{1+\|H\|^2} - 2\frac{|\nabla\|H\|^2|}{(1+\|H\|^2)^3} \\ =& \frac{\|H\|^2 + 2\langle H, B_{ik}\rangle^2 + 2\|\nabla H\|^2}{1+\|H\|^2} - 2\frac{|\nabla\|H\|^2|}{(1+\|H\|^2)^3}.\end{aligned} \tag{2.23}$$

Then applying Theorem 1 to $-\frac{1}{1+\|H\|^2}$, we can conclude that there exist points $\{x_j\} \subset M$, such that

$$\lim_{j\to\infty}\left(-\frac{1}{1+\|H\|^2}\right)(x_j) = \sup\left(-\frac{1}{1+\|H\|^2}\right),$$

$$\lim_{j\to\infty}\left|\nabla\left(-\frac{1}{1+\|H\|^2}\right)\right|(x_j) = 0,$$

$$\lim_{j\to\infty}\mathcal{L}\left(-\frac{1}{1+\|H\|^2}\right)(x_j) \le 0.$$

The above equalities imply that

$$0 \ge \lim_{j\to\infty}\mathcal{L}\left(-\frac{1}{1+\|H\|^2}\right)(x_j) \ge \frac{\sup_M \|H\|^2}{1+\sup_M \|H\|^2}.$$

This forces $H \equiv 0$. We then can conclude that $\|B\|^2 = 0$ (see (2.9) in [24]). Hence M is an m-plane. ■

By Remark 1, we can choose $F(r) = c(r+2)^2(\log(r+2)+1)^2$, then we obtain the following result.

Corollary 1. *Let $X : M^m \to \mathbb{R}^{m+n}_n$ be a complete noncompact spacelike self-shrinker. Let r be the distance function on M from some fixed point. If X^T satisfies*

$$|X^T| = O((r+2)\log(r+2)+1) \qquad \text{as} \quad r \to +\infty$$

then M has to be an affine m-plane.

Remark 3. *Actually, the Omori–Yau maximum principle holds under the condition of Corollary 1. More precisely:*
Let $X : M^m \to \mathbb{R}^{m+n}_n$ be a complete noncompact spacelike self-shrinker with $|X^T| = O((r+2)\log(r+2)+1)$ (as $r \to +\infty$), where r is the distance function on M from some fixed point. Let $u \in C^2(M)$ with $\lim\limits_{r\to+\infty}\frac{u(x)}{\varphi(r(x))} = 0$, here $\varphi(t) = \log(\log(\log(t+2)+1)+1)$. Then there exist points $\{x_j\} \subset M$, such that

$$\lim_{j\to+\infty} u(x_j) = \sup u, \qquad \lim_{j\to+\infty}|\nabla u|(x_j) = 0, \qquad \lim_{j\to+\infty}\mathcal{L}u(x_j) \le 0.$$

Remark 4. *Choose $F(r) = c(r+2)^2(\log(r+2)+1)^2$. If the metric of M has a positive lower bound, it is easy to see that the condition (2.20) is satisfied. Hence M must be an affine m-plane. This also shows that Proposition 20 in* [17] *holds.*

3. Liouville theorems for V-harmonic maps and geometric applications

In this section, we will use Liouville type theorems for V-harmonic maps to obtain Bernstein theorems for self-shrinkers.

For any $p \in M$, let $\{e_1, ..., e_m\}$ be a local frame field near p. Define the Gauss map $\gamma : p \to \gamma(p)$ which is obtained by parallel translation of T_pM to the origin in the ambient space $\mathbb{R}^{m+n}_n$. The image of the Gauss map lies in a pseudo-Grassmannian $G^m_{m,n}$-the totality of all the spacelike m-planes in $\mathbb{R}^{m+n}_n$. It is a specific Cartan–Hadamard manifold.

Theorem 8. *For an m-dimensional spacelike submanifold $X : M \to \mathbb{R}^{m+n}_n$, its Gauss map $\gamma : M \to G^n_{m,n}$ is a $-\frac{1}{2}X^T$-harmonic map iff $H + \frac{1}{2}X^N$ is a parallel vector field in the normal bundle NM.*

Proof. Let $\{e_1, ..., e_m\}$ be a local tangent orthonormal frame field on M and $\nu_1, ..., \nu_n$ be a local normal orthonormal frame field on M such that $\nabla e_i = 0$ and $\nabla \nu_\alpha = 0$ at the considered point. For any $P \in G^n_{m,n}$, there are m vectors $v_1, ..., v_m$ spanning P. Then we have Plücker coordinates $v_1 \wedge \cdots \wedge v_m$ for P up to a constant. The Gauss map γ can be described by $P \to e_1 \wedge \cdots \wedge e_m$, thus

$$d\gamma(e_i) = \overline{\nabla}_{e_i}(e_1 \wedge \cdots \wedge e_m) = \sum_j e_1 \wedge \cdots \wedge B_{e_ie_j} \wedge \cdots \wedge e_m = \sum_{j,\alpha} h^\alpha_{ij} e_{\alpha j}, \tag{3.1}$$

where $h^\alpha_{ij} = \langle B_{e_ie_j}, \nu_\alpha \rangle$ and $\{e_{\alpha i}\} = \{e_1 \wedge \cdots \wedge e_{i-1} \wedge e_\alpha \wedge e_{i+1} \wedge \cdots \wedge e_m\}$ is an orthonormal basis for $TG^n_{m,n}$. which means that the relation of the norm of the second fundamental form and the energy density of the Gauss map

$$e(\gamma) = \langle d\gamma(e_i), d\gamma(e_i) \rangle = \langle h^\alpha_{ij} e_{\alpha j}, h^\beta_{ik} e_{\beta k} \rangle = \|B\|^2. \tag{3.2}$$

Hence a direct calculation yields

$$\begin{aligned}
\tau_{-\frac{1}{2}X^T}(\gamma) =& \tau(\gamma) + d\gamma(-\frac{1}{2}X^T) = (\nabla_{e_i} d\gamma)(e_i) + d\gamma(-\frac{1}{2}X^T) \\
=& \nabla_{e_i}(d\gamma(e_i)) - \frac{1}{2}\langle X, e_i \rangle d\gamma(e_i) = \nabla_{e_i}(h^\alpha_{ij} e_{\alpha j}) - \frac{1}{2}\langle X, e_i \rangle h^\alpha_{ij} e_{\alpha j} \\
=& (\nabla_{e_i} h^\alpha_{ij}) e_{\alpha j} - \frac{1}{2}\langle \langle X, e_i \rangle e_i, A_{e_\alpha}(e_j) \rangle e_{\alpha j} \\
=& (\nabla_{e_j} h^\alpha_{ii}) e_{\alpha j} - \frac{1}{2}\langle X, A_{e_\alpha}(e_j) \rangle e_{\alpha j} \\
=& (\nabla_{e_j} H^\alpha) e_{\alpha j} + \frac{1}{2}\langle X, \overline{\nabla}_{e_j} e_\alpha \rangle e_{\alpha j} \\
=& (\nabla_{e_j} H^\alpha) e_{\alpha j} + \frac{1}{2}\nabla_{e_j}\langle X, e_\alpha \rangle e_{\alpha j} \\
=& (\nabla_{e_j}(H^\alpha + \frac{X^\alpha}{2})) e_{\alpha j},
\end{aligned} \tag{3.3}$$

where we have used the Codazzi equation. Hence we obtain

$$\tau_{-\frac{1}{2}X^T}(\gamma) = 0 \quad \Leftrightarrow \quad \nabla(H + \frac{1}{2}X^N) = 0.$$

∎

Corollary 2. *If M^m is a self-shrinker in $\mathbb{R}^{m+n}_n$, then its Gauss map $\gamma : M \to G^n_{m,n}$ is a $-\frac{1}{2}X^T$-harmonic map.*

By a similar method, we can also get the following (see also [18])

Corollary 3. *If M^m is a self-shrinker in $\mathbb{R}^{m+n}$, then its Gauss map $\gamma : M \to G_{m,n}$ is a $-\frac{1}{2}X^T$-harmonic map.*

In view of these facts, Liouville theorems for V-harmonic maps can be applied to self-shrinkers in Euclidean space and pseudo-Euclidean space.

We recall the following Liouville theorem for V-harmonic maps:

Theorem A *(1) (c.f. [9], Theorem 2.) Let (M, g) be a complete noncompact Riemannian manifold with*

$$\mathrm{Ric_V} := \mathrm{Ric}^M - \frac{1}{2}L_V g \geq -A,$$

where $A \geq 0$ is a constant, Ric^M is the Ricci curvature of M and L_V is the Lie derivative. Let (X, h) be a complete Riemannian manifold with sectional curvature bounded above by a positive constant κ. Let $u : M \to X$ be a V-harmonic map such that $u(M) \subset B_R(p)$, where $B_R(p)$ is a regular ball in X, i.e., disjoint from the cut-locus of p and $R < \frac{\pi}{2\sqrt{\kappa}}$. If V satisfies

$$\langle V, \nabla r\rangle \leq v(r) \tag{3.4}$$

for some nondecreasing function $v(\cdot)$ satisfying $\lim_{r\to+\infty} \frac{|v(r)|}{r} = 0$, where r denotes the distance function on M from a fixed point $\tilde{p} \in M$, then $e(u)$ is bounded by a constant depending only on A, κ and R. Furthermore, if $A = 0$, namely,

$$\mathrm{Ric}^M \geq \frac{1}{2}L_V g,$$

then u must be a constant map.

(2) (c.f. [39], [36]) Let M^m be a complete noncompact manifold with $\mathrm{Ric}_f \geq 0$ and N be a complete Riemannian manifold with nonpositive sectional curvature. If $u : M \to N$ is a f-harmonic maps with finite weighted energy, then $e(u)$ must be constant.

Using this we can obtain the following Bernstein theorems for self-shrinkers:

Theorem 9. *Let $X : M^m \to \mathbb{R}^{m+n}_n$ be a complete noncompact spacelike self-shrinker with infinite volume. If $\int_M \|B\|^2 < \infty$, then M must be a plane. Here $\|B\|$ is the absolute value of the squared norm of the second fundamental form of M.*

Proof. Computing as in (2.11), we have

$$\begin{aligned}\mathrm{Ric}_{-\frac{X^T}{2}}(e_i, e_i) =& \mathrm{Ric}(e_i, e_i) - \frac{1}{2}(L_{-\frac{X^T}{2}}g)(e_i, e_i)\\ =& -\sum_\alpha H^\alpha h^\alpha_{ii} + \sum_{\alpha,j}(h^\alpha_{ij})^2 + \frac{1}{2} + \sum_\alpha H^\alpha h^\alpha_{ii}\\ =& \frac{1}{2} + \sum_{\alpha,j}(h^\alpha_{ij})^2 \geq \frac{1}{2},\end{aligned} \tag{3.5}$$

where $\{e_i\}$ is a local orthonormal normal frame field of M^m.

From Corollary 2, we know that the Gauss map $\gamma : M^m \to G^n_{m,n}$ is an f-harmonic map with $f = \frac{|X|^2}{4}$. By (3.5) and Theorem A (2) we can conclude that the energy density $e(\gamma)$ of the

Gauss map γ must be constant. Then (3.2), the assumptions $\int_M \|B\|^2 < \infty$ and that M has infinite volume imply $\|B\| = 0$, i.e., M must be a plane. ∎

Theorem 10. *Let $X : M^m \to \mathbb{R}^{m+n}_n$ be a complete noncompact spacelike self-shrinker. If the image under the Gauss map lies in a regular ball $B_R(p) \subset G^n_{m,n}$ and X satisfies*

$$\langle X, \nabla r\rangle \geq -v(r)$$

for some nondecreasing function $v(\cdot)$ satisfying $\lim_{r\to+\infty} \frac{|v(r)|}{r} = 0$, where r denotes the distance function on M from a fixed point $p \in M$. Then M has to be a plane.

Proof. By (3.5), Theorem A (1) and Corollary 2, we conclude that the Gauss map $\gamma : M^m \to G^n_{m,n}$ must be constant. Therefore M is a plane.

∎

Let N be a complete Riemannian manifold of nonpositive sectional curvature and $c : \mathbb{R} \to N$ be a geodesic with the arc length parameter. The union of open balls

$$B_c := \bigcup_{t>0} B_t(c(t))$$

is called the horoball with center $c(+\infty)$, where $B_c(c(t))$ is an open geodesic ball of radius t and centered at $c(t)$. Let ρ be the distance function on N. For each $x \in N$, the function $t \mapsto t - \rho(x, c(t))$ is bounded from above and monotonically increasing. Therefore, the function

$$B(x) := \lim_{t\to+\infty} (t - \rho(x, c(t)))$$

is well defined on N and is called the Busemann function for the geodesic c. It is easily seen that $B(x) > 0$ on B_c. We know that B is a C^2-function with $|\nabla B| = 1$ (see [4] [37] [43]). We may also assume that $B \geq 1$ since Busemann functions are horo-functions.

Let $B^n(x) := n - \rho(x, c(n))$, $n \in \mathbb{Z}^+$. We know that $B^n(x)$ converges to $B(x)$ uniformly up to second derivatives on any compact subset when $n \to \infty$.

Theorem 11. *Let (M^m, g) be a complete noncompact Riemannian manifold with*

$$\mathrm{Ric}_V := \mathrm{Ric}^M - \frac{1}{2} L_V g \geq -A,$$

where $A \geq 0$ is a constant, Ric^M is the Ricci curvature of M and L_V is the Lie derivative. Let (N^n, h) be a complete Riemannian manifold with sectional curvature bounded above by a negative constant $-\kappa^2 (\kappa > 0)$. Let $u : M \to N$ be a V-harmonic map such that $u(M) \subset B_c$, where B_c is a horoball centered at $c(+\infty)$ with respect to a geodesic $c(t)$ with the arc length parameter. Suppose that $\|V\|_{L^\infty(M)} < +\infty$. Then

$$\frac{e(u)}{(B\circ u)^2} \leq C(A + \frac{\|V\|^2_{L^\infty}}{m-1}), \tag{3.6}$$

where $C > 0$ is a constant depending only on m, κ.

Proof. Multiplying the metric tensor by a suitable constant we may assume the upper bound of the sectional curvature of N to be -1.

Let B be the Busemann function on B_c and $B^n(p) = n - \rho(p, c(n))$, where ρ is the distance function from $c(n)$ in N. By the Hessian comparison theorem, we have

$$\mathrm{Hess}(B^n) = -\mathrm{Hess}(\rho(p, c(n))) \leq -(\coth \rho)(h - d\rho \otimes d\rho)$$

and

$$\begin{aligned}\Delta_V(B^n\circ u) =&\mathrm{Hess}(B^n)(du(e_i),du(e_i))+d\rho(\tau_V(u))\\ \le& -(\coth\rho)(h-d\rho\otimes d\rho)(du(e_i),du(e_i)).\end{aligned}$$

Let the $\{i_\alpha\}$ constitute that subset of the i, with complementary subset $\{i_s\}$, for which the $du(e_{i_\alpha})$ are parallel to $\nabla\rho$ and the $du(e_{i_s})$ are perpendicular to $\nabla\rho$. Then we obtain that

$$\Delta_V(B^n\circ u)\le -(\coth\rho)h(du(e_{i_s}),du(e_{i_s}))\le -h(du(e_{i_s}),du(e_{i_s})) \tag{3.7}$$

and

$$\begin{aligned}h(du(e_{i_\alpha}),du(e_{i_\alpha})) =&h(du(e_{i_\alpha}),\nabla\rho)h(du(e_{i_\alpha}),\nabla\rho)\\ =&h(\nabla(\rho\circ u),e_{i_\alpha})h(\nabla(\rho\circ u),e_{i_\alpha})\\ \le&|\nabla(B^n\circ u)|^2.\end{aligned}\tag{3.8}$$

From (3.7) and (3.8), we have that for any $\varepsilon\in(0,1)$

$$\begin{aligned}\varepsilon\frac{|\nabla(B^n\circ u)|^2}{(B^n\circ u)^2}-\frac{\Delta_V(B^n\circ u)}{B^n\circ u}\ge&\frac{\varepsilon}{(B^n\circ u)^2}\left(|\nabla(B^n\circ u)|^2-\Delta_V(B^n\circ u)\right)\\ \ge&\varepsilon\frac{|\nabla u|^2}{(B^n\circ u)^2}.\end{aligned}$$

Since $B^n(x)$ converges to $B(x)$ uniformly up to second derivatives on any compact subset when $n\to\infty$, then the above inequality implies that

$$\varepsilon\frac{|\nabla(B\circ u)|^2}{(B\circ u)^2}-\frac{\Delta_V(B\circ u)}{B\circ u}\ge\varepsilon\frac{|\nabla u|^2}{(B\circ u)^2}.\tag{3.9}$$

On the other hand, the Bochner type formula in [9] gives us that

$$\frac{1}{2}\Delta_V e(u)\ge|\nabla du|^2-Ae(u).\tag{3.10}$$

Let e_i be an orthonormal basis of M, e^i be the dual basis of e_i, and let f_α be an orthonormal basis of N. Let $du=u_i^\alpha e^i\otimes f_\alpha$. We choose e_1 such that $e_1=\frac{\nabla|du|}{|\nabla|du||}$ at the considered point. Then we have

$$\begin{aligned}|\nabla du|^2=&\sum_{\alpha,i,j}(u_{ij}^\alpha)^2\ge\sum_\alpha(u_{11}^\alpha)^2+2\sum_{\alpha,j\ge2}(u_{1j}^\alpha)^2+\sum_{\alpha,j\ge2}(u_{jj}^\alpha)^2\\ \ge&\sum_\alpha(u_{11}^\alpha)^2+2\sum_{\alpha,j\ge2}(u_{1j}^\alpha)^2+\frac{1}{m-1}\sum_\alpha\left(\sum_{j\ge2}u_{jj}^\alpha\right)^2\\ =&\sum_\alpha(u_{11}^\alpha)^2+2\sum_{\alpha,j\ge2}(u_{1j}^\alpha)^2+\frac{1}{m-1}\sum_\alpha(\tau(u)^\alpha-u_{11}^\alpha)^2\\ \ge&\frac{1}{m-1}|\tau(u)|^2-\frac{2}{m-1}\sum_\alpha\tau(u)^\alpha u_{11}^\alpha+\frac{m}{m-1}\sum_{\alpha,j}(u_{1j}^\alpha)^2.\end{aligned}\tag{3.11}$$

Then the Cauchy–Schwartz inequality implies that for any $\epsilon>0$

$$|\nabla du|^2\ge\frac{1-\epsilon}{m-1}|\tau(u)|^2+\left(\frac{m}{m-1}-\frac{1}{(m-1)\epsilon}\right)\sum_{\alpha,j}(u_{1j}^\alpha)^2.\tag{3.12}$$

It is easy to check that $\sum_{\alpha,j}(u^\alpha_{1j})^2 \geq |\nabla\sqrt{e(u)}|^2$. Letting $\epsilon = 2$ in (3.12), we obtain

$$|\nabla du|^2 \geq \left(1+\frac{1}{2(m-1)}\right)|\nabla\sqrt{e(u)}|^2 - \frac{1}{m-1}|\tau(u)|^2.$$

Since u is V-harmonic, we have $\tau(u)+\langle V,\nabla u\rangle = \tau_V(u) = 0$. Hence the above inequality implies that

$$|\nabla du|^2 \geq \left(1+\frac{1}{2(m-1)}\right)|\nabla\sqrt{e(u)}|^2 - \frac{\|V\|^2_{L^\infty}}{m-1}e(u). \tag{3.13}$$

Let $\varepsilon = \frac{1}{2(m-1)}$, from (3.10) and (3.13), we get

$$\frac{1}{2}\Delta_V e(u) \geq (1+\varepsilon)|\nabla\sqrt{e(u)}|^2 - \left(A+\frac{\|V\|^2_{L^\infty}}{m-1}\right)e(u).$$

Namely,

$$|\nabla u|\Delta_V|\nabla u| \geq \varepsilon|\nabla|\nabla u||^2 - \tilde{A}|\nabla u|^2, \tag{3.14}$$

where $\tilde{A} = A + \frac{\|V\|^2_{L^\infty}}{m-1}$. Let $\psi = \frac{|\nabla u|}{B\circ u}$, then

$$\nabla\psi = \frac{\nabla|\nabla u|}{B\circ u} - \frac{|\nabla u|\cdot\nabla(B\circ u)}{(B\circ u)^2},$$

$$\begin{aligned}\Delta_V\psi =& \frac{\Delta_V|\nabla u|}{B\circ u} - \frac{2\langle\nabla(B\circ u),\nabla\psi\rangle}{B\circ u} - \frac{\psi\Delta_V(B\circ u)}{B\circ u}\\ \geq& \varepsilon\frac{|\nabla|\nabla u||^2}{(B\circ u)|\nabla u|} - \tilde{A}\frac{|\nabla u|}{B\circ u} - \frac{2\langle\nabla(B\circ u),\nabla\psi\rangle}{B\circ u} - \frac{\psi\Delta_V(B\circ u)}{B\circ u}.\end{aligned}$$

Since

$$\begin{aligned}-2\frac{\langle\nabla(B\circ u),\nabla\psi\rangle}{B\circ u} =& -(2-2\varepsilon)\frac{\langle\nabla(B\circ u),\nabla\psi\rangle}{B\circ u} - 2\varepsilon\frac{\langle\nabla(B\circ u),\nabla\psi\rangle}{B\circ u}\\ =& -(2-2\varepsilon)\frac{\langle\nabla(B\circ u),\nabla\psi\rangle}{B\circ u} - 2\varepsilon\frac{\langle\nabla(B\circ u),\nabla|\nabla u|\rangle}{(B\circ u)^2} + 2\varepsilon\frac{|\nabla(B\circ u)|^2|\nabla u|}{(B\circ u)^3}\\ \geq& -(2-2\varepsilon)\frac{\langle\nabla(B\circ u),\nabla\psi\rangle}{B\circ u} - \varepsilon\frac{|\nabla|\nabla u||^2}{(B\circ u)|\nabla u|} + \varepsilon\frac{|\nabla(B\circ u)|^2|\nabla u|}{(B\circ u)^3},\end{aligned}$$

we conclude that

$$\Delta_V\psi \geq -\tilde{A}\psi - \frac{\psi\Delta_V(B\circ u)}{B\circ u} + \varepsilon\frac{|\nabla(B\circ u)|^2|\nabla u|}{(B\circ u)^3} - (2-2\varepsilon)\frac{\langle\nabla(B\circ u),\nabla\psi\rangle}{B\circ u}. \tag{3.15}$$

Let r be the distance function on M from a fixed point $\tilde{p}$ and $B_a(\tilde{p})$ be a geodesic ball of radius around $\tilde{p}$. Define $f : B_a(\tilde{p}) \to \mathbb{R}$ by

$$f := (a^2-r^2)\psi(x) = (a^2-r^2)\frac{|\nabla u|}{B\circ u}.$$

Since $f|_{\partial B_a(\tilde{p})} = 0$, f achieves an absolute maximum in the interior of $B_a(\tilde{p})$, say $f \leq f(q)$, for some q inside $B_a(\tilde{p})$. By using the technique of support function we may assume that f is smooth near q. We may also assume $\nabla u(q) \neq 0$. Then from

$$\begin{aligned}\nabla f(q) =& 0,\\ \Delta_V f(q) \leq& 0.\end{aligned}$$

we obtain the following at the point q:

$$\frac{\nabla r^2}{a^2-r^2}=\frac{\nabla\psi}{\psi},$$

$$-\frac{\Delta_V r^2}{a^2-r^2}+\frac{\Delta_V\psi}{\psi}-\frac{2\langle\nabla r^2,\nabla\psi\rangle}{(a^2-r^2)\psi}\le 0.$$

From the above two inequalities we can easily seen that

$$\frac{\Delta_V\psi}{\psi}-\frac{\Delta_V r^2}{a^2-r^2}-\frac{2|\nabla r^2|^2}{(a^2-r^2)^2}\le 0.$$

By the V-Laplacian comparison theorem (Lemma 1), we have

$$\Delta_V r^2 = 2|\nabla r|^2+2r\Delta_V r\le c_1+c_2 r+2r\|V\|_{L^\infty},$$

where c_1 depends only on m, c_2 depends on m and A. It follows that

$$\begin{aligned}-\tilde{A}-\frac{\Delta_V(B\circ u)}{B\circ u}+\varepsilon\frac{|\nabla(B\circ u)|^2}{(B\circ u)^2}-2(2-2\varepsilon)r\frac{\langle\nabla(B\circ u),\nabla r\rangle}{(a^2-r^2)(B\circ u)}\\ -\frac{c_1+c_2r+2r\|V\|_{L^\infty}}{a^2-r^2}-\frac{8r^2}{(a^2-r^2)^2}\le 0.\end{aligned}\tag{3.16}$$

By the Schwartz inequality and noting that $|\nabla B|=1$, we get

$$\left|\frac{\langle\nabla(B\circ u),\nabla r\rangle}{(a^2-r^2)(B\circ u)}\right|\le\frac{|\nabla(B\circ u)|\cdot|\nabla r|}{(a^2-r^2)(B\circ u)}\le\frac{|\nabla u|}{(a^2-r^2)(B\circ u)}.\tag{3.17}$$

The inequalities (3.9), (3.16) and (3.17) imply that

$$\varepsilon\frac{|\nabla u|^2}{(B\circ u)^2}-\frac{2(2-2\varepsilon)r}{a^2-r^2}\frac{|\nabla u|}{B\circ u}-\frac{c_1+c_2r+2r\|V\|_{L^\infty}}{a^2-r^2}-\frac{8r^2}{(a^2-r^2)^2}-\tilde{A}\le 0.\tag{3.18}$$

Hence

$$\frac{|\nabla u|}{B\circ u}\le\max\left\{\frac{4(2-2\varepsilon)r}{\varepsilon(a^2-r^2)},\frac{2}{\sqrt{\varepsilon}}\sqrt{\frac{c_1+c_2r+2r\|V\|_{L^\infty}}{a^2-r^2}+\frac{8r^2}{(a^2-r^2)^2}+\tilde{A}}\right\},$$

From this we can derive the upper bound of f, and it is easy to conclude that at every point of $B_{\frac{a}{2}}(\tilde{p})$, we have

$$\frac{|\nabla u|}{B\circ u}\le C\left(\frac{1}{a}+\sqrt{\frac{1+\|V\|_{L^\infty}}{a}+\frac{1}{a^2}+\tilde{A}}\right).$$

Here C is a positive constant depending only on m and κ.

For any fixed $x\in M$, letting $a\to\infty$ in the above inequality, we obtain $\frac{|\nabla u|}{B\circ u}\le C\sqrt{A+\frac{\|V\|^2_{L^\infty}}{m-1}}$. ■

Furthermore, by a similar proof, one can show that when Ric_V is bounded below by a positive constant, we have the following:

Theorem 12. *Let (M^m,g) be a complete noncompact Riemannian manifold with*

$$\mathrm{Ric}_V:=\mathrm{Ric}^M-\frac{1}{2}L_Vg\ge A,$$

where $A\ge 0$ is a constant, Ric^M is the Ricci curvature of M and L_V is the Lie derivative. Let (N^n,h) be a complete Riemannian manifold with sectional curvature bounded above by a *negative constant* $-\kappa^2(\kappa > 0)$*. Let* $u : M \to N$ *be a* V*-harmonic map such that* $u(M) \subset B_c$*, where* B_c *is a horoball centered at* $c(+\infty)$ *with respect to a geodesic* $c(t)$ *parametrized by arc length. Suppose that* $\|V\|_{L^\infty(M)} < +\infty$*. If* $A \geq \frac{\|V\|^2_{L^\infty}}{m-1}$*, then* u *must be a constant map. If* $A < \frac{\|V\|^2_{L^\infty}}{m-1}$*, then* $\frac{e(u)}{(B\circ u)^2}$ *is bounded by a constant depending only on* A, m, κ *and* $\|V\|_{L^\infty(M)}$*.*

■

By (3.5) and Theorem 12, one can immediately obtain the following:

Corollary 4. *Let* $X : M^m \to \mathbb{R}^{m+1}_1$ *be a complete noncompact spacelike self-shrinker. If the image under the Gauss map* $\gamma : M \to \mathbb{H}^m(-1)$ *lies in a horoball in* $\mathbb{H}^m(-1)$*, then* $\|X^T\|_{L^\infty(M)} > m-1$*.*

Proof. We assume that $\|X^T\|_{L^\infty(M)} \leq m-1$, then by (3.5) and Theorem 12, we have $e(u) = 0$, which together with (3.2), implies that M is a hyperplane. On the other hand, it follows that M must be compact by Theorem 1 in [19]. This is a contradiction. ■

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